LAGRANGIAN SURFACES IN THE COMPLEX EUCLIDEAN PLANE WITH CONFORMAL MASLOV FORM

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Abstract. We completely classify the compact orientable Lagrangian surfaces in the complex Euclidean plane with conformal Maslov form, obtaining a new family of embedded Lagrangian tori.

1. Introduction. The Lagrangian surfaces of the complex Euclidean plane C^2 have been extensively studied from topological and analytical points of view ([G], [We]). The topology of them can be described in the following result of Gromov [G]: "*A surface admits a Lagrangian immersion in C^2 if and only if its complexified tangent bundle is trivial*". But only few results are known about these surfaces, from a Riemannian point of view. In this paper we study a family of Lagrangian surfaces of C^2 defined in terms of a regular behaviour of its Gauss map. Lagrangian surfaces with parallel mean curvature vector (i.e. Lagrangian surfaces with harmonic Gauss map [R-V]) appear as a particular case of our family.

Let $\phi: M \to C^2$ be a Lagrangian immersion of an orientable surface M into C^2. Let $(\langle \, , \, \rangle, J)$ be the Kähler structure of C^2 with respect to which ϕ is Lagrangian. Then J defines an isomorphism between the tangent and the normal bundles of M. If v is the Gauss map of ϕ, it is known that v lies in the product $S^2 \times S^1$ of a two-dimensional sphere S^2 and a circle S^1. If $v_1: M \to S^2$ denotes the first component of the Gauss map v, then our family is defined as those Lagrangian immersions such that v_1 is a harmonic map. (Lagrangian surfaces such that $v_2: M \to S^1$ is a harmonic map will be studied in a forthcoming paper.)

In §2, we obtain some characterizations of the harmonicity of v_1. Corollary 2 says that v_1 is a harmonic map if and only if the Maslov form of ϕ is conformal, i.e. the associated vector field JH is conformal, H being the mean curvature vector of ϕ. In Corollary 1, it is proved that the harmonicity of v_1 is also equivalent to the fact that a differential cubic form Θ, or a complex vector field $\mathcal{X}$ which appear in a natural way, will be holomorphic. This allows us to reduce the case in which M is compact only to the cases in which M is a torus or a sphere (Proposition 1).

In §3, we make a local study of this kind of surfaces. In a isothermal coordinate system, we obtain the Frenet equations of such surfaces, and we prove that if the

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induced metric is given by $e^{2u}|dz|^2$, and H has no zeroes, then u satisfies an O.D.E. (the equation (3.6)). Conversely, for each solution of that equation, and integrating the Frenet equations of the immersion, we obtain a Lagrangian immersion with conformal Maslov form (Theorem 1).

In §4, using the above local theorem, we completely classify the Lagrangian immersions with holomorphic cubic form Θ identically zero (Theorem 2). In this case, the first component of the Gauss map v_1 is a conformal map. This result is parallel to the classification of the umbilical surfaces in R^3 and allows us to characterize the Whitney sphere in C^2 ([Wn]) as the only Lagrangian sphere in C^2 with conformal Maslov form (Corollary 4). It can be considered as a version of Hopf's theorem for this family of surfaces.

Finally, in §5, we also completely classify the Lagrangian immersions of a torus in C^2 with conformal Maslov form. We observe that, in this case, the equation (3.6) is the sinh-Gordon equation $u'' + \sinh 4u = 0$, whose solutions can be expressed in terms of elliptic functions. This shows the parallelism between this family of surfaces and the constant mean curvature surfaces in R^3 described by Delaunay [De]. In Theorem 3, we study the immersions $\psi : R^2 \to C^2$ associated to the solutions of the equation $u'' + \sinh 4u = 0$. We prove that for any $x \in R$ the curves $y \mapsto \psi(x, y)$ are circles in C^2 centred at the same point, and that the immersions ψ are doubly-periodic except in an exceptional case. Also, we prove that the immersion induced by ψ in the corresponding torus is always an embedding. In Proposition 2, we compute the area and estimate the Willmore functional of these tori. Finally, we classify the Lagrangian immersions of flat surfaces with conformal Maslov form (Proposition 3). Besides the flat tori, there appears a new example which does not have parallel mean curvature vector and which is described as the product of two Cornu's spirals.

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2. Lagrangian surfaces and the Gauss map. Let C^2 be the two-dimensional complex Euclidean plane endowed with a canonical structure of Kähler manifold. We denote by $\langle \, , \, \rangle$ the Euclidean metric in $C^2 \equiv R^4$ and by J a canonical complex structure. The Kähler form Ω is defined by $\Omega(X, Y) = \langle X, JY \rangle$ for any tangent vector fields X and Y.

Let M be an orientable surface and $\phi : M \to C^2$ a Lagrangian immersion, i.e. an immersion with $\phi^*\Omega = 0$. We also denote by $\langle \, , \, \rangle$ the induced metric. Then, the most elementary properties of ϕ are:

(a) The tangent and normal bundles of ϕ are isomorphisms from Riemannian point of view; this means that if ∇ and $\nabla^\perp$ denote the Levi-Civita connection of $\langle \, , \, \rangle$ and the normal connection, respectively, then

$$(2.1) \qquad\qquad J \circ \nabla = \nabla^\perp \circ J.$$

(b) If σ is the second fundamental form of ϕ and A_η is the Weingarten endo-

morphism associated to a normal vector field η, then, for any tangent vector fields X and Y,

$$(2.2) \qquad\qquad \sigma(X, Y) = JA_{JX}Y .$$

From this we deduce that if C is the trilinear map on TM given by $C(X, Y, Z) = \Omega(\sigma(X, Y), Z)$, then C is symmetric. Moreover, if ∇C denotes the covariant derivative of C, then ∇C is also symmetric.

(c)   Finally, let H be the mean curvature vector of ϕ. As JH is a tangent vector field on M, if ϖ is the 1-form on M given by $\varpi(V) = (1/\pi)\Omega(V, H)$, with V tangent to M, then ϖ is closed. ϖ is called the *Maslov form* on M.

DEFINITION 1.   Let $\phi: M \to C^2$ a Lagrangian immersion. The *Maslov* form ϖ is said to be *conformal* if the tangent vector field JH is conformal.

REMARK 1.   If ϖ is conformal, i.e. JH is a conformal vector field, then we have $\nabla_V JH = \pi(\delta\varpi/2)V$ for any tangent vector V, where $\delta\varpi$ is the codifferential of ϖ. Taking derivative in this expression, it is easy to get $\Delta\varpi = -2K\varpi$, where K is the Gauss curvature of the surface and Δ is the Laplacian. As ϖ is a closed form, we observe that the *Maslov form* is *harmonic* if and only if $\delta\varpi = 0$.

