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ABSTRACT KAZHDAN-LUSZTIG THEORIES

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Abstract. In this paper, we prove two main results. The first establishes that Lusztig's conjecture for the characters of the irreducible representations of a semisimple algebraic group in positive characteristic is equivalent to a simple assertion that certain pairs of irreducible modules have non-split extensions. The pairs of irreducible modules in question are those with regular dominant weights which are mirror images of each other in adjacent alcoves (in the Jantzen region). Secondly, we establish that the validity of the Lusztig conjecture yields a complete calculation of all Yoneda Ext groups between irreducible modules having regular dominant weights in the Jantzen region. These results arise from a general theory involving so-called Kazhdan-Lusztig theories in an abstract highest weight category. Accordingly, our results are applicable to a number of other situations, including the Bernstein-Gelfand-Gelfand category for a complex Lie algebra and the category of modules for a quantum group at a root of unity.

A major unsolved problem in finite group theory centers on determining the characters and degrees of the irreducible modular representations of finite groups of Lie type in the defining characteristic. Lusztig took a significant step toward a solution in 1979 by formulating his celebrated conjecture [L1] for the characters of simple modules for semisimple algebraic groups. Since that time, mathematicians have devoted considerable effort to establishing this conjecture, which would completely solve the above problem as long as the characteristic is not too small relative to the root system. A similar conjecture, by Kazhdan and Lusztig [KL1], for the composition factor multiplicities of Verma modules for semisimple complex Lie algebras, has already been settled [BB], [BK]. For some time, we have worked to develop algebraic techniques in positive characteristic capturing some of the geometric methods used in the characteristic zero Lie algebra theory.

Let G be a semisimple, simply connected algebraic group defined over an algebraically closed field k of positive characteristic p. For a dominant weight λ, let $L(\lambda)$ be the corresponding irreducible rational G-module of highest weight λ. This paper contains two main results. The first, given in Theorem 5.3, establishes that the Lusztig conjecture is equivalent to the simple assertion that $\mathrm{Ext}^1_G(L(\lambda), L(\lambda')) \neq 0$ for p-regular

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dominant weights λ and λ' which are mirror images of one another in *adjacent* p-alcoves and which satisfy the Jantzen condition (5.0.2). (Using the translation principle, one can assume that λ and λ' lie in the orbit of 0 under the "dot" action of the affine Weyl group.) The possibility of such a reduction can be traced back to questions raised by Vogan in a characteristic zero context; see (5.8) and the remarks above (5.3). Section 5 contains other equivalent forms of this reduction. We note especially (5.4a), an assertion that certain explicit quotients of Weyl modules have just two composition factors. Also, (5.4b) recasts the conjecture in terms of the *vanishing* of Ext^1 between Weyl modules and simple modules having the same "parity". In addition, (5.5) observes that a very special case (involving a single filtration term) of the Jantzen conjecture for Weyl module filtrations implies the Lusztig conjecture. (This improves on an earlier reduction [A3; (6.13)] of Andersen.)

The second main result, given in Theorem 3.5, gives a new way of computing Ext-groups in highest weight categories. As a corollary, the validity of the Lusztig conjecture would give a complete determination of *all* groups $\mathrm{Ext}^n_G(L(\gamma), L(\tau))$ for irreducible G-modules $L(\gamma)$ and $L(\tau)$ having p-regular highest weights γ, τ satisfying the Jantzen condition; see (5.8).

These results emerge from a general theory. Related to the above, an Ext^1 vanishing condition is precisely what we need to establish a limited but purely algebraic decomposition theory, somewhat analogous to, but much more elementary than, Gabber's theory for perverse sheaves [BBD]. See Section 4. There are further analogies with perverse sheaf theory (though our proofs require no geometric methods). Also, the theory of derived categories, together with its interplay with the theory of highest weight categories (as developed in [CPS1], [CPS2], [CPS3], [PS]), plays a central role in this paper. Other key ideas, given in §§ 2, 3, involve the introduction of new concepts, namely, that of *enriched* Grothendieck groups and "Kazhdan-Lusztig" theories for abstract highest weight categories. These new Grothendieck groups may be viewed as "q-analogs" of the classical Grothendieck group associated to a highest weight category.

Our reduction of the Lusztig conjecture also holds for the Kazhdan-Lusztig conjecture for the category $\mathcal{O}$ associated to a complex semisimple Lie algebra. That conjecture has, of course, been proved, but our work settles an old question as to whether "even-odd" vanishing of certain Ext^n-groups formally implies the "Vogan conjecture". See Remark 5.8. The calculation of Ext^n-groups mentioned above also applies to the category $\mathcal{O}$ to yield a complete determination of the groups $\mathrm{Ext}^n_{\mathcal{O}}(L(\lambda), L(\tau))$ for all n and all integral weights λ, τ; cf. (3.8). We require, at least presently, the work of Soergel [So] to obtain this computation in the case of singular weights. For regular weights, our method is more elementary and provides a less involved proof of a result of Beilinson-Ginzburg [BG].

Finally, similar remarks apply to quantum groups at a root of unity. For example, our algebraic reduction of Lusztig's conjecture applies as well to his quantum group conjecture [L3; (8.2)] (with some restrictions). See (5.8).

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1. Preliminaries. We begin by establishing some conventions used throughout this paper. Let k be a fixed field, and let $\mathscr{C}$ be a highest weight category over k with weight poset Λ, required to be *interval finite*. We do not repeat the definition of a highest weight category here, although several examples will be given in §§ 3, 5. We will generally adhere to the notation and conventions of [CPS1; Defn. 3.1], supplemented by the following two assumptions (1.1)–(1.2):

(1.1) The abelian category $\mathscr{C}$ is finite (i.e., every object has finite length). If $\lambda \in \Lambda$, the corresponding simple object $L(\lambda)$ satisfies the condition that $\mathrm{End}_{\mathscr{C}}(L(\lambda)) \cong k$.

(1.2) The opposite category $\mathscr{C}^{\mathrm{op}}$ is a highest weight category with weight poset Λ.

