

## OPERATING FUNCTIONS ON FOURIER MULTIPLIERS

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**Abstract.** We consider the algebra of translation invariant operators of  $L^p(T)$  to  $L^q(T)$ . If  $1 \leq p < q \leq \infty$ , the spectrum of such an operator coincides with the closure of the range of the corresponding multiplier. Furthermore, if  $p \leq 2 < q$ , the operating functions on the algebra of multipliers are characterized.

**1. Introduction.** Let  $G$  be a compact abelian group and  $\Gamma$  be the dual to  $G$ . Let  $1 \leq p, q \leq \infty$ . A complex valued function  $a$  on  $\Gamma$  is called an  $(L^p, L^q)$ -multiplier if the operator  $T = T_a$  defined by the Fourier transform

$$(\widehat{Tf})(\gamma) = a(\gamma)\hat{f}(\gamma), \quad \gamma \in \Gamma$$

for trigonometric polynomials  $f(x) = \sum \hat{f}(\gamma)\gamma(x)$  has a bounded extension of  $L^p$  to  $L^q$ . The multiplier  $a$  is identified with an  $(L^p, L^q)$  bounded convolution operator associated with a pseudomeasure  $T$  such that  $\hat{T} = a$ . Such an operator is also denoted by  $T$ . The set of  $(L^p, L^q)$ -multipliers is denoted by  $M(p, q)$ .

By duality we have  $M(p, q) = M(q', p')$ , where  $1 = 1/p + 1/p'$  and  $1 = 1/q + 1/q'$ . If  $1 \leq q < p \leq 2$ , then  $M(p, q) \subset M(s, s)$  for all  $p < s < p'$  (see Doss [2]). Therefore we restrict our attention to the case  $1 \leq p \leq q \leq \infty$ .

$M(p, q)$  is a commutative Banach algebra with pointwise multiplication. It has unit if  $p = q$  and does not if  $p < q$  and  $G$  is infinite.

It is not difficult to see that  $M(2, 2) = L^\infty(\Gamma)$  and the set  $M(1, p)$  is identified with  $L^p(G)$  if  $1 < p \leq \infty$  and with  $M(G)$ , the set of Borel measures, if  $p = 1$ . In the other cases no effective characterization of  $M(p, q)$  seems to be known. However, some sufficient conditions for a sequence on  $\Gamma$  to belong to  $M(p, q)$ , and some properties and examples of  $(L^p, L^q)$ -multipliers are studied by many authors.

A function  $\Phi$  on the domain  $\Omega$  of the complex plane is said to operate on  $M(p, q)$  if  $\Phi(a) \in M(p, q)$  for  $a \in M(p, q)$  such that the range of  $a$  is contained in  $\Omega$ . When  $p = q = 1$ , such a function is characterized by Kahane-Rudin [6], that is,  $\Phi$  is an operating function if and only if it has an entire analytic extension and their result is extended by Igari [5] to the case  $p \neq 2$  (cf. also Sarnak [9], Sato [10]).

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There are distinctive properties on  $M(p, q)$  between the cases  $p = q$  and  $p \neq q$ . A characterization of the operating functions for the case  $p = 1$  and  $2 \leq q \leq \infty$  is given by Rudin [8] and Rider [7]. By their results the only operating functions on  $[-1, 1]$  of  $\hat{L}^q$  to  $\hat{L}^q$  are of the form  $\alpha t + |t|^{2(1-1/q)}\Phi_0(t)$ , where  $\Phi_0(t)$  is any function bounded near the origin.

In this paper we shall give some sufficient conditions for a function to operate on  $(L^p, L^q)$ -multipliers when  $p \neq q$ . Furthermore we shall show in §3 that our condition is also necessary if  $1 \leq p \leq 2 < q \leq \infty$ . For the results and examples of multipliers related to our paper we refer to Bonami [1], Hare [3], [4], Zygmund [11; Chap. XII] and their references.

**2. Spectra of Fourier multipliers in  $M(p, q)$ .** In this section we shall show some algebraic properties of multipliers for general orthogonal expansion. Let  $L^p, 1 \leq p \leq \infty$ , be infinite dimensional Lebesgue spaces on a finite measure space. Let  $\{\phi_n(x); n = 0, \pm 1, \pm 2, \dots\}$  be a complete orthonormal system in  $L^2$  such that  $\|\phi_n\|_\infty < \infty$ . For a function  $f \in L^1$ , let  $\hat{f}_n$  be the  $n$ -th Fourier coefficient with respect to  $\{\phi_n\}$ .

