SEXTIC CURVES WITH SIMPLE SINGULARITIES

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Abstract. Nikulin's lattice embedding theory is used to determine the structure of reduced sextic curves.

1. Introduction. One of the classical problems for plane algebraic curves over the complex number field is to determine all possible configurations for a given degree d. When $d \leq 3$ the answers are trivial. When d increases the problem becomes more complicated. The classical method can be applied to the cases $d = 4, 5$ to determine the combinations of singularities of irreducible curves of degree d (cf. [1]). In [6], [7] and [8] Urabe studied the combination of simple singularities of reduced sextic curves using the surjectivity of the period map for $K3$ surfaces. However the complete solution was not reached.

In this paper, we try to work out all configurations of reduced sextic curves. As the first step we adopt the method in [6] to determine all possible combinations of simple singularities. Here we use the full version of Nikulin's embedding theory of even lattices to obtain a computer-aided solution of the problem.

The second step is to determine the configurations of curves for each combination of singularities obtained in the first step. This requires further analysis of the Picard groups of the corresponding double sextics (double coverings of the plane branched along sextic curves). The idea is that every irreducible component of a reduced sextic curve determines an element in the lattice satisfying certain numerical conditions. It turns out that in turn these numerical conditions are sufficient to determine the irreducible component (cf. Theorem 3.16). Basing ourselves on this we present a practical method to achieve our purpose. In particular, we completely worked out the most interesting case of maximizing sextics.

We also give a proof that there exist smooth sextic curves whose corresponding double sextics are singular $K3$ surfaces with arbitrarily large discriminants.

Throughout this paper we work in the category of complex analytic varieties.

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tions, especially for pointing out an essential gap in the first version of our manuscript. We are indebted to Professor C. T. C. Wall for his helpful comments on the figures of sextic curves.

2. Combinations of singularities. A singularity p of a reduced curve C on a smooth surface S is called a *simple singularity* if p is a double point of C or p is a triple point of C but there is no infinitely near triple point of C over p. All Dynkin graphs in this paper are restricted to those of types A_i $(i \geq 1)$, D_j $(j \geq 4)$ and E_k $(k = 6, 7, 8)$. They classify simple singularities for curves and rational double points for surfaces.

A *subgraph* of a Dynkin graph G (not necessarily connected) is a graph which can be obtained from G by deleting some vertices and the edges issuing from them.

Let C be a reduced sextic curve with at most simple singularities. For each non-negative integer k and each integer $l \geq 4$ let a_k and b_l denote the numbers of simple singularities of type A_k and type D_l on C, respectively. For $m = 6, 7, 8$ let c_m denote the number of simple singularities of type E_m on C. Then we say that the combination of singularities on C corresponds to the Dynkin graph $G = \sum a_k A_k + \sum b_l D_l + \sum c_m E_m$. The number $r = r(G) = \sum a_k k + \sum b_l l + \sum c_m m$ is called the *rank* of G. Our main result in this section is the following theorem.

THEOREM 2.1. *Let* $G = \sum a_k A_k + \sum b_l D_l + \sum c_m E_m$ *be a finite Dynkin graph. The following conditions are equivalent.*

1. *There exists a reduced sextic curve in* P^2 *whose singularities are all simple and the combination of singularities corresponds to* G.
2. *One of the following holds:*
    (a) *The Dynkin graph* G *is a subgraph of a graph in Table 2 at the end of the paper.*
    (b) *The Dynkin graph* G *is one of the following*

$$3D_5 + A_3, \ 6A_3, \ 5A_3 + 3A_1, \ 4A_3 + 6A_1, \ 9A_2 .$$

There are 519 *(resp.* 987, 975, 782*) Dynkin graphs of rank* 19 *(*18, 17, 16*) satisfying the above conditions.*

REMARK 2.2. For rank less than 16 the complete list has already been obtained by Urabe [6]. Some Dynkin graphs for higher ranks are described in [7] and [8].

The main theorem is obtained by a complete search for solutions. Most calculations were done by computer. A similar algorithm has been briefly explained in [9, §3]. In the remaining of this section we explain the main points of the approach.

It is well-known that double sextics are $K3$ surfaces. Using the surjectivity of the period map for $K3$ surfaces, Urabe [6] proved the following theorem.

THEOREM 2.3 (cf. [6]). *Let* G *be as in Theorem* 2.1 *with arbitrary rank* $r(G)$. *Then the following conditions are equivalent.*

1. *There exists a reduced sextic curve in $\mathbf{P}^2$ whose singularities are all simple and the combination of singularities corresponds to G.*
2. *Let $Q = Q(G)$ be the negative definite root lattice of type G. Let Λ denote the unimodular even lattice with signature $(3, 19)$. Let $\mathbf{Z}\lambda$ be a free lattice of rank one with $\lambda^2 = 2$. The orthogonal sum $S = \mathbf{Z}\lambda \oplus Q$ has an embedding $S \subset \Lambda$ satisfying the following two conditions:*
   (a) *If $\eta \in \Lambda$, $\eta \cdot \lambda = 0$, $\eta^2 = -2$ and $m\eta \in S$ for some positive integer m, then $\eta \in Q$.*
   (b) *If $\eta \in \Lambda$, $\eta \cdot \lambda = 1$ and $\eta^2 = 0$, then $m\eta \notin S$ for every positive integer m.*

REMARK 2.4.   The primitive hull of S in Λ is defined to be $\tilde{S} = \{\alpha \in \Lambda \mid m\alpha \in S$ for some non-zero integer $m\}$. For every G satisfying the conditions in the theorem, the corresponding sextic curve can be so chosen that the Picard group of the double sextic (a $K3$ surface) is isomorphic to $\tilde{S}$.

Let us fix some notions in lattice theory (refer to [4] for basic definitions). Let S be a sublattice of a lattice S'. If S'/S is a finite group, then S' is called an *overlattice* of S.

Let S be an even lattice. Let $S^* = \text{Hon}(S, \mathbf{Z})$ be the dual of S. Then S^*/S is a finite abelian group. The bilinear form and the quadratic form extend canonically to a $\mathbf{Q}$-valued bilinear form and a $\mathbf{Q}$-valued quadratic form on S^*. They induce a finite bilinear form $b: S^*/S \times S^*/S \to \mathbf{Q}/\mathbf{Z}$ and a finite quadratic form $q: S^*/S \to \mathbf{Q}/2\mathbf{Z}$ satisfying the conditions:

1. $q(na) = n^2 q(a)$ for all $n \in \mathbf{Z}$ and $a \in S^*/S$.
2. $q(a + a') - q(a) - q(a') \equiv 2b(a, a') \pmod{2\mathbf{Z}}$ for all $a, a' \in S^*/S$.

The form q is called the *discriminantal form* of the lattice S.

For a finite abelian group A, let $l(A)$ denote the minimal number of generators of A and let $|A|$ denote the order of A. For a prime number p, let A_p denote the p-Sylow subgroup of A. For an even lattice S let q_p denote the restriction of the discriminantal form q of S to $(S^*/S)_p$.

The ring of p-adic integers is denoted by $\mathbf{Z}_p$. A p-adic lattice is a free $\mathbf{Z}_p$-module of finite rank equipped with a symmetric nondegenerate bilinear form with values in $\mathbf{Z}_p$. Its discriminantal form is defined in a similar manner. For every finite quadratic form q on a finite abelian p-group A, there exists a p-adic lattice of rank $l(A)$ whose discriminantal form is isomorphic to q. Suppose T is a p-adic lattice with this property. The isomorphism class of T is uniquely determined by q. Choose an arbitrary basis of T. Then a matrix is determined by applying the bilinear form on this basis. The determinant of this matrix is denoted by $\text{discr}(T)$, which does not depend upon the choice of the basis up to a factor in $(\mathbf{Z}_p^*)^2$. We define $\text{discr}(q) := \text{discr}(T)$ up to a factor in $(\mathbf{Z}_p^*)^2$.

The following theorem is a key result in Nikulin's theory of embeddings of even lattices.

THEOREM 2.5 (cf. [2, 1.12.2]). *The following conditions are equivalent*:

1. *There exists a primitive embedding of an even lattice with signature (t_+, t_-) and the discriminantal form q into some even unimodular lattice of signature (l_+, l_-).*

2. *There exists an even lattice with signature $(l_+ - t_+, l_- - t_-)$ and the discriminantal form $-q$.*

3. *There exists an even lattice with signature $(l_- - t_-, l_+ - t_+)$ and the discriminantal form q.*

4. *There exists an even lattice S with signature (t_+, t_-) and the discriminantal form q such that the following conditions are simultaneously satisfied*:

    (a) $l_+ - l_- \equiv 0 \pmod 8$.

    (b) $l_- - t_- \geq 0$, $l_+ - t_+ \geq 0$, $l_+ + l_- - t_+ - t_- \geq l(S^*/S)$.

    (c) $(-1)^{(l_+ - t_+)} |S^*/S| \equiv \mathrm{discr}(q_p) \pmod{\mathbf{Z}_p^{*2}}$ *for all odd prime numbers p for which $l_+ + l_- - t_+ - t_- = l((S^*/S)_p)$.*

    (d) $|S^*/S| \equiv \pm \mathrm{discr}(q_2) \pmod{\mathbf{Z}_2^{*2}}$ *if $l_+ + l_- - t_+ - t_- = l((S^*/S)_2)$ and there do not exist 2-adic lattices T_1 and T_2 such that $T_1^*/T_1 \cong \mathbf{Z}/2\mathbf{Z}$ and the discriminantal form of $T_1 \oplus T_2$ is isomorphic to q_2.*

Suppose that a Dynkin graph $G = \sum a_k A_k + \sum b_l D_l + \sum c_m E_m$ is given. Our task is to decide whether it satisfies the condition 2 of Theorem 2.3. Let $S = \mathbf{Z}\lambda \oplus Q(G)$. Let $q: S^*/S \to \mathbf{Q}/2\mathbf{Z}$ be the discriminantal form of S. The finite group $A = S^*/S$ and the map q can be easily computed, once G is given. There is also an associated bilinear form $b: A \times A \to \mathbf{Q}/\mathbf{Z}$. Let $\tilde{\lambda} = \lambda/2$ be the dual of λ in S^*. Let $m: A \to \mathbf{Q}$ be a function defined in the following way: Assume that $[\eta] \in A$ is represented by $\eta = n\tilde{\lambda} + \phi$, where $0 \leq n \leq 1$ and $\phi \in Q(G)^*$. Then $m([\eta]) = n^2/2 + \max\{(\phi + \psi)^2 \,|\, \psi \in Q(G)\}$. This function is well-defined since the lattice $Q(G)$ is negative definite.

LEMMA 2.6.   *Let $m: A \to \mathbf{Q}$ be the function as defined above.*

1. *If $\phi \in Q(G)^* \setminus Q(G)$ and $m([\phi]) \neq -2$, then $(\phi + \psi)^2 \neq -2$ for every $\psi \in Q(G)$.*

2. *If $t \in Q(G)^*$ and $m([\tilde{\lambda} + t]) \neq 0$, then $(\tilde{\lambda} + t + \psi)^2 \neq 0$ for every $\psi \in Q(G)$.*

PROOF.   1. Assume that $\phi^2 = m([\phi]) > -2$. So $-2 < \phi^2 < 0$. For any $\psi \in Q(G)$, $(\phi + \psi)^2 = \phi^2 + 2\phi\psi + \psi^2$. Since $\phi\psi \in \mathbf{Z}$ and $\psi^2 \in 2\mathbf{Z}$, we have $(\phi + \psi)^2 \in \phi^2 + 2\mathbf{Z}$. Hence $(\phi + \psi)^2 \neq -2$.

2. Assume that $(\tilde{\lambda} + t)^2 = m([\tilde{\lambda} + t]) > 0$. Since $(\tilde{\lambda} + t)^2 = 1/2 + t^2$, we have $-1/2 < t^2 \leq 0$. For any $\psi \in Q(G)$, $(\tilde{\lambda} + t + \psi)^2 = 1/2 + (t + \psi)^2 = 1/2 + t^2 + 2t\psi + \psi^2 \in 1/2 + t^2 + 2\mathbf{Z}$. Hence $(\tilde{\lambda} + t + \psi)^2 \neq 0$. $\square$

The next step is to find all possible subgroups H of A satisfying the following two conditions:

1. $q(x) \in 2\mathbf{Z}$ for all $x \in H$. This means that H is isotropic.

2. There is no non-zero element h in H satisfying any of the following conditions:

    (a) $h = [\phi]$ for some $\phi \in Q(G)^*$ and $m(h) = -2$.

    (b) $h = [\tilde{\lambda} + t]$ with $t \in Q(G)^*$ and $m(h) = 0$.

The set of all such H is denoted by $\mathfrak{H}$.