This last condition implies that, in addition to "induced objects" $A(\lambda)$, $\lambda \in \Lambda$, the highest weight category $\mathscr{C}$ has "Weyl objects" $V(\lambda)$, $\lambda \in \Lambda$. (Thus, $\mathscr{C}$ satisfies both conditions (A) and (V) of [CPS2]. Also, the category $\mathscr{C}$ has both enough projective and injective objects.) When the weight poset Λ is finite, condition (1.2) is automatic.

From now on, unless otherwise explicitly mentioned, "highest weight category" will always means a highest weight category satisfying the above conditions (1.1)–(1.2).

Often $\mathscr{C}$ has a duality $D : \mathscr{C} \to \mathscr{C}^{\mathrm{op}}$ in the sense of [CPS2]. Thus, D is a contravariant functor fixing simple objects (up to isomorphism) and satisfying $D^2 \cong \mathrm{id}_{\mathscr{C}}$. When $\mathscr{C}$ has a duality D, $D(A(\lambda)) \cong V(\lambda)$ for all weights λ.

(1.3) Recall that an ideal Γ in a poset Λ is a subset such that $\lambda \leq \gamma \in \Gamma$ implies that $\lambda \in \Gamma$. (Similarly, a subset Ω is a coideal if $\lambda \geq \omega \in \Omega$ implies that $\lambda \in \Omega$.)

Given an ideal Γ of the weight poset Λ of a highest weight category $\mathscr{C}$, let $\mathscr{C}[\Gamma] \subset \mathscr{C}$ be the full subcategory consisting of all objects having composition factors of the form $L(\gamma)$ with $\gamma \in \Gamma$. Then $\mathscr{C}[\Gamma]$ is a highest weight category having weight poset Γ (satisfying conditions (1.1)–(1.2)). The natural full embedding functor $i_* : \mathscr{C}[\Gamma] \to \mathscr{C}$ carries induced objects $A(\gamma)$ and Weyl objects $V(\gamma)$ in $\mathscr{C}[\Gamma]$ to the corresponding objects in $\mathscr{C}$. These results are proved in [CPS1; (3.5)] when Γ is finitely generated. However, when $\mathscr{C}$ is finite (as in this paper), the arguments given there readily apply to any Γ.

We often use the fact that if λ is a *maximal* weight in Λ, then $A(\lambda)$ (resp., $V(\lambda)$) is the injective hull (resp., projective cover) of $L(\lambda)$. This result follows immediately from the axioms for a highest weight category. In particular, if λ is a maximal element in an ideal $\Gamma \subset \Lambda$, $A(\lambda)$ is the injective hull of $L(\lambda)$ in the category $\mathscr{C}[\Gamma]$.

(1.4) Keeping the notation of (1.3), let $\Omega \subset \Gamma$ be a finite coideal. Then the quotient category $\mathscr{C}(\Omega) \equiv \mathscr{C}[\Gamma]/\mathscr{C}[\Gamma \backslash \Omega]$ is a highest weight category with weight poset Ω. The quotient functor $j^* : \mathscr{C}[\Gamma] \to \mathscr{C}(\Omega)$ sends induced and Weyl objects to the corresponding induced and Weyl objects, respectively, in $\mathscr{C}(\Omega)$. For $\lambda \in \Omega$, j^* maps the simple object $L(\lambda)$ in $\mathscr{C}$ to the corresponding simple object in $\mathscr{C}(\Omega)$. If $\lambda \notin \Omega$, we have $j^* L(\lambda) \cong 0$. Thus,

$j^*A(\lambda)\cong j^*V(\lambda)\cong 0$, if $\lambda\in\Lambda-\Omega$. For more details, see [CPS2; (1.4)] and [P; §1]. The fact that these quotient categories provide new examples of highest weight categories illustrates the versatility of the highest weight category notion. Our notion of a Kazhdan-Lusztig theory in §3 will also be inherited by these quotients; see (3.10).

(1.5) Let $D^b(\mathscr{C})$ be the bounded derived category associated to the highest weight category $\mathscr{C}$. We mention several facts often used implicitly. First, if $\Gamma\subset\Lambda$ is an ideal, then i_* induces a full embedding $i_*: D^b(\mathscr{C}[\Gamma])\to D^b(\mathscr{C})$ with strict image the relative derived category $D^b_{\mathscr{C}[\Gamma]}(\mathscr{C})$ (i.e., the full subcategory of $D^b(\mathscr{C})$ consisting of all complexes with cohomology in $\mathscr{C}[\Gamma]$). See [CPS1; (3.9), (1.3)]. Second, if $\Omega\subset\Gamma$ is a finite coideal, j^* induces an exact quotient functor $j^*: D^b(\mathscr{C}[\Gamma])\to D^b(\mathscr{C}(\Omega))$. Finally, a duality D on $\mathscr{C}$ induces an equivalence $D: D^b(\mathscr{C})\to D^b(\mathscr{C})^{\mathrm{op}}$ of triangulated categories. To see this, observe that if $\mathscr{T}$ is a triangulated category, then the dual category $\mathscr{T}^{\mathrm{op}}$ carries a natural structure as a triangulated category: for $X\in\mathrm{Ob}(\mathscr{T})$, let $\tilde{X}$ denote the corresponding object in $\mathscr{T}^{\mathrm{op}}$, and adopt a similar notation for morphisms. The shift in $\mathscr{T}^{\mathrm{op}}$ is now defined by $\tilde{X}[1]=\widetilde{X[-1]}$. A diagram $\tilde{X}\xrightarrow{\tilde{f}}\tilde{Y}\xrightarrow{\tilde{g}}\tilde{Z}\xrightarrow{\tilde{h}}$ is a distinguished triangle in $\mathscr{T}^{\mathrm{op}}$ if and only if $Z\xrightarrow{g}Y\xrightarrow{f}X\xrightarrow{h[1]}$ is a distinguished triangle in $\mathscr{T}$. (The verification that $\mathscr{T}^{\mathrm{op}}$ becomes a triangulated category with these definitions is routine.) Now let D be a duality on $\mathscr{C}$. Thus, $D:\mathscr{C}\to\mathscr{C}^{\mathrm{op}}$ is a (covariant) equivalence, and so induces an equivalence $D^b(\mathscr{C})\to D^b(\mathscr{C}^{\mathrm{op}})$. However, $D^b(\mathscr{C}^{\mathrm{op}})$ is naturally equivalent to $D^b(\mathscr{C})^{\mathrm{op}}$ by the functor taking a complex $\cdots\to\tilde{X}^n\xrightarrow{\tilde{d}^n}\tilde{X}^{n+1}\to\cdots$ in $D^b(\mathscr{C})$ to the complex $\cdots\to Y^n\xrightarrow{d'^n}Y^{n+1}\to\cdots$, where $Y^n=X^{-n}$ and $d'^n=(-1)^n d^{-n-1}$.