For a sequence  $a = \{a(n)\}$  the Fourier multiplier operator  $T = T_a$  with respect to  $\{\phi_n\}$  is defined by

$$Tf = \sum a(n)\hat{f}_n\phi_n$$

for any finite sum  $f = \sum \hat{f}_n\phi_n$ . Let  $1 \leq p \leq q \leq \infty$ .  $M(p, q)$  stands for the set of all operators  $T$  which have the bounded extension from  $L^p$  to  $L^q$ .

$C$  will denote a positive constant which will be different in each occurrence.

**PROPOSITION 1.**  *$M(p, q)$  is a commutative Banach algebra with operator norm. If  $1 \leq p < q \leq \infty$ , then it has no unit.*

**PROOF.** It suffices to show the last part for  $q < \infty$ .

If  $\|f\|_q \leq C\|f\|_p$  for any finite sum  $f = \sum \hat{f}_n\phi_n$ , then we have, for any non-negative function  $g$ ,

$$\|g^\theta\|_1 = \|g^{\theta/q}\|_q^q \leq C^q \|g^{\theta/q}\|_p^q \leq C^q \|g^{\theta p/q}\|_1^{q/p} \leq C^{q(1+(q/p)+\dots+(q/p)^{n-1})} \|g^{\theta(p/q)^n}\|_1^{(q/p)^n}.$$

Put  $\theta = (q/p)^n$ . Then

$$\|g\|_\theta \leq C^{pq(\theta^{-1})/(q-p)\theta} \|g\|_1$$

for every  $n$  and  $g \geq 0$ . Hence  $\|g\|_\infty \leq C^{pq/(q-p)} \|g\|_1$ , which is absurd, since  $L^1$  is assumed to be infinite dimensional.

We can identify the operator  $T_a$  with the multiplier  $a$ . The multiplier associated with  $T$  is denoted by  $\hat{T}$  and we use the notation  $T$  for  $\hat{T}$  if there is no confusion.

**DEFINITION 1.** A function  $\Phi(z)$  in a domain  $\Omega$  of the complex plane is said to operate on  $M(p, q)$ , if  $\Phi(T) \in M(p, q)$  for every  $T \in M(p, q)$  such that the range of  $\hat{T}$  is contained in  $\Omega$ .

**THEOREM 1.** *Let  $1 \leq p < \infty$  and  $\Phi_0$  be a function in  $[-1, 1]$ . Assume that  $\Phi_0$  is bounded near the origin if  $p = 1$  or  $q = \infty$  and uniformly bounded in  $[-1, 1]$  if  $p > 1$ .*

(i) *Suppose  $1 \leq p < q \leq 2$  or  $2 \leq p < q \leq \infty$ . Let  $\beta_0 = (1/q - 1/2)/(1/p - 1/q)$  or  $(1/2 - 1/p)/(1/p - 1/q)$  respectively and  $n_0$  be the smallest integer such that  $n_0 \geq \beta_0$ . Then for any constants  $\alpha_1, \alpha_2, \dots, \alpha_{n_0}$*

$$\Phi(t) = \alpha_1 t + \alpha_2 t^2 + \dots + \alpha_{n_0} t^{n_0} + |t|^{\beta_0+1} \Phi_0(t)$$

*operates on  $M(p, q)$ .*

(ii) *Suppose  $1 \leq p < 2 \leq q \leq \infty$ . Let  $\beta_1 = \min \{ (1/2 - 1/q)/(1/p - 1/2), (1/p - 1/2)/(1/2 - 1/q) \}$ . Then for any constant  $\alpha$*

$$\Phi(t) = \alpha t + |t|^{\beta_1+1} \Phi_0(t)$$

*operates on  $M(p, q)$ .*

In §3, Theorem 3 we shall show that the converse of (ii) holds for the trigonometric expansion case.

**LEMMA 1.** *Let  $1 \leq p < q < 2$ . Suppose that  $T \in M(p, q)$  and the corresponding multiplier to  $T$  is real valued. Put*

$$1/q_n = 1/p - n(1/p - 1/q) \quad \text{and} \quad 1/s_n = 1/2 + n(1/p - 1/q).$$

*Let  $n_0$  be the integer defined in Theorem 1, (i). Then  $T^n \in M(s_n, 2)$  and*

$$\|T^n\|_{M(s_n, 2)} \leq \|T\|_{M(p, q)}^n \quad \text{for } n = 1, 2, \dots, n_0.$$

