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SURFACES WITH EXTREME VALUE OF CURVATURE IN ALEXANDROV SPACES

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Abstract. In an Alexandrov space with curvature bound, we prove that a curvature takes the extreme value over some specially constructed surfaces if and only if each of the surfaces is totally geodesic and locally isometric to a surface with constant curvature.

Introduction. An Alexandrov space is a locally compact complete length space (i.e., a space in which distance is measured by the infimum of lengths of curves) with curvature bounded either below or above in the distance comparison sense, that is, the Alexandrov-Toponogov comparison theorem holds for all small geodesic triangles. A complete Riemannian manifold with sectional curvature bounded either below or above is an Alexandrov space and in fact the difference lies in the differentiability. Until recently people discussed only C^{∞}-Riemannian manifolds and forgot about other important aspects of metric spaces. It was the work of Gromov that ended this long sleeping period. Inspired by the idea developed by Gromov [11], [12], Alexandrov spaces got footlights, and it became known that they can be obtained as the so-called Gromov-Hausdorff limits (cf. [13], [15], [23]) of sequences of Riemannian manifolds belonging to a certain class determined by geometric quantities; curvature, diameter, and volume (cf. [17], [18], [24]).

Since the notion of Alexandrov spaces is a generalization of Riemannian manifolds, is seems natural to consider the problem: To what extent can one extend results in Riemannian geometry to Alexandrov spaces? It is known that some well-known results in Riemannian geometry can be extended to finite Hausdorff dimensional Alexandrov spaces of curvature bounded below. For example, the Myers-Toponogov compactness theorem [6], the Diameter sphere theorem of Grove and Shiohama [19], [22], the fibration theorem of Yamaguchi [27], [28], and the Soul theorem of Cheeger and Gromoll [9], [22] can be generalized. It should also be mentioned that the isometry group of a finite Hausdorff dimensional Alexandrov space with lower curvature bounded is a Lie group [10].

In this paper, we will show that for specially constructed surfaces Σ_i ($i=1, 2$) in an Alexandrov space X with curvature bounded either below or above, the curvature

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of X takes the extreme value over the surface Σ_i if and only if Σ_i is totally geodesic in X and locally isometric to a surface with constant curvature. The surface Σ_1 is an exponential image of a plane and Σ_2 is a ruled surface produced by a parallel line field along a geodesic. This can be proved in the Riemannian case with the help of the Jacobi equation and the curvature tensor. However, a generalization to an Alexandrov space without a differential structure has a quite different character, and we will show how it can be done.

We refer the reader to [3], [6] and [26] for basic tools and notation on Alexandrov spaces.

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1. Preliminaries. In this section, we present some well-known facts about Alexandrov spaces. Let (X, d) be a complete locally compact length space, i.e., a complete locally compact metric space such that any two points $p, q \in X$ is joined by a minimal geodesic whose length is equal to the distance $d(p, q)$ between p and q, where the length of a continuous curve $\alpha: [a, b] \to X$ is defined to be

$$\sup_{a=t_0<\cdots<t_n=b} \sum_{i=0}^{n-1} d(\alpha(t_i), \alpha(t_{i+1}))\,.$$

From now on, we will assume that a minimal geodesic α is parametrized by arclength (i.e., the length of $\alpha|_{[a,s]}$ is $|s-a|$ for all $s \in (a, b]$). For $p, q \in X$ we denote by pq a minimal geodesic joining p and q. A limit of minimal geodesics is again a minimal geodesic and moreover with the limit length. For any three points $p, q, r \in X$, the union of three minimal geodesics pq, qr, rp is called a geodesic triangle in X and denoted by $\triangle(pqr)$.

For a fixed real number k, we denote by $M^2(k)$ the 2-dimensional complete simply connected Riemannian manifold of sectional curvature k. More precisely, $M^2(k)$ is a Euclidean plane when $k=0$, a sphere when $k>0$, and a hyperbolic space when $k<0$. For a geodesic triangle $\triangle(pqr)$ in X we denote by $\triangle(\tilde{p}\tilde{q}\tilde{r})$ a geodesic triangle sketched in $M^2(k)$ whose corresponding edges have equal lengths as $\triangle(pqr)$. For $k \le 0$ the geodesic triangle $\triangle(\tilde{p}\tilde{q}\tilde{r})$ always exists and is unique up to rigid motion, and for $k>0$ it exists only with the additional assumption that the perimeter $d(p, q)+d(q, r)+d(r, p)$ of $\triangle(pqr)$ is less than $2\pi/\sqrt{k}$. We denote by $\sphericalangle\tilde{p}\tilde{q}\tilde{r}$ the angle at $\tilde{q}$ of $\triangle(\tilde{p}\tilde{q}\tilde{r})$. For the sake of simplicity we often write $\tilde{\triangle}(pqr)$ instead of $\triangle(\tilde{p}\tilde{q}\tilde{r})$ and also $\tilde{\sphericalangle}pqr$ instead of $\sphericalangle\tilde{p}\tilde{q}\tilde{r}$.

DEFINITION 1.1. A complete locally compact length space X is said to have *curvature bounded below* (*above, resp.*) *by* k (written, $\mathrm{Curv}(X) \ge k$ ($\mathrm{Curv}(X) \le k$, resp.)) if for each point $x \in X$ there exists an open neighborhood U_x satisfying the following condition: For any geodesic triangle $\triangle(pqr)$ with vertices in U_x and any point $u \in qr$ the inequality $d(p, u) \ge d(\tilde{p}, \hat{u})$ ($d(p, u) \le d(\tilde{p}, \hat{u})$, resp.) is satisfied, where $\hat{u} \in \tilde{q}\tilde{r}$ is the point of the geodesic triangle $\tilde{\triangle}(pqr)$ corresponding to u, that is, such that $d(q, u) = d(\tilde{q}, \hat{u})$.

This definition is taken from [3] and [6]. We will sometimes call a geodesic triangle $\triangle(pqr)$ a small geodesic triangle if it is contained in an open neighborhood U_x satisfying appropriate conditions. An Alexandrov space is, by definition, a complete locally compact length space with curvature bounded either below or above. Then of course a complete Riemannian manifold with sectional curvature bounded either below or above is an Alexandrov space.