2. Enriched Grothendieck groups. If $\mathscr{C}$ is an abelian category, there is a standard procedure, first appearing in [SGA5; VIII.2] (see also [H; III.1]), for defining a Grothendieck group $K_0(D^b(\mathscr{C}))$ of the bounded derived category $D^b(\mathscr{C})$. One takes the free abelian group generated by elements $[X]$, $X\in\mathrm{Ob}(D^b(\mathscr{C}))$, modulo the natural additive relations $[X]+[Z]=[Y]$ associated to distinguished triangles $X\to Y\to Z\to$. Then $K_0(D^b(\mathscr{C}))$ is isomorphic to the usual Grothendieck group $K_0(\mathscr{C})$ of $\mathscr{C}$, since, in addition to the relations defining the latter, one has for the former the relation $[X[1]]=-[X]$ defined by the distinguished triangle $X\to 0\to X[1]\to$(obtained by rotating the distinguished triangle $X\xrightarrow{\mathrm{id}}X\to 0\to$).

Unfortunately, the isomorphism $K_0(D^b(\mathscr{C}))\cong K_0(\mathscr{C})$ means that all degree information in $K_0(D^b(\mathscr{C}))$ is lost. For a general abelian category $\mathscr{C}$ there seems to be little hope to remedy this deficiency. However, for a highest weight category $\mathscr{C}$ with a little additional structure, there are very natural "enriched" Grothendieck groups which do keep track of degree information.

In this section, let $\mathscr{C}$ be a fixed highest weight category with weight poset Λ (as in §1). Let $t\equiv q^{1/2}$ be an indeterminate and form the ring $\boldsymbol{Z}[t,t^{-1}]$ of Laurent polynomials.

By abuse of notation, we also use the symbol "t" to denote the exact functor

$$t: D^b(\mathscr{C}) \to D^b(\mathscr{C})$$

defined by putting $t(X) = X[-1]$ on objects X and $t(f) = f[-1]$ on morphisms $f: X \to Y$ in $D^b(\mathscr{C})$. Thus, the indeterminate $q = t^2$ defines an exact functor $q: D^b(\mathscr{C}) \to D^b(\mathscr{C})$ which is just the "double upward" shift operator on the derived category.

We suppose that we are given a "length" function

$$\ell: \Lambda \to \boldsymbol{Z}$$

defined on the weights of $\mathscr{C}$. The function ℓ can be quite arbitrary; we shall only use the fact that it assigns to each weight λ a *parity* depending on whether $\ell(\lambda)$ is even or odd.

Define a full additive subcategory $\mathscr{E}^L \equiv \mathscr{E}^L(\mathscr{C}, \ell)$ of $D^b(\mathscr{C})$ recursively as follows. Let $\mathscr{E}_0^L \subset D^b(\mathscr{C})$ be the full additive subcategory whose objects are finite direct sums of objects of the form $V(\lambda)[s]$ for $\lambda \in \Lambda$ and integers $s \equiv \ell(\lambda) \pmod 2$. Having defined $\mathscr{E}_i^L$, let $\mathscr{E}_{i+1}^L \subset D^b(\mathscr{C})$ be the full additive subcategory with objects X for which there exists a distinguished triangle:

$$E_i \to X \to E_i' \to$$

with $E_i, E_i' \in \mathrm{Ob}(\mathscr{E}_i^L)$. Finally, $\mathscr{E}^L$ is defined by setting

$$\mathrm{Ob}(\mathscr{E}^L) = \bigcup_i \mathrm{Ob}(\mathscr{E}_i^L).$$

Clearly, $\mathscr{E}^L$ is invariant under all even translations $X \mapsto X[2m]$, $m \in \boldsymbol{Z}$. Also, $\mathscr{E}^L$ is a strict subcategory of $D^b(\mathscr{C})$ in the sense that any object in $D^b(\mathscr{C})$ isomorphic to an object in $\mathscr{E}^L$ already belongs to $\mathscr{E}^L$. (We remark that the objects X of $\mathscr{E}^L$ are those that may be written $X = X_1 * \cdots * X_n$, using the $*$-operation of [BBD: §3], with each X_i of the form $V(\lambda_i)[k]$ for $k \equiv \ell(\lambda_i) \pmod 2$; intuitively, X is filtered with sections X_i.)

Having defined $\mathscr{E}^L$, let $\mathscr{E}^L[1] \subset D^b(\mathscr{C})$ denote the full subcategory with objects $X[1]$, $X \in \mathrm{Ob}(\mathscr{E}^L)$. Then define $\hat{\mathscr{E}}^L = \mathscr{E}^L \oplus \mathscr{E}^L[1]$, the full subcategory of $D^b(\mathscr{C})$ having objects of the form $X \oplus X'$, where $X \in \mathrm{Ob}(\mathscr{E}^L)$ and $X' \in \mathrm{Ob}(\mathscr{E}^L[1])$.

Now define the "left" Grothendieck group $K_0^L(\mathscr{C}, \ell)$ of $\mathscr{C}$ with respect to the length function ℓ to be the free abelian group generated by symbols $[X] \equiv [X]_L$, $X \in \mathrm{Ob}(\hat{\mathscr{E}}^L)$, subject to the relations $[X] + [Z] = [Y]$ provided there exists a distinguished triangle $X \to Y \to Z \to$ in $\mathscr{E}^L$ or in $\mathscr{E}^L[1]$. Finally, for $X = X' \oplus X''$, with $X' \in \mathscr{E}^L$ and $X'' \in \mathscr{E}^L[1]$, we require that the relation $[X] = [X'] + [X'']$ hold.