**PROOF OF LEMMA 1.** We have that  $T \in M(q_n, q_{n+1})$  for  $n = 0, \dots, n_0$ . In fact, since  $T \in M(p, q)$  and  $\hat{T}$  is real valued,  $\|T\|_{M(p, q)} = \|T\|_{M(q', p')}$ . Thus  $T \in M(u, v)$  and  $\|T\|_{M(u, v)} \leq \|T\|_{M(p, q)}$  by interpolation, where  $1/u = (1 - \theta)/p + \theta/q'$ ,  $1/v = (1 - \theta)/q + \theta/p'$  and  $0 \leq \theta \leq 1$ . Choose  $\theta$  so that  $\theta = n(1/p - 1/q)/(1/p + 1/q - 1)$ . Then we get our assertion.

Now we have that  $T^n \in M(q_0, q_1)M(q_1, q_2) \dots M(q_{n-1}, q_n)$  and  $\|T^n\|_{M(p, q_n)} \leq \|T\|_{M(p, q)}^n$  for  $n = 1, \dots, n_0$ . Remark that  $q_{n_0+1} \geq 2 > q_{n_0}$  by definition. Since  $\|T^n\|_{M(p, q_n)} = \|T^n\|_{M(q'_n, p')}$  and  $q_n < 2$ , we have  $T^n \in M(s_n, 2)$  where  $1/s_n = (1 - \tau)/p + \tau/q'_n$  and  $0 \leq \tau \leq 1$  is chosen so that  $1/2 = (1 - \tau)/q_n + \tau/p'$ , which proves our lemma.

**LEMMA 2.** *Let  $1 < p \leq 2 \leq q \leq p'$ . Let  $a$  be the multiplier operator defined by a bounded sequence  $\{a(n)\}$  and  $T \in M(p, q)$  with real valued multiplier. Then*

(i) 
$$\|Ta\|_{M(p, 2)} \leq \|a(\cdot)\|_\infty \|T\|_{M(p, q)},$$

*and*

(ii) 
$$\|T^2a\|_{M(p, p')} \leq \|a(\cdot)\|_\infty \|T\|_{M(p, q)}^2.$$

**PROOF OF LEMMA 2.** For  $f \in L^p$  we have

$$\|Taf\|_2 \leq \|a(\cdot)\|_\infty \|T\|_{M(p, 2)} \|f\|_p.$$

Since  $\|T\|_{M(p,2)} \leq \|T\|_{M(p,q)}$ , we get (i).

Remark that  $\|T^2 a\|_{M(p,p')} \leq \|Ta\|_{M(p,2)} \|T\|_{M(2,p')}$  and  $\|T\|_{M(2,p')} = \|T\|_{M(p,2)}$ . Then (ii) follows from (i).

PROOF OF THEOREM 1. By a simple consideration we can assume that  $\Phi_0$  is uniformly bounded.

(i) Suppose that  $1 \leq p < q < 2$  and  $n_0 \geq 2$ . Let  $T$  be a multiplier in  $M(p, q)$  with range( $\hat{T}$ )  $\subset [-1, 1]$ . To prove (i) we apply Lemmas 1 and 2. We may assume that  $\alpha_1 = \dots = \alpha_{n_0} = 0$ . Thus  $\Phi(t) = |t|^{\beta_0+1} \Phi_0(t)$ . For  $0 \leq \Re z \leq 1$  define  $S^z$  by

$$\hat{S}^z(n) = \text{sign } \hat{T}(n) |\hat{T}(n)|^{(n_0-1)z+1} \Phi_0(\hat{T}(n)).$$

Then with notations given in Lemma 1

$$\|S^{iy}\|_{M(s_1,2)} \leq \|\Phi_0\|_\infty \|T\|_{M(s_1,2)} \leq \|\Phi_0\|_\infty \|T\|_{M(p,q)}$$

and

$$\|S^{1+iy}\|_{M(s_{n_0},2)} \leq \|\Phi_0\|_\infty \|T^{n_0}\|_{M(s_{n_0},2)} \leq \|\Phi_0\|_\infty \|T\|_{M(p,q)}^{n_0}.$$

Thus by Stein's interpolation theorem

$$\|S^\theta\|_{M(q,2)} \leq \|\Phi_0\|_\infty \|T\|_{M(p,q)}^{1+\theta(n_0-1)},$$

where  $\theta$  is chosen so that  $1/q = (1-\theta)/s_1 + \theta/s_{n_0}$  which is equivalent to say that  $\theta(n_0-1)+1 = \beta_0$ . Thus

$$\hat{S}^{\theta+1}(n) = |\hat{T}(n)|^{\beta_0+1} \Phi_0(\hat{T}(n))$$

and

$$\|S^{\theta+1}\|_{M(p,q)} \leq \|S^{\theta+1}\|_{M(p,2)} \leq \|T\|_{M(p,q)} \|S^\theta\|_{M(q,2)} \leq \|\Phi_0\|_\infty \|T\|_{M(p,q)}^{\beta_0+1},$$

which proves (i) for  $n_0 \geq 2$ .