There are several equivalent ways to describe the curvature bound in an Alexandrov space. They sometimes involve the concept of angle, which we will explain now. Let α and β be minimal geodesics having a common origin p. Then $\alpha(s)$ and $\beta(t)$ are points on α and β, respectively, such that $s=d(p, \alpha(s))$ and $t=d(p, \beta(t))$. If $\mathrm{Curv}(X)\geq k$ ($\mathrm{Curv}(X)\leq k$, resp.), then the angle $\tilde{\sphericalangle}\alpha(s)p\beta(t)$ is monotone non-increasing (non-decreasing, resp.) for sufficiently small $s, t>0$ in the following sense: $\tilde{\sphericalangle}\alpha(s_1)p\beta(t_1)\geq \tilde{\sphericalangle}\alpha(s_2)p\beta(t_2)$ ($\tilde{\sphericalangle}\alpha(s_1)p\beta(t_1)\leq \tilde{\sphericalangle}\alpha(s_2)p\beta(t_2)$, resp.) for $0\leq s_1\leq s_2\leq s$, $0\leq t_1\leq t_2\leq t$ (cf. [2], [3]). This property is called the local version of the *Alexandrov convexity* (*concavity*, *resp.*) *property* for $p\alpha(s)$ and $p\beta(t)$ for sufficiently small $s, t>0$, and equivalent to curvature bounded below (above, resp). We will simply say an open set U to have the Alexandrov property if any two minimal geodesics pq and pr in U have the Alexandrov convexity or concavity property. Then any open set with the Alexandrov property satisfies the condition of Definition 1.1. Since $\tilde{\sphericalangle}\alpha(s)p\beta(t)$ is monotone for sufficiently small, $s, t>0$, the limit of $\tilde{\sphericalangle}\alpha(s)p\beta(t)$ as $s, t\to 0$ exists. We can define the natural angle at p between α and β, $\sphericalangle(\alpha, \beta)=\lim_{s,t\to 0} \tilde{\sphericalangle}\alpha(s)p\beta(t)$. When there is no danger of ambiguity, the angle at p between α and β may be written as $\sphericalangle qpr$, where $q\in\alpha$, $r\in\beta$.

Making use of the property of angle as stated above, we can easily obtain the local verision of the *Toponogov theorem* and the *Hinge theorem*.

TOPONOGOV THEOREM. *If* $\mathrm{Curv}(X)\geq k$ ($\mathrm{Curv}(X)\leq k$, *resp.*) *and if* $\triangle(pqr)$ *is a small geodesic triangle, then* $\sphericalangle pqr\geq \tilde{\sphericalangle}pqr$, $\sphericalangle qrp\geq \tilde{\sphericalangle}qrp$, and $\sphericalangle rpq\geq \tilde{\sphericalangle}rpq$ ($\sphericalangle pqr\leq \tilde{\sphericalangle}pqr$, $\sphericalangle qrp\leq \tilde{\sphericalangle}qrp$, *and* $\sphericalangle rpq\leq \tilde{\sphericalangle}rpq$, *resp.*).

HINGE THEOREM. *If* $\mathrm{Curv}(X)\geq k$ ($\mathrm{Curv}(X)\leq k$, *resp.*) *and if* $\hat{\alpha}$, $\hat{\beta}$ *are minimal geodesics on* $M^2(k)$ *with the same starting point and the same lengths as* α, β *and the same angle at* p, *then* $d(\alpha(s), \beta(t))\leq d(\hat{\alpha}(s), \hat{\beta}(t))$ ($d(\alpha(s), \beta(t))\geq d(\hat{\alpha}(s), \hat{\beta}(t))$, *resp.*) *for sufficiently small* $s, t>0$.

We have the following properties of angle for later use which are taken directly from [3] and [6].

PROPOSITION 1.2. *Let* X *be an Alexandrov space.*

(i) *Suppose that* α, β, γ *are minimal geodesics emanating from* $p\in X$. *Then* $\sphericalangle(\alpha, \gamma)\leq \sphericalangle(\alpha, \beta)+\sphericalangle(\beta, \gamma)$.

(ii) *Suppose* $\mathrm{Curv}(X)\geq k$ ($\mathrm{Curv}(X)\leq k$, *resp.*). *If* x *is an interior point of a minimal geodesic* pq, *then for any* $r(\neq x)\in X$ *we have* $\sphericalangle pxr+\sphericalangle qxr=\pi$ ($\geq\pi$, *resp.*).

In the above proposition, it should be noted that with an upper curvature bound the sum of adjacent angles is not equal to but greater than or equal to π. This is due to the fact that without a lower curvature bound we may have a so-called branch point x of minimal geodesics pr and pq (i.e., x belongs to interior points of minimal geodesics pr and pq such that $pr \cap pq = px$, $xr \subset pr$, $xq \subset pq$, and $xr \cap xq = \{x\}$) and this may cause some difficulties. We can take as an example a flat cone with the total vertex angle greater than 2π, which is an Alexandrov space of curvature bounded above by zero. At the vertex, the sum of adjacent angles along a geodesic is greater than π. We will take care of this problem in Section 2.

Let α, β be minimal geodesics emanating from p in X. It is called (the global version of) the Alexandrov convexity (concavity, resp.) property for $p\alpha(s)$ and $p\beta(t)$ that the angle $\tilde{\angle}\alpha(s)p\beta(t)$ is monotone non-increasing (non-decreasing, resp.) for s, $t>0$ not necessarily small. Then any geodesic triangle in Alexandrov spaces with curvature bounded below has the Alexandrov convexity property (cf. [6]). However, the Alexandrov concavity property does not hold even for Riemannian manifolds with curvature bounded above, and we need some extra conditions on Alexandrov spaces with curvatue bounded above. For an Alexandrov space X with curvature bounded above by k, we consider the following conditions:

(1) The minimal geodesics depend continuously on their ends in X (i.e., if pq and p_nq_n are minimal geodesics in X such that $p_n \to p$, $q_n \to q$, as $n \to \infty$, then for a point r on pq and a point r_n on p_nq_n such that $d(p, r):d(r, q) = d(p_n, r_n):d(r_n, q_n)$, we have $r_n \to r$ as $n \to \infty$.).

(2) If $k>0$, then the perimeter of any geodesic triangle in X is less than $2\pi/\sqrt{k}$.