If $\phi: M \to C^2$ is a Lagrangian immersion, we can associate to ϕ a cubic differential form Θ, a complex vector field $\mathcal{X}$ and a differential form Υ on M defined by

$$\Theta(z) = f(z)(dz)^3 , \quad \text{with} \quad f(z) = 4C(\partial_z, \partial_z, \partial_z) ,$$

$$\mathcal{X}(z) = g(z)\partial_z , \quad \text{with} \quad g(z) = 2e^{-2u}\Omega(H, \partial_{\bar{z}}) ,$$

$$\Upsilon(z) = h(z)dz , \quad \text{with} \quad h(z) = 2\Omega(H, \partial_z) ,$$

where $z = x + iy$ is a local isothermal coordinate on M, $\langle \ , \ \rangle = e^{2u}ds^2$ with ds^2 the Euclidean metric, and C and Ω are extended C-linearly to the complexified tangent bundle. We observe that $h(z) = e^{2u}\bar{g}(z)$.

LEMMA 1.   *If $\phi: M \to C^2$ is a Lagrangian immersion, then in the above notation we have*:

   (i) $f_{\bar{z}} = e^{4u}\overline{g_{\bar{z}}}$.
   (ii) $\operatorname{Im}(h_{\bar{z}}) = 0$.
   (iii) $f_{\bar{z}} = -e^{2u}\mathcal{L}_{JH}\langle \ , \ \rangle(\partial_z, \partial_z)$.
   (iv) $h_{\bar{z}} = -(\pi/2)e^{2u}\delta\varpi$.
   (v) $|f|^2 = e^{6u}(|\sigma|^2 - 3|H|^2)$, $|g|^2 = e^{-2u}|H|^2$, $|h|^2 = e^{2u}|H|^2$.
   (vi) $\Delta_0 u + (|g|^2 e^{4u} - |f|^2 e^{-4u})/2 = 0$.

Here $\mathcal{L}$ is the Lie derivative, $|\sigma|$ (resp. $|H|$) is the length of σ (resp. H) with respect to the induced metric and Δ_0 is the Laplacian of the Euclidean metric ds^2.

PROOF.   Let $z = x + iy$ be a local isothermal coordinate on M. Then we have $f_{\bar{z}} = 4(\nabla C)(\partial_z, \partial_z, \partial_z, \partial_{\bar{z}})$. Taking derivative with respect to z in the equality $2\sigma(\partial_z, \partial_{\bar{z}}) =$

$e^{2u}H$ we obtain $2(\nabla\sigma)(\partial_z, \partial_z, \partial_{\bar{z}}) = e^{2u}\nabla_z^\perp H$ and so

(2.3) $$f_{\bar{z}} = 2e^{2u}\Omega(\nabla_z^\perp H, \partial_z) .$$

It is not difficult to see that $g_{\bar{z}} = 2e^{-2u}\overline{\Omega(\nabla_z^\perp H, \partial_z)}$, which proves (i).

In the same way, we get

(2.4) $$h_{\bar{z}} = 2\Omega(\nabla_{\bar{z}}^\perp H, \partial_z)$$

and it is immediate to check (ii) taking into account the closedness of ϖ.

On the other hand, using the properties of ϕ mentioned before, we have

(2.5) $$\mathscr{L}_{JH}\langle\ ,\ \rangle(X, Y) = -2\Omega(\nabla_X^\perp H, Y) ,$$

(2.6) $$(\delta\varpi)(p) = -\frac{1}{\pi}\sum_{i=1}^2 \Omega(\nabla_{e_i}^\perp H, e_i) ,$$

where X and Y are tangent vector fields, and $\{e_1, e_2\}$ is an orthonormal basis of T_pM, $p \in M$.

So, by (2.3) to (2.6), we obtain (iii) and (iv). Next, a straightforward computation leads to the expressions given in (v) for $|f|^2$, $|g|^2$ and $|h|^2$.

Finally, using (v) and the Gauss equation $K = 2|H|^2 - |\sigma|^2/2$ in the equality $\Delta_0 u = -e^{2u}K$, we see that u satisfies the differential equation given in (vi). □

As a consequence of this lemma, we can characterize the holomorphy of Θ, $\mathscr{X}$ and Υ by means of the Maslov form of the immersion ϕ.

COROLLARY 1.   *If $\phi: M \to \mathbf{C}^2$ is a Lagrangian immersion of an orientable surface, then the following properties are equivalent:*

(a)   *The Maslov form ϖ on M is conformal.*

(b)   *Θ is a holomorphic cubic form.*

(c)   *$\mathscr{X}$ is a holomorphic vector field.*

Also, the Maslov form ϖ on M is harmonic if and only if Υ is a holomorphic differential form.

REMARK 2.   Chern and Wolfson [Ch-W] observed that Θ is a holomorphic cubic form when M is a Lagrangian minimal surface of the complex projective plane.

PROOF.   From Lemma 1-(i), we get that (b) and (c) are equivalent. Using Lemma 1-(iii), Θ is holomorphic, i.e. $f_{\bar{z}} = 0$, if and only if

$$\mathscr{L}_{JH}\langle\ ,\ \rangle(\partial_x, \partial_x) = \mathscr{L}_{JH}\langle\ ,\ \rangle(\partial_y, \partial_y) \quad \text{and} \quad \mathscr{L}_{JH}\langle\ ,\ \rangle(\partial_x, \partial_y) = 0 .$$

These expressions mean that JH is a conformal vector field. Finally, from Lemma 1-(iv), we conclude the proof. □

Suppose now that $\phi: M \to \mathbf{C}^2$ is a Lagrangian immersion of a *compact* orientable surface into $\mathbf{C}^2$. If the Maslov form ϖ is conformal (resp. harmonic), from Corollary 1,

Θ and $\mathcal{X}$ (resp. Υ) are holomorphic and the Riemann-Roch Theorem imposes strong restrictions. In fact, when ϖ is conformal, if the genus of M is at least two, then the above theorem says that $\mathcal{X} \equiv 0$. When ϖ is harmonic, if the genus of M is zero, the same theorem says that $\Upsilon \equiv 0$. In both cases this means that ϕ is a minimal immersion, which is impossible because the surface M is compact. So, we obtain the following result.

PROPOSITION 1. *Let $\phi: M \to C^2$ be a Lagrangian immersion of a compact orientable surface. If the Maslov form is conformal, then* genus$(M) \le 1$. *If the Maslov form is harmonic, then* genus$(M) \ge 1$.

Now, we are going to study the Gauss map of a Lagrangian surface in the complex Euclidean plane. Firstly, let $G(2, 4)$ be the Grassmannian of oriented two-planes in $C^2 \equiv R^4$. We will identify $G(2, 4)$ with the set of unit decomposable 2-vectors in $\bigwedge^2 R^4$. We describe briefly this identification (see [C-M] for details).