When  $n_0 = 1$ , define  $R^z$  by

$$\hat{R}^z(n) = \text{sign } \hat{T}(n) |\hat{T}(n)|^z \Phi_0(\hat{T}(n)).$$

Then

$$\|R^{iy}\|_{M(2,2)} \leq \|\Phi_0\|_\infty \quad \text{and} \quad \|R^{1+iy}\|_{M(s_1,2)} \leq \|\Phi_0\|_\infty \|T\|_{M(p,q)}.$$

Thus

$$\|R^\theta\|_{M(q,2)} \leq \|\Phi_0\|_\infty \|T\|_{M(p,q)}^\theta,$$

where  $1/q = (1-\theta)/2 + \theta/s_1$ , that is,  $\theta = \beta_0$ . The rest of the proof is similar to the case  $n_0 \geq 2$ . Thus (i) is proved.

(ii) Suppose  $T \in M(p, q)$  and range( $\hat{T}$ )  $\subset [-1, 1]$ . Since  $\|T\|_{M(p,q)} = \|T\|_{M(q',p')}$ , we consider the case  $q \leq p'$ . We assume  $\Phi(t) = |t|^{\beta_1+1} \Phi_0(z)$ .

Applying Lemma 2 with  $a_n = \hat{R}^z(n)$  we get

$$\|R^z\|_{M(p,2)} \leq \| \Phi_0 \|_\infty \|T\|_{M(p,q)}, \quad \Re z = 0$$

and

$$\|R^z\|_{M(p,p')} \leq \| \Phi_0 \|_\infty \|T\|_{M(p,q)}^2, \quad \Re z = 1.$$

By interpolation again we have

$$\|TR^\theta\|_{M(p,q)} \leq \| \Phi_0 \|_\infty \|T\|_{M(p,q)}^{\theta+1},$$

where  $1/q = (1 - \theta)/2 + \theta/p'$ , i.e.,  $\theta = \beta_1$ . This proves that  $\Phi$  operates on  $M(p, q)$ .

REMARK 1. Lemmas 1 and 2 are valid for complex valued multipliers under some conditions, for example, that the orthogonal system  $\{\phi_n\}$  consists of real valued functions or the characters of a compact abelian group. Under such a condition the domain  $[-1, 1]$  of  $\Phi_0$  in Theorem 1 is replaced by a domain  $\Omega$  containing the origin of the complex plane.

THEOREM 2. Let  $X$  be a set and  $B$  be a Banach algebra of bounded functions on  $X$  with pointwise multiplication.

Let  $N$  be a non-negative integer. Suppose that analytic functions with zero of order  $N$  at 0 operate in  $B$ , that is, if  $f \in B$  and  $\Phi(z) = z^N \Phi_0(z)$  where  $\Phi_0$  is an analytic function in a domain containing the range of  $f$ , then  $\Phi(f) \in B$ . Then the spectrum of  $f$  in  $B$  coincides with the closure of the range of  $f$ .

PROOF. It suffices to show that  $\text{sp}(f, B) \subset \overline{\text{range}(f)}$ .

Suppose that  $\lambda \notin \overline{\text{range}(f)}$  and  $\lambda \in \text{sp}(f, B)$ . We may assume  $N \geq 1$ . Choose a homomorphism  $\xi$  such that  $\xi(f) = \lambda$ . Put

$$\Phi(z) = \left( \frac{1}{z - \lambda} + \frac{1}{\lambda} \right)^N = z^N \Phi_0(z).$$

Then  $\Phi_0(z)$  is analytic on  $\overline{\text{range}(f)}$ . Thus  $\Phi(f) = g \in B$ . Then  $f^N = \lambda^N (f - \lambda)^N g$ . Hence  $\lambda^N = \lambda^N (\xi(f) - \lambda)^N \xi(g) = 0$ . Thus  $\lambda = 0$ . By assumption,  $\Psi(z) = z^N (1/z^{N+1})$  operates on  $f$ . Thus  $\Psi(f)f = 1 \in B$ . This is impossible if  $B$  has no unit. If  $B$  has unit, this contradicts that  $0 \neq \xi(f) = \lambda$ .