Then any geodesic triangle in X has the Alexandrov concavity property if and only if X satisfies the conditions (1) and (2) (cf. [2], Theorem 5.1 in [3]).

A map f between metric spaces is called a local isometry if it is locally distance preserving (i.e., $d(f(x), f(y)) = d(x, y)$, for x, y in a neighborhood of every point in a metric space). We denote by $\blacktriangle(pqr)$ a ruled surface of $\triangle(pqr)$ on X, which is by definition the union of minimal geodesics px for all $x \in qr$. Unfortunately, for a point $x \in qr$, the minimal geodesic px may not be unique, and hence we may not have a well-defined unique ruled surface. In $M^2(k)$, of course, the ruled surface is uniquely determined by its boundary unless the geodesic triangle is the equator of a sphere.

On occasion, we will hope that locally a minimal geodesic connecting two points in Alexandrov spaces is unique. In fact, in the case where $\mathrm{Curv}(X) \le k$, for any point $x \in X$, there exists an open convex ball B_x in the sense that any two points in B_x can be joined by a unique minimal geodesic lying on B_x (cf. [2]). Without an upper curvature bound, this is not true in general. We take as an example a flat cone with total vertex angle less than π. For any neighborhood of the vertex we can always find two points with more than one minimal geodesics. In the case where $\mathrm{Curv}(X) \ge k$, however, we know that a minimal geodesic connecting two interior points of a minimal geodesic in X is unique because otherwise it would produce a branch point, which is impossible with a lower curvature bound. It is also known that if X has Wald curvature bounded below, then for any point $p \in X$, there exists a dense subset J_p of X such that for all $x \in J_p$ there is a unique, almost extendable minimal geodesic from p to x (see Theorem 1.4 in [25]). The Wald curvature condition looks stronger than ours, but if X is a locally compact complete length space, then they coincide (see 2.3–2.5 in [6]). Furthermore, in the case of finite Hausdorff dimensional Alexandrov spaces with curvature bounded below, a concept of the cut locus of a point $p \in X$ can be defined and its Hausdorff dimension is not greater than $\dim_H X - 1$ (cf. [21]).

Without a lower curvature bound we may have a branch point, and without an upper curvature bound we may have this problem with the local uniqueness of minimal geodesics. In Riemannian geometry, we do not have these problems, and we see that even in local scale an Alexandrov space can be much more complicated than a Riemannian manifold. Depending on whether we have an upper curvature bound or a lower bound, we encounter different kinds of difficulties, and maybe this is the reason why there are hardly any theorems which can apply to both cases simultaneously. We will also have to handle them separately sometimes in order to obtain the same conclusion.

DEFINITION 1.3. A subset Y is called *totally geodesic* in an intrinsic metric space X if for every point $y \in Y$ there exists a neighborhood U_y around y in X such that every pair of points in $Y \cap U_y$ is joined by a minimal geodesic in U_y which is, in fact, contained in Y.

This definition of a totally geodesic subset in an intrinsic metric space is clearly a generalization of totally geodesic submanifolds in Riemannian manifolds. Y is metrically embedded in X in the sense that the induced metric on Y is an intrinsic metric. In Riemannian geometry, the equality case of the Toponogov theorem gives rise to a kind of rigidity theorem for a subset. The same kind of property has been observed for Alexandrov spaces. We will have only a sketch of the proof (see [26] for more details of the proof).

PROPOSITION 1.4. *Let X be an Alexandrov space with curvature bounded either below or above by k. Assume that a geodesic triangle $\triangle(pqr)$ is contained in an open set with the Alexandrov property and that the angle $\sphericalangle pqr$ is equal to $\tilde{\sphericalangle} pqr$. Then there exists a smooth ruled surface $\blacktriangle(pqr)$ which is totally geodesic in X and isometric to the ruled surface $\tilde{\blacktriangle}(pqr)$ in $M^2(k)$.*

PROOF. We will first prove the proposition in the case where $\mathrm{Curv}(X) \geq k$. If $u \in qr$ and $\hat{u} \in \tilde{q}\tilde{r}$ are taken so that $d(q, u) = d(\tilde{q}, \hat{u})$, then by the Hinge theorem and the curvature condition we have $d(p, u) = d(\tilde{p}, \hat{u})$. Similarly, there exist $v \in pq$ and $\hat{v} \in \tilde{p}\tilde{q}$ such that $d(q, v) = d(\tilde{q}, \hat{v})$ and $d(v, u) = d(\hat{v}, \hat{u})$.

By Proposition 1.2 (ii) and the Toponogov theorem, we have

$$\pi = \sphericalangle uvq + \sphericalangle uvp \geq \sphericalangle \hat{u}\hat{v}\tilde{q} + \sphericalangle \hat{u}\hat{v}\tilde{p} = \pi .$$

Thus we can obtain that $\sphericalangle uvq = \sphericalangle \hat{u}\hat{v}\tilde{q}$, and if the limit of uv as $v \to q$ is a minimal geodesic uq, then we have $\sphericalangle upq = \tilde{\sphericalangle} upq$.

The minimal geodesic $\tilde{r}\hat{v}$ intersects $\tilde{p}\hat{u}$ at a unique point $\hat{w}$ in $M^2(k)$. We will verify that if $w \in pu$ is a point with $d(p, w) = d(\tilde{p}, \hat{w})$, then $d(r, w) = d(\tilde{r}, \hat{w})$ and $d(w, v) = d(\hat{w}, \hat{v})$. From the fact that $\sphericalangle upq = \tilde{\sphericalangle} upq$ and the curvature condition, we see that $d(w, v) = d(\hat{w}, \hat{v})$. In view of the properties of angle and the Toponogov theorem, we have

$$\pi = \sphericalangle puq + \sphericalangle pur \geq \sphericalangle \tilde{p}\hat{u}\tilde{q} + \sphericalangle \tilde{p}\hat{u}\tilde{r} = \pi .$$

Thus we can obtain that $\sphericalangle pur = \sphericalangle \tilde{p}\hat{u}\tilde{r}$, and hence we have $d(r, w) = d(\tilde{r}, \hat{w})$.