The dual notation of the "right" Grothendieck group $K_0^R(\mathscr{C}, \ell)$ is defined by analogy with $K_0^L(\mathscr{C}, \ell)$ above, but using the objects $A(\lambda)[s]$ for $\lambda \in \Lambda$ and integers $s \equiv \ell(\lambda) \pmod 2$. We denote the subcategories corresponding to $\mathscr{E}^L$ and $\hat{\mathscr{E}}^L$ by $\mathscr{E}^R$ and $\hat{\mathscr{E}}^R$, respectively. For an object X in $\hat{\mathscr{E}}^R$ we write $[X]$ (or, more precisely, $[X]_R$) for the corresponding element in $K_0^R(\mathscr{C}, \ell)$.

Denote by $f \mapsto \bar{f}$ the automorphism of $\boldsymbol{Z}[t, t^{-1}]$ which sends t to t^{-1}. The exact functor $t: D^b(\mathscr{C}) \to D^b(\mathscr{C})$ above induces a natural action of the ring $\boldsymbol{Z}[t, t^{-1}]$ on $K_0^L(\mathscr{C}, \ell)$ and on $K_0^R(\mathscr{C}, \ell)$. In both cases, the action of t is given by

$$t[X] = [tX] = [X[-1]] .$$

Clearly, a duality D on $\mathscr{C}$ induces contravariant functors $D: \hat{\mathscr{E}}^L \to \hat{\mathscr{E}}^R$ and $D: \hat{\mathscr{E}}^R \to \hat{\mathscr{E}}^L$ as well as additive isomorphisms $D: K_0^L(\mathscr{C}, \ell) \to K_0^R(\mathscr{C}, \ell)$ and $D: K_0^R(\mathscr{C}, \ell) \to K_0^L(\mathscr{C}, \ell)$, semi-linear relative to the automorphism $f \mapsto \bar{f}$ of $\boldsymbol{Z}[t, t^{-1}]$ defined above.

(2.1) EXAMPLE. The category Vect_k of finite dimensional vector spaces over the field k is a highest weight category with a unique simple object (up to isomorphism). Taking the length function to be identically equal to 0, every object in $\mathscr{E}^L = \mathscr{E}^R$ is a direct sum of objects of the form $V[2m]$, where V is a one-dimensional vector space. It follows easily that

$$K_0^L(\mathrm{Vect}_k, 0) = K_0^R(\mathrm{Vect}_k, 0) \cong \boldsymbol{Z}[t, t^{-1}]$$

as $\boldsymbol{Z}[t, t^{-1}]$-modules. In fact, $D^b(\mathrm{Vect}_k) = \hat{\mathscr{E}}^L = \hat{\mathscr{E}}^R$. Given X in $D^b(\mathrm{Vect}_k)$, we have

$$X \cong \bigoplus_n H^n(X)[-n] , \tag{2.1.1}$$

and in the left and right Grothendieck groups an identification

$$[X]_L = [X]_R = \sum_n \dim_k H^n(X) t^n \in \boldsymbol{Z}[t, t^{-1}] . \tag{2.1.2}$$

Returning to the general case of the highest weight category $\mathscr{C}$, the specialization homomorphism $\boldsymbol{Z}[t, t^{-1}] \to \boldsymbol{Z}$ given by $t \mapsto -1$ defines isomorphisms

$$K^0(D^b(\mathscr{C})) \cong K_0(\mathscr{C}) \cong K_0^L(\mathscr{C}, \ell) \otimes_{\boldsymbol{Z}[t,t^{-1}]} \boldsymbol{Z} \cong K_0^R(\mathscr{C}, \ell) \otimes_{\boldsymbol{Z}[t,t^{-1}]} \boldsymbol{Z} . \tag{2.1.3}$$

Thus, both $K_0^L(\mathscr{C}, \ell)$ and $K_0^R(\mathscr{C}, \ell)$ can be viewed as "q-analogs" (actually "t-analogs"!) of the ordinary Grothendieck group $K_0(\mathscr{C})$.

We now show there is a natural pairing between these two Grothendieck groups into $\boldsymbol{Z}[t, t^{-1}]$. This requires the following lemma, which is immediate from the proof of [CPS1; (3.11)].

(2.2) LEMMA. *Let $\mathscr{C}$ be a highest weight category as above. For $\lambda, \nu \in \Lambda$ and $r, s \in \boldsymbol{Z}$, we have*

$$\dim_k \mathrm{Hom}_{D^b(\mathscr{C})}(V(\lambda)[r], A(\nu)[s]) = \delta_{rs} \delta_{\lambda\nu} .$$

(2.3) PROPOSITION. *Let $\mathscr{C}$ be a highest weight category with poset Λ and length function $\ell: \Lambda \to \boldsymbol{Z}$. There is a natural non-degenerate sesquilinear pairing*

$$\langle\, , \rangle : K_0^L(\mathscr{C}, \ell) \times K_0^R(\mathscr{C}, \ell) \to \boldsymbol{Z}[t, t^{-1}]$$

given by setting

$$\langle [V], [W] \rangle = [\boldsymbol{R}\operatorname{Hom}^{\bullet}(V, W)]$$

with the right-hand side computed in $K_0^L(\mathrm{Vect}_k, 0) = K_0^R(\mathrm{Vect}_k, 0)$ (*cf. Example* 2.1).

If $\mathscr{C}$ *has a duality* D, *then*

$$\langle x, Dy \rangle = \langle y, Dx \rangle \ \forall x, y \in K_0^L(\mathscr{C}, \ell) .$$

In general, $K_0^L(\mathscr{C}, \ell)$ *is a free* $\boldsymbol{Z}[t, t^{-1}]$*-module with basis given by* $\{[V(\lambda)] \mid \lambda \in \Lambda\}$, *while* $K_0^R(\mathscr{C}, \ell)$ *is free with corresponding dual basis given by* $\{[A(\lambda)] \mid \lambda \in \Lambda\}$.