3. **Operating functions on  $M(p, q)$ .** In this section we shall restrict our argument to the most typical trigonometric expansion case, that is,  $G = [-\pi, \pi]$  and  $\phi_n(x) = e^{inx}$ . The converse of Theorem 3, (ii) is valid.

THEOREM 3. Let  $1 \leq p < 2 \leq q \leq \infty$  and  $\Phi$  be a function on  $[-1, 1]$ . If  $\Phi$  operates on  $M(p, q)$ , then  $\Phi$  is of the form that  $\Phi(t) = \alpha t + |t|^{\beta_1+1} \Phi_0(t)$ , where  $\alpha$  is any complex number and  $\beta_1$  is the number given in Theorem 1, and  $\Phi_0$  is a function in  $[-1, 1]$  bounded near the origin if  $p = 1$  or  $q = \infty$  and uniformly bounded in  $[-1, 1]$  if  $p > 1$ .

PROOF. As mentioned in the proof of Lemma 2, we may assume that  $1/2 - 1/q \leq$

$1/p - 1/2$ . Our proof is divided into several steps.

First step. *If  $\Phi$  operates on  $M(p, q)$ , then there exist two constants  $C$  and  $\eta > 0$  such that if  $\|T\|_{M(p, q)} < \eta$ , then  $\|\Phi(T)\|_{M(p, q)} \leq C$ .*

In fact, if this does not hold, for any positive integer  $m$  there is a multiplier  $T_m$  such that

$$\|T_m\|_{M(p, q)} < \frac{1}{m^2} \quad \text{and} \quad \|\Phi(T_m)\|_{M(p, q)} > m.$$

By a simple consideration we can assume that  $T_m$  are polynomials. Let  $N_m$  be the degree of  $T_m$ . Put  $T(x) = \sum_{m=1}^{\infty} e^{in_m x} T_m(x)$ , where  $\{n_m\}$  is chosen so that  $n_m + 3N_m < n_{m+1} - 3N_{m+1}$ . Then  $T \in M(p, q)$ . Thus  $\Phi(T) \in M(p, q)$ . Let  $H_m(x) = e^{in_m x} V_{N_m}(x)$ , where  $V_N$  is the de la Vallée Poussin kernel of degree  $N$ . Then  $(\Phi(T)H_m)(x) = e^{in_m x} (\Phi(T_m))(x)$ . Thus

$$\|\Phi(T_m)\|_{M(p, q)} \leq \|H_m\|_{M(p, q)} \|\Phi(T)\|_{M(p, q)} \leq 3 \|\Phi(T)\|_{M(p, q)} < \infty,$$

which is absurd.

We remark that if  $\Phi$  operates on  $M(p, q)$ , the odd part  $\Phi(t) - \Phi(-t)$  and the even part  $\Phi(t) + \Phi(-t)$  have a similar property. Therefore we consider two cases separately.

Second step. *If  $\Phi$  is even, the condition is necessary for  $\Phi$  to operate on  $M(p, q)$ .*

The following lemma is due to Rudin-Shapiro when  $r=2$  (cf. [8]) and to Rider [7] for  $r > 2$ .

LEMMA A. *For a prime number  $r$  there is a sequence  $\{\delta^{(r)}(n)\}$  with  $\delta^{(r)}(n) = r - 1$  or  $-1$  such that*

$$\left| \sum_{n=1}^N \delta^{(r)}(n) e^{inx} \right| < (r-1)r(1 + \sqrt{r})\sqrt{N} \quad (0 \leq x \leq 2\pi; N = 1, 2, \dots).$$

Let

$$\Delta_N^{(r)}(x) = \sum_{n=1}^N \delta^{(r)}(n) e^{inx}.$$

LEMMA 3. *For a prime number  $r$  we have that*

$$(1) \quad \|\Delta_N^{(r)}\|_{M(p, q)} \leq (r-1)r(1 + \sqrt{r})N^{1/p - 1/2}.$$

PROOF OF LEMMA 3. We apply the Riesz-Thorin interpolation theorem to the inequalities  $\|\Delta_N^{(r)}\|_{M(2, 2)} = \|\Delta_N^{(r)}\|_{\infty} \leq r-1$  and  $\|\Delta_N^{(r)}\|_{M(1, s)} = \|\Delta_N^{(r)}\|_s \leq \|\Delta_N^{(r)}\|_{\infty} \leq (r-1)r(1 + \sqrt{r})\sqrt{N}$  for any  $s \geq 1$ . Define  $0 < \theta < 1$  by  $1/p = \theta/1 + (1-\theta)/2$  and  $2 \leq s \leq \infty$  by  $1/q = \theta/s + (1-\theta)/2$ . Then  $\theta/2 = 1/p - 1/2$ . Thus we get the lemma.