This means that if $\alpha: [0, d(q, r)] \to X$ and $\beta: [0, d(q, p)] \to X$ are the edges with $\alpha(0) = \beta(0) = q$, $\alpha(d(q, r)) = r$, $\beta(d(q, p)) = p$, then natural maps $f: [0, d(q, r)] \times [0, d(q, p)] \to X$ and $\tilde{f}: [0, d(q, r)] \times [0, d(q, p)] \to \tilde{\blacktriangle}(pqr)$ are defined as follows: To each $(s, t) \in [0, d(q, r)] \times [0, d(q, p)]$ a point $f(s, t)$ ($\tilde{f}(s, t)$, resp.) is assigned as the intersection of geodesic $p\alpha(s) \cap r\beta(t)$ ($\tilde{p}\tilde{\alpha}(s) \cap \tilde{r}\tilde{\beta}(t)$, resp.), where $\tilde{\alpha}$ and $\tilde{\beta}$ are the edges of $\tilde{\blacktriangle}(pqr)$ corresponding to α and β. Then it is easy to see that $f \circ \tilde{f}^{-1}: \tilde{\blacktriangle}(pqr) \to X$ is an isometric embedding.

In the case where $\mathrm{Curv}(X) \leq k$, all the inequalities in the above proof should be replaced by the opposite inequalities and also the sum of adjacent angles can only be not less than π. Namely, $\pi = \sphericalangle uvq + \sphericalangle uvp$ should be replaced by $\pi \leq \sphericalangle uvq + \sphericalangle uvp$ and we should have $\pi \leq \sphericalangle puq + \sphericalangle pur$. Then we have

$$\pi \leq \sphericalangle uvq + \sphericalangle uvp \leq \sphericalangle \hat{u}\hat{v}\tilde{q} + \sphericalangle \hat{u}\hat{v}\tilde{q} = \pi ,$$

$$\pi \leq \sphericalangle puq + \sphericalangle pur \leq \sphericalangle \tilde{p}\hat{u}\tilde{q} + \sphericalangle \tilde{p}\hat{u}\tilde{r} = \pi ,$$

and hence we obtain the same conclusion as in the lower bound case. □

Unless the geodesic triangel $\triangle(pqr)$ is in a convex ball, the ruled surface $\blacktriangle(pqr)$ may not be unique and we may have several sheets of surfaces with the vertices of $\triangle(pqr)$. The above proof demonstrates that if any of them satisfies the angle condition of Proposition 1.4 then it is isometric to the standard one no matter how many of them there are. Applying the same idea as the proof of the above proposition a little more carefully, we can in fact witness the following stronger statement, which we need for the main theorem (see Theorem 5.1 in [3] and Lemma 6.4 in [26] for the proof).

PROPOSITION 1.5. *Let X be an Alexandrov space with curvature bounded below (above, resp.) by k. Let $\alpha: [0, a] \to X$ and $\beta: [0, b] \to X$ be minimal geodesics emanating from p such that*

$$\alpha([0, a]) \cup \beta([0, b]) \subset \left(\bigcup_{a \in [0,a]} U_{\alpha(s)} \right) \cap \left(\bigcup_{t \in [0,b]} U_{\beta(t)} \right),$$

where U_x is an open neighborhood around x with the Alexandrov property.

Assume that there are $s_0 \in (0, a)$, $t_0 \in (0, b)$, *and a minimal geodesic* $\alpha(s_0)\beta(t_0)$ *such that* $\sphericalangle p\alpha(s_0)\beta(t_0) = \tilde{\sphericalangle} p\alpha(s_0)\beta(t_0)$. *Then there exists a smooth ruled surface* $\blacktriangle(p\alpha(s_0)\beta(t_0))$ *which is totally geodesic in* X *and isometric to the ruled surface* $\tilde{\blacktriangle}(p\alpha(s_0)\beta(t_0))$ *in* $M^2(k)$.

Again there may be several minimal geodesics joining $\alpha(s_0)$ and $\beta(t_0)$. As long as they satisfy (2), however, we have the same conclusion. In the case when $\mathrm{Curv}(X) \leq k$, if we assume that $U_{\alpha(s_0)}$ or $U_{\beta(t_0)}$ is contained in open convex balls, then the ruled surface $\blacktriangle(p\alpha(s_0)\beta(t_0))$ is unique.

2. Subsets with the extreme value of curvature. Throughout this section X is an Alexandrov space with curvature bounded either below or above by k, and Y is a subset in X. In this section we will discuss a new concept that the curvature of an Alexandrov space takes the extreme value over a subset, and we also discuss properties of such subsets. By assumption, the extreme value of the curvature means that $\mathrm{Curv}(X) = k$ over Y.

If X is a complete Riemannian manifold with $\mathrm{Curv}(X) \geq 0$, we have the splitting theorem (cf. [8], [9]) in the universal covering space and it produces flat subsets in X. If X is a compact complete Riemannian manifold with $\mathrm{Curv}(X) \leq 0$, then a solvable subgroup of the fundamental group $\pi_1(X)$ will also produce flat subsets in X (cf. [8]). There are results in Alexandrov spaces corresponding to these phenomena under suitable conditions (cf. [5], [16]). We are in fact interested in this kind of subsets in the case of arbitrary curvature bound.

DEFINITION 2.1. For an Alexandrov space X with curvature bounded either below or above by k, we say $\mathrm{Curv}(X) = k$ over Y if for each point $x \in X$ there exists an open neighborhood U_x such that for any geodesic triangle $\triangle(pqr)$ with vertices in $U_x \cap Y$, there exists a point v on an edge, say qr, different from q, r such that for $\hat{v} \in \tilde{q}\tilde{r}$ with $d(q, v) = d(\tilde{q}, \hat{v})$ we have $d(p, v) = d(\tilde{p}, \hat{v})$.

In fact, this definition is the equality case of Definition 1.1. In the following proposition, we will show what this definition means in terms of angles and prove that it in fact implies seemingly stronger conditions.

PROPOSITION 2.2. *Let* Y *be a subset of an Alexandrov space* X *with a curvature bound* k. *Suppose that* $\mathrm{Curv}(X) = k$ *over* Y. *Then for each point* $x \in Y$ *there exists an open neighborhood* U_x *around* x *such that if three points* p, q, r *are contained in* $U_x \cap Y$, *then the following hold*:

(i) *There exists a smooth ruled surface* $\blacktriangle(pqr)$ *which is totally geodesic in* X *and isometric to the ruled surface* $\tilde{\blacktriangle}(pqr)$ *in* $M^2(k)$.