If $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of R^4, then we consider the subspaces $\mathcal{P}$ and $\mathcal{N}$ of $\bigwedge^2 R^4$ given by

$$\mathcal{P} = \text{span} \left\{ \frac{1}{2}(e_1 \wedge e_2 + e_3 \wedge e_4), \frac{1}{2}(e_1 \wedge e_3 - e_2 \wedge e_4), \frac{1}{2}(e_1 \wedge e_4 + e_2 \wedge e_3) \right\},$$

$$\mathcal{N} = \text{span} \left\{ \frac{1}{2}(e_1 \wedge e_2 - e_3 \wedge e_4), \frac{1}{2}(e_1 \wedge e_3 + e_2 \wedge e_4), \frac{1}{2}(e_1 \wedge e_4 - e_2 \wedge e_3) \right\}.$$

If S_+^2 and S_-^2 are the two-dimensional spheres of $\mathcal{P}$ and $\mathcal{N}$ of radius $1/\sqrt{2}$, then $G(2, 4)$ can be identified with $S_+^2 \times S_-^2$ by the map

$$P \in G(2, 4) \mapsto \left(\frac{1}{2}(v_1 \wedge v_2 + v_3 \wedge v_4), \frac{1}{2}(v_1 \wedge v_2 - v_3 \wedge v_4) \right),$$

where $\{v_1, v_2\}$ is an orthonormal basis of the plane P and $\{v_1, v_2, v_3, v_4\}$ is an orthonormal basis of R^4 defining the same orientation as $\{e_1, e_2, e_3, e_4\}$.

In this way, if $v: M \to G(2, 4)$ is the Gauss map of ϕ and if we take the basis $e_3 = Je_1, e_4 = Je_2$, then v lies in the product of S_+^2 and a great circle S_-^1 of S_-^2. Hence the Gauss map $v: M \to S_+^2 \times S_-^1$ is given by

$$v(p) = \left(\frac{1}{2}(v \wedge w + Jv \wedge Jw), \frac{1}{2}(v \wedge w - Jv \wedge Jw) \right),$$

where $\{v, w\}$ is an orthonormal basis of $T_p M$ such that $\{v, w, Jv, Jw\}$ defines the same orientation as $\{e_1, e_2, Je_1, Je_2\}$.

If $z = x + iy$ is a local isothermal coordinate on M, we can write the components of the Gauss map $v = (v_1, v_2)$ of ϕ as

$$v_1 = -ie^{-2u}(\phi_z \wedge \phi_{\bar{z}} + J\phi_z \wedge J\phi_{\bar{z}}), \qquad v_2 = -ie^{-2u}(\phi_z \wedge \phi_{\bar{z}} - J\phi_z \wedge J\phi_{\bar{z}}),$$

where $\wedge$ is extended C-linearly to the complexified tangent bundle.

Secondly, by standard arguments, the Frenet equations of a Lagrangian immersion $\phi \colon M \to \mathbf{C}^2$ are given by

$$\phi_{zz} = 2u_z\phi_z + \frac{h}{2}J\phi_z + \frac{e^{-2u}}{2}fJ\phi_{\bar{z}},$$

(2.7) $$\qquad \phi_{\bar{z}\bar{z}} = 2u_{\bar{z}}\phi_{\bar{z}} + \frac{e^{-2u}}{2}\bar{f}J\phi_z + \frac{\bar{h}}{2}J\phi_{\bar{z}},$$

$$\phi_{z\bar{z}} = \frac{\bar{h}}{2}J\phi_z + \frac{h}{2}J\phi_{\bar{z}}.$$

A well-known result of Ruh and Vilms [R-V] says that "ϕ *has parallel mean curvature vector if and only if v is a harmonic map*". In this case, using (2.1), we have that JH is a parallel vector field on M. So, either $JH \equiv 0$ and ϕ is a minimal immersion, or JH is non-null. In the latter case, M is flat and the second fundamental form is parallel. These surfaces are well-known to be cylinders and tori.

We are going to generalize the above property studying separately the harmonicity of the components of the Gauss map.

LEMMA 2. *Let $\phi \colon M \to \mathbf{C}^2$ be a Lagrangian immersion of an orientable surface and $v = (v_1, v_2) \colon M \to S^2_+ \times S^1_-$ its Gauss map. If $\tau(v_i)$ denote the tension fields of v_i, $i = 1$, 2, then*

$$\tau(v_1) = i(e^{-4u}f_{\bar{z}}\phi_{\bar{z}} \wedge J\phi_{\bar{z}} - g_{\bar{z}}\phi_z \wedge J\phi_z),$$

$$\tau(v_2) = ie^{-2u}h_{\bar{z}}(\phi_{\bar{z}} \wedge J\phi_z - \phi_z \wedge J\phi_{\bar{z}}),$$

where $z = x + iy$ is a local isothermal coordinate on M.

PROOF. Using the expressions for v_i, $i = 1$, 2 given above together with (2.7), we obtain

(2.8) $$\qquad v_{1_z} = i(-g\phi_z \wedge J\phi_z + fe^{-4u}\phi_{\bar{z}} \wedge J\phi_{\bar{z}}),$$

$$v_{2_z} = i\bar{g}(\phi_{\bar{z}} \wedge J\phi_z - \phi_z \wedge J\phi_{\bar{z}}).$$

Hence, taking into account that τ_i is the tangential component of $v_{i_{z\bar{z}}}$, $i = 1$, 2, and using again (2.7), we easily get the result. $\qquad\qquad\square$

The two families of Lagrangian surfaces described in Corollary 1 by means of its Maslov form can also be determined by a regular behaviour of its Gauss map.

COROLLARY 2. *Let $\phi \colon M \to \mathbf{C}^2$ be a Lagrangian immersion of an orientable surface and $v = (v_1, v_2) \colon M \to S^2_+ \times S^1_-$ its Gauss map. Then:*

(a) *The Maslov form on M is conformal if and only if v_1 is a harmonic map.*

(b) *The Maslov form on M is harmonic if and only if v_2 is a harmonic map.*

3. Local study. Let $\phi: M \to C^2$ be a Lagrangian immersion of an orientable surface M and $z = x + iy$ an isothermal coordinate in a connected neighbourhood U of M. In U, the induced metric $\langle\, ,\, \rangle$ is given by $\langle\, ,\, \rangle = e^{2u}ds^2$, where ds^2 is the Euclidean metric. If we consider the matrices

$$X = \begin{pmatrix} \phi_z \\ \phi_{\bar{z}} \end{pmatrix}, \quad A = \begin{pmatrix} 2u_z & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} 0 & 0 \\ 0 & 2u_{\bar{z}} \end{pmatrix}, \quad B = \frac{1}{2}\begin{pmatrix} h & e^{-2u}f \\ \bar{h} & h \end{pmatrix},$$

the Frenet equations (2.7) can be rewritten as

$$(3.1) \qquad\qquad X_z = AX + BJX, \qquad X_{\bar{z}} = \hat{A}X + B^*JX,$$

where B^* denotes the conjugate transpose of B. In this way, the integrability conditions for (3.1) are given by the differential equations

$$(3.2) \qquad A_{\bar{z}} - \hat{A}_z + [B^*, B] = 0, \qquad B_{\bar{z}} - B_z^* + [A, B^*] + [B, \hat{A}] = 0.$$

Since $h = e^{2u}\bar{g}$ (see §2), it is straightforward to see that (3.2) is equivalent to the differential equations

$$(3.3) \qquad \begin{aligned} &4u_{z\bar{z}} + \frac{e^{4u}|g|^2 - e^{-4u}|f|^2}{2} = 0, \\ &\mathrm{Im}(h_{\bar{z}}) = 0, \\ &f_{\bar{z}} - e^{4u}\overline{g_{\bar{z}}} = 0. \end{aligned}$$