Since  $\Delta_N^{(2)} N^{-(1/p - 1/2)}$  is uniformly bounded in  $M(p, q)$ , by the first step there is a constant  $C > 0$  such that

$$\left\| \Phi \left( \frac{\Delta_N^{(2)}}{CN^{1/p-1/2}} \right) \right\|_{M(p,q)} \leq C.$$

Since  $\Phi$  is even,

$$\left\| \sum_{n=1}^N \Phi \left( \frac{1}{CN^{1/p-1/2}} \right) e^{inx} \right\|_q \leq C \left\| \sum_{n=1}^N e^{inx} \right\|_p.$$

Thus

$$\left| \Phi \left( \frac{1}{CN^{1/p-1/2}} \right) \right| N^{1-1/q} \leq CN^{1-1/p}.$$

If  $\Phi(t)$  is replaced by  $\Phi(\eta t)$ ,  $1/2 < \eta < 1$ , the last inequality holds uniformly in  $\eta$ . Thus

$$|\Phi(t)t^{-1}| \leq C|t|^{\beta_1} \text{ near the origin.}$$

Assume that  $p > 1$  and  $\Phi$  is not bounded. Then there exists a sequence  $t_n \in [-1, 1]$  such that  $|\Phi(t_n)| \rightarrow \infty$ . Let  $m(k) = t_n$  if  $k = 2^n$  and  $= 0$  otherwise. Then  $\{m(k)\} \in M(p, q)$ . Thus  $\Phi(m(\cdot)) \in M(p, q) \subset M(2, 2)$ , which is absurd. Thus  $\Phi_0$  is bounded in  $[-1, 1]$ .

REMARK. If  $\Phi$  is odd, then in the same way we can prove that

$$|\Phi(t)| \leq C|t|,$$

which will be used later.

In fact,  $\sqrt{N} = \|\Delta_N^{(2)}\|_2 \leq (\|\Delta_N^{(2)}\|_\infty \|\Delta_N^{(2)}\|_1)^{1/2}$ . Thus

$$5^{-1}\sqrt{N} \leq \|\Delta_N^{(2)}\|_1.$$

Now suppose that  $\Phi$  is odd. Then

$$\left\| \sum_{n=1}^N \Phi \left( \frac{1}{CN^{1/p-1/2}} \delta^{(2)}(n) \right) e^{inx} \right\|_q \leq C \left\| \sum_{n=1}^N e^{inx} \right\|_p.$$

Thus

$$\begin{aligned} 5^{-1}\sqrt{N} \left| \Phi \left( \frac{1}{CN^{1/p-1/2}} \right) \right| &\leq \left\| \sum_{n=1}^N \Phi \left( \frac{1}{CN^{1/p-1/2}} \right) \delta^{(2)}(n) e^{inx} \right\|_1 \\ &\leq \left\| \sum_{n=1}^N \Phi \left( \frac{1}{CN^{1/p-1/2}} \delta^{(2)}(n) \right) e^{inx} \right\|_q \leq C \left\| \sum_{n=1}^N e^{inx} \right\|_p \leq CN^{1-1/p}. \end{aligned}$$

Therefore we get  $|\Phi(1/CN^{1/p-1/2})| \leq CN^{1/2-1/p}$ , which proves our assertion.

Thus in the following we assume  $\beta_1 > 0$ .

Third step. *In the following  $\Phi$  is assumed to be odd. Then we have that*

$$(2) \quad \left| \Phi(t) - 2\Phi\left(\frac{t}{2}\right) \right| \leq C|t|^{1+\beta_1} \quad \text{for } t \in [-1, 1].$$