Let T be the inverse image of H under the canonical map $S^* \to A$. Then the condition 1 implies that T is an even overlattice of S. If T has a primitive embedding into Λ, it induces an embedding of S into Λ. By Lemma 2.6 the conditions 2a and 2b imply that this embedding satisfies the conditions 2a and 2b in Theorem 2.3. Conversely, if S has an embedding into Λ satisfying the conditions 2a and 2b in Theorem 2.3, then the primitive hull $\tilde{S}$ of S in Λ is an even overlattice of S and $\tilde{S} \subset S^*$. The image of $\tilde{S}$ in $A = S^*/S$ obviously satisfies the conditions 1 and 2.

We have seen that each member H in $\mathfrak{H}$ corresponds to an overlattice T of S. Let $H^\perp = \{x \in A \mid b(x, h) = 0$ for every $h \in H\}$. Then q induces a map $q' : H^\perp/H \to \mathbf{Q}/2\mathbf{Z}$, which is exactly the discriminantal form of the lattice T. Then we can apply Theorem 2.5 to determine whether T has a primitive embedding into Λ.

Following the procedure just explained we wrote a program to generate a list of all Dynkin graphs satisfying the conditions in Theorem 2.3. We divided the whole list into two parts. The first part contains the Dynkin graphs of rank 19, while the second one contains all others. Then we wrote another program to generate a third list, which contains all proper subgraphs of the members of the first list. It turns out that all members of the third list belong to the second one. The only Dynkin graphs which belong to the second list but not the third are those listed in 2b of Theorem 2.1.

Upon the referee's suggestion we make a comparison between our computer-aided results and the results in Theorem A.3 and Theorem A.4 of [8]. By Urabe's simple principle we can generate a large list of Dynkin graphs which can be realized as a combination of singularities on a plane sextic curve.

PROPOSITION 2.7. *The Dynkin graphs with* 19 *vertices which can be realized as a combination of singularities on a plane sextic curve but do not satisfy the conditions of Theorems A.3 and A.4 of* [8] *are the following nine ones*:

$$E_6 + A_4 + 2A_3 + A_2 + A_1, \quad D_9 + 2A_4 + A_2, \quad D_6 + 2A_4 + A_3 + A_2,$$

$$2D_5 + A_6 + A_2 + A_1, \quad 2D_5 + 2A_3 + A_2 + A_1, \quad D_5 + A_5 + A_4 + A_3 + A_2,$$

$$2A_8 + A_3, \quad A_8 + A_5 + A_4 + A_2, \quad 3A_5 + 2A_2.$$

3. **Irreducible decomposition of reduced sextic curves.** In this section we will solve the following problem:

Assume that there exist reduced sextic curves in $\mathbf{P}^2$ *whose singularities are all simple and the combination of singularities corresponds to* $G = \sum a_k A_k + \sum b_l D_l + \sum c_m E_m$. *Find all possible configurations of such curves.*

A configuration of a reduced sextic curve means the information concerning the number of irreducible components, the degrees of irreducible components and the way each irreducible component passes through the singularities.

In order to give a more precise description of the configurations of a plane curve,

we need the concept of local components of a curve at a singularity. Let Y be a smooth surface, C a reduced curve on Y and p a singularity of C. Let (x, y) denote the local coordinates of Y with $p = (0, 0)$. Then in a neighborhood of p, C is the zero locus of $f(x, y) = 0$, where $f(x, y)$ is a convergent power series without multiple factors. In the local ring $\mathcal{O}_{Y,p}$ the power series $f(x, y)$ has a unique factorization $f(x, y) = f_1(x, y) \cdots f_r(x, y)$ up to order and invertible factors. Take a small neighborhood U of p in Y such that $f_1(x, y), \ldots, f_r(x, y)$ are regular in U. Let $C_1, \ldots, C_r$ be the zero loci of $f_1(x, y), \ldots, f_r(x, y)$ in U, respectively. Then the germs of curves at p represented by $C_1, \ldots, C_r$ are the *local components* of C at p.

We use the Roman numerals I, II, III to denote the local components of a simple singularity of type T by the following convention:

(1) When T is A_{2n}, E_6 or E_8, there is only one local component, denoted by I.

(2) When T is A_{2n-1}, the local equation can be written as $(y - x^n)(y + x^n) = 0$. There are two local components. Let I and II denote the ones with defining equations $y - x^n = 0$ and $y + x^n = 0$, respectively.

(3) When T is D_{2n+1}, the local equation can be written as $y(x^2 - y^{2n-1}) = 0$. There are two local components. Let I and II denote the ones with defining equations $y = 0$ and $x^2 - y^{2n-1} = 0$, respectively.

(4) When T is D_{2n}, the local equation can be written as $y(x - y^{n-1})(x + y^{n-1}) = 0$. There are three local components. Let I, II and III denote the ones with defining equations $x - y^{n-1} = 0$, $x + y^{n-1}$ and $y = 0$, respectively.

(5) When T is E_7, the local equation can be written as $y(y^2 - x^3) = 0$. There are two local components. Let I and II denote the ones with defining equations $y = 0$ and $y^2 - x^3 = 0$, respectively.
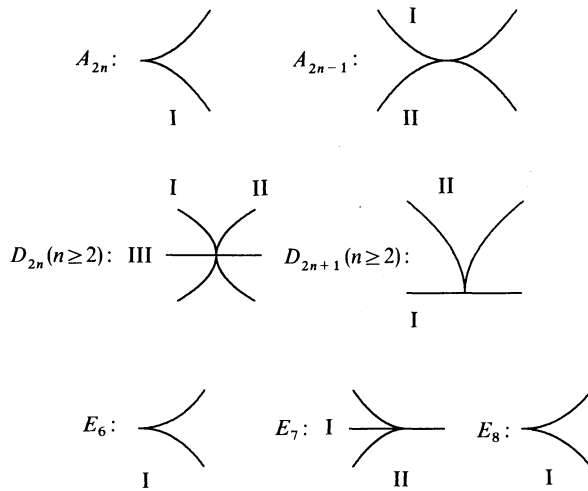
The notations is illustrated in Figure 1.

FIGURE 1. Local components of simple singularities.

Let p be a singularity of type T, where T is A_n, D_m or E_l. The set of local components of p is denoted by $\mathscr{A}(p)$ or $\mathscr{A}(T)$. According to Figure 1, $\mathscr{A}(A_{2n}) = \{I\}$, $\mathscr{A}(A_{2n-1}) = \{I, II\}$, $\mathscr{A}(D_{2n}) = \{I, II, III\}$, etc. For a subset $\mathscr{E}$ of $\mathscr{A}(T)$ we define a non-negative integer $\delta(\mathscr{E}, T)$ by the following rules:

1. $\delta(\varnothing, T) = 0$ for any T;
2. $\delta(\{I\}, A_{2n}) = n$;
3. $\delta(\{I\}, A_{2n-1}) = \delta(\{II\}, A_{2n-1}) = 0$, $\delta(\{I, II\}, A_{2n-1}) = n$;
4. $\delta(\{I\}, D_{2n}) = \delta(\{II\}, D_{2n}) = \delta(\{III\}, D_{2n}) = 0$, $\delta(\{I, II\}, D_{2n}) = n-1$, $\delta(\{I, III\}, D_{2n})$
   $= \delta(\{II, III\}, D_{2n}) = 1$, $\delta(\{I, II, III\}, D_{2n}) = n+1$;
5. $\delta(\{I\}, D_{2n+1}) = 0$, $\delta(\{II\}, D_{2n+1}) = n-1$, $\delta(\{I, II\}, D_{2n+1}) = n+1$;
6. $\delta(\{I\}, E_6) = 3$;
7. $\delta(\{I\}, E_7) = 0$, $\delta(\{II\}, E_7) = 1$, $\delta(\{I, II\}, E_7) = 4$;
8. $\delta(\{I\}, E_8) = 4$.

Let C be a reduced sextic curve on $\boldsymbol{P}^2$ with simple singularities as its only singularities and P the set of singularities of C. Let C' be a union of some components of C. For each $p \in P$, let $\mathscr{A}_{C'}(p)$ denote the set of all local components in $\mathscr{A}(p)$ that are contained in C'. Let $d(C')$ denote the degree of C'. Then the degree $d(C')$ together with the collection $\{\mathscr{A}_{C'}(p)\}_{p \in P}$ are called the *characteristic data* of C'. The characteristic data of all irreducible components of C give the configuration of C.

Assume that C' is an irreducible component of C with characteristic data $(d(C')$, $\{\mathscr{A}_{C'}(p)\}_{p \in P})$. Then the geometric genus of C' can be calculated by the well-known formula

$$(1) \qquad g(C') = (d(C')-1)(d(C')-2)/2 - \sum_{p \in P} \delta(\mathscr{A}_{C'}(p), T(p)),$$

where $T(p)$ is the type of p.

Let $\pi_0 : X_0 \to \boldsymbol{P}^2$ be the double cover of $\boldsymbol{P}^2$ branched over a reduced sextic curve C with simple singularities as its only singularities. Let us recall the *canonical resolution* of the singularities of X_0.

The construction is inductive. Let Y_0 denote $\boldsymbol{P}^2$ and C_0 denote C. Start with $j = 0$. If C_j is a smooth curve on Y_j then stop the process. Otherwise choose an arbitrary singular point p_j of C_j. Let m_j be the multiplicity of the curve C_j at p_j. Let $\sigma_{j+1} : Y_{j+1} \to Y_j$ be the blowing-up of Y_j at p_j and let $E_{j+1} = \sigma_{j+1}^{-1}(p_j)$. Then $C_{j+1} = \sigma_{j+1}^*(C_j) - 2[m_j/2] E_{j+1}$ is a reduced curve, whose divisor class is even. Here $[m_j/2]$ denotes the largest integer not exceeding $m_j/2$. Let $\pi_{j+1} : X_{j+1} \to Y_{j+1}$ be the double cover of Y_{j+1} branched over C_{j+1}. Then there is a unique morphism $\tau_{j+1} : X_{j+1} \to X_j$ such that the diagram

$$
\begin{array}{ccc}
X_{j+1} & \xrightarrow{\ \tau_{j+1}\ } & X_j \\
\downarrow{\scriptstyle \pi_{j+1}} & & \downarrow{\scriptstyle \pi_j} \\
Y_{j+1} & \xrightarrow[\ \sigma_{j+1}\ ]{} & Y_j
\end{array}
$$

is commutative. This process terminates after a finite number of steps (say r steps). The smooth surface X_r is a $K3$ surface, known as a double sextic corresponding to the sextic curve C. Let $X = X_r$, $\tau = \tau_1 \cdots \tau_r$, $f = \pi_r$, $Y = Y_r$, $\sigma = \sigma_1 \cdots \sigma_r$ and $\eta = \pi_0 \tau$. Then we have a commutative diagram:

$$(2) \qquad \begin{array}{ccc} X & \overset{\tau}{\longrightarrow} & X_0 \\ f \downarrow & \diagdown \eta & \downarrow \pi_0 \\ Y & \underset{\sigma}{\longrightarrow} & \boldsymbol{P}^2 . \end{array}$$

Note that τ is the minimal resolution of rational double points on X_0.

DEFINITION 3.1. Let $G = \sum_1^r G_i$ be a Dynkin graph with rank less than or equal to 19, where G_i is A_n, D_m or E_l. Assume that there is a set of data $\mathscr{D}_j = (d_j, \{\mathscr{A}_j(G_i)\}_{1 \le i \le r})$, $j = 1, \ldots, s$, in which d_j is a positive integer and $\mathscr{A}_j(G_i)$ is a subset of $\mathscr{A}(G_i)$ for every j. We say that these data form a virtual decomposition of G if the following conditions are satisfied:

(1) $\sum_{j=1}^s d_j = 6$;

(2) for each $i = 1, \ldots, r$, $(\mathscr{A}_1(G_i), \ldots, \mathscr{A}_s(G_i))$ is a partition of the set $\mathscr{A}(G_i)$;

(3) for each $j = 1, \ldots, s$, $(d_j - 1)(d_j - 2)/2 - \sum_{i=1}^r \delta(\mathscr{A}_j(G_i), G_i) \ge 0$,

(4) for each pair $1 \le j < k \le s$, $d_j d_k = \sum_{i=1}^r \mathscr{A}_j(G_i) \cdot \mathscr{A}_k(G_i)$, where $\mathscr{A}_j(G_i) \cdot \mathscr{A}_k(G_i)$ is the intersection number of the local curves $\sum_{D \in \mathscr{A}_j(G_i)} D$ and $\sum_{D \in \mathscr{A}_k(G_i)} D$.

DEFINITION 3.2. Let $\{\mathscr{D}_j = (d_j, \{\mathscr{A}_j(G_i)\}_{1 \le i \le r})\}_{j=1,\ldots,s}$ be a virtual decomposition of G. For $1 \le k \le 6$, let n_k denote the cardinality of the set $\{j \,|\, 1 \le j \le s, d_j = k\}$. Then the vector $(n_1, \ldots, n_6)$ is defined to be the total degree of the virtual decomposition.