From now on we consider the case where the Maslov form of ϕ is conformal and assume that H has no zeroes on U. In this case, f and g are holomorphic functions (Corollary 1). Since g has no zeroes (Lemma 1-(v)), we can normalize it as $g \equiv 1$. Indeed, $1/g(z)$ is well-defined and never vanishes, and if $w(z)$ is a solution of $dw/dz = 1/g(z)$ then, at least locally, w is a good coordinate on U. In this coordinate $\mathscr{X}(w) = \partial_w$. Rename w as z and let $g \equiv 1$. Then $h = e^{2u}$ and Lemma 1-(ii) says that $u_y = 0$, and so $u(x, y) = u(x)$, where, by Lemma 1-(vi), u satisfies the O.D.E.

$$(3.4) \qquad\qquad u'' + \frac{e^{4u} - |f|^2 e^{-4u}}{2} = 0.$$

If U_0 is the set of zeroes of f, $\log|f|$ is a harmonic map on $U - U_0$ and from (3.4), $|f|_y = 0$. So, $|f| = \mu e^{\lambda x}$ on $U - U_0$, where $\lambda, \mu \in R$ with $\mu > 0$. Since $|f|$ is a continuous function on U, either $U_0 = U$ or $U_0 = \varnothing$. Hence, we see that on U

$$|f|^2 = \mu^2 e^{2\lambda x}, \quad \lambda, \mu \in R, \quad \mu \geq 0.$$

Now, using this fact jointly with the holomorphy of f, it is easy to see that f has the following form:

$$f(z) = \mu e^{i\alpha}e^{\lambda z}, \quad \alpha \in R.$$

In this case (when ϖ is conformal and H has no zeroes) the matrices that appear in the Frenet equations (3.1) of the immersion ϕ are reduced to

$$(3.5) \qquad A=\begin{pmatrix} u' & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{A}=\begin{pmatrix} 0 & 0 \\ 0 & u' \end{pmatrix}, \quad B=\frac{e^{2u}}{2}\begin{pmatrix} 1 & \mu e^{-4u}e^{i\alpha}e^{\lambda z} \\ 1 & 1 \end{pmatrix}$$

and the integrability conditions (3.3) of the Frenet equations are equivalent to the condition that the function $u(x)$ satisfies the O.D.E.

$$(3.6) \qquad u'' + \frac{e^{4u} - \mu^2 e^{2\lambda x}e^{-4u}}{2} = 0 \ .$$

We can summarize this in the following result.

THEOREM 1. *Let* $\phi: M \rightarrow \mathbf{C}^2$ *be a Lagrangian immersion of an orientable surface with conformal Maslov form such that* H *has no zeroes on* M. *Then, around each point, there is a conformal parametrization of* M, $(U, z = x+iy)$, *and real numbers* λ, μ *with* $\mu \geq 0$, *such that the induced metric* $\langle \ , \ \rangle$ *is given by* $\langle \ , \ \rangle = e^{2u(x)}ds^2$, *where* u *is a solution of* (3.6).

Conversely, let α, λ *and* μ *be real numbers with* $\mu \geq 0$. *If* $u: \mathbf{R} \rightarrow \mathbf{R}$ *is a solution of* (3.6) *and if the initial conditions* $\phi(z_0), \phi_z(z_0), \phi_{\bar{z}}(z_0)$ *are compatible with the conditions* $\phi = \bar{\phi}, \ \phi_{\bar{z}} = \bar{\phi}_z$, *then by integrating the equations* (3.1) *and* (3.5) *we obtain a unique Lagrangian immersion with conformal Maslov form*

$$\phi: (\mathbf{C}, e^{2u}ds^2) \rightarrow \mathbf{C}^2 \ ,$$

where ds^2 *is the Euclidean metric such that* $\Theta(z) = \mu e^{i\alpha}e^{\lambda z}(dz)^3$ *and* $\mathscr{X}(z) = \partial_z$ *are the corresponding holomorphic cubic form and the holomorphic vector field associated to* ϕ, *respectively*.

REMARK 3. We note that, if $\mu > 0$, then varying α for each solution u of (3.6), we can obtain a one-parameter family of Lagrangian immersions. We also notice that in this case it is sufficient to consider $\alpha \in [0, \pi]$ because if one takes the immersion corresponding to $-\alpha$ and write its Frenet equations in the coordinate $\bar{z} = x - iy$, one obtains exactly the same Frenet equations.

Using the definition of $\mathscr{X}$, JH can be written as $JH = \partial_x$. By Remark 1, we have $\nabla_V JH = u'V$ for any tangent vector V at $(x, y) \in \mathbf{C}$.

REMARK 4. The immersion ϕ of Theorem 1 has parallel mean curvature vector if and only if u is constant (and then, necessarily $\mu > 0$ and $\lambda = 0$).

4. Lagrangian surfaces with cubic form Θ identically zero.
First, we are going to study the immersions given in Theorem 1 with $\Theta \equiv 0$, i.e. $\mu = 0$. In this case, the equation (3.6) is reduced to

$$(4.1) \qquad\qquad u'' + \frac{e^{4u}}{2} = 0 \, ,$$

and we can write the Frenet equations (3.1) and (3.5) in the following easy form:

$$\psi_{xx} = u'\psi_x + \frac{3e^{2u}}{2} J\psi_x \, ,$$

$$(4.2) \qquad\qquad \psi_{xy} = u'\psi_y + \frac{e^{2u}}{2} J\psi_y \, ,$$

$$\psi_{yy} = -u'\psi_x + \frac{e^{2u}}{2} J\psi_x \, .$$

The general solution of the equation (4.1) is given by $\hat{u}(x) = u(bx + a) + (1/2)\log b$, where

$$(4.3) \qquad\qquad u(x) = \frac{1}{2} \log \frac{4e^{2x}}{1 + e^{4x}}$$

and a, b are real numbers, with $b > 0$. So, it is easy to see that, up to dilatations, Theorem 1 only gives one immersion when $\Theta \equiv 0$.

We are going to determine the Lagrangian immersion ψ associated to the solution given in (4.3). From (4.2) and (4.3) it is easy to see that $\psi_{yy} + \psi$ is constant. Up to translations we can take it zero and finally conclude that ψ satisfies $\psi_{yy} + \psi = 0$ and so $\psi(x, y) = \cos y \psi(x, 0) + \sin y \psi_y(x, 0)$. Hence:

$$(4.4) \qquad \psi(x, y) = \cos y \{u'(x)\psi_x(x, 0) - (e^{2u(x)}/2)J\psi_x(x, 0)\} + \sin y \psi_y(x, 0) \, .$$

If we fix the initial conditions $\psi_x(0, 0) = (0, 0, \sqrt{2}, 0)$ and $\psi_y(0, 0) = (0, \sqrt{2}, 0, 0)$ with respect to any orthonormal basis of $C^2 \equiv R^4$, $B = \{e_1, e_2, Je_1, Je_2\}$, by integrating the first two equations of (4.2) and after a long computation, we see from (4.4) that ψ is given by

$$\psi(x, y) = \frac{\sqrt{2}\, e^x}{1 + e^{4x}} (\cos y(1 + e^{2x}), \sin y(1 + e^{2x}), \cos y(e^{2x} - 1), \sin y(e^{2x} - 1)) \, .$$