First we remark that

$$(3) \quad \Phi\left(\frac{t}{2} \delta^{(3)}(n)\right) = \frac{1}{3} \left[ \Phi(t) - 2\Phi\left(\frac{t}{2}\right) \right] + \frac{1}{3} \left[ \Phi(t) + \Phi\left(\frac{t}{2}\right) \right] \delta^{(3)}(n), \quad t \in [-1, 1]$$

$$= \Psi_1(t) + \Psi_2(t) \delta^{(3)}(n), \quad \text{say.}$$

Since  $\|\Phi(\Delta_N^{(3)}/CN^{1/p-1/2})\|_{M(p,q)} \leq C$  by the first step and since  $|\Phi(t)| \leq C|t|$ ,

$$\left\| \sum_{n=1}^N \Psi_2\left(\frac{1}{CN^{1/p-1/2}}\right) \delta^{(3)}(n) e^{inx} \right\|_{M(p,q)} = \left| \Psi_2\left(\frac{1}{CN^{1/p-1/2}}\right) \right| \|\Delta_N^{(3)}\|_{M(p,q)} \leq C.$$

Thus by (3)

$$\left\| \sum_{n=1}^N \Psi_1\left(\frac{1}{CN^{1/p-1/2}}\right) e^{inx} \right\|_{M(p,q)} \leq C, \quad N=1, 2, \dots$$

Now we apply the multiplier  $\sum_{n=1}^N \Psi_1(1/CN^{1/p-1/2})e^{inx}$  to  $f(x) = \sum_{n=1}^N e^{inx}$ . Then

$$\left\| \sum_{n=1}^N \Psi_1\left(\frac{1}{CN^{1/p-1/2}}\right) e^{inx} \right\|_q \leq C \left\| \sum_{n=1}^N e^{inx} \right\|_p.$$

Thus

$$\left| \Psi_1\left(\frac{1}{CN^{1/p-1/2}}\right) \right| N^{1-1/q} \leq CN^{1-1/p}.$$

Therefore we get our assertion as in the third step.

(2) shows that  $\{2^n \Phi(t/2^n); n=1, 2, \dots\}$  is a Cauchy sequence for every  $t \in [-1, 1]$ . Let  $\Phi_1(t) = \lim_{n \rightarrow \infty} 2^n \Phi(t/2^n)$  and put

$$\Phi(t) = \Phi_1(t) + \Phi_2(t).$$

Fourth step. *We have*

$$|\Phi_2(t)| \leq C|t|^{\beta_1+1}.$$

In fact,

$$|\Phi_2(t)| = \left| \lim_{n \rightarrow \infty} 2^n \Phi\left(\frac{t}{2^n}\right) - \Phi(t) \right| \leq \sum_{n=1}^{\infty} \left| 2^n \Phi\left(\frac{t}{2^n}\right) - 2^{n-1} \Phi\left(\frac{t}{2^{n-1}}\right) \right| \leq C|t|^{\beta_1+1}.$$

To prove that  $\Phi_1(t) = \alpha t$  we apply the technique in [7].

Fifth step.  $\Phi_1$  is continuous in  $[-1, 1]$ .

It suffices to prove that  $\Phi_1(t)$  is continuous at  $t=1$  replacing  $\Phi_1(t)$  by  $\Phi_1(\eta t)$ . Let  $\{t_m\}$  be a sequence such that  $|1-t_m| < 2^{-m}$ . Put  $N_m = [2^{m/(1/p-1/2)}]$  and let

$$T_m = \frac{1}{2^m} \sum_{n=1}^{N_m} \left\{ \delta^{(2)}(n) + \frac{1-t_m}{2} (1-\delta^{(2)}(n)) \right\} e^{inx}.$$



Then

$$\|T_m\|_{M(p,q)} \leq 2^{-m} \left\| \sum_{n=1}^{N_m} \delta^{(2)}(n) e^{inx} \right\|_{M(p,q)} + C 2^{-2m} \left\| \sum_{n=1}^{N_m} e^{inx} \right\|_{M(p,q)}$$

To the first term on the right hand side apply (1) and note that the second term is bounded since  $1/2 - 1/q \leq 1/p - 1/2$ . Then there exists an integer  $m_0$  such that  $\|T_m\|_{M(p,q)} \leq 2^{m_0}$  for all  $m \geq 1$ .

Since  $\Phi_1$  is odd,

$$\Phi_1 \left( \delta^{(2)}(n) + \frac{1-t_m}{2} (1 - \delta^{(2)}(n)) \right) = \frac{1}{2} [\Phi_1(1) - \Phi_1(t_m)] + \frac{1}{2} [\Phi_1(1) + \Phi_1(t_m)] \delta^{(2)}(n).$$

Applying successively the relation  $2\Phi_1(t/2) = \Phi_1(t)$ ,

$$\begin{aligned} C &\geq \|\Phi_1(2^{-m_0} T_m)\|_{M(p,q)} = 2^{-m-m_0} \|\Phi_1(2^m T_m)\|_{M(p,q)} \\ &\geq C |\Phi_1(1) - \Phi_1(t_m)| 2^{-m} N_m^{1/p-1/q} - C 2^{-m} N_m^{1/p-1/2}. \end{aligned}$$

Since  $2^{-m} N_m^{1/p-1/2} \approx 1$  and  $2^{-m} N_m^{1/p-1/q} \rightarrow \infty$ , we have  $\Phi_1(t_m) \rightarrow \Phi_1(1)$  as  $m \rightarrow \infty$ , which implies that  $\Phi_1(t)$  is continuous at  $t=1$ .