PROOF. First, we claim that $\operatorname{Hom}^n(X, Y) = 0$ for $X \in \operatorname{Ob}(\mathscr{E}^L)$, $Y \in \operatorname{Ob}(\mathscr{E}^R)$ and all sufficiently large $|n|$. From the definitions of $\mathscr{E}^L$ and $\mathscr{E}^R$, it suffices to verify this assertion in the special case when $X = V(\lambda)$ and $Y = A(\nu)$ for weights λ, ν. This follows from Lemma 2.2. Thus, by (2.1.1), we have the following identity in $K_0^L(\mathrm{Vect}_k, 0) \cong \boldsymbol{Z}[t, t^{-1}]$:

(2.3.2)
$$\begin{aligned}[\boldsymbol{R}\operatorname{Hom}^{\bullet}(X, Y)] &= \sum_n \dim_k \operatorname{Hom}^n_{D^b(\mathscr{C})}(X, Y) t^n \\ &= \sum_n [\operatorname{Hom}^n_{D^b(\mathscr{C})}(X, Y)[-n]] \in \boldsymbol{Z}[t, t^{-1}] .\end{aligned}$$

To establish that $\langle , \rangle$ defines a sesquilinear pairing it suffices to define it on the subgroups of $K_0^L(\mathscr{C}, \ell)$ and $K_0^R(\mathscr{C}, \ell)$ generated, respectively, by $[X]$, $X \in \operatorname{Ob}(\mathscr{E}^L)$, and by $[Y]$, $Y \in \operatorname{Ob}(\mathscr{E}^R)$. By (2.3.2), it suffices to consider the bifunctors $\operatorname{Hom}^n_{D^b(\mathscr{C})}(-, -)$ for each n separately.

If n is *odd*, the orthogonality relations (2.2) and an easy induction based on the recursive definition of $\mathscr{E}^L$ yields, for $X \in \operatorname{Ob}(\mathscr{E}^L)$ and $Y \in \operatorname{Ob}(\mathscr{E}^R)$:

(2.3.3)
$$\operatorname{Hom}^n(X, Y) = 0 .$$

For any $n \in \boldsymbol{Z}$, any distinguished triangle $X \to Y \to Z \to$ in $\mathscr{E}^L$, and any $W \in \operatorname{Ob}(\mathscr{E}^R)$, (2.3.3) leads to an exact sequence:

(2.3.4)
$$0 \to \operatorname{Hom}^n(Z, W) \to \operatorname{Hom}^n(Y, W) \to \operatorname{Hom}^n(X, W) \to 0 .$$

Let LK_0 be the free abelian group with basis elements $[X]$, $X \in \operatorname{Ob}(\mathscr{E}^L)$. For each $Y \in \operatorname{Ob}(\mathscr{E}^R)$, there is a natural map $LK_0 \to K_0(\mathrm{Vect}_k)$ given by $[X] \mapsto [\operatorname{Hom}^n(X, Y)]$ which, by (2.3.4), factors through $K_0^L(\mathscr{C}, \ell)$. Thus, we obtain a well-defined homomorphism

$$[\operatorname{Hom}^n(-, Y)] : K_0^L(\mathscr{C}, \ell) \to K_0(\mathrm{Vect}_k) \cong \boldsymbol{Z}$$

of abelian groups for any object Y in $\mathscr{E}^R$. Similarly, we have a well-defined homomorphism

$$[\operatorname{Hom}^n(X, -)] : K_0^R(\mathscr{C}, \ell) \to \boldsymbol{Z}$$

for any object X in $\mathscr{E}^L$. Observe that for any such X, Y we have

$$[\mathrm{Hom}^n(X, -)]([Y]) = [\mathrm{Hom}^n(X, Y)] = [\mathrm{Hom}^n(-, Y)]([X]) .$$

Let RK_0 be the free abelian group with basis elements $[Y]$, $Y \in \mathrm{Ob}(\mathscr{E}^R)$. It follows easily that there is, for each $n \in \boldsymbol{Z}$, a pairing $\langle\, ,\rangle_n : LK_0 \times RK_0 \to \boldsymbol{Z}$ defined by

$$\langle \sum_i m_i[X_i], \sum_j n_j[Y_j] \rangle_n = \sum_{i,j} m_i n_j [\mathrm{Hom}^n(X_i, Y_j)] .$$

These pairings induce the desired pairing $\langle\, ,\rangle = \sum_n \langle\, ,\rangle_n t^n$ on Grothendieck groups.

The adjoint formula (2.3.1) follows from the properties of D and the definition of the pairing. The statement about dual bases is clear from Lemma 2.2 $\square$

The following "recognition" theorem gives a homological criterion for determining when objects belong to $\mathscr{E}^L$ (or, dually, to $\mathscr{E}^R$).

(2.4) THEOREM. *Let $\mathscr{C}$ be a highest weight category with weight poset Λ and length function $\ell : \Lambda \to \boldsymbol{Z}$. Let $M \in \mathrm{Ob}(D^b(\mathscr{C}))$. If M belongs to $\mathscr{E}^L$ (resp., $\mathscr{E}^R$) then*

(2.4.1) $$\mathrm{Hom}^n(M, A(\lambda)) \neq 0 \Rightarrow n \equiv \ell(\lambda) \pmod 2 \ \ \forall \lambda \in \Lambda$$

(resp.,

(2.4.2) $$\mathrm{Hom}^n(V(\lambda), M) \neq 0 \Rightarrow n \equiv \ell(\lambda) \pmod 2 \ \ \forall \lambda \in \Lambda .)$$

Conversely, if Λ is finite and the above condition holds, then M belongs to $\mathscr{E}^L$ (resp., $\mathscr{E}^R$).

PROOF. We will only give the argument for $\mathscr{E}^L$. A dual argument applies to $\mathscr{E}^R$. Clearly, if M belongs to $\mathscr{E}^L$, (2.4.1) holds by Lemma 2.2.

Conversely, suppose $M \in \mathrm{Ob}(D^b(\mathscr{C}))$ satisfies (2.4.1) and Λ is finite. Let $\Gamma \subset \Lambda$ be the ideal generated by the weights of the composition factors of the cohomology objects $H^\bullet(M)$. By (1.5), the full embedding $D^b(\mathscr{C}[\Gamma]) \to D^b(\mathscr{C})$ has as strict image the relative derived category $D^b_{\mathscr{C}[\Gamma]}(\mathscr{C})$. Thus, we can assume M belongs to $D^b(\mathscr{C}[\Gamma])$, i.e., we can assume $\Gamma = \Lambda$ in what follows.