(ii) *If* $\tilde{p}\hat{q}$, $\tilde{p}\hat{r}$ *are minimal geodesics on* $M^2(k)$ *with the same lengths as* pq, pr *and the same angle at* p, *then* $d(q, r) = d(\hat{q}, \hat{r})$.

PROOF. We will first prove the proposition in the case where $\mathrm{Curv}(X) \geq k$.

(i) If Curv$(X)=k$ over Y, then for each point $x \in X$ we can take an open neighborhood V_x around x which satisfies the condition of Definition 2.1 as well as the Alexandrov property. Without loss of generality, we assume that the point v in Definition 2.1 is on the minimal geodesic qr. By Proposition 1.2 (ii) and the Toponogov theorem, we have

$$\pi = \sphericalangle pvq + \sphericalangle pvr \geq \sphericalangle \tilde{p}\tilde{v}\tilde{q} + \sphericalangle \tilde{p}\tilde{v}\tilde{r} = \pi .$$

Thus we can conclude that $\sphericalangle pvq = \sphericalangle \tilde{p}\tilde{v}\tilde{q}$ and $\sphericalangle pvr = \sphericalangle \tilde{p}\tilde{v}\tilde{r}$. For an arbitrary point $u \in qv$ and $\hat{u} \in \tilde{q}\tilde{v}$ with $d(q, u) = d(\tilde{q}, \hat{u})$, by the Hinge theorem for pv, vu, and $\sphericalangle pvu$, we have $d(p, u) \leq d(\tilde{p}, \hat{u})$. By the curvature condition, we already have $d(p, u) \geq d(\tilde{p}, \hat{u})$, and hence we can conclude that $d(p, u) = d(\tilde{p}, \hat{u})$. In view of the properties of angle and the Toponogov theorem, we have

$$\pi = \sphericalangle puq + \sphericalangle pur \geq \sphericalangle \tilde{p}\hat{u}\tilde{q} + \sphericalangle \tilde{p}\hat{u}\tilde{r} = \pi .$$

Thus we can obtain that $\sphericalangle puq = \sphericalangle \tilde{p}\hat{u}\tilde{q}$, and if the limit of pu as $u \to q$ is a minimal geodesic pq, then we have $\sphericalangle pqr = \tilde{\sphericalangle} pqr$. Then, by Proposition 1.4, (i) follows.

(ii) Suppose that $d(q, r) \neq d(\hat{q}, \hat{r})$. Let $\tilde{\triangle}(pqr)$ be a geodesic triangle in $M^2(k)$ such that $\hat{q} = \tilde{q}$. Then we have $\tilde{r} \neq \hat{r}$, and hence $\sphericalangle \hat{q}\tilde{p}\hat{r} \neq \tilde{\sphericalangle} qpr$. By (i) there exists a minimal geodesic pr satisfying $\tilde{\sphericalangle} qpr = \sphericalangle qpr$, and we can conclude that $\sphericalangle \hat{q}\tilde{p}\hat{r} \neq \sphericalangle qpr$, a contradiction.

In the case where Curv$(X) \leq k$, all the inequalities in the above proof should be replaced by the opposite inequalities and also the sum of adjacent angles can only be not less than π. Hence we obtain the same conclusion as in the lower bound case. The proof for (ii) is almost identical and omitted. □

If the open set U_x in the proposition can be chosen to be convex, for example when Curv$(X) \leq k$, the geodesic triangle $\triangle(pqr)$ is unique and we can obtain the same conclusion for any geodesic triangle with vertices in $U_x \cap Y$. Therefore even when the curvature is bounded above we can obtain $\pi = \sphericalangle pvq + \sphericalangle pvr$ as a result. In fact, for a minimal geodesic pq in Y, if we had a strict inequality in Proposition 1.2 (i) (Curv$(X) \leq k$), it would lead to a contradiction, and hence we have:

COROLLARY 2.3. *If* Curv$(X)=k$ *over* Y, *then the sum of adjacent angles of any minimal geodesic in* Y *is equal to* π.

The above Proposition 2.2 is the main ingredient we need for the proof of our main theorem.

3. Main result. In this section, we will construct surfaces Σ_i in Alexandrov spaces and prove the main theorem. Throughout this section X is an Alexandrov space with curvature bounded either below or above by k.

In order to construct surface Σ_1, we need the notion that each of the points in Σ_1

has no conjugate points. Let G be a set of minimal geodesics $\alpha: [0, l] \to X$ having the uniform metric d_H defined by $d_H(\alpha, \beta) = \sup d(\alpha(t), \beta(t))$. We then define the endpoint map End: $G \to X$ by $\mathrm{End}(\alpha) = \alpha(l)$.

DEFINITION 3.1. Let G_p be the set of all minimal geodesics starting from p. We say q is not *conjugate* to p along a minimal geodesic pq if the endpoint map End on G_p maps some neighborhood of pq in G_p homeomorphically onto a neighborhood of q in X.

In the case of complete Riemannian manifolds, this definition is equivalent to the usual one (cf. [1], [29]).

The subset Σ_1 in X is, by definition, the union of the traces of minimal geodesics in G_1 described below. For a fixed point $p \in X$, let G_1 be the maximal set of minimal geodesics emanating from p satisfying the following conditions:

(1a) If α, β, γ are minimal geodesics in G_1, then $\sphericalangle(\alpha, \gamma) = \sphericalangle(\alpha, \beta) + \sphericalangle(\beta, \gamma)$, $\sphericalangle(\beta, \alpha) = \sphericalangle(\beta, \gamma) + \sphericalangle(\gamma, \alpha)$, or $\sphericalangle(\gamma, \beta) = \sphericalangle(\gamma, \alpha) + \sphericalangle(\alpha, \beta)$.

(1b) The point p has no conjugate points in Σ_1.

If $\mathrm{Curv}(X) \leq k$ for $k > 0$, then it is further required that the perimeter of any geodesic triangle with vertex p in Σ_1 is less thant $2\pi/\sqrt{k}$. By the condition (1b) of G_1, for any point $x \in \Sigma_1$, there exists a unique minimal geodesic px in G_1. Moreover, for any two points y and z on the minimal geodesic px, the minimal geodesic yz is unique in X.