Hence, we deduce that the immersion ψ is singly-periodic of period 2π. So, ψ induces an immersion from C^* into C^2 via the exponential map $e^z \colon C \to C^*$. This new immersion can be extended regularly to 0 and ∞. By the stereographic projection, it is easy to see that the above immersion defines an immersion η from the whole sphere. It can be written as

$$(4.5) \qquad\qquad \eta(x, y, z) = \frac{\sqrt{2}}{1 + z^2} (x, y, xz, yz) \, .$$

This immersion is well-known and there is a deformation by Lagrangian immersions of η to the Whitney immersion ([Wn]) given by

$$\eta_t(x, y, z) = \frac{\sqrt{2}}{1 + tz^2} \, (x, y, (2-t)xz, (2-t)yz), \quad 0 \leq t \leq 1 \,.$$

The immersion $\eta : S^2 \to C^2$ given in (4.5) will be also called the *Whitney immersion*. As its cubic form $\Theta \equiv 0$, the Maslov form ϖ is conformal and the vector field $\mathscr{X}$ is holomorphic. $\mathscr{X}$ has its zeroes at the poles of S^2 and the immersion η is an embedding except at the poles. Also, from (2.8), it is easy to see that the harmonic map v_1 is a conformal map.

We can summarize the above reasoning in the following result.

THEOREM 2.   *Let $\phi : M \to C^2$ be a Lagrangian immersion of an orientable surface. If the cubic form Θ is identically zero, then $\phi(M)$ is an open set of a plane or of the Whitney sphere.*

PROOF. $\Theta \equiv 0$ is holomorphic and, from Corollary 1, $\mathscr{X}$ is a holomorphic vector field on M. If $\mathscr{X} \equiv 0$, then from Lemma 1-(v) ϕ is a totally geodesic immersion and hence $\phi(M)$ is an open set of a plane in C^2 which is Lagrangian for certain symplectic structure of C^2. Otherwise, $\mathscr{X}$ has only isolated zeroes and $M' = \{p \in M; \mathscr{X}(p) \neq 0\}$ is also connected. So, by Theorem 1 and the above reasoning, $\phi(M')$ is open in $\eta(S^2) - \{0\}$. Hence, $\phi(M)$ is open in $\eta(S^2)$. $\square$

From Lemma 1-(v), the Gauss equation and Theorem 2, we obtain the following result.

COROLLARY 3.   *Let $\phi : M \to C^2$ be a Lagrangian immersion of an orientable surface. Then $2K \leq |H|^2$ and the equality holds if and only if $\phi(M)$ is an open set of a plane or of the Whitney sphere.*

Now, using a consequence of the Riemann-Roch theorem which says that a holomorphic cubic form on a sphere is identically zero, we obtain a result similar to Hopf's theorem.

COROLLARY 4.   *The only Lagrangian immersion of a sphere in C^2 with conformal Maslov form is the Whitney immersion.*

The Whitney immersion has a very nice behaviour with respect to the Willmore functional $W(\phi) = \int_M |H|^2 dM$. We can prove easily that the immersion η is a Willmore immersion, i.e. it is a critical point for the above functional. In fact, we can prove a stronger property using Corollaries 3 and 4 and the Gauss-Bonnet theorem.

COROLLARY 5.   *Let $\phi : M \to C^2$ be a Lagrangian immersion of a sphere in C^2. Then*

$$W(\phi) = \int_M |H|^2 dM \geq 8\pi$$

and the equality holds if and only if ϕ is the Whitney immersion.

Finally, following the ideas of Ejiri [E], if F is an inversion of a sphere in C^2 centred at the origin and $\hat{\eta} = F \circ \eta: S^2 - \eta^{-1}(0) \to C^2$, then $\hat{\eta}$ is a complete minimal immersion of total curvature -4π. This example was described by Hoffman and Osserman [H-Os]. The inequality given in Corollary 5 was obtained by Wintgen [Wi]. Also, the Whitney immersion is one of the examples described in [W].

5. All Lagrangian tori with conformal Maslow form. Let $\phi: M \to C^2$ be a Lagrangian immersion with conformal Maslov form of a torus M into C^2. Let $\psi: C \to C^2$ be the lift of ϕ to the universal covering of M. Then, ψ is also a Lagrangian immersion and, if $\tilde{\omega}$ denotes its Maslov form, then $\tilde{\omega}$ is also conformal. Let Θ and $\mathcal{X}$ be the holomorphic cubic form and the holomorphic vector field associated to ψ. If $z = x + iy$ is the standard coordinate in C, by expressing Θ and $\mathcal{X}$ in this coordinate system, the functions associated to Θ and $\mathcal{X}$ are bounded holomorphic functions defined on the whole plane C and hence they are constant. So, we can suppose that $\Theta(z) = \rho e^{i\beta}(dz)^3$ and $\mathcal{X}(z) = \varrho e^{i\vartheta}\partial_z$. If $\varrho = 0$, ψ is a minimal immersion and then ϕ is also a minimal immersion. If $\rho = 0$, using Lemma 1-(v) jointly with the Gauss equation of ψ, we get $\tilde{K} = |\tilde{H}|^2/2$, where $\tilde{K}$ and $\tilde{H}$ are the Gauss curvature and the mean curvature of ψ. So, if K and H denote the Gauss curvature and the mean curvature of the torus, $K = |H|^2/2$ too. The Gauss-Bonnet theorem says that ϕ is a minimal immersion. In both cases we get a contradiction by the compactness of the torus. Hence $\rho, \varrho > 0$. Changing the coordinate in C in a suitable way, we can rewrite Θ and $\mathcal{X}$ as

$$\Theta(z) = \mu e^{i\alpha}(dz)^3, \quad \mu > 0, \quad \text{and} \quad \mathcal{X}(z) = \partial_z.$$

So, *in order to classify the Lagrangian immersion of a torus in C^2 with conformal Maslov form, we need to know which of the immersions given in Theorem 1 for $\lambda = 0$ are doubly-periodic.*

By standard arguments, we observe that in this case, i.e. when $\lambda = 0$, it is sufficient, up to dilatations, to study the Lagrangian immersions associated to the solutions of the sinh-Gordon equation

(5.1) $$u'' + \sinh 4u = 0,$$

and, without loss of generality, to those with $u'(0) = 0$.

We first see that the only constant solution of (5.1) is $u \equiv 0$. The solutions of (5.1) are well-known since they can be expressed in terms of a certain class of elliptic functions (see [D, Chap. 7] for details) and, consequently, they are periodic. So, there exists a positive real number B (see [A-S, Chap. 16]; [D, Chap. 6]) such that for any real

number x,

(5.2) $u(x+2B)=u(x)$, $\quad u(B-x)=u(B+x)$, $\quad u(-x)=u(x)$.

Moreover, if F and G are the functions given by $F(x)=\int_0^x e^{2u(s)}ds$ and $G(x)=\int_0^x e^{-2u(s)}ds$, then for any real number x,

(5.3) $F(x+2B)=F(x)+\pi$, $\qquad G(x+2B)=G(x)+\pi$.