Let $\gamma \in \Lambda$ be maximal, and fix an integer r such that $H^r(M)$ has a composition factor $L(\gamma)$. Since $A(\gamma)$ (resp., $V(\gamma)$) is injective (resp., projective) in the category $\mathscr{C}$

$$0 \neq \mathrm{Hom}_{\mathscr{C}}(H^r(M), A(\gamma)) \cong \mathrm{Hom}_{D^b(\mathscr{C})}(M, A(\gamma)[-r]) .$$

Choose a morphism $V(\gamma)[-r] \xrightarrow{f} M$ which induces a surjection

(2.4.3) $$\mathrm{Hom}_{D^b(\mathscr{C})}(M, A(\gamma)[-r]) \to \mathrm{Hom}_{D^b(\mathscr{C})}(V(\gamma)[-r], A(\gamma)[-r]) ,$$

and form the distinguished triangle $V(\gamma)[-r] \xrightarrow{f} M \to M' \to$. The surjection (2.4.3) implies that $\mathrm{Hom}^n(M', A(\lambda))$ embeds into $\mathrm{Hom}^n(M, A(\lambda))$ for all n and all $\lambda \in \Lambda$. Thus M' satisfies the condition: $\mathrm{Hom}^n(M', A(\lambda)) \neq 0 \Rightarrow n \equiv \ell(\lambda) \pmod 2$ $\forall \lambda \in \Lambda$. Also, we have an inequality

$$[H^r(M') : L(\gamma)] < [H^r(M) : L(\gamma)]$$

of multiplicities, while $[H^n(M') : L(\gamma)] = [H^n(M) : L(\gamma)]$ for $n \neq r$. An evident induction argument on multiplicities and $|\Gamma|$ now completes the proof. □

3. Abstract Kazhdan-Lusztig polynomials. Fix a highest weight category $\mathscr{C}$ as in §1 and a length function $\ell: \Lambda \to \boldsymbol{Z}$ on its weight poset. If $X \in \mathrm{Ob}(D^b(\mathscr{C}))$ and $\nu \in \Lambda$, define left and right Poincaré "polynomials" $p^L_{\nu,X}, p^R_{\nu,X} \in \boldsymbol{Z}[t, t^{-1}]$ by the formulas:

$$p^L_{\nu,X} = \sum_n \dim_k \mathrm{Hom}^n(X, A(\nu))t^n \tag{3.0.1}$$

and

$$p^R_{\nu,X} = \sum_n \dim_k \mathrm{Hom}^n(V(\nu), X)t^n . \tag{3.0.2}$$

To see that $p^L_{\nu,X}$ and $p^R_{\nu,X}$ are in $\boldsymbol{Z}[t, t^{-1}]$ (i.e., there is a bounded range of nonzero coefficients $\mathrm{Hom}^n(X, A(\nu))$ and $\mathrm{Hom}^n(V(\nu), X)$), we argue as follows. A truncation argument using the distinguished triangles $\tau_{\leq n}X \to X \to \tau_{>n}X \to$ [BBD; §1.3] reduces to the case in which $X \in \mathrm{Ob}(\mathscr{C})$. From there, induction allows us to assume that $X \cong L(\lambda)$ for some weight λ. The desired conclusion then follows from [CPS1; (3.8b)] (and its dual statement) which asserts that, for fixed λ and ν, $\mathrm{Ext}^n_{\mathscr{C}}(L(\lambda), A(\nu))$ and $\mathrm{Ext}^n_{\mathscr{C}}(V(\nu), L(\lambda))$ vanish for sufficiently large integers n.

We record that, for $X \in \mathrm{Ob}(D^b(\mathscr{C}))$ and $\nu \in \Lambda$, the coefficients of $p^L_{\nu,X}, p^R_{\nu,X}$ are all *non-negative*. Also, if $X \in \mathrm{Ob}(\mathscr{C})$, then $p^L_{\nu,X}$ and $p^R_{\nu,X}$ belong to $\boldsymbol{Z}[t]$, i.e., they are *polynomials* in t. This follows since $\mathrm{Hom}^n(Z, W) \cong \mathrm{Ext}^n(Z, W) = 0$ for all integers $n < 0$ when both Z and W belong to $\mathscr{C}$. This fact will be often used without further comment.

We call the polynomials $p^L_{\nu,\lambda} \equiv p^L_{\nu,L(\lambda)}$ and $p^R_{\nu,\lambda} \equiv p^R_{\nu,L(\lambda)}$ the *Poincaré polynomials* of $L(\lambda)$. Define the corresponding *Kazhdan-Lusztig polynomials* $P_{\nu,\lambda}$ by the formula

$$P_{\nu,\lambda} \equiv P^L_{\nu,\lambda} = t^{\ell(\lambda)-\ell(\nu)}\bar{p}^L_{\nu,\lambda} , \tag{3.0.3}$$

with $\bar{p}^L_{\nu,\lambda}$ as above (2.1). (*Right* Kazhdan-Lusztig polynomials will not be needed.) The definition (3.0.3) is motivated by work of Vogan [V1; II, 3.4] on the Kazhdan-Lusztig conjecture for the category $\mathcal{O}$.

In the cases of interest, $P_{\nu,\lambda}$ *is* a polynomial in t^2, which can then be regarded as a polynomial in $q = t^2$. This is an often used convention which we shall *not* adhere to in this paper. Also, our Kazhdan-Lusztig polynomials $P_{\nu,\lambda}$ (3.0.3) are indexed by pairs of weights $\lambda, \nu \in \Lambda$, while the classical Kazhdan-Lusztig polynomials $P_{x,y}$ [KL1] are indexed by pairs of elements x, y in a Coxeter group.