Let $(n'_1, \ldots, n'_6)$ be the total degree of another virtual decomposition. Then we define $(n_1, \ldots, n_6) < (n'_1, \ldots, n'_6)$ if there is some $1 \le m \le 6$ such that $n_j = n'_j$ for $1 \le j \le m - 1$ and $n_m > n'_m$.

If $C = C_1 + \cdots + C_s$ is a decomposition of the sextic curve, then the characteristic data of $C_1, \ldots, C_s$ obviously form a virtual decomposition of the Dynkin graph of the singularities of C, but an arbitrary virtual decomposition of a Dynkin graph G certainly need not come from the decomposition of a sextic curve.

Let $G = \sum_i G_i$. For each i, let $\mathrm{Aut}(G_i)$ be the permutation group on $\mathscr{A}(G_i)$ defined as follows: If $G_i = D_4$, then $\mathrm{Aut}(G_i)$ is the full symmetric group of $\mathscr{A}(G_i) = \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$; if $G_i = A_{2n-1}$ or $G_i = D_{2n}$ with $n > 2$ then $\mathrm{Aut}(G_i)$ is the group generated by the transposition of local components I and II; in all other cases $\mathrm{Aut}(G_i) = \{1\}$. Then $\mathrm{Aut}(G_i)$ acts on the set of all subsets of $\mathscr{A}(G_i)$. This action induces an action of $\mathrm{Aut}(G) = \prod_i \mathrm{Aut}(G_i)$ on the set of virtual decompositions of G. If two virtual decompositions are in the same orbit then they are said to be equal. Obviously if two equal virtual decompositions come from two irreducible decompositions of sextic curves, then these two sextic curves have the same configuration.

Let p be a simple singularity on C. Let X be the double sextic constructed in the diagram (2). The point $\pi_0^{-1}(p)$ on X_0 is a rational double point of X_0. The dual graph of the exceptional divisor $\eta^{-1}(p)$ is a connected Dynkin graph of type A, D or E, as shown in Figure 2.
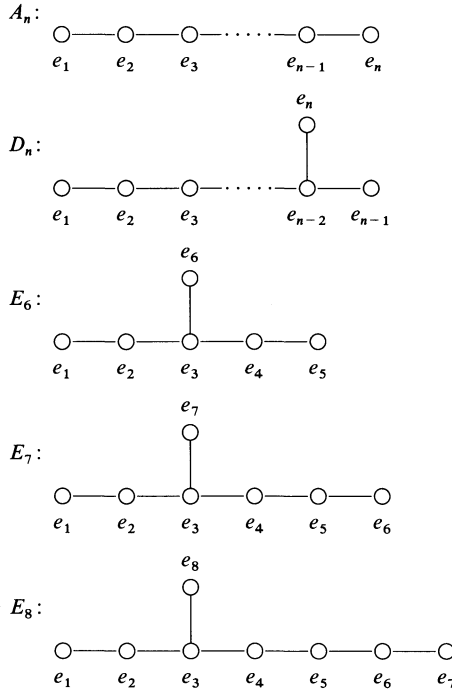

FIGURE 2. Dynkin graphs of rational double points.

For each Dynkin graph G in Figure 2, the root lattice $Q(G)$ is generated by the vertices $e_1, e_2, \ldots$. They correspond to the (-2)-curves in the exceptional divisor. The product of each e_i with itself is -2 and if $i \neq j$ then $e_i \cdot e_j = 1$ or 0 depending on whether there is an edge connecting e_i and e_j. Let β be an element of $Q(G) \otimes_{\mathbf{Z}} \mathbf{Q}$. Then $\beta = \sum_i c_i e_i$ with $c_i \in \mathbf{Q}$. We defined the *weight* of β to be the cardinality of the set $\{i \mid c_i \notin \mathbf{Z}\}$. Obviously the weight of β is zero if and only if $\beta \in Q(G)$.

Let $\beta = \sum_i c_i e_i$, $\beta' = \sum_i c_i' e_i$ with $c_i, c_i' \in \mathbf{Q}$. We define $\beta \geq \beta'$ if and only if $c_i \geq c_i'$ for every $1 \leq i \leq r$.

Let U be a small open neighborhood of p such that the irreducible decomposition of $C \cap U$ coincides with the local decomposition of C at p. Let C' be an irreducible component of $C \cap U$. Then

$$(3) \qquad \eta^*(C') \sim 2\bar{C}' + \sum a_i e_i,$$

where $\bar{C}'$ is the proper transform of C' in X. The divisor $\sum a_i e_i$ is called the *associated*

divisor of the local component C' at p. The associated divisors for the local components in Figure 1 can be easily computed using the canonical resolution described before. They are listed in Table 1. If T is one of A_n, D_m and E_l, and $B \in \mathscr{A}(T)$, then we denote by $\alpha(B)$ the associated divisor of B.

By (3) we have $\alpha(B)/2 \in Q(G)^*$ and $\iota(\alpha(B)) = \alpha(B)$, where ι denotes the involution of X associated with the double cover $f: X \to Y$.

TABLE 1.   Associated divisors.

A_{2n}	I:	$2(e_1 + 2e_2 + \cdots + ne_n + ne_{n+1} + \cdots + 2e_{2n-1} + e_{2n})$
A_{2n-1}	I:	$e_1 + 2e_2 + \cdots + ne_n + (n-1)e_{n+1} + \cdots + 2e_{2n-2} + e_{2n-1}$
	II:	$e_1 + 2e_2 + \cdots + ne_n + (n-1)e_{n+1} + \cdots + 2e_{2n-2} + e_{2n-1}$
D_{2n}	I:	$e_1 + 2e_2 + \cdots + (2n-2)e_{2n-2} + (n-1)e_{2n-1} + ne_{2n}$
	II:	$e_1 + 2e_2 + \cdots + (2n-2)e_{2n-2} + ne_{2n-1} + (n-1)e_{2n}$
	III:	$2e_1 + 2e_2 + \cdots + 2e_{2n-2} + e_{2n-1} + e_{2n}$
D_{2n+1}	I:	$2e_1 + 2e_2 + \cdots + 2e_{2n-1} + e_{2n} + e_{2n+1}$
	II:	$2(e_1 + e_2 + \cdots + (2n-1)e_{2n-1}) + (2n-1)e_{2n} + (2n-1)e_{2n-1}$
E_6	I:	$2(2e_1 + 4e_2 + 6e_3 + 4e_4 + 2e_5 + 3e_6)$
E_7	I:	$2e_1 + 4e_2 + 6e_3 + 5e_4 + 4e_5 + 3e_6 + 3e_7$
	II:	$4e_1 + 8e_2 + 12e_3 + 9e_4 + 6e_5 + 3e_6 + 7e_7$
E_8	I:	$2(5e_1 + 10e_2 + 15e_3 + 12e_4 + 9e_5 + 6e_6 + 3e_7 + 8e_8)$

The following three lemmas are easy to check in terms of Table 1.

LEMMA 3.3.   *Let T be A_n, D_m or E_l. Let $\mathscr{E}$ be a subset of $\mathscr{A}(T)$. Let $\mathscr{E}' = \mathscr{A}(T) \setminus \mathscr{E}$. Let $E = \sum_{B \in \mathscr{E}} \alpha(B)$ and $E' = \sum_{B' \in \mathscr{E}'} \alpha(B')$. Then*
1. $EE' = -2\mathscr{E} \cdot \mathscr{E}'$.
2. $E(E' - E) = 8\delta(\mathscr{E}, T)$.
3. *The weight of $E/2$ is equal to $\mathscr{E} \cdot \mathscr{E}'$.*
4. $E > E'$ *if $\delta(\mathscr{E}, T) > 0$.*

LEMMA 3.4.   *Let T be A_n, D_m or E_l. Let $\mathscr{E}$ and $\mathscr{E}'$ be subsets of $\mathscr{A}(T)$. Assume that*

$$\sum_{B \in \mathscr{E}} \alpha(B) \geq \sum_{B \in \mathscr{E}'} \alpha(B) .$$

Then $\delta(\mathscr{E}, T) \geq \delta(\mathscr{E}', T)$.

LEMMA 3.5.   *Let T be A_n, D_m or E_l. Let $\mathscr{E}$ and $\mathscr{E}'$ be subsets of $\mathscr{A}(T)$. Assume that the following conditions are satisfied:*
1. $\sum_{B \in \mathscr{E}} \alpha(B) \geq \sum_{B \in \mathscr{E}'} \alpha(B)$;
2. $\sum_{B \in \mathscr{E}} \alpha(B)/2 - \sum_{B \in \mathscr{E}'} \alpha(B)/2 \in Q(T)$;
3. *Either $\delta(\mathscr{E}, T) = 0$ or $\delta(\mathscr{E}', T) > 0$.*

Then $\sum_{B \in \mathscr{E}} \alpha(B) = \sum_{B \in \mathscr{E}'} \alpha(B)$.

Now we discuss the global decompositions of sextic curves. In the remaining of this section, C always denotes a reduced plane sextic curve with only simple singularities,

$P = \{p_1, \ldots, p_r\}$ denotes the set of singularities of C and G_i denotes the Dynkin graph of p_i for $i = 1, \ldots, r$. Let λ be the divisor class of the pull-back of a line in $\boldsymbol{P}^2$ under η. Then $\lambda^2 = 2$. Let $S = \boldsymbol{Z}\lambda + \bigoplus_{i=1}^{r} Q(G_i)$ and let $\widetilde{Q(G_i)}$ and $\tilde{S}$ denote the primitive hulls of $Q(G_i)$ and S in $H^2(X, \boldsymbol{Z})$, respectively. They coincide with their primitive hulls in $\text{Pic}(X)$, since $\text{Pic}(X)$ is always primitive in $H^2(X, \boldsymbol{Z})$. For any divisor D on X, its canonical image in $H^2(X, \boldsymbol{Z})$ is denoted by $[D]$.

Assume that $\beta \in \tilde{S}$. Then β can be expressed as $d\lambda + \sum_{i=1}^{r} \beta_i$, where $d \in \boldsymbol{Q}$ and $\beta_i \in Q(G_i) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$. Let w_i be the weight of β_i. The *weight* of β is defined to be $\sum_{i=1}^{r} w_i$.

Let $\{\mathcal{D}_j = (d_j, \{\mathcal{A}_j(G_i)\}_{1 \leq i \leq r})\}_{j=1, \ldots, s}$ be a virtual decomposition of G. Then for each $j = 1, \ldots, s$, $D_j = d_j\lambda/2 - \sum_{i=1}^{r} \sum_{B \in \mathcal{A}_j(G_i)} \alpha(B)/2$ is an element of S^*, where S^* is the dual lattice of $S = \boldsymbol{Z}\lambda + \bigoplus_i Q(G_i)$.

LEMMA 3.6. *Let* $G = \sum_{i=1}^{r} G_i$ *be the Dynkin graph corresponding to the simple singularities of* C. *Let* $\{\mathcal{D}_j = (d_j, \{\mathcal{A}_j(G_i)\}_{1 \leq i \leq r})\}_{j=1, \ldots, s}$ *be a virtual decomposition of* G. *Let* $D_j = D_j\lambda/2 - \sum_{i=1}^{r} \sum_{B \in \mathcal{A}_j(G_i)} \alpha(B)/2$ *for* $j = 1, \ldots, s$. *Then* $D_j^2 \geq -2$ *for every* j *and the weight of* D_j *is equal to* $d_j(6 - d_j)$.

PROOF. By (3) of Definition 3.1 and Lemma 3.3 we have

$$4(d_j - 1)(d_j - 2) \geq \sum_{i=1}^{r} 8\delta(\mathcal{A}_j(G_i), G_i)$$

$$= \sum_{i=1}^{r} \left(\left(\sum_{B \in \mathcal{A}_j(G_i)} \alpha(B) \right) \left(\sum_{B' \in \mathcal{A}(G_i) \setminus \mathcal{A}_j(G_i)} \alpha(B') \right) - \left(\sum_{B \in \mathcal{A}_j(G_i)} \alpha(B) \right)^2 \right)$$

$$= -2d_j(6 - d_j) - \sum_{i=1}^{r} \left(\sum_{B \in \mathcal{A}_j(G_i)} \alpha(B) \right)^2 .$$

Hence

$$D_j^2 = d_j^2/2 + \sum_{i=1}^{r} \left(\sum_{B \in \mathcal{A}_j(G_i)} \alpha(B) \right)^2 \Big/ 4 \geq -2 .$$

The second statement follows from Lemma 3.3 and the condition (4) of Definition 3.1. □

COROLLARY 3.7. *Let* $\{\mathcal{D}_j = (d_j, \{\mathcal{A}_j(G_i)\}_{1 \leq i \leq r})\}_{j=1, \ldots, s}$ *be a virtual decomposition of* $G = \sum_{1}^{r} G_i$. *Let* $D_j = d_j\lambda/2 - (\sum_{i=1}^{r} \sum_{B \in \mathcal{A}_j(G_i)} \alpha(B))/2$ *for* $j = 1, \ldots, s$. *If* $s > 1$, *then* $\sum_{i=1}^{r} \sum_{B \in \mathcal{A}_j(G_i)} \alpha(B)/2 \notin Q(G)$ *for every* $1 \leq j \leq s$.