In this case we can write the Frenet equations (3.1) and (3.5) as follows:

$$\psi_{xx}=u'\psi_x+\frac{3e^{2u}+e^{-2u}\cos\alpha}{2}J\psi_x-\frac{e^{-2u}\sin\alpha}{2}J\psi_y,$$

(5.4) $$\psi_{xy}=u'\psi_y-\frac{e^{-2u}\sin\alpha}{2}J\psi_x+\frac{e^{2u}-e^{-2u}\cos\alpha}{2}J\psi_y,$$

$$\psi_{yy}=-u'\psi_x+\frac{e^{2u}-e^{-2u}\cos\alpha}{2}J\psi_x+\frac{e^{-2u}\sin\alpha}{2}J\psi_y.$$

The properties of the immersions associated to the solutions of the sinh-Gordon equation (5.1) are studied in the following result.

THEOREM 3.   *Let $\psi_{u,\alpha}$ be the Lagrangian immersion with conformal Maslov form associated to a solution u of* (5.1) *such that $u'(0)=0$ and $\alpha\in[0,\pi]$ (see Remark* 3).

For $(u,\alpha)=(0,0)$, the immersion $\psi_{0,0}$ defines a right circular cylinder in a hyperplane of C^2.

For $(u,\alpha)\neq(0,0)$, the immersions $\psi_{u,\alpha}$ satisfy the following properties:
(a)   *For all $x,y\in R$,*

$$\psi_{u,\alpha}(x,y+2\pi/l)=\psi_{u,\alpha}(x,y),$$

where l is the positive constant (depending on u and α) given by $l=(1/2)(e^{4a}-2\cos\alpha+e^{-4a})^{1/2}$, where $a=u(0)$.
(b)   *The curves $\{c_x: y\mapsto\psi_{u,\alpha}(x,y), x\in R\}$ are circles in C^2 of radius $e^{u(x)}/l$ centred at the same point P_0.*
(c)   *There exists a real number τ (depending on u and α) such that*

$$\psi_{u,\alpha}(x+2B,y+\tau)=\psi_{u,\alpha}(x,y),$$

where $2B$ is the period of u.
(d)   *The circles c_x and $c_{x'}$, with $x,x'\in(0,2B)$, $x\neq x'$, do not intersect.*
(e)   *Let $F:C^2-\{P_0\}\rightarrow C^2$ be the inversion centred at P_0 given by*

$$F(P)=\frac{P-P_0}{l^2|P-P_0|^2}.$$

Then $\psi_{-u,\pi}$ is congruent to $F\circ\psi_{u,\pi}$.

REMARK 5. Note that $l=0$ if and only if $\alpha=0$ and $u\equiv0$.

PROOF. The singular case $(u,\alpha)=(0,0)$ is clear.

In the general case, to simplify expressions we introduce the following functions:

$$(5.5) \qquad b=\frac{e^{2u}-e^{-2u}\cos\alpha}{2}\,,\qquad c=\frac{3e^{2u}+e^{-2u}\cos\alpha}{2}\,,\qquad d=\frac{e^{-2u}\sin\alpha}{2}\,.$$

So, (5.4) can be rewritten as

$$
\begin{aligned}
\psi_{xx}&=u'\psi_x+cJ\psi_x-dJ\psi_y\,,\\
(5.6)\qquad \psi_{xy}&=u'\psi_y-dJ\psi_x+bJ\psi_y\,,\\
\psi_{yy}&=-u'\psi_x+bJ\psi_x+dJ\psi_y\,.
\end{aligned}
$$

Using (5.1) and (5.5), it is straightforward to see that the derivative of $u'^2+b^2+d^2$ is zero, and so

$$(5.7) \qquad u'^2+b^2+d^2=(b^2+d^2)(0)=l^2\,.$$

From (5.1), (5.5), (5.6) and (5.7), it is easy to see that $\psi_{yy}+l^2\psi$ is independent of x and y. Hence, this vectorial function is a constant P_0, and up to translations we can take $P_0\equiv0$ and, in this way, $\psi_{yy}+l^2\psi=0$. By integration we get

$$(5.8) \qquad \psi(x,y)=\cos(ly)\psi(x,0)+(1/l)\sin(ly)\psi_y(x,0)\,.$$

This equation proves (a).

By (5.6) and (5.7), we easily see that $|\psi(x,0)|^2=|\psi_y(x,0)/l|^2=e^{2u(x)}/l^2$ and $\langle\psi(x,0),\psi_y(x,0)\rangle=0$. This fact, together with (5.8), proves (b).

Now we are going to study the variation of the planes which contain the circles $\{c_x,x\in R\}$. So we consider the curve

$$\gamma=(\gamma_1,\gamma_2)\colon R\to G(2,4)\equiv S_+^2\times S_-^2$$

such that $\gamma(x)$ is the plane $\Pi(x)$ which contains the circle c_x. We are going to study this curve following the description of $G(2,4)$ given in §2. Let $\mathscr{B}(x)=\{e_1(x),e_2(x),e_3(x),e_4(x)\}$, $x\in R$, be the orthonormal basis of C^2 given by

$$e_1(x)=\frac{\psi_x(x,0)}{e^{u(x)}}\,,\qquad e_2(x)=\frac{\psi_y(x,0)}{e^{u(x)}}\,,\qquad e_3(x)=\frac{J\psi_x(x,0)}{e^{u(x)}}\,,\qquad e_4(x)=\frac{J\psi_y(x,0)}{e^{u(x)}}\,.$$

We denote $e_i\equiv e_i(0)$, $1\leq i\leq4$. It is not difficult to see that

$$\{v_1(x)=(1/l)(u'e_1(x)-be_3(x)-de_4(x),\ v_2(x)=e_2(x)\}$$

is an orthonormal basis of the plane $\Pi(x)$. If we define

$$v_3(x)=(u'^2+b^2)^{-1/2}(be_1(x)+u'e_3(x))\,,$$

$$v_4(x) = \frac{(u'^2 + b^2)^{-1/2}}{l}\,(u'de_1(x) - bde_3(x) + (u'^2 + b^2)e_4(x))\,,$$

then for any $x \in \boldsymbol{R}$, $\{v_1(x), v_2(x), v_3(x), v_4(x)\}$ is an orthonornal basis of $\boldsymbol{C}^2$ defining the same orientation as $\mathscr{B}(0)$.