An object X in $D^b(\mathscr{C})$ determines a row "vector" $\boldsymbol{p}^L_X = (\cdots, p^L_{\nu,X}, \cdots)$ and a column "vector" $\boldsymbol{p}^R_X = (\cdots, p^R_{\nu,X}, \cdots)^T$ with entries indexed by Λ. Unless Λ is finite, $\boldsymbol{p}^L_X$ and $\boldsymbol{p}^R_X$ may have infinite support. However, if $X \in \mathrm{Ob}(\hat{\mathscr{E}}^L)$, then in $K^L_0(\mathscr{C}, \ell)$ there is a finite expression

$$[X] = \sum_{\nu \in \Lambda} \bar{p}^L_{\nu,X}[V(\nu)] , \tag{3.0.4}$$

and $\boldsymbol{p}_X^L$ has finite support. In terms of the sesquilinear pairing of Proposition 2.3, we have

(3.0.5) $$p^L_{\nu,X} = \langle [X], [A(\nu)] \rangle .$$

If, on the other hand, $Y \in \mathrm{Ob}(\hat{\mathscr{E}}^R)$, we have the finite expression

(3.0.6) $$[Y] = \sum_\nu p^R_{\nu,Y}[A(\nu)]$$

in $K_0^R(\mathscr{C}, \ell)$, as well as the equation

(3.0.7) $$p^R_{\nu,Y} = \langle [V(\nu)], [Y] \rangle .$$

Clearly, (3.0.5) (resp., (3.0.7)) implies that, for any fixed ν, $p^L_{\nu,X}$ (resp., $p^R_{\nu,Y}$) is *additive* in X (resp., Y) over distinguished triangles defining $K_0^L(\mathscr{C}, \ell)$ (resp., $K_0^R(\mathscr{C}, \ell)$). Also, for $X \in \mathrm{Ob}(\hat{\mathscr{E}}^L)$, $X = 0$ if and only if $[X] = 0$ (or equivalently, if and only if $p^L_{\nu,X} = 0$ for all $\nu \in \Lambda$). Similar remarks apply to $\hat{\mathscr{E}}^R$.

The following lemma is easily verified, using the above definitions and (2.3.1).

(3.1) LEMMA. *If $\mathscr{C}$ has a duality D, then $p^L_{\nu,X} = p^R_{\nu,DX}$ for all weights ν and all objects $X \in \mathrm{Ob}(D^b(\mathscr{C}))$. Hence, for any such X, we have $\boldsymbol{p}^R_{DX} = (\boldsymbol{p}^L_X)^T$.*

For $X \in \mathrm{Ob}(\mathscr{C})$, let $\mathrm{ch}(X)$ be the image of X in the Grothendieck group $K_0(\mathscr{C})$. The following proposition is easily established using the definitions.

(3.2) PROPOSITION. *Let $X \in \mathrm{Ob}(\mathscr{C})$ for a highest weight category $\mathscr{C}$. Then in the Grothendieck group $K_0(\mathscr{C})$ we have the following formula for the "character" of X:*

$$\mathrm{ch}(X) = \sum_{\nu \in \Lambda} p^L_{\nu,X}(-1)\mathrm{ch}\, V(\nu) = \sum_{\nu \in \Lambda} p^R_{\nu,X}(-1)\,\mathrm{ch}\, A(\nu) .$$

The formula in (3.2) is an abstract version of Delorme's result for the category $\mathcal{O}$; see [KL1]. However, if $X \in \mathrm{Ob}(\hat{\mathscr{E}}^L)$, it follows from an actual formula in $K_0^L(\mathscr{C}, \ell)$, as observed above. In this case, much more information can be obtained, suggesting (3.3) and (3.4) below.

(3.3) DEFINITION. Let $\mathscr{C}$ be a highest weight category having a length function $\ell : \Lambda \to \boldsymbol{Z}$. An object $X \in D^b(\mathscr{C})$ has a *left parity* if, for some integer ε,

$$\mathrm{Hom}^n(X, A(\nu)) \neq 0 \Rightarrow n \equiv \ell(\nu) + \varepsilon \pmod 2 \quad \forall \nu \in \Lambda .$$

If ε is even (resp., odd), we say X has even (resp., odd) left parity. The notation of *right parity* is defined dually. When X has both a left parity and a right parity and they both agree, we say X has a *two-sided parity*.

Finally, $\mathscr{C}$ has a *Kazhdan-Lusztig theory* (with respect to ℓ) if every simple object has a two-sided parity.

The next result is immediate using Theorem 2.4 and the fact that there are nonzero morphisms $L(\lambda)[-\ell(\lambda)]\to A(\lambda)[-\ell(\lambda)]$ and $V(\lambda)[-\ell(\lambda)]\to L(\lambda)[-\ell(\lambda)]$. When Λ is finite (or even bounded below, i.e., when all finitely generated ideals are finite), the result gives an alternative definition of a Kazhdan-Lusztig theory: $L(\lambda)[-\ell(\lambda)]\in \mathrm{Ob}(\mathscr{E}^L\cap\mathscr{E}^R)$ for all λ. This approach was taken in an earlier version of this paper, though the 'even-odd vanishing' characterization (3.3) seems better suited in general. We explore more sophisticated aspects of the infinite poset case in a future paper.

(3.4) PROPOSITION. *Let $\mathscr{C}$ be a highest weight category with finite weight poset Λ and length function ℓ. For $\lambda\in\Lambda$, $L(\lambda)$ has a left (resp., right) parity if and only if $L(\lambda)[-\ell(\lambda)]\in\mathrm{Ob}(\mathscr{E}^L)$ (resp., $\mathrm{Ob}(\mathscr{E}^R)$). Also, $L(\lambda)$ has a two-sided parity if and only if it has both a left parity and a right parity.*

Proposition 2.3 and the expressions (3.0.4) and (3.0.5) imply the next result.

(3.5) THEOREM. *Assume $\mathscr{C}$ is a highest weight category with a length function $\ell:\Lambda\to\boldsymbol{Z}$. For $X\in\mathrm{Ob}(\hat{\mathscr{E}}^L)$ and $Y\in\mathrm{Ob}(\hat{\mathscr{E}}^R)$, we have $\langle[X],[Y]\rangle=\boldsymbol{p}_X^L\cdot\boldsymbol{p}_Y^R$. In particular, if $X, Y\in\mathrm{Ob}(\mathscr{C})$, we obtain*

$$\sum_n \dim_k \mathrm{Ext}^n(X,Y)t^n=\boldsymbol{p}_X^L\cdot\boldsymbol{p}_Y^R\,.$$

An immediate consequence of the theorem is the following result.