COROLLARY 3.8. *Let* $\{\mathcal{D}_j = (d_j, \{\mathcal{A}_j(G_i)\}_{1 \leq i \leq r})\}_{j=1, \ldots, s}$ *be a virtual decomposition of* $G = \sum_{1}^{r} G_i$. *Let* $D_j = d_j\lambda/2 - (\sum_{i=1}^{r} \sum_{B \in \mathcal{A}_j(G_i)} \alpha(B))/2$ *for* $j = 1, \ldots, s$. *If* $D_j \in \text{Pic}(X)$, *then* $|D_j| \neq \emptyset$.

PROOF. Since $D_j^2 \geq -2$, the Riemann-Roch theorem implies that either $|D_j| \neq \emptyset$ or $|-D_j| \neq \emptyset$. However, $|-D_j| = \emptyset$ for $D_j\lambda = d_j > 0$. □

LEMMA 3.9.   *Assume that C' is an irreducible component of C with characteristic data $(d(C'), \{\mathscr{A}_{C'}(G_i)\}_{1 \le i \le r})$. Denote the proper transform of C' in X by $\bar{C}'$. Then*

$$(4) \qquad\qquad \bar{C}' \sim \eta^*(C')/2 - \sum_{i=1}^{r} \sum_{B \in \mathscr{A}_{C'}(G_i)} \alpha(B) \Big/ 2 \ .$$

PROOF.   The definition of associated divisors for local components implies the equality (4) immediately.   □

LEMMA 3.10.   *Let X be the K3 surface in the diagram (2). Let Z be an effective divisor on X such that $C_1 = \eta_*(Z)$ is an irreducible component of C. Assume that $\deg(C_1) = d$. Then $[Z] \in \tilde{S}$ and the weight of $[Z]$ is equal to $d(6-d)$.*

PROOF.   Let Z_1 be the irreducible component of Z such that $\eta(Z_1) = C_1$. Then Z_1 is contained in the ramification divisor of f. So $[Z_1] \in \tilde{S}$. Since $\lambda(Z - Z_1) = 0$, we have $[Z - Z_1] \in Q(G)$. Hence $[Z] \in \tilde{S}$ and the weight of Z is equal to that of Z_1, which is equal to $d(6-d)$ by the results of Lemmas 3.3 and 3.9.   □

LEMMA 3.11.   *Let $f: X \to Y$ be the double cover as in the diagram (2). Let ι denote the involution of X associated with the double cover. Let $D = d\lambda/2 - \sum_{i=1}^{r} \mu_i \in \mathrm{Pic}(X)$, where $d > 0$ and $\mu_i \in Q(\widetilde{G}_i)$. Assume that $\iota(D) \sim D$ and one of the following two conditions is satisfied:*

  (1) $d < 3$;
  (2) $d = 3$ and $4\mu_i > \sum_{B \in \mathscr{A}(G_i)} \alpha(B)$ for some i.
Then $\iota(Z) = Z$ for every $Z \in |D|$.

PROOF.   Since $\iota(D) \sim D \sim Z$, we have $2Z \sim Z + \iota(Z) = f^*(W)$ for some effective divisor W on Y. Let B be the branch locus of f on Y, and L a line on $\mathbf{P}^2$. By the definition of $\sigma: Y \to \mathbf{P}^2$, there is an effective divisor Δ on Y with $B + 2\Delta \sim 6\sigma^*L$ and $\sigma_*\Delta = 0$. Then

$$H^0(X, \mathcal{O}_X(2Z)) \cong H^0(X, f^*(\mathcal{O}_Y(W))) \cong H^0(Y, \mathcal{O}(W)) \oplus H^0(Y, \mathcal{O}(W - 3\sigma^*L + \Delta)) \ .$$

If $d < 3$ then $(W - 3\sigma^*L + \Delta)\sigma^*(L) = d - 3 < 0$. So $H^0(Y, \mathcal{O}(W - 3\sigma^*L + \Delta)) = 0$. Let us consider the case $d = 3$. Let $\bar{B}$ denote the proper transform of B in X. We can write $\bar{B} = \bar{C} + \sum_{i=1}^{r} \bar{A}_i$, where $\bar{C}$ denotes the proper transform of C in X and $\bar{A}_i$ is an effective divisor with support contained in $\eta^{-1}(p_i)$. We have $f^*(W - 3\sigma^*L + \Delta) \sim 2Z - \bar{B} = 2Z - \bar{C} - \sum_{i=1}^{r} \bar{A}_i \sim 2(3\lambda/2 - \sum_{i=1}^{r} \mu_i) - (3\lambda - \sum_{i=1}^{r} \sum_{B \in \mathscr{A}(G_i)} \alpha(B)/2) - \sum_{i=1}^{r} \bar{A}_i = -\sum_{i=1}^{r}(2\mu_i + \bar{A}_i - \sum_{B \in \mathscr{A}(G_i)} \alpha(B)/2)$, which is not effective by the condition (2). So $H^0(Y, \mathcal{O}(W - 3\sigma^*L + \Delta)) = 0$ too. Hence $H^0(X, \mathcal{O}_X(2Z)) \cong H^0(Y, \mathcal{O}(W))$. This implies that every member of $|2Z|$ is the pull-back of a member of $|W|$. In particular, $\iota(Z) = Z$.   □

LEMMA 3.12.   *Let C be a reduced sextic curve with simple singularities as its only singularities. Let G be the Dynkin graph corresponding to the singularities of C. Let $\{\mathscr{D}_j = (d_j, \{\mathscr{A}_j(G_i)\}_{1 \le i \le r})\}_{j = 1, \ldots, s}$ be a virtual decomposition of G such that $D_1 = d_1\lambda/2 - $*

$\sum_{i=1}^{r} \sum_{B \in \mathcal{A}_1(G_i)} \alpha(B)/2 \in \mathrm{Pic}(X)$. Let $Z \in |D_1|$. If $1 \le d_1 \le 2$, then $\eta(Z)$ is a curve of degree d_1 contained in C.

PROOF. Let Z_1 be the sum of all irreducible components of Z with non-zero intersections with λ. Then $\iota(Z_1) = Z_1$, since $\iota(Z) = Z$ by Lemma 3.11. It follows that $[Z_1] \in \tilde{S}$.

Let $C_1 = \eta(Z) = \eta(Z_1)$. If we can show that $\deg(C_1) = d_1$, then Z_1 is contained in the ramification divisor of f, since $\iota(Z_1) = Z_1$.

Suppose that $\deg(C_1) < d_1$. Then $\deg(C_1) = 1$ and $d_1 = 2$. In this case we have $Z_1 \sim f^*(\bar{C}_1)$, where $\bar{C}_1$ is the proper transform of C_1 in Y. So the weight of $[Z_1]$ is zero. Since $Z - Z_1$ is effective and $\lambda(Z - Z_1) = 0$, we have $[Z - Z_1] \in Q(G)$. This implies that the weight of $[Z]$ is zero, contradicting Corollary 3.7. Hence $\deg(C_1) = d_1$. □

LEMMA 3.13. Let $\{\mathcal{D}_j = (d_j, \{\mathcal{A}_j(G_i)\}_{1 \le i \le r})\}_{j=1,\ldots,s}$ be a virtual decomposition of $G = \sum_1^r G_i$. Let $D_j = d_j \lambda/2 - \sum_{i=1}^{r} \sum_{B \in \mathcal{A}_j(G_i)} \alpha(B)/2$ for $j = 1, \ldots, s$. Assume that $D_1, D_2 \in \mathrm{Pic}(X)$ and $d_2 < 3$. Then $|D_2 - D_1| = \varnothing$.

PROOF. If $d_1 > d_2$ then the conclusion is obvious. So we discuss the following two cases:

(1) $d_1 < d_2$. In this case we have $d_1 = 1$ and $d_2 = 2$. Suppose that there exists $Z \in |D_2 - D_1|$. Since $Z\lambda = 1$, Lemma 3.11 implies that $\eta(Z)$ is an irreducible component of C. By Lemma 3.10 the weight of $D_2 - D_1$ is equal to 5. However the weight of $D_2 - D_1$ is equal to that of $D_2 + D_1$, which is equal to 9 by Lemma 3.3 and the condition (4) of Definition 3.1, a contradiction.

(2) $d_1 = d_2$. Suppose that there exists $Z \in |D_2 - D_1|$. Since $Z\lambda = 0$, Z is the sum of (-2)-curves. Hence $Z \in Q(G)$, which implies that the weight of $D_2 - D_1$ is zero. However the weight of $D_2 - D_1$ is equal to that of $D_2 + D_1$, which is equal to $2d_1(6 - 2d_1)$ by Lemma 3.3 and the condition (4) of Definition 3.1, a contradiction. □

LEMMA 3.14. Let $G = \sum_1^r G_i$. Let $\{\mathcal{D}_j = (d_j, \{\mathcal{A}_j(G_i)\}_{1 \le i \le r})\}_{j=1,\ldots,s}$ and $\{\mathcal{D}'_j = (d'_j, \{\mathcal{A}'_j(G_i)\}_{1 \le i \le r})\}_{j=1,\ldots,s'}$ be two virtual decompositions of G. Assume that $s = s'$ and $d_j = d'_j$ for $1 \le j \le s$. If $\sum_{i=1}^{r} \sum_{B \in \mathcal{A}_j(G_i)} \alpha(B) = \sum_{i=1}^{r} \sum_{B \in \mathcal{A}'_j(G_i)} \alpha(B)$ for $1 \le j \le s$, then these two virtual decompositions are equal.

PROOF. It suffices to show that for any $1 \le i \le r$ there exists $g_i \in \mathrm{Aut}(G_i)$ such that $g_i(\mathcal{A}_j(G_i)) = \mathcal{A}'_j(G_i)$ for $1 \le j \le s$.

If G_i is not of type A_{2n-1}, then it is obvious from Table 1 that $\sum_{B \in \mathcal{A}_j(G_i)} \alpha(B) = \sum_{B \in \mathcal{A}'_j(G_i)} \alpha(B)$ implies $\mathcal{A}_j(G_i) = \mathcal{A}'_j(G_i)$. So in this case we may take $g_i = 1$.

It remains to consider the case $G_i = A_{2n-1}$. If $\mathcal{A}_l(G_i) = \{I, II\}$ for some $1 \le l \le s$, then $\mathcal{A}'_l(G_i)$ is also $\{I, II\}$. It is obvious that $\mathcal{A}_j(G_i) = \mathcal{A}'_j(G_i) = \varnothing$ for all $j \ne l$. So we may take $g_i = 1$. If there exist $l \ne k$ such that $\mathcal{A}_l(G_i) = \{I\}$ and $\mathcal{A}_k(G_i) = \{II\}$, then either $\mathcal{A}'_l(G_i) = \{I\}$, $\mathcal{A}'_k(G_i) = \{II\}$ or $\mathcal{A}'_l(G_i) = \{II\}$, $\mathcal{A}'_k(G_i) = \{I\}$. In the former case, we take $g_i = 1$ while in the latter case we take g_i to be the transposition of I and II. □

LEMMA 3.15. *Let* $\{\mathscr{D}_j = (d_j, \{\mathscr{A}_j(G_i)\}_{1 \le i \le r})\}_{j=1,\ldots,s}$ *be a virtual decomposition of* $G = \sum_1^r G_i$. *Let* $D_j = d_j\lambda/2 - \sum_{i=1}^r \sum_{B \in \mathscr{A}_j(G_i)} \alpha(B)/2$ *for some* $1 \le j \le s$. *Assume that* $D_j \in$ Pic(X). *Let* Z_j *be a member of* $|D_j|$ *such that the image of* Z_j *in* $\boldsymbol{P}^2$ *is an irreducible component* C_j *of the sextic curve* C *with* $\deg(C_j) = d_j$. *Let* $(d_j, \{\mathscr{A}_{C_j}(G_i)\}_{1 \le i \le r})$ *be the characteristic data of* C_j. *Then*

$$\sum_{B \in \mathscr{A}_j(G_i)} \alpha(B) = \sum_{B \in \mathscr{A}_{C_j}(G_i)} \alpha(B)$$

for every $1 \le i \le r$ *if one of the following conditions is satisfied*:
1. $d_j < 3$;
2. $d_j = 3$ *and* $\delta(\mathscr{A}_j(G_k), G_k) > 0$ *for some* $1 \le k \le r$.