In this way, we can write

(5.9) $\gamma_1(x) = \dfrac{1}{2}\,(v_1 \wedge v_2 + v_3 \wedge v_4)(x)\,,$ $\gamma_2(x) = \dfrac{1}{2}\,(v_1 \wedge v_2 - v_3 \wedge v_4)(x)\,.$

In order to study both curves, we introduce the following notation. Let $U(x)$, $V(x)$, $W(x)$ (resp. $\hat{U}(x)$, $\hat{V}(x)$, $\hat{W}(x)$) be orthonormal basis of the subspaces $\mathscr{P}(x)$ (resp. $\mathscr{N}(x)$) of $\bigwedge^2 \boldsymbol{R}^2$ given by (see §2):

$$2U(x) = (e_1 \wedge e_2 + e_3 \wedge e_4)(x),\ 2V(x) = (e_1 \wedge e_3 - e_2 \wedge e_4)(x),\ 2W(x) = (e_1 \wedge e_4 + e_2 \wedge e_3)(x)$$

$$2\hat{U}(x) = (e_1 \wedge e_2 - e_3 \wedge e_4)(x),\ 2\hat{V}(x) = (e_1 \wedge e_3 + e_2 \wedge e_4)(x),\ 2\hat{W}(x) = (e_1 \wedge e_4 - e_2 \wedge e_3)(x)\,.$$

So, γ_1 and γ_2 can be written as

(5.10) $\gamma_1(x) = \dfrac{1}{l}\,(u'U - dV + bW)(x)\,,$ $\gamma_2(x) = \dfrac{1}{l}\,(u'\hat{U} + d\hat{V} - b\hat{W})(x)\,.$

By (5.6), U, V and W (resp. $\hat{U}$, $\hat{V}$, $\hat{W}$) satisfy the following O.D.E. systems:

$$U' = -2dV + (b - c)W, \qquad V' = 2dU, \qquad W' = (c - b)U$$

$$(\text{resp. } \hat{U}' = (b + c)\hat{W},\ \hat{V}' = 0,\ \hat{W}' = -(b + c)\hat{U})\,.$$

By (5.1), (5.5), (5.6), (5.10) and the above equations, we show that $\gamma_1' \equiv 0$ and determine $\hat{U}$, $\hat{V}$ and $\hat{W}$. Hence, from (5.10):

(5.11) $\gamma_1(x) = \gamma_1(0) = (1/l)(-d(0)V(0) + b(0)W(0))\,,$

(5.12) $\gamma_2(x) = (1/l)[(u'\cos k + b\sin k)(x)\hat{U}(0) + d(x)\hat{V}(0) + (u'\sin k - b\cos k)(x)\hat{W}(0)]\,,$

where $k(x) = \int_0^x (b(s) + c(s))ds$.

By (5.5), $k(x) = 2F(x)$ and by (5.3), $k(x + 2B) = k(x) + 2\pi$. Moreover, from (5.2) and (5.5), u', b and d are also periodic of period $2B$, and so $\gamma_2(x + 2B) = \gamma_2(x)$. Since γ_1 is a constant function, we see that for any $x \in \boldsymbol{R}$, $\Pi(x)$ and $\Pi(x + 2B)$ are the same plane. We also note that $2B$ is the smallest possible period for $\Pi(x)$. So, the circles c_x and c_{x+2B} are in the same plane and have the same radius because u is periodic of period $2B$. Then, there exists $\tau(x)$ such that $c_x(y) = c_{x+2B}(y + \tau(x))$. This can be written as $\psi(x + 2B, y + \tau(x)) = \psi(x, y)$. Taking derivative with respect to x in the last expression and computing the length of the result, we have $\tau'(x) = 0$ and so $\tau(x)$ is constant. This proves (c).

Suppose now that the circles c_x and $c_{x'}$, $x, x' \in (0, 2B)$ intersect. Necessarily they

must have the same radius, hence $u(x) = u(x')$ according to (b). By (5.2), $x' = 2B - x$ with $x \in (0, B)$. Moreover, the planes $\Pi(x)$ and $\Pi(2B-x)$ have a non-trivial intersection. As $\{v_1(x), v_2(x)\}$ is an orthonormal basis of the plane $\Pi(x)$, we have $q(x) = v_1(x) \wedge v_2(x) \wedge v_1(2B-x) \wedge v_2(2B-x) = 0$. By (5.9) and (5.11), for any $x \in R$, we have

(5.13) $\qquad \Pi(x) \equiv (\gamma_1(x), \gamma_2(x)) = (\gamma_1(0), v_1(x) \wedge v_2(x) - \gamma_1(0))$.

From (5.9), (5.11) and (5.12) we get

(5.14) $\qquad v_1(x) \wedge v_2(x) = (1/l)[(u' \cos k + b \sin k)(x)\hat{U}(0) + d(x)\hat{V}(0)$

$\qquad\qquad\qquad + (u' \sin k - b \cos k)(x)\hat{W}(0) - d(0)V(0) + b(0)W(0)]$.

Using (5.2), (5.3) and (5.5) we obtain $u'(2B-x) = -u'(x)$, $b(2B-x) = b(x)$, $d(2B-x) = d(x)$ and $k(2B-x) = 2\pi - k(x)$. Finally, by (5.7) and (5.14), we can write explicitly

$$ q(x) = (1/l^2)(u' \cos k + b \sin k)^2(x)e_1 \wedge e_2 \wedge e_3 \wedge e_4 \ . $$

So, $q(x) = 0$ means, by (5.14), the $v_1(x) \wedge v_2(x) = v_1(2B-x) \wedge v_2(2B-x)$ and so, from (5.13), $\Pi(x) = \Pi(2B-x)$, which is impossible because $x \in (0, B)$. This reasoning proves (d).

In order to prove (e), we may assume that P_0 is the origin of C^2 as above. As $|\psi_{u,\pi}|^2 = e^{2u}/l^2$, let Ψ be the map given by $\Psi(x, y) = e^{-2u(x)}\psi_{u,\pi}(x, y)$. Then, by (5.1), (5.4), (5.5) and (5.7), it is easy to prove that Ψ satisfies the Frenet equations of the immersion $\psi_{-u,\pi}$, which proves (e). $\qquad\qquad\qquad\square$

From Theorem 3, (a) and (c) we deduce that the immersions $\psi_{u,\alpha}$ are doubly-periodic, except for $(u, \alpha) = (0, 0)$. So they provide examples of Lagrangian tori with conformal Maslov form, all of which are embedded by (d).

COROLLARY 6. *Let u be a solution of the equation (5.1) with $u'(0) = 0$ and $\alpha \in [0, \pi]$ with $(u, \alpha) \neq (0, 0)$. Let $\Lambda_{u,\alpha}$ be the lattice in C generated by $\{(0, 2\pi/l), (2B, \tau)\}$, $T_{u,\alpha}$ the torus $C/\Lambda_{u,\alpha}$ and $\phi_{u,\alpha} : T_{u,\alpha} \to C^2$ the immersion induced by $\psi_{u,\alpha}$. Then $\phi_{u,\alpha}$ is a Lagrangian embedding with conformal Maslov form of $T_{u,\alpha}$ into C^2.*

REMARK 6. By Theorem 3, the embeddings $\phi_{0,\alpha}$, $\alpha \neq 0$, define a family of constant mean curvature $(\hat{H}(\alpha))$ flat tori embedded in a hypersphere in C^2 of radius $r(\alpha)$, where $\hat{H}(\alpha) = \cos(\alpha/2)$ and $r(\alpha) = \csc(\alpha/2)$. We note that $\phi_{0,\pi}$ is the Clifford torus. $\phi_{0,\alpha}$ with $0 \leq \alpha \leq \pi$ is a deformation from the right circular cylinder to the Clifford torus by Lagrangian surfaces with parallel mean curvature vector.

In this way, we have classified all the Lagrangian immersions with conformal Maslov form of a compact orientable surface of genus one into C^2.