When a tangent space of X at p can be defined, for example in the case of Riemannian manifolds, the condition (1a) above means that the initial vectors of the minimal geodesics in G_1 are contained in a 2-dimensional subspace of the tangent space, and hence Σ_1 is a surface. If X is a finite Hausdorff dimensional Alexandrov space with curvature bounded below, then we can define the tangent cone and the exponential map at a point in X (cf. [6]). Two minimal geodesics emanating from a point are by definition equivalent if one is a subarc of the other. For a point $p \in X$, let Ω'_p be the set of all equivalence classes of minimal geodesics emanating from p. The space of directions Ω_p at p is the completion of Ω'_p with respect to the angle distance (Proposition 1.2 (i)). We denoted by x' the set consisting of all directions represented by minimal geodesics joining p to x. If $\xi \in x'$, we define the exponential map, $\exp_p \xi t$, as the minimal geodesic px parametrized by the arclength. The tangent cone at $p \in X$ is defined to be the cone over the space of directions Ω_p. In fact, the construction of Σ_1 is modelled on the exponential image of a plane in a tangent cone.

LEMMA 3.2. *Let X be an Alexandrov space with curvature bounded either below or above by k and let Σ_1 be as constructed as above. If* $\mathrm{Curv}(X) = k$ *over Σ_1 then Σ_1 is totally geodesic in X.*

PROOF. If $\mathrm{Curv}(X) = k$ over Σ_1, for a fixed point $y \in \Sigma_1$ let py be the unique minimal geodesic in Σ_1. For any point $z \in py$, we can find an open neighborhood U_z around z which satisfies the condition of Definition 2.1 and the Alexandrov property. We further assume that there are no conjugate points of p in U_z. Since $\{U_z | z \in py\}$ is an open covering of the compact set py, by the Lebesgue number lemma and the condition (1b), it is easy to see that there exists an open neighborhood V_y such that for any two distinct points $q, r \in \Sigma_1 \cap V_y$, we have

$$pq \cup pr \subset \bigcup_{z \in py} U_z.$$

We can now choose points $p = q_0, \ldots, q_i, \ldots, q_n = q$ on the minimal geodesic pq and $p = r_0, \ldots, r_i, \ldots, r_n = r$ on pr so that, by joining these points on the sides of the geodesic triangle $\triangle(pqr)$, we obtain small geodesic triangles $\triangle(q_i q_{i+1} r_{i+1})$ and $\triangle(q_i r_{i+1} r_i)$, $i = 0, 1, 2, \ldots, n-1$, in the sense that each of them is contained in U_z for some $z \in py$. Of course, these geodesic triangles may not be unique for given vertices. In fact, the edges $q_i q_{i+1}$ and $r_i r_{i+1}$ are unique because of the condition (1b), but the edges $q_i r_i$ and $q_i r_{i+1}$ may not be unique. By Proposition 2.2 (i), there exists a smooth ruled surface $\blacktriangle(pq_1 r_1)$ which is totally geodesic in X and isometric to the ruled surface $\tilde{\blacktriangle}(pq_1 r_1)$ in $M^2(k)$, and hence there exists a minimal geodesic $q_1 r_1$ satisfying $\sphericalangle pq_1 r_1 = \tilde{\sphericalangle} pq_1 r_1$.

Now we can find extensions $\tilde{p}\hat{q}$ of $\tilde{p}\tilde{q}_1$ and $\tilde{p}\hat{r}$ of $\tilde{p}\tilde{r}_1$ in $M^2(k)$ so that $d(p, q) = d(\tilde{p}, \hat{q})$ and $d(p, r) = d(\tilde{p}, \hat{r})$. We first take a point $\hat{q}_2$ in $\tilde{p}\hat{q}$ with $d(q_1, q_2) = d(\tilde{q}_1, \hat{q}_2)$. Since $\sphericalangle pq_1 r_1 = \tilde{\sphericalangle} pq_1 r_1$, by Corollary 2.3, we have $\sphericalangle q_2 q_1 r_1 = \sphericalangle \hat{q}_2 \tilde{q}_1 \tilde{r}_1$. Then, by Proposition 2.2 (ii), we obtain that $d(r_1, q_2) = d(\tilde{r}_1, \hat{q}_2)$ (i.e. $\triangle(\tilde{q}_1 \tilde{r}_1 \hat{q}_2) = \tilde{\triangle}(q_1 r_1 q_2)$). From Proposition 2.2 (i) there exists a minimal geodesic $r_1 q_2$ such that $\sphericalangle pq_2 r_1 = \sphericalangle q_1 q_2 r_1 = \tilde{\sphericalangle} q_1 q_2 r_1 = \tilde{\sphericalangle} pq_2 r_1$. Thus the geodesic triangle $\triangle(pq_2 r_1)$ satisfies the conditions in Proposition 1.5, which implies that for the minimal geodesic $r_1 q_2$ we have $\sphericalangle rr_1 q_2 = \sphericalangle \hat{r}\tilde{r}_1 \tilde{q}_2$.

By induction on $i = 0, 1, 2, \ldots, n-1$, we may now conclude that the geodesic triangle $\triangle(pqr)$ satisfies the hypothesis in Proposition 1.5, and hence there exists a smooth ruled surface $\blacktriangle(pqr)$ which is totally geodesic and isometric to $\tilde{\blacktriangle}(pqr)$ in $M^2(k)$. We note that by the condition (1b) the given minimal geodesics pq and pr are the edges of $\blacktriangle(pqr)$. Since the minimal geodesic qr in contained in U_z for some z, there are no conjugate points of p on qr, and hence the condition (1b) is satisfied. Furthermore, for any point $x \in qr$, we have

$$\sphericalangle qpx + \sphericalangle xpr = \tilde{\sphericalangle} qpx + \tilde{\sphericalangle} xpr = \tilde{\sphericalangle} qpr = \sphericalangle qpr ,$$

which is the condition (1a). Therefore we can conclude that the minimal geodesic qr is contained in Σ_1, and hence Σ_1 is totally geodesic. □

In order to construct the second surface Σ_2, we need the notion of parallel translation in an Alexandrov space. We say that an intrinsic metric space X satisfies the condition of the local extendibility of minimal geodesics if for each point of X there is a ball of sufficiently small radius with center at this point such that, if two points lying inside the ball can be joined by a minimal geodesic, then this can be extended so that these points become interior points of the extended minimal geodesic.