PROOF. Let Z be the irreducible component of Z_j such that the image of Z in $\boldsymbol{P}^2$ is C_j. Then $Z \sim d_j\lambda/2 - \sum_{i=1}^r \sum_{B \in \mathscr{A}_{C_j}(G_i)} \alpha(B)/2$ by Lemma 3.9. Since the effective divisor $Z_j - Z$ is in $Q(G)$, we have $\sum_{B \in \mathscr{A}_{C_j}(G_i)} \alpha(B)/2 - \sum_{B \in \mathscr{A}_j(G_i)} \alpha(B)/2 \in Q(G_i)$ and $\sum_{B \in \mathscr{A}_{C_j}(G_i)} \alpha(B) \ge \sum_{B \in \mathscr{A}_j(G_i)} \alpha(B)$ for all $1 \le i \le r$.

If $d_j < 3$, then $\delta(\mathscr{A}_{C_j}(G_i), G_i) = 0$ for every $1 \le i \le r$. Lemma 3.5 implies $\sum_{B \in \mathscr{A}_j(G_i)} \alpha(B) = \sum_{B \in \mathscr{A}_{C_j}(G_i)} \alpha(B)$.

If $d_j = 3$ and $\delta(\mathscr{A}_j(G_k), G_k) > 0$ for some $1 \le k \le r$, then $\delta(\mathscr{A}_{C_j}(G_k), G_k) > 0$ by Lemma 3.4. Hence $\delta(\mathscr{A}_{C_j}(G_i), G_i) = 0$ for all $i \ne k$, since C_j has at most one singularity. The result follows from Lemma 3.5 immediately. $\square$

THEOREM 3.16. *Let* $G = \sum_{i=1}^r G_i$ *be a Dynkin graph with rank less than or equal to* 19. *Let* $S = \boldsymbol{Z}\lambda \oplus Q(G)$, $\lambda^2 = 2$. *Let* $\{\mathscr{D}_j = (d_j, \{\mathscr{A}_j(G_i)\}_{1 \le i \le r})\}_{j=1,\ldots,s}$ *be a virtual decomposition of* G. *Let* $D_j = d_j\lambda/2 - \sum_{i=1}^r \sum_{B \in \mathscr{A}_j(G_i)} \alpha(B)/2$ *for* $j = 1, \ldots, s$. *Then there is a reduced sextic curve* C *such that the characteristic data of its irreducible decomposition is exactly equal to* $\{\mathscr{D}_j\}_{j=1,\ldots,s}$ *if and only if there is an overlattice* T *of* S *satisfying the following three conditions*:
1. *The lattice* T *has a primitive embedding in the even unimodular lattice with signature* (3, 19) *and has the following properties* (a) *and* (b):
    (a) *there is no* $t \in T - S$ *such that* $t\lambda = 0$ *and* $t^2 = -2$;
    (b) *there is no* $t \in T$ *such that* $t\lambda = 1$ *and* $t^2 = 0$;
2. *The canonical images of* $D_1, \ldots, D_s$ *in* S^*/S *are in* T/S;
3. *For every possible virtual decomposition* $\{\mathscr{D}'_j = (d'_j, \{\mathscr{A}'_j(G_i)\}_{1 \le i \le r})\}_{j=1,\ldots,s'}$ *of smaller total degree, there is some* $1 \le k \le s'$ *such that the canonical image of* $D'_k = d'_k\lambda/2 - \sum_{i=1}^r \sum_{B \in \mathscr{A}'_k(G_i)} \alpha(B)/2$ *in* S^*/S *is not in* T/S.

PROOF. Sufficiency: Assume that the conditions 1–3 are satisfied. By Theorem 2.3 and the condition 1 there exists a reduced sextic curve C whose combination of singularities corresponds to the Dynkin graph G. Make a canonical resolution of the double cover of $\boldsymbol{P}^2$ branched over C and let X be the $K3$ surface in the diagram (2). We may assume that $T \cong$ Pic(X).

Consider the following three cases.

Case I: $s = 1$. It suffices to show that C is irreducible. Otherwise, the characteristic data of all irreducible components of C would form a virtual decomposition of G satisfying the condition 2 but violating the condition 3.

Case II: $s = 2$ and $d_1 = d_2 = 3$. First we assume that $\delta(\mathscr{A}_1(G_k), G_k) > 0$ for some $1 \le k \le r$. By 4 of Lemma 3.3 we can check that D_1 satisfies the condition 2 in Lemma 3.11. Take $Z \in |D_1|$. Then $\iota(Z) = Z$ by Lemma 3.11. Since $d_1 = 3$, there exists an irreducible component Z_1 of Z such that λZ_1 is an odd number and Z_1 is contained in the ramification divisor of f. If $\lambda Z_1 = 1$, then the image of Z_1 in $\mathbf{P}^2$ is a line component of C. So the set of characteristic data of the irreducible components of C is a virtual decomposition with smaller total degree. This would violate the condition 3. Hence $\lambda Z_1 = 3$. This implies that the image C_1 of Z in $\mathbf{P}^2$ is an irreducible cubic curve which is a component of C. By Lemma 3.15 we have

$$(5) \qquad \sum_{i=1}^{r} \sum_{B \in \mathscr{A}_{C_1}(G_i)} \alpha(B) = \sum_{i=1}^{r} \sum_{B \in \mathscr{A}_1(G_i)} \alpha(B).$$

Let $C_2 = C - C_1$. Then C_2 is irreducible, for otherwise the characteristic data of the irreducible components of C would be a virtual decomposition of smaller total degree. Hence

$$\sum_{i=1}^{r} \sum_{B \in \mathscr{A}_{C_2}(G_i)} \alpha(B) = \sum_{i=1}^{r} \sum_{B \in \mathscr{A}(G_i)} \alpha(B) - \sum_{i=1}^{r} \sum_{B \in \mathscr{A}_{C_1}(G_i)} \alpha(B)$$

$$= \sum_{i=1}^{r} \sum_{B \in \mathscr{A}(G_i)} \alpha(B) - \sum_{i=1}^{r} \sum_{B \in \mathscr{A}_1(G_i)} \alpha(B)$$

$$= \sum_{i=1}^{r} \sum_{B \in \mathscr{A}_2(G_i)} \alpha(B)$$

by (5). Lemma 3.14 implies that the virtual decomposition $\{\mathscr{D}_1, \mathscr{D}_2\}$ is equal to the characteristic data of the irreducible components of C.

It remains to consider the case where $\delta(\mathscr{A}_1(G_i), G_i) = \delta(\mathscr{A}_2(G_i), G_i) = 0$ for all $1 \le i \le r$. Then all G_i's are of type A_{2n-1}. In this case, we have $D_1 \sim D_2$ and $D_1^2 = 0$. Hence $|D_1|$ is a pencil of elliptic curves on X. Let R denote the ramification divisor of f. Then $R \sim 2D_1$. Since R is a smooth reduced curve, R is the union of two nonsingular elliptic curves, each of which is linearly equivalent to D_1. It is easy to see that the characteristic data of their images in $\mathbf{P}^2$ are equal to $\mathscr{D}_1$ and $\mathscr{D}_2$ respectively.

Case III: $d_j < 3$ for some j. After a rearrangement of the indices we may assume that $1 \le d_1 \le \cdots \le d_s$. Since $d_1 + \cdots + d_s = 6$, we have $d_{s-1} < 3$. Take $Z_j \in |D_j|$ and let $C_j = \eta(Z_j)$ for $j = 1, \ldots, s-1$. Then every C_j is a curve of degree d_j contained in C by Lemma 3.12.

First of all, C_1 is irreducible, for otherwise the characteristic data of the irreducible components of C would give rise to a virtual decomposition with smaller total degree,

which would contradict the condition 3. Hence $\sum_{i=1}^{r}\sum_{B\in\mathscr{A}_{C_1(G_i)}}\alpha(B)=\sum_{i=1}^{r}\sum_{B\in\mathscr{A}_1(G_i)}\alpha(B)$ by Lemma 3.15.

Let k be the largest integer such that

(i) $k \le s-1$;

(ii) $C_1, \ldots, C_k$ are distinct irreducible components of C;

(iii) $\sum_{i=1}^{r}\sum_{B\in\mathscr{A}_{C_j(G_i)}}\alpha(B)=\sum_{i=1}^{r}\sum_{B\in\mathscr{A}_j(G_i)}\alpha(B)$ for $1\le j\le k$.

We would like to show that $k=s-1$. We have already seen that $k\ge 1$. Suppose that $k<s-1$. First we show that C_{k+1} does not contain any of $C_1, \ldots, C_k$. Indeed, if C_j were a component of C_{k+1} for some $j\le k$, then there would be an irreducible component W of Z_{k+1} such that $\eta(W)=C_j$. But $W\sim d_j\lambda/2-\sum_{i=1}^{r}\sum_{B\in\mathscr{A}_{C_j(G_i)}}\alpha(B)/2$ by Lemma 3.9. So $W\sim D_j$ by (iii). This would imply that $Z_{k+1}-W\in|D_{k+1}-D_j|$, contradicting Lemma 3.13. Secondly, we see that C_{k+1} is irreducible, for otherwise the characteristic data of the irreducible components of C would give rise to a virtual decomposition with smaller total degree. Finally Lemma 3.15 implies that $\sum_{i=1}^{r}\sum_{B\in\mathscr{A}_{C_{k+1}(G_i)}}\alpha(B)=\sum_{i=1}^{r}\sum_{B\in\mathscr{A}_{k+1}(G_i)}\alpha(B)$. This would contradict the maximality of k. We conclude that $k=s-1$. Hence $C_1, \ldots, C_{s-1}$ are distinct irreducible components of the sextic curve C and $\sum_{i=1}^{r}\sum_{B\in\mathscr{A}_{C_j(G_i)}}\alpha(B)=\sum_{i=1}^{r}\sum_{B\in\mathscr{A}_j(G_i)}\alpha(B)$ for every $1\le j\le s-1$.

Let $C_s=C-C_1-\cdots-C_{s-1}$. Then C_s is an irreducible component of C by the condition 3. It is obvious that

$$\sum_{i=1}^{r}\sum_{B\in\mathscr{A}_{C_s(G_i)}}\alpha(B)=\sum_{i=1}^{r}\sum_{B\in\mathscr{A}_s(G_i)}\alpha(B).$$

The result follows from Lemma 3.14.

Necessity: Let C be a reduced sextic curve and $C=C_1+\cdots+C_s$. Take the virtual decomposition $\{\mathscr{D}_j\}_{j=1,\ldots,s}$ to be the characteristic data of $C_1, \ldots, C_s$. Let $T=\mathrm{Pic}(X)\cap\tilde{S}$. Then the conditions 1 and 2 are satisfied. By the proof of the first part of the theorem the condition 3 is also satisfied. $\square$

COROLLARY 3.17.   *Let X be the K3 surface in the diagram 2. Let $S\in\mathrm{Pic}(X)$ be the subgroup generated by the divisor classes of the pull-back of a line in $\mathbf{P}^2$ and the exceptional divisors of τ. Let T be the primitive hull of S in $\mathrm{Pic}(X)$. Then the virtual decomposition associated with the characteristic data on irreducible components of C is uniquely characterized by the conditions 2 and 3 in Theorem 3.16.*

COROLLARY 3.18.   *Let $G=\sum a_k A_k+\sum b_l D_l+\sum c_m E_m$ be a finite Dynkin graph of rank less than or equal to 19. Let $Q=Q(G)$ be the root lattice of type G. Let Λ denote the unimodular even lattice with signature $(3,19)$. Let $\mathbf{Z}\lambda$ be a free lattice of rank one with $\lambda^2=2$. Assume that the orthogonal sum $S=\mathbf{Z}\lambda\oplus Q$ has a primitive embedding into Λ. Then there exists an irreducible sextic curve in $\mathbf{P}^2$ whose combination of singularities corresponds to G.*

PROOF. Take $T = S$ and apply Lemma 3.9. $\square$

COROLLARY 3.19. *If $r(G) \leq 10$, then there exists an irreducible sextic curve whose combination of singularities corresponds to G.*

PROOF. If $G \neq 10A_1$, then $S = \mathbf{Z}\lambda \oplus Q(G)$ satisfies the conditions of Theorem 2.5, since $l(S^*/S) < \text{rank}(S) \leq 11 \leq 22 - \text{rank}(S) = l_+ + l_- - t_+ - t_-$. Hence S has a primitive embedding into Λ. The result follows from Corollary 3.18. If $G = 10A_1$, then a generic g_6^2 of a rational curve A gives rise to a map from A into $\mathbf{P}^2$ whose image is a sextic curve with ten ordinary double points. $\square$

The significance of Theorem 3.16 is that it provides a method to determine all configurations of reduced sextic curves. This can be achieved by the following steps:

(1) For a given Dynkin graph G, find the set $\mathfrak{N}$ of all possible virtual decompositions of G.