Sixth step.  $\Phi_1(t) = \alpha t$  for  $t \in [-1, 1]$ .

Fix an arbitrary  $0 < t \leq 1$  and a prime number  $r$ . Since  $\Phi_1(t) = 2\Phi_1(t/2)$ , we have

(4)

$$\Phi_1 \left( \frac{t \delta^{(r)}(n)}{2^m(r-1)} \right) = \frac{1}{2^m r} \left[ \Phi_1(t) - (r-1) \Phi_1 \left( \frac{t}{r-1} \right) \right] + \frac{1}{2^m r} \left[ \Phi_1(t) + \Phi_1 \left( \frac{t}{r-1} \right) \right] \delta^{(r)}(n).$$

By Lemma 3  $\|t 2^{-m} (r-1)^{-1} \Delta_{N_m}^{(r)}\|_{M(p,q)} \leq C t 2^{-m} (r-1)^{-1} N_m^{1/p-1/2}$ . We choose  $N_m = [2^{m/(1/p-1/2)}]$  as before. Then the last term is bounded by  $2^{m_0}$  uniformly in  $m$ , where  $m_0$  is a positive integer. Thus

$$\left\| \Phi_1 \left( \frac{t}{(r-1) 2^{m+m_0}} \Delta_{N_m}^{(r)} \right) \right\|_{M(p,q)} \leq C.$$

Thus

$$\left\| \Phi_1 \left( \frac{t}{(r-1) 2^m} \Delta_{N_m}^{(r)} \right) \right\|_{M(p,q)} \leq C 2^{m_0}.$$

By (4) we get

$$C 2^{m_0} \geq \frac{1}{2^m r} \left| \Phi_1(t) - (r-1) \Phi_1 \left( \frac{t}{r-1} \right) \right| N_m^{1/p-1/q} - \frac{C}{2^m r} \left| \Phi_1(t) + \Phi_1 \left( \frac{t}{r-1} \right) \right| N_m^{1/p-1/2}.$$

If we note as before that  $2^{-m} N_m^{1/p-1/2} \approx 1$  and  $2^{-m} N_m^{1/p-1/q} \rightarrow \infty$ , we get

(5) 
$$\Phi_1(t) = (r-1) \Phi_1 \left( \frac{t}{r-1} \right).$$

(5) holds for all  $0 < t \leq 1$  and prime numbers  $r$ . Thus for all prime numbers  $q, r$  such that  $q > r$  and for  $n = 1, 2, \dots$ , we have

$$\Phi_1\left(\left(\frac{r-1}{q-1}\right)^n\right) = \left(\frac{r-1}{q-1}\right)^n \Phi_1(1).$$

Remark that the set

$$\left\{\left(\frac{r-1}{q-1}\right)^n; r \text{ and } q \text{ are prime and } n = 1, 2, \dots\right\}$$

is dense in  $(0, 1)$  and that  $\Phi_1$  is continuous. Then we have  $\Phi_1(t) = t\Phi_1(1)$  for  $0 < t \leq 1$ . This proves the sixth step.

The proof of Theorem 2 is complete.

REMARK 2. Under the conditions in Theorem 1, (i), for trigonometric expansion case,  $\Phi$  is an operating function of  $M(p, q)$  to  $M(p, 2)$  if and only if  $\Phi$  has the form  $|t|^{\beta_0+1}\Phi_0(t)$ . In fact, the if part is proved by the last inequalities in the proof of Theorem 1, (i). The only if part is shown by the same way to the proof of Theorem 3 applying the  $N$ -th Dirichlet kernel in place of  $\Delta_N^{(2)}$ .

REMARK 3. Theorem 3 will hold for infinite compact abelian groups  $G$ . If  $\Gamma$ , the dual to  $G$ , has an element of arbitrarily large order, our proof can be applied by approximation. If  $\Gamma$  is of bounded order, it will be needed to modify the proof to get an analogue of Lemma A and the other part of the proof will be almost similar.

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