Let X be an Alexandrov space with the local extendibility of minimal geodesics. We use the following modification of the construction of parallel translation due to É. Cartan (cf. [20]). It is sufficient to consider a small open ball U of X in which one can carry out all the constructions mentioned below. Consider a minimal geodesic $\gamma: [0, l] \to U$; by dividing it in half m times we separate it into 2^m segments of equal length by points $\gamma(0) = p_0, \ldots, p_i, \ldots, p_{2^m} = \gamma(l)$. We put $l_m = l/2^m$. Then $p_i = \gamma(il_m)$. For a minimal geodesic $\sigma: [0, s] \to X$ with $\sigma(0) = p_0$, we choose m large enough that $l_m \leq s$. We first join the point $h = \sigma(l_m)$ to the midpoint $o = \gamma(l_m/2)$ of the minimal geodesic $p_0 p_1$, and then extend the minimal geodesic ho beyond o to a minimal geodesic hh_1 so that o is the midpoint of hh_1. We again join h_1 to $o_1 = \gamma(3l_m/2)$, the midpoint $p_1 p_2$, and then extend it to a minimal geodesic $h_1 h_2$ beyond o_1 so that o_1 is the midpoint of $h_1 h_2$. By connecting p_2 and h_2 we obtain a geodesic σ_1 starting from p_2. We may say that the direction of σ_1 at p_2 is approximately parallel to that of σ. We start from h_2 and repeat this process. Then, in m steps we arrive at $\gamma(l)$ and obtain a minimal geodesic σ_m starting from $\gamma(l)$. We adjust the length of σ_m so that it is same as that of σ, and denoted it by $\Pi_m(\sigma)$. If there exists a limit of $\Pi_m(\sigma)$ as $m \to \infty$, then by definition it is called a *parallel translation* of σ along the minimal geodesic γ.

In general, a direction parallel to a given direction may not be unique, and therefore this parallel translation may not preserve angles. As an example, we consider a cone and two rays starting from the vertex with the maximum angle. Then we will have two directions along a ray which are parallel to the other ray. However, in a space with curvature bounded both below and above, a parallel translation along an arbitrary rectifiable curve is an isometric map of the corresponding tangent spaces (cf. [20]). In the case of a Riemannian manifold, the above construction of parallel translation coincides with the usual one determined by the Riemannian connection (cf. [2], [3], [20]).

We now define the subset Σ_2 in X. The subset Σ_2 in X is, by definition, the union of the traces of minimal geodesics in G_2 described below. Let $\gamma: [a, b] \to X$ be a minimal geodesic and let σ be a minimal geodesic with $\gamma(a) = \sigma(0)$. Let G_2 be the maximal set of minimal geodesics σ_s emanating from $\gamma(s)$ for each $s \in [a, b]$ satisfying the following conditions:

(2a) For reach $s \in [a, b]$, a minimal geodesic σ_s is a parallel translation of σ along $\gamma|_{[a, s]}$.

(2b) For a minimal geodesic $\alpha: [0, l] \to X$ in G_2 with $\alpha(0) = \gamma(s_0)$, there exists an open neighborhood U of $\alpha(l)$ in X such that the coordinate map $\mathrm{Cor}: U \cap \Sigma_2 \to [a, b] \times \boldsymbol{R}$, $\mathrm{Cor}(\sigma_s(t)) = (s, t)$, is a well-defined homeomorphism onto an open neighborhood of $(s_0, l) \in [a, b] \times \boldsymbol{R}$.

If $\mathrm{Curv}(X) \leq k$ for $k > 0$, then it is further required that the perimeter of any geodesic triangle in Σ_2 is less than $2\pi/\sqrt{k}$. The condition (2b) insures that for each $p \in \Sigma_2$ there exist a unique geodesic $p_0 p \in G_2$ connecting γ and p, and in a small open neighborhood the parallel translation is unique. In a Riemannian manifold, Σ_2 can be regarded as a ruled surface produced by a parallel line field along a geodesic. Recall that we assume the local extendibility of minimal geodesics in order to define the parallel translation in Σ_2. This fact will be used again in the proof of the following lemma.

LEMMA 3.3. *Let X be an Alexandrov space with curvature bounded either below or above by k and let Σ_2 be as constructed as above. If* $\mathrm{Curv}(X)=k$ *over Σ_2 then Σ_2 is totally geodesic in X.*

PROOF. If $\mathrm{Curv}(X)=k$ over Σ_2, for a fixed point $y\in\Sigma_2$, let y_0y be the unique minimal geodesic which is parallel to σ along γ and hence contained in Σ_2. By the same idea as in the case of Σ_1, we first cover y_0y by open sets U_z, $z\in y_0y$, satisfying the condition of Definition 2.1, the Alexandrov property and the condition (2b). Then there exists an open neighborhood V_y such that for any two distinct points $q, r\in\Sigma_2\cap V_y$, the unique minimal geodesics $q_0q, r_0r\in G_2$ are contained in a small neighborhood of y_0y. Then we can take points $q_0\in\gamma, q_1,\dots,q_i,\dots,q_n=q$ on the minimal geodesic q_0q and $r_0\in\gamma, r_1,\dots,r_i,\dots,r_n=r$ on r_0r so that, by joining these points, we obtain small geodesic triangles $\triangle(q_iq_{i+1}r_i)$ and $\triangle(q_{i+1}r_{i+1}r_i)$, for $i=0, 1, 2,\dots,n-1$, each of which is contained in U_z for some z.