COROLLARY 7. *Let $\phi : M \to C^2$ be a Lagrangian immersion with conformal Maslov form of a torus M in the complex Euclidean plane. Then ϕ is congruent to $\phi_{u,\alpha}$ for some solution u of (5.1) with $u'(0) = 0$ and some $\alpha \in [0, \pi]$, except when $(u, \alpha) = (0, 0)$.*

We study now some geometric properties of the tori $\phi_{u,\alpha}$.

PROPOSITION 2.   *If $A(u, \alpha)$ and $W(u, \alpha)$ denote the area and the Willmore functional of the torus $T_{u,\alpha}$, then:*
(i) $A(u, \alpha) = 2\pi^2/l$.
(ii) $W(u, \alpha) \geq 2\pi^2$ *and the equality holds if and only if $T_{u,\alpha}$ is the Clifford torus.*

PROOF.   (i) is a direct consequence of (5.3) and the definition of area.
From Lemma 1-(v) we have that $|H|^2 = e^{2u}$. So

$$(5.15) \qquad\qquad W(u, \alpha) = \frac{2\pi}{l} \int_0^{2B} e^{4u(x)} dx \; .$$

Let $E = E(p^2)$ be the complete elliptic integral of second class, with $p^2 = 1 - e^{-8|a|}$ (see [A-S, Chap. 17]; [D, Chap. 6]). Then, using [A-S, 17.2.10, 17.3.4, 17.4.4], we can compute that $\int_0^{2B} e^{4u} dx = 2e^{2|a|} E$. So, from (5.15), we obtain that $W(u, \alpha) = (4\pi e^{2|a|}/l)E$. Hence, we estimate that

$$(5.16) \qquad\qquad W(u, \alpha) \geq W(u, \pi) = \frac{8\pi E}{1 + \sqrt{1-p^2}} \; .$$

Now, from [A-S, Figure 17.2]; [D, p. 143], we have that $\pi/4 \leq E/(1 + \sqrt{1-p^2})$ and the equality holds if and only if $p^2 = 0$, i.e. $u \equiv 0$. Using this inequality in (5.16) we prove (ii). $\square$

The study of the remaining case, $\mu > 0$ and $\lambda \neq 0$ in Theorem 1, can be simplified in the following way. If u is a solution of (3.6) with $\mu > 0$ and $\lambda \neq 0$ and $\phi_\alpha : (C, e^{2u}ds^2) \to C^2$ is the one-parameter family of Lagrangian immersions associated to u, then it is clear that $\phi_\alpha(z) = \phi_0(z + (\alpha/\lambda)i)$. So, in this case, there is only one immersion associated to each solution of the equation (3.6). Also, it is easy to check that, up to dilatations, it is sufficient to study the Lagrangian immersions associated to the solutions of the equation

$$(5.17) \qquad\qquad u'' + \frac{e^{4u} - e^{8\varepsilon x} e^{-4u}}{2} = 0 \; , \quad \varepsilon = \operatorname{sig} \lambda = \pm 1 \; .$$

We first observe that u is a solution of (5.17) for $\varepsilon = 1$ if and only if $v(x) = u(-x)$ is a solution of (5.17) for $\varepsilon = -1$, and that the immersion ϕ_{-1} associated to $v(x)$ is given by $\phi_{-1}(z) = \phi_1(-z)$ changing the complex structure J by $-J$, where ϕ_1 is the Lagrangian immersion associated to u.

Secondly, we note that $u(x) = x$ is the unique solution of the equation (5.17) for $\varepsilon = 1$, which gives a Lagrangian immersion ϕ of C with a *flat* metric into C^2. We can easily integrate the corresponding Frenet equations and see that this immersion ϕ is given by

$$\phi(x, y) = (C(e^x \cos y), C(e^x \sin y), S(e^x \cos y), S(e^x \sin y)) ,$$

where C and S are the Fresnel integrals defined by $C(a) = \int_0^a \cos t^2 dt$ and $S(a) = \int_0^a \sin t^2 dt$. The cubic form of this immersion is $\Theta(z) = e^{4z}(dz)^3$. As ϕ is singly-periodic of period 2π, it induces an immersion from C^* into C^2 via the exponential map which can be extended regularly to 0. Since its induced metric is the Euclidean metric, we get a Lagrangian immersion with conformal Maslov form $\Phi: (C, ds^2) \to C^2$ given by

$$\Phi(x, y) = (C(x), C(y), S(x), S(y)) .$$

We remark that Φ is the product of two Cornu's spirals and it is an embedding.

Finally, this last study allows us to classify the Lagrangian immersions of flat surfaces in C^2 with conformal Maslov form. Indeed, from Theorem 1, Corollary 7, Remark 6 and a reasoning similar to that in Theorem 2, we obtain the following result.

PROPOSITION 3.   *A Lagrangian immersion of a flat orientable surface with conformal Maslov form in C^2 is locally congruent to $\phi_{0,\alpha}$ for some $\alpha \in [0, \pi]$, or to the Cornu immersion Φ, or to a plane.*

REFERENCES

[A-S]   M. A. ABRAMOWITZ AND I. A. STEGUN, Handbook of mathematical functions, Dover Publications, New York, 1965.

[C-M]   B.-Y. CHEN AND J. M. MORVAN, Géométrie des surfaces lagrangiennes de C^2, J. Math. Pures Appl. 66 (1987), 321–325.

[Ch-Wo]   S. S. CHERN AND J. G. WOLFSON, Minimal surfaces by moving frames, Amer. J. Math. 105 (1983), 59–83.

[D]   H. T. DAVIS, Introduction to nonlinear differential and integral equations, Dover Publications, New York, 1962.

[De]   C. DELAUNAY, Sur la surface de revolution dont la courbure moyenne est constante, J. Math. Pures Appl. Ser. 1 (6) (1841), 309–320.

[E]   N. EJIRI, Wilmore surfaces with a duality in $S^n(1)$, Proc. London Math. Soc. (3) 57 (1988), 383–416.

[G]   M. GROMOV, A topological technique for the construction of solutions of differential equations and inequalities, Int. Congr. of Mathematiciens (Nice 1970), Vol. 2, Gauthier-Villars, Paris, 1971, 221–225.

[H-Os]   D. A. HOFFMAN AND R. OSSERMAN, The geometry of the generalized Gauss map, Memoirs Amer. Math. Soc. 236 (1980).

[R-V]   E. A. RUH AND J. VILMS, The tension field of the Gauss map, Trans. Amer. Math. Soc. 149 (1970), 569–573.

[W]   J. L. WEINER, On an inequality of P. Wintgen for the integral of the square of the mean curvature, J. London Math. Soc. (2) 34 (1986), 148–158.

[We]   R. O. WELLS, Compact real submanifolds of a complex manifold with non-degenerated holomorphic tangent bundles, Math. Ann. 179 (1969), 123–129.

[Wn]   A. WEINSTEIN, Lectures on symplectic manifolds, Conference Board of the Mathematicals Sciences 29 (1977).

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[Wi]   P. WINTGEN, Sur l'inegalite de Chen-Willmore, C. R. Acad. Sci. Paris 288 (1979), 993–995.

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