(2) Use the method in the previous section to find the set $\mathfrak{H}$ of all possible overlattices of $\lambda\mathbf{Z} \oplus Q(G)$ satisfying the conditions in 1 of Theorem 3.16.

(3) Find the subset $\mathfrak{N}'$ of $\mathfrak{N}$ consisting of all virtual decompositions of G such that all conditions in Theorem 3.16 are satisfied for some $T \in \mathfrak{H}$. This set $\mathfrak{N}'$ is the set of all configurations of reduced sextic curves whose combination of singularities corresponds to G.

EXAMPLE. Let $G = \sum_{i=1}^{4} G_i$, where $G_1 = D_6$, $G_2 = D_5$, $G_3 = A_7$, $G_4 = A_1$.

Step 1: It is easy to find that there are three virtual decompositions:

(i) $\mathscr{D}_1 = (6, \{\text{I, II, III}\}, \{\text{I, II}\}, \{\text{I, II}\}, \{\text{I, II}\})$.

(ii) $\mathscr{D}_1 = (4, \{\text{I, II}\}, \{\text{II}\}, \{\text{I}\}, \varnothing)$.
$\mathscr{D}_2 = (1, \{\text{III}\}, \{\text{I}\}, \varnothing, \{\text{I}\})$.
$\mathscr{D}_3 = (1, \varnothing, \varnothing, \{\text{II}\}, \{\text{II}\})$.

(iii) $\mathscr{D}_1 = (3, \{\text{III}\}, \{\text{II}\}, \{\text{I}\}, \{\text{I}\})$.
$\mathscr{D}_2 = (2, \{\text{I}\}, \varnothing, \{\text{II}\}, \{\text{II}\})$.
$\mathscr{D}_3 = (1, \{\text{II}\}, \{\text{I}\}, \varnothing, \varnothing)$.

Step 2: Determine lattice embeddings.
We have $S = \mathbf{Z}\lambda \oplus \sum_{i=1}^{4} Q(G_i)$ and

(6)
$$S^*/S = Q(\mathbf{Z}\lambda)^*/Q(\mathbf{Z}\lambda) \oplus Q(G_1)^*/Q(G_1) \oplus Q(G_2)^*/Q(G_2)$$
$$\oplus Q(G_3)^*/Q(G_3) \oplus Q(G_4)^*/Q(G_4).$$

The discriminantal form of S is determined as follows:

1. $Q(\mathbf{Z}\lambda)^*/Q(\mathbf{Z}\lambda) \cong \mathbf{Z}/2\mathbf{Z}$ is generated by $b_0 = \lambda/2$ with $b_0^2 = 1/2$.

2. $Q(G_1)^*/Q(G_1) \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ is generated by $b_1 = (e_1 + 2e_2 + 3e_3 + 4e_4 + 2e_5 + 3e_6)/2$ and $b_2 = (e_1 + 2e_2 + 3e_3 + 4e_4 + 3e_5 + 2e_6)/2$ with $b_1^2 = b_2^2 = -3/2$ and $b_1 b_2 = -1$.

3. $Q(G_2)^*/Q(G_2) \cong \mathbf{Z}/4\mathbf{Z}$ is generated by $b_3 = (2e_1 + 4e_2 + 6e_3 + 3e_4 + 5e_5)/4$ with $b_3^2 = -5/4$.

4. $Q(G_3)^*/Q(G_3) \cong Z/8Z$ is generated by $b_4 = (e_1 + 2e_2 + 3e_3 + 4e_4 + 5e_5 + 6e_6 + 7e_7)/8$ with $b_4^2 = -7/8$.

5. $Q(G_4)^*/Q(G_4) \cong Z/2Z$ is generated by $b_5 = e_1/2$ with $b_5^2 = -1/2$.

6. The direct sum in (6) is orthogonal.

Calculation shows that there is only one overlattice T of S up to automorphisms of S satisfying the conditions in 1 or Theorem 3.16. The quotient T/S is generated by $\beta_1 = b_0 + b_1 + b_2 + 2b_3 + 4b_4 + b_5$ and $\beta_2 = b_2 + 4b_4 + b_5$.

Step 3: Check the conditions in Theorem 3.16.

For the virtual decomposition (ii) in Step 1, we have $D_1 \equiv b_0 + b_1 + b_2 + 2b_3 + 4b_4$ (mod S). One sees that the canonical image of D_1 in S^*/S is not in T/S. Hence the condition 2 in Theorem 3.16 is not satisfied. Thus the virtual decomposition (ii) does not correspond to an actual decomposition.

For the virtual decomposition (iii) one easily sees that D_1, D_2 and D_3 are congruent to β_1, β_2 and $\beta_1 + \beta_2$, respectively, modulo S. Thus the condition 2 in Theorem 3.16 is satisfied. The other conditions are easily checked. It is obvious that the virtual decomposition (i) does not satisfy the last condition of Theorem 3.16.

Hence the sextic curve C with $D_6 + D_5 + A_7 + A_1$ as its singularities consists of three irreducible components: a cuspidal cubic, a conic and a line. Their intersections are illustrated in Figure 3.
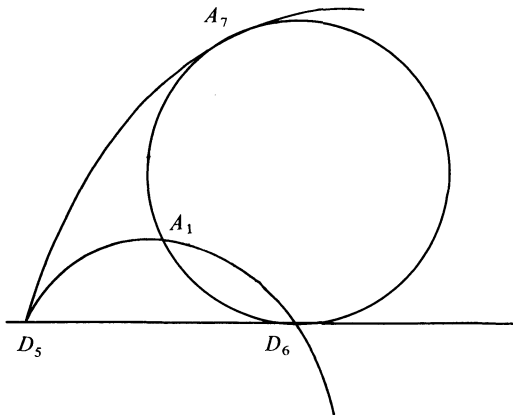
FIGURE 3. An example.

4. Maximizing sextics.

Adopting the terminology in [3], we call a plane sextic curve a *maximizing sextic* if all its singularities are simple singularities and the rank of the Dynkin graph of the singularities is equal to 19. The double cover of the plane branched over a maximizing sextic curve is birational to a singular $K3$ surface in the sense of [5].

By using the method described in the previous section, we determine all possible

TABLE 2.

*$2E_8 + A_3$,	*$2E_8 + A_2 + A_1$,	$E_8 + E_7 + D_4$,
*$E_8 + E_7 + A_4$,	$E_8 + E_7 + A_3 + A_1$,	*$E_8 + E_7 + 2A_2$,
*$E_8 + E_6 + D_5$,	*$E_8 + E_6 + A_5$,	*$E_8 + E_6 + A_4 + A_1$,
*$E_8 + E_6 + A_3 + A_2$,	*$E_8 + D_{11}$,	$E_8 + D_{10} + A_1$,
*$E_8 + D_9 + A_2$,	$E_8 + D_8 + A_2 + A_1$,	*$E_8 + D_7 + A_4$,
$E_8 + D_7 + A_3 + A_1$,	$E_8 + D_6 + D_5$,	$E_8 + D_6 + A_5$,
$E_8 + D_6 + A_3 + A_2$,	*$E_8 + D_5 + A_6$,	$E_8 + D_5 + A_5 + A_1$,
*$E_8 + D_5 + A_4 + A_2$,	*$E_8 + A_{11}$,	*$E_8 + A_{10} + A_1$,
*$E_8 + A_9 + A_2[2]$,	$E_8 + A_9 + 2A_1$,	*$E_8 + A_8 + A_3$,
*$E_8 + A_8 + A_2 + A_1$,	*$E_8 + A_7 + A_4$,	$E_8 + A_7 + A_3 + A_1$,
*$E_8 + A_7 + 2A_2$,	$E_8 + A_7 + A_2 + 2A_1$,	*$E_8 + A_6 + A_5$,
*$E_8 + A_6 + A_4 + A_1$,	*$E_8 + A_6 + A_3 + A_2$,	*$E_8 + A_6 + 2A_2 + A_1$,
*$E_8 + A_5 + A_4 + A_2$,	$E_8 + A_5 + A_4 + 2A_1$,	$E_8 + A_5 + 2A_3$,
$E_8 + A_5 + A_3 + A_2 + A_1$,	$E_8 + A_4 + 2A_3 + A_1$,	*$E_8 + A_4 + A_3 + 2A_2$,
$2E_7 + D_5[2]$,	$2E_7 + D_4 + A_1$,	$2E_7 + A_5$,
$2E_7 + A_3 + A_2[2]$,	$2E_7 + A_2 + 3A_1$,	*$E_7 + 2E_6$,
$E_7 + E_6 + D_6$,	$E_7 + E_6 + D_5 + A_1$,	*$E_7 + E_6 + A_6$,
*$E_7 + E_6 + A_4 + A_2$,	$E_7 + E_6 + A_4 + 2A_1$,	$E_7 + E_6 + 2A_3$,
$E_7 + E_6 + A_3 + A_2 + A_1$,	$E_7 + D_{12}[2]$,	$E_7 + D_{11} + A_1$,
$E_7 + D_{10} + A_2[3]$,	$E_7 + D_{10} + 2A_1$,	$E_7 + D_9 + A_3$,
$E_7 + D_9 + A_2 + A_1$,	$E_7 + D_8 + A_4$,	$E_7 + D_8 + A_3 + A_1[3]$,
$E_7 + D_8 + 2A_2$,	$E_7 + D_8 + A_2 + 2A_1$,	$E_7 + D_8 + 4A_1$,
$E_7 + D_7 + D_5$,	$E_7 + D_7 + D_4 + A_1$,	$E_7 + D_7 + A_5[3]$,
$E_7 + D_7 + A_4 + A_1$,	$E_7 + 2D_6[2]$,	$E_7 + D_6 + D_5 + A_1$,
$E_7 + D_6 + D_4 + A_2$,	$E_7 + D_6 + A_4 + 2A_1$,	$E_7 + D_6 + 2A_3$,
$E_7 + D_6 + A_3 + A_2 + A_1$,	$E_7 + D_6 + A_3 + 3A_1$,	$E_7 + D_6 + 2A_2 + 2A_1$,
$E_7 + 2D_5 + 2A_1$,	$E_7 + D_5 + D_4 + A_2 + A_1$,	$E_7 + D_5 + A_7$,
$E_7 + D_5 + A_6 + A_1$,	$E_7 + D_5 + A_5 + A_2[3]$,	$E_7 + D_5 + A_4 + A_3$,
$E_7 + D_5 + A_4 + A_2 + A_1$,	$E_7 + D_5 + A_3 + 2A_2$,	$E_7 + D_4 + A_8$,
$E_7 + D_4 + A_6 + A_2$,	$E_7 + D_4 + A_5 + A_3$,	$E_7 + D_4 + A_5 + 3A_1$,
$E_7 + D_4 + 2A_4$,	$E_7 + D_4 + A_4 + A_3 + A_1$,	$E_7 + D_4 + 2A_3 + 2A_1$,
*$E_7 + A_{12}$,	$E_7 + A_{11} + A_1$,	*$E_7 + A_{10} + A_2$,
$E_7 + A_9 + A_3[3]$,	$E_7 + A_9 + A_2 + A_1[3]$,	$E_7 + A_9 + 3A_1$,
*$E_7 + A_8 + A_4$,	$E_7 + A_8 + A_3 + A_1$,	$E_7 + A_7 + A_4 + A_1[2]$,
$E_7 + A_7 + A_3 + A_2$,	$E_7 + A_7 + A_3 + 2A_1$,	$E_7 + A_7 + 2A_2 + A_1[2]$,
$E_7 + A_7 + A_2 + 3A_1$,	*$E_7 + 2A_6$,	*$E_7 + A_6 + A_4 + A_2$,
$E_7 + A_6 + A_3 + A_2 + A_1$,	$E_7 + A_6 + 2A_2 + 2A_1$,	$E_7 + 2A_5 + 2A_1[2]$,
$E_7 + A_5 + A_4 + A_3[3]$,	$E_7 + A_5 + A_4 + 3A_1$,	$E_7 + A_5 + 2A_3 + A_1[3]$,
$E_7 + 2A_3 + 2A_2 + 2A_1$,	*$E_7 + 2A_4 + 2A_2$,	$E_7 + A_4 + A_3 + 2A_2 + A_1$,
*$2E_6 + A_6 + A_1$,	*$3E_6 + A_1$,	*$2E_6 + A_7$,
*$2E_6 + A_3 + 2A_2$,	*$2E_6 + A_5 + A_2$,	*$2E_6 + A_4 + A_3$,
$E_6 + D_{10} + A_3$,	*$E_6 + D_{13}$,	*$E_6 + D_{11} + A_2$,
$E_6 + D_9 + A_3 + A_1$,	$E_6 + D_{10} + 3A_1$,	*$E_6 + D_9 + A_4$,
$E_6 + D_8 + A_3 + 2A_1$,	$E_6 + D_8 + A_5$,	$E_6 + D_8 + A_4 + A_1$,
*$E_6 + D_7 + A_6$,	$E_6 + D_7 + D_6$,	$E_6 + D_7 + D_5 + A_1$,
$E_6 + D_6 + A_4 + A_3$,	$E_6 + D_6 + D_5 + A_2$,	$E_6 + D_6 + A_5 + 2A_1$,
*$E_6 + D_5 + A_6 + A_2$,	$E_6 + D_6 + 2A_3 + A_1$,	*$E_6 + D_5 + A_8$,
	$E_6 + D_5 + A_5 + A_3$,	*$E_6 + D_5 + 2A_4$,

TABLE 2.   (cont.)