We first consider the small geodesic triangle $\triangle(q_0q_1r_0)$. By Proposition 2.2 (i) we know that there exists a minimal geodesic q_1r_0 satisfying $\sphericalangle r_0q_1q_0=\tilde{\sphericalangle} r_0q_1q_0$. We claim that the minimal geodesic q_1r_0 lies on Σ_2. We can first extend the minimal geodesic qq_0 to qq_{-1} beyond q_0 so that $\triangle(q_{-1}q_1r_0)$ is a small geodesic triangle. If we take a point $\hat{q}_{-1}\in M^2(k)$ with $d(\hat{q}_{-1},\tilde{q}_0)=d(q_{-1},q_0)$, then by Proposition 2.2 (i) there exists a smooth ruled surface $\blacktriangle(q_{-1}q_1r_0)$ which is totally geodesic in X and isometric to $\tilde{\blacktriangle}(q_{-1}q_1r_0)$ in $M^2(k)$. Since X has a convex ball if $\mathrm{Curv}(X)\le k$ and has no branch points if $\mathrm{Curv}(X)\ge k$, without loss of generality, we may assume that the minimal geodesic q_0r_0 is unique in X. Therefore the minimal geodesic q_0r_0 is contained in the ruled surface $\blacktriangle(q_{-1}q_1r_0)$. Then Cartan's process of parallel translation along $\tilde{q}_0\tilde{r}_0$ in $\tilde{\blacktriangle}(q_{-1}q_1r_0)$ can be carried over to the ruled surface $\blacktriangle(q_{-1}q_1r_0)$, and produce the same process on it. Therefore every point on q_1r_0 is an end point a geodesic parallel to σ along γ. Since the condition (2b) is clearly satisfied for a small geodesic triangle, we can conclude that the minimal geodesic q_1r_0 is contained in Σ_2.

Similarly, if we choose a point $\hat{r}_1\in M^2(k)$ with $d(r_0,r_1)=d(\tilde{r}_0,\hat{r}_1)$ and $\sphericalangle q_0r_0r_1=\sphericalangle\tilde{q}_0\tilde{r}_0\hat{r}_1$, then from the same parallel translation argument as above we see that there exists a minimal geodesic q_0r_1 which lies on Σ_2. Since parallel translation is unique in a small ball, the ruled surfaces $\blacktriangle(q_0q_1r_0)$ and $\blacktriangle(q_0r_1r_0)$ overlap and hence $\sphericalangle q_1r_0r_1=\sphericalangle\tilde{q}_1\tilde{r}_0\hat{r}_1$. From Proposition 2.2 (ii), we have $d(q_1,r_1)=d(\tilde{q}_1,\hat{r}_1)$, and the region $\blacktriangle(q_0q_1r_0)\cup\blacktriangle(q_1r_1r_0)$ is totally geodesic in X and isometric to $\tilde{\blacktriangle}(q_0q_1r_0)\cup\tilde{\blacktriangle}(q_1r_1r_0)$ in $M^2(k)$.

Now we can find the extensions $\tilde{p}_0\hat{q}$ of $\tilde{p}_0\tilde{q}_1$ and $\tilde{r}_0\hat{r}$ of $\tilde{r}_0\hat{r}_1$ in $M^2(k)$ so that $d(q_0,q)=d(\tilde{q}_0,\hat{q})$ and $d(r_0,r)=d(\tilde{r}_0,\hat{r})$. We take a point $\hat{q}_2$ on $\tilde{q}_0\hat{q}$ with $d(q_1,q_2)=$

$d(\tilde{q}_1, \hat{q}_2)$. By Corollary 2.3, there exists a minimal geodesic r_1q_1 satisfying $\sphericalangle r_1q_1q_2 = \sphericalangle \tilde{r}_1\tilde{q}_1\hat{q}_2$, and, by Proposition 2.2 (ii), we have $d(r_1, q_2) = d(\tilde{r}_1, \hat{q}_2)$ (i.e., $\triangle(\tilde{q}_1\tilde{q}_1\tilde{r}_1) = \tilde{\triangle}(q_1q_2r_1)$). Thus from Proposition 2.2 (i) we have $\sphericalangle q_0q_2r_1 = \sphericalangle q_1q_2r_1 = \tilde{\sphericalangle} q_1q_2r_1 = \tilde{\sphericalangle} q_0q_2r_1$, and hence the geodesic triangle $\triangle(q_0q_2r_1)$ satisfies the conditions in Proposition 1.5. Thus there exists a ruled surface $\blacktriangle(q_0q_2r_1)$ which is totally geodesic in X and isometric to $\tilde{\blacktriangle}(q_0q_2r_1)$ in $M^2(k)$. Therefore the region $\blacktriangle(q_0r_1r_0) \cup \blacktriangle(q_0q_2r_1)$ is totally geodesic in X and isometric to $\tilde{\blacktriangle}(q_0r_1r_0) \cup \tilde{\blacktriangle}(q_0q_2r_1)$ in $M^2(k)$.

By induction on $i = 0, 1, 2, \ldots, n-1$, we may now conclude that the region $\blacktriangle(q_0qr_0) \cup \blacktriangle(r_0qr)$ is totally geodesic in X and isometric to $\tilde{\blacktriangle}(q_0qr_0) \cup \tilde{\blacktriangle}(r_0qr)$ in $M^2(k)$. Thus from Cartan's process of parallel translation as above we see that every point on the minimal geodesic qr is an end point of a geodesic parallel to σ along γ. Again, in a small ball, the condition (2b) is satisfied, and hence we can conclude that qr lies on Σ_2 and therefore Σ_2 is totally geodesic in X. □

We are now ready to prove our main theorem.

THEOREM 3.4. *Let X be an Alexandrov space with curvature bounded either below or above by k and Σ_i $(i = 1, 2)$ be constructed as above. Then* $\mathrm{Curv}(X) = k$ *over Σ_i if and only if Σ_i is totally geodesic in X and locally isometric to $M^2(k)$.*

PROOF. Suppose Σ_i $(i = 1, 2)$ is totally geodesic in X and locally isometric to $M^2(k)$. Then for any small geodesic triangle $\triangle(pqr)$ with vertices in Y we know that there exists a ruled surface $\blacktriangle(pqr)$ which is contained in Σ_i and isometric to the ruled surface $\tilde{\blacktriangle}(pqr)$ in $M^2(k)$. Thus it clearly satisfies the required condition of Definition 2.1, and hence we have $\mathrm{Curv}(X) = k$ over Σ_i.

Suppose $\mathrm{Curv}(X) = k$ over Σ_i $(i = 1, 2)$. Then, by Lemmas 3.2 and 3.3, we can show that Σ_i is totally geodesic in X, and we can easily show that it is also locally isometric to $M^2(k)$. □

REMARK. Let X be a 3-dimensional Euclidean space. Then X is an Alexandrov space with curvature bounded both above and below by zero and the curvature of X takes the extreme value over any subset in X. However we can easily construct a surface which is not totally geodesic unless it is of the type of Σ_i. Therefore the construction of Σ_i as we do is necessary.

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