$E_6+D_5+A_4+A_3+A_1$,	$*E_6+A_{13}$,	$*E_6+A_{12}+A_1$,
$*E_6+A_{11}+A_2$,	$E_6+A_{11}+2A_1$,	$*E_6+A_{10}+A_3$,
$*E_6+A_{10}+A_2+A_1$,	$*E_6+A_9+A_4[2]$,	$E_6+A_9+A_3+A_1$,
$E_6+A_9+A_2+2A_1$,	$*E_6+A_8+A_4+A_1$,	$*E_6+A_8+A_3+A_2$,
$*E_6+A_8+2A_2+A_1$,	$*E_6+A_7+A_6$,	$E_6+A_7+A_5+A_1[2]$,
$*E_6+A_7+A_4+A_2$,	$E_6+A_7+A_4+2A_1$,	$E_6+A_7+A_3+A_2+A_1$,
$E_6+A_7+A_3+3A_1$,	$E_6+A_6+A_5+2A_1$,	$*E_6+A_6+A_4+A_3$,
$*E_6+A_6+A_4+A_2+A_1$,	$E_6+A_6+2A_3+A_1$,	$E_6+2A_5+A_3[2]$,
$E_6+2A_5+3A_1$,	$*E_6+A_5+2A_4$,	$E_6+A_5+A_4+A_3+A_1$,
$*E_6+A_5+A_4+2A_2$,	$E_6+A_5+2A_3+2A_1$,	$E_6+A_5+A_3+2A_2+A_1$,
$*E_6+A_5+4A_2$,	$E_6+A_4+2A_3+A_2+A_1$,	$*D_{19}$,
$D_{18}+A_1$,	$*D_{17}+A_2$,	$D_{16}+A_3$,
$D_{16}+A_2+A_1[2]$,	$*D_{15}+A_4$,	$D_{15}+A_3+A_1$,
$D_{14}+D_5$,	$D_{14}+D_4+A_1$,	$D_{14}+A_5$,
$D_{14}+A_4+A_1$,	$D_{14}+A_3+A_2$,	$D_{14}+A_3+2A_1$,
$D_{14}+2A_2+A_1$,	$D_{13}+D_6$,	$*D_{13}+A_6$,
$D_{13}+A_5+A_1$,	$*D_{13}+A_4+A_2$,	$D_{12}+D_7$,
$D_{12}+D_6+A_1$,	$D_{12}+D_5+A_2$,	$D_{12}+D_5+2A_1$,
$D_{12}+D_4+A_2+A_1$,	$D_{12}+A_5+2A_1$,	$D_{12}+A_4+A_3$,
$D_{12}+2A_3+A_1$,	$D_{12}+A_3+A_2+2A_1$,	$D_{11}+D_7+A_1$,
$D_{11}+D_6+A_2$,	$*D_{11}+A_8$,	$D_{11}+A_7+A_1$,
$*D_{11}+A_6+A_2$,	$D_{11}+A_5+A_3[2]$,	$D_{11}+A_5+A_2+A_1$,
$D_{11}+A_4+A_3+A_1$,	$*D_{11}+A_4+2A_2$,	$D_{10}+D_9$,
$D_{10}+D_8+A_1[2]$,	$D_{10}+D_7+A_2$,	$D_{10}+D_7+2A_1$,
$D_{10}+D_6+A_3[2]$,	$D_{10}+D_6+A_2+A_1[2]$,	$D_{10}+D_6+3A_1$,
$D_{10}+D_5+A_4$,	$D_{10}+D_5+A_3+A_1[2]$,	$D_{10}+D_5+A_2+2A_1$,
$D_{10}+D_4+A_5$,	$D_{10}+D_4+A_4+A_1$,	$D_{10}+D_4+A_3+2A_1$,
$D_{10}+D_4+2A_2+A_1$,	$D_{10}+A_9[2]$,	$D_{10}+A_8+A_1$,
$D_{10}+A_7+A_2[2]$,	$D_{10}+A_6+A_3$,	$D_{10}+A_6+A_2+A_1$,
$D_{10}+A_5+A_4[4]$,	$D_{10}+A_5+A_3+A_1[4]$,	$D_{10}+2A_4+A_1$,
$D_{10}+A_4+A_3+A_2$,	$D_{10}+A_4+A_3+2A_1$,	$D_9+D_8+2A_1$,
$D_9+D_6+A_4$,	$D_9+D_6+A_3+A_1$,	$D_9+D_5+A_5[2]$,
$D_9+D_5+2A_2+A_1$,	$D_9+D_4+A_5+A_1$,	$*D_9+A_{10}$,
$D_9+A_9+A_1[2]$,	$D_9+A_7+A_3$,	$D_9+A_7+A_2+A_1$,
$*D_9+A_6+A_4$,	$D_9+A_5+A_4+A_1$,	$D_9+A_5+A_3+2A_1$,
$*D_9+2A_4+A_2$,	$2D_8+A_3$,	$2D_8+A_2+A_1[2]$,
$2D_8+3A_1$,	$D_8+D_7+A_3+A_1$,	$D_8+D_7+A_2+2A_1$,
$D_8+D_6+D_5$,	$D_8+D_6+D_4+A_1[2]$,	$D_8+D_6+A_5$,
$D_8+D_6+A_4+A_1$,	$D_8+D_6+A_3+A_2$,	$D_8+D_6+A_3+2A_1[3]$,
$D_8+D_6+2A_2+A_1$,	$D_8+D_6+A_2+3A_1$,	$D_8+D_5+D_4+2A_1$,
$D_8+D_5+A_5+A_1[3]$,	$D_8+D_5+A_4+2A_1$,	$D_8+D_5+A_3+3A_1$,
$D_8+D_4+A_5+2A_1$,	$D_8+D_4+2A_3+A_1$,	$D_8+D_4+A_3+A_2+2A_1$,
D_8+A_{11},	$D_8+A_{10}+A_1$,	$D_8+A_9+A_2[2]$,
$D_8+A_9+2A_1$,	$D_8+A_8+A_2+A_1$,	$D_8+A_7+A_4$,
$D_8+A_7+A_3+A_1[2]$,	$D_8+A_7+2A_2$,	$D_8+A_7+A_2+2A_1[2]$,
$D_8+A_6+A_5$,	$D_8+A_6+A_4+A_1$,	$D_8+A_6+A_3+2A_1$,
$D_8+A_6+2A_2+A_1$,	$D_8+2A_5+A_1[2]$,	$D_8+A_5+A_4+A_2$,
$D_8+A_5+A_4+2A_1[2]$,	$D_8+A_5+2A_3$,	$D_8+A_5+A_3+A_2+A_1[3]$,

TABLE 2. (cont.)

$D_8 + A_5 + A_3 + 3A_1,$	$D_8 + A_4 + 2A_3 + A_1,$	$D_8 + A_4 + A_3 + A_2 + 2A_1,$
$2D_7 + A_5,$	$2D_7 + A_4 + A_1,$	$D_7 + 2D_6,$
$D_7 + D_6 + D_5 + A_1,$	$D_7 + D_6 + A_6,$	$D_7 + D_6 + A_5 + A_1,$
$D_7 + D_6 + A_4 + A_2,$	$D_7 + D_5 + A_7,$	$D_7 + D_5 + A_5 + 2A_1,$
$D_7 + D_4 + A_7 + A_1,$	$D_7 + D_4 + A_5 + A_2 + A_1,$	$*D_7 + A_{12},$
$D_7 + A_{11} + A_1,$	$*D_7 + A_{10} + A_2,$	$D_7 + A_9 + A_3,$
$D_7 + A_9 + A_2 + A_1[2],$	$*D_7 + A_8 + A_4,$	$D_7 + A_8 + A_3 + A_1,$
$D_7 + A_7 + A_5,$	$D_7 + A_7 + A_4 + A_1,$	$D_7 + A_7 + A_3 + A_2,$
$D_7 + A_7 + A_3 + 2A_1,$	$*D_7 + 2A_6,$	$D_7 + A_6 + A_5 + A_1,$
$*D_7 + A_6 + A_4 + A_2,$	$D_7 + A_5 + A_4 + A_3[2],$	$D_7 + 2A_4 + A_3 + A_1,$
$3D_6 + A_1,$	$2D_6 + D_5 + A_2,$	$2D_6 + D_5 + 2A_1,$
$2D_6 + D_4 + A_3,$	$2D_6 + D_4 + A_2 + A_1,$	$2D_6 + D_4 + 3A_1[2],$
$2D_6 + A_5 + 2A_1,$	$2D_6 + A_4 + A_3,$	$2D_6 + 2A_3 + A_1,$
$2D_6 + A_3 + A_2 + 2A_1,$	$2D_6 + A_3 + 4A_1,$	$D_6 + D_5 + D_4 + A_3 + A_1,$
$D_6 + D_5 + A_8,$	$D_6 + D_5 + A_7 + A_1,$	$D_6 + D_5 + A_6 + A_2,$
$D_6 + D_5 + A_5 + A_3[4],$	$D_6 + D_5 + A_5 + A_2 + A_1,$	$D_6 + D_5 + A_5 + 3A_1,$
$D_6 + D_5 + A_4 + A_3 + A_1,$	$D_6 + D_5 + A_4 + 2A_2,$	$D_6 + D_5 + A_3 + 2A_2 + A_1,$
$D_6 + 2D_4 + A_3 + 2A_1,$	$D_6 + D_4 + A_9,$	$D_6 + D_4 + A_7 + A_2,$
$D_6 + D_4 + A_5 + A_4[2],$	$D_6 + D_4 + A_5 + A_3 + A_1[3],$	$D_6 + D_4 + 2A_3 + 3A_1,$
$D_6 + A_{11} + 2A_1,$	$D_6 + A_{10} + A_3,$	$D_6 + A_9 + A_3 + A_1[2],$
$D_6 + A_9 + A_2 + 2A_1[2],$	$D_6 + A_7 + A_4 + 2A_1,$	$D_6 + A_7 + 2A_3,$
$D_6 + A_7 + A_3 + A_2 + A_1,$	$D_6 + A_7 + A_3 + 3A_1,$	$D_6 + A_7 + 2A_2 + 2A_1,$
$D_6 + A_6 + A_5 + 2A_1,$	$D_6 + A_6 + A_4 + A_3,$	$D_6 + 2A_5 + 3A_1,$
$D_6 + A_5 + A_4 + A_3 + A_1,$	$D_6 + A_5 + A_4 + A_2 + 2A_1,$	$D_6 + A_5 + 2A_3 + 2A_1[2],$
$D_6 + A_5 + A_3 + A_2 + 3A_1,$	$D_6 + 2A_4 + A_3 + A_2,$	$2D_5 + A_9,$
$2D_5 + A_7 + A_2,$	$2D_5 + A_7 + 2A_1,$	$2D_5 + A_6 + A_2 + A_1,$
$2D_5 + A_5 + A_4,$	$2D_5 + 2A_3 + A_2 + A_1,$	$D_5 + D_4 + A_9 + A_1,$
$D_5 + D_4 + A_7 + A_2 + A_1,$	$D_5 + D_4 + A_5 + A_4 + A_1,$	$D_5 + D_4 + A_5 + A_3 + 2A_1[2],$
$*D_5 + A_{14},$	$D_5 + A_{13} + A_1[2],$	$*D_5 + A_{12} + A_2,$
$D_5 + A_{11} + A_3[2],$	$D_5 + A_{11} + A_2 + A_1[2],$	$*D_5 + A_{10} + A_4,$
$*D_5 + A_{10} + 2A_2,$	$D_5 + A_9 + A_5[3],$	$D_5 + A_9 + A_4 + A_1[2],$
$D_5 + A_9 + A_3 + A_2[2],$	$D_5 + A_9 + A_3 + 2A_1,$	$D_5 + A_9 + 2A_2 + A_1[2],$
$*D_5 + A_8 + A_6,$	$D_5 + A_8 + A_5 + A_1,$	$*D_5 + A_8 + A_4 + A_2,$
$*D_5 + A_8 + 3A_2,$	$D_5 + 2A_7,$	$D_5 + A_7 + A_6 + A_1,$
$D_5 + A_7 + A_5 + A_2,$	$D_5 + A_7 + A_5 + 2A_1[2],$	$D_5 + A_7 + A_4 + A_2 + A_1,$
$D_5 + A_7 + 2A_3 + A_1[2],$	$D_5 + A_7 + A_3 + A_2 + 2A_1,$	$D_5 + A_6 + A_5 + A_3,$
$D_5 + A_6 + A_5 + A_2 + A_1,$	$*D_5 + A_6 + 2A_4,$	$*D_5 + A_6 + A_4 + 2A_2,$
$D_5 + A_6 + A_3 + 2A_2 + A_1,$	$D_5 + 2A_5 + A_4[2],$	$D_5 + 2A_5 + A_3 + A_1[2],$
$D_5 + 2A_5 + 2A_2[2],$	$D_5 + A_5 + 2A_4 + A_1,$	$D_5 + A_5 + A_4 + A_3 + A_2,$
$D_5 + A_5 + A_4 + A_3 + 2A_1,$	$4D_4 + 3A_1,$	$2D_4 + 2A_5 + A_1,$
$D_4 + A_{15},$	$D_4 + A_{13} + A_2,$	$D_4 + A_{11} + A_4,$
$D_4 + A_{11} + A_3 + A_1,$	$D_4 + A_{11} + 2A_2,$	$D_4 + A_{10} + A_5,$
$D_4 + A_9 + A_6,$	$D_4 + A_9 + A_4 + A_2,$	$D_4 + A_9 + A_3 + A_2 + A_1,$
$D_4 + A_8 + A_5 + A_2,$	$D_4 + 2A_7 + A_1,$	$D_4 + A_7 + A_5 + A_3,$
$D_4 + A_7 + A_4 + A_3 + A_1,$	$D_4 + A_7 + 2A_3 + 2A_1,$	$D_4 + A_6 + A_5 + A_4,$
$D_4 + A_6 + A_5 + A_3 + A_1,$	$D_4 + 3A_5[3],$	$*A_{19},$
$*A_{18} + A_1,$	$*A_{17} + A_2[2],$	$A_{17} + 2A_1,$
$*A_{16} + A_3,$	$*A_{16} + A_2 + A_1,$	$*A_{15} + A_4[2],$

TABLE 2. (cont.)

$A_{15}+A_3+A_1,$	$A_{15}+A_2+2A_1[2],$	$*A_{14}+A_4+A_1,$
$*A_{14}+A_3+A_2,$	$*A_{14}+2A_2+A_1,$	$*A_{13}+A_6,$
$A_{13}+A_5+A_1,$	$*A_{13}+A_4+A_2,$	$A_{13}+A_4+2A_1,$
$A_{13}+A_3+A_2+A_1[2],$	$A_{13}+2A_2+2A_1[2],$	$*A_{12}+A_7,$
$*A_{12}+A_6+A_1,$	$A_{12}+A_5+2A_1,$	$*A_{12}+A_4+A_3,$
$*A_{12}+A_4+A_2+A_1,$	$A_{12}+2A_3+A_1,$	$A_{11}+A_7+A_1,$
$A_{11}+A_6+2A_1,$	$A_{11}+A_5+A_3[3],$	$A_{11}+A_5+A_2+A_1,$
$A_{11}+A_5+3A_1,$	$*A_{11}+2A_4,$	$*A_{11}+A_4+2A_2,$
$A_{11}+A_4+A_2+2A_1,$	$A_{11}+2A_3+2A_1,$	$A_{11}+A_3+2A_2+A_1,$
$A_{11}+3A_2+2A_1,$	$*A_{10}+A_9[2],$	$*A_{10}+A_8+A_1,$
$*A_{10}+A_7+A_2,$	$A_{10}+A_7+2A_1,$	$*A_{10}+A_6+A_3,$
$*A_{10}+A_6+A_2+A_1,$	$*A_{10}+A_5+A_4,$	$A_{10}+A_5+A_3+A_1,$
$*A_{10}+2A_4+A_1,$	$*A_{10}+A_4+A_3+A_2,$	$*A_{10}+A_4+2A_2+A_1,$
$2A_9+A_1,$	$A_9+A_8+2A_1,$	$A_9+A_7+A_3,$
$A_9+A_7+A_2+A_1[3],$	$*A_9+A_6+A_4[2],$	$A_9+A_6+A_3+A_1[2],$
$A_9+A_6+A_2+2A_1,$	$A_9+A_5+A_4+A_1[2],$	$A_9+A_5+A_3+2A_1,$
$A_9+A_5+A_2+3A_1,$	$*A_9+2A_4+A_2[2],$	$A_9+2A_4+2A_1,$
$A_9+2A_3+A_2+2A_1,$	$*2A_8+A_3,$	$*A_8+A_7+A_4,$
$A_8+A_7+A_3+A_1,$	$A_8+A_7+A_2+2A_1,$	$*A_8+A_6+A_4+A_1,$
$*A_8+A_5+A_4+A_2,$	$A_8+A_5+A_4+2A_1,$	$A_8+A_5+A_3+A_2+A_1,$
$A_8+A_4+2A_3+A_1,$	$*A_8+A_4+3A_2+A_1,$	$2A_7+A_5,$
$2A_7+A_4+A_1[2],$	$2A_7+A_3+A_2,$	$2A_7+A_3+2A_1[2],$
$2A_7+2A_2+A_1,$	$*A_7+2A_6,$	$A_7+A_6+A_5+A_1[2],$
$*A_7+A_6+A_4+A_2,$	$A_7+A_6+A_4+2A_1,$	$A_7+A_6+2A_2+2A_1,$
$A_7+2A_5+2A_1[2],$	$A_7+A_5+A_4+A_3,$	$A_7+A_5+A_4+A_2+A_1[2],$
$A_7+A_5+A_4+3A_1,$	$A_7+A_5+A_3+A_2+2A_1,$	$*A_7+2A_4+2A_2,$
$*3A_6+A_1,$	$*2A_6+A_4+A_2+A_1,$	$A_6+2A_5+3A_1,$
$*A_6+A_5+2A_4,$	$A_6+A_5+A_4+A_3+A_1,$	$3A_5+A_3+A_1,$
$3A_5+2A_2,$	$3A_5+A_2+2A_1,$	$3A_5+4A_1.$

configurations of all maximizing sextics. Due to the limit of space, we do not give the list of configurations (available from the author upon request). Instead, we include the number of configurations for each item in Table 2 in the bracket right after it. If the number is omitted, it means that it is equal to 1. For example, $E_8+A_9+A_2$ has two configurations. Moreover, if an irreducible maximizing sextic exists for a given Dynkin graph in Table 2 then it is marked with an asterisk. There are 128 irreducible maximizing sextics.

REMARK 4.1. The same methods can be applied to determine the configurations of other reduced sextic curves with simple singularities. The list is too long to be printed here.

Once the enumeration of maximizing sextics has been settled, some questions raised in [3] can be answered. Here are two of them.

(1) *The exact upper bound for the discriminants of maximizing sextics.*

In the process of the computation, the discriminants of overlattices are evaluated. So without much additional effort, we can find maximizing sextics with large discriminant. The largest discriminant is 3600, the sextic curve is irreducible and its singularities correspond to $A_7 + 2A_4 + 2A_2$. This is the only sextic curve whose discriminant reaches 3600. The irreducible curve corresponding to $2A_6 + A_4 + A_2 + A_1$ has discriminant 2940, which is the second largest value.

(2) *The existence of a smooth sextic whose double cover is a singular K3 surface with arbitrarily large discriminant.*

By definition, a singular $K3$ surface is a $K3$ surface whose Picard group has rank 20.

THEOREM 4.2. *For any integer N, there exists a smooth sextic curve C in $\mathbf{P}^2$ such that*

1. *the double cover X of $\mathbf{P}^2$ branched over C is a singular $K3$ surface;*
2. *the discriminant of X is greater than N.*

PROOF. Obviously, we may assume that $N > 2$. By Dirichlet's arithmetic progression theorem there are infinitely many prime numbers of the form $16n - 1$. Take nine distinct prime numbers $p_1 = 16n_1 - 1, \ldots, p_9 = 16n_9 - 1$. For $i = 1, \ldots, 9$, let U_i be a lattice of rank two generated by v_i and w_i such that $v_i^2 = -4$, $w_i^2 = -4n_i$ and $v_i w_i = 1$. Let $\mathbf{Z}\theta$ be a lattice of rank one with $\theta^2 = -2^N$. Let S be the orthogonal sum of $\mathbf{Z}\lambda$, $U_1, \ldots, U_9$ and $\mathbf{Z}\theta$, where $\lambda^2 = 2$. It is an even lattice. The abelian group S^*/S is the direct sum of $\mathbf{Z}/2\mathbf{Z}$, $\mathbf{Z}/p_1\mathbf{Z}, \ldots, \mathbf{Z}/p_9\mathbf{Z}$ and $\mathbf{Z}/2^N\mathbf{Z}$. Then S has a primitive embedding into the even unimodular lattice Λ of signature (3, 19) by Theorem 2.5. By the surjectivity of the period map for $K3$ surfaces there exists a $K3$ surface X whose Picard group is isomorphic to S. Since S does not contain an element η such that $\eta\lambda = 1$ and $\eta^2 = 0$, the complete linear system $|\lambda|$ determines a double cover from X to $\mathbf{P}^2$ branched over a sextic curve C (see [6] for detailed arguments). We claim that C is smooth. Otherwise there would exist a (-2)-curve on X which has zero intersection with λ, but this is impossible because the orthogonal complement of $\mathbf{Z}\lambda$ in S does not contain an element η with $\eta^2 = -2$. $\square$

5. Other applications.

The result we obtained so far has many potential applications. For example, one can consider the triple cover of 6-tuple cover over a sextic curve. Since there are a lot of sextic curves at our disposal we may obtain many examples of surfaces of general type with small invariants.

Here we mention two other simple applications.

5.1. Contact of two cuspidal cubics.

Consider two cuspidal cubics C_1 and C_2 on the plane. Here we always assume that C_1 and C_2 do not intersect at cusps. We ask what are the possible ways of contact for C_1 and C_2.

Since $C_1 C_2 = 9$, the contact of C_1 and C_2 can be described by a partition of 9. We use a sequence $(n_1, \ldots, n_r)$ to denote a partition $9 = n_1 + \cdots + n_r$. We say that this partition is admissible if there exist C_1 and C_2 meeting at r distinct points with contact

numbers $n_1, \ldots, n_r$.

If a partition $(n_1, \ldots, n_r)$ of 9 is admissible, then the Dynkin graph of the singularities of the sextic curve $C_1 + C_2$ will be $2A_2 + \sum_i A_{2n_i - 1}$. The rank of this Dynkin graph is $22 - r$. This implies that $r \geq 3$. When $r = 3$ the sextic curve $C_1 + C_2$ is maximizing. Here we list all admissible partitions with $r = 3$:

$$(7, 1, 1), \ (6, 2, 1), \ (4, 4, 1), \ (3, 3, 3).$$

Of course this is only a sample. For other curves C_1 and C_2 with $\deg(C_1) + \deg(C_2) = 6$ one may determine their contacts in a similar way.

5.2. Sextics with cusps.   The irreducible curves with cusps of multiplicity 2 or 3 were discussed in [10] and [11]. Here we restrict to sextics with cusps of type A_{2n}, E_6 and E_8 as its only singularities. Using our result all such curves can be enumerated. Here we only give the list of all elliptic curves as follows.

$$2E_8 + A_2, \ E_8 + E_6 + A_4, \ E_8 + E_6 + 2A_2, \ E_8 + A_{10}, \ E_8 + A_8 + A_2,$$

$$E_8 + A_6 + A_4, \ E_8 + A_6 + 2A_2, \ E_8 + 2A_4 + A_2, \ E_8 + A_4 + 3A_2, \ 3E_6,$$

$$2E_6 + A_6, \ 2E_6 + A_4 + A_2, \ E_6 + A_{12}, \ E_6 + A_{10} + A_2, \ E_6 + A_8 + A_4,$$

$$E_6 + 2A_6, \ E_6 + A_6 + A_4 + A_2, \ E_6 + 3A_4, \ E_6 + 2A_4 + 2A_2, \ A_{18}, \ A_{16} + A_2,$$

$$A_{14} + A_4, \ A_{12} + A_6, \ A_{12} + A_4 + A_2, \ A_{10} + A_8, \ A_{10} + A_6 + A_2,$$

$$A_{10} + 2A_4, \ A_{10} + A_4 + 2A_2, \ A_8 + A_6 + A_4, \ A_8 + 2A_4 + A_2,$$

$$2A_6 + A_4 + A_2, \ A_6 + 3A_4, \ A_6 + 2A_4 + 2A_2, \ 3A_4 + 3A_2, \ 9A_2.$$

The last one is well-known, and is the dual curve of a smooth cubic. Note that there are no rational curves with this property.

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