(3.6) COROLLARY. *Let $\mathscr{C}$ be a highest weight category with finite weight poset Λ. Suppose that $\mathscr{C}$ has a Kazhdan-Lusztig theory relative to a length function ℓ.*

(a) *Let $\boldsymbol{P}_{\mathscr{C}}(t)$ denote the $|\Lambda|\times|\Lambda|$ "Poincaré" matrix whose (λ,ν)-entry is the polynomial $\sum_n \dim_k\mathrm{Ext}^n(L(\lambda),L(\nu))t^n$. Then $\boldsymbol{P}_{\mathscr{C}}(t)=\boldsymbol{p}^L\cdot\boldsymbol{p}^R$, where $\boldsymbol{p}^L$ (resp., $\boldsymbol{p}^R$) is the matrix whose rows (resp., columns) are the $\boldsymbol{p}_{L(\lambda)}^L$ (resp., $\boldsymbol{p}_{L(\lambda)}^R$).*

(b) *Defining $\mu(\nu,\lambda)=\dim_k\mathrm{Ext}^1(L(\lambda),A(\nu))$ for $\nu<\lambda$, we have an equality*

$$\mu(\nu,\lambda)=\dim_k\mathrm{Ext}^1(L(\lambda),L(\nu))\,.$$

(c) *Assume $\mathscr{C}$ has a duality. Given weights $\nu<\lambda$, we have*

$$\dim_k\mathrm{Ext}^2(L(\nu),L(\lambda))\geq\sum_{\omega<\nu}\mu(\omega,\nu)\cdot\mu(\omega,\lambda)\,.$$

PROOF. Theorem 3.5 implies (a) and the fact that $\dim_k\mathrm{Ext}^1(L(\lambda),L(\nu))$ is the coefficient of t in the polynomial $\sum_\tau p_{\tau,\lambda}^L p_{\tau,\nu}^R$. Also, $p_{\tau,\lambda}^L$ (resp., $p_{\tau,\nu}^R$) has nonzero constant term (equal to 1 by (1.1)) if and only if $\tau=\lambda$ (resp., $\tau=\nu$). If $\tau=\lambda$, the $p_{\tau,\nu}^R=0$ since the inequality $\nu<\lambda$ implies that $\mathrm{Hom}^\bullet(V(\lambda),L(\nu))=0$ by (the dual of) [CPS1; (3.8b)]. Assertion (b) is now immediate. Finally, (c) follows from (a) and (b). □

(3.7) REMARK. Suppose $\mathscr{C}$ has a Kazhdan-Lusztig theory with respect to ℓ. Then (3.6a) implies that, if $\lambda,\nu\in\Lambda$ and $n\in\boldsymbol{Z}$ with $\mathrm{Ext}^n(L(\lambda),L(\nu))\neq 0$, then $\ell(\lambda)-\ell(\nu)\equiv n$ (mod 2). In particular, if $n=1$, $L(\lambda)$ and $L(\nu)$ have opposite parity. Using this, we see the *relative* parity of any two irreducibles in a single "block" of $\mathscr{C}$ is uniquely determined. Also, the groups $K_0^L(\mathscr{C}, \ell)$, $K_0^R(\mathscr{C}, \ell)$ are *independent* of ℓ (assuming only that ℓ gives rise to a Kazhdan-Lusztig theory).

We now give several examples of explicit calculations.

(3.8) EXAMPLE: Ext groups for the category $\mathcal{O}$. Let $\mathfrak{g}$ be complex semisimple Lie algebra, with Cartan subalgebra $\mathfrak{h}$ and Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$. Let $\rho \in \mathfrak{h}^*$ be the half-sum of the positive roots defined by $\mathfrak{b}$. Fix $\lambda \in \mathfrak{h}^*$ such that $\lambda + \rho$ is an anti-dominant integral weight. Let W^λ be the set of distinguished left coset representatives for the stabilizer W_λ of λ in the Weyl group W (relative to the "dot" action). (Thus, $w \in W^\lambda$ has the minimal length among elements in the coset wW_λ.) The usual ordering on W induces a poset structure on $\Gamma(\lambda) \equiv W \cdot \lambda = \{w(\lambda+\rho)-\rho) \mid w \in W\} : \gamma, \gamma' \in \Gamma(v)$ have unique representations $\gamma = w \cdot \lambda$ and $\gamma' = w' \cdot \lambda$, $w, w' \in W^\lambda$, and we put $\gamma \leq \gamma' \Leftrightarrow w \leq w'$.

Consider the [BGG] category $\mathcal{O}$ (corresponding to $\mathfrak{g}$) and the highest weight category $\mathcal{O}_\lambda \equiv \mathcal{O}[\Gamma(\lambda)]$ defined as in (1.3). Here $V(y \cdot \lambda)$ (resp., $A(y \cdot \lambda)$) is the Verma (resp., dual Verma) module of highest weight $y \cdot \lambda$. We can establish the following result.

(3.8.1) THEOREM. *The highest weight category $\mathcal{O}_\lambda$ has a Kazhdan-Lusztig theory with respect to the length function ℓ defined by $\ell(w \cdot \lambda) = \ell(w)$, $w \in W^\lambda$.*

PROOF. By Theorem 2.4 and an evident duality argument, it suffices to show, for $v, w \in W^\lambda$, that $\mathrm{Hom}^n(V(v \cdot \lambda), L(w \cdot \lambda)[-\ell(w)]) \neq 0 \Rightarrow n \equiv \ell(v) \pmod 2$.

If λ is regular, this follows from the truth of the Kazhdan-Lusztig conjecture, Vogan's homological interpretation [V1; II, 3.4] of the Kazhdan-Lusztig polynomials, and Proposition 3.2.

If λ is singular, we use Soergel's "nil-cohomology theorem" [So; p. 566], which, in our notation, calculates $\dim_{\mathbf{C}} \mathrm{Hom}^n(V(v \cdot \lambda), L(w \cdot \lambda)[-\ell(w)])$ as

$$\sum_{z \in W_\lambda} (-1)^{\ell(z)} \dim_{\mathbf{C}} \mathrm{Hom}^{n-\ell(z)}(V(vz \cdot (-2\rho)), L(w \cdot (-2\rho))[-\ell(w)]).$$

(The translation between Soergel's notation and ours is as follows: $M(v(\lambda+\rho)) = V(v \cdot \lambda)$, where $M(\theta)$ is, in Soergel's notation, the Verma module with highest weight $\theta - \rho$. Also, his $L(\lambda)$ identifies with our $L(\lambda - \rho)$.) Since $\ell(vz) = \ell(v) + \ell(z)$ for $v \in W^\lambda$, $z \in W_\lambda$, if the above expression is non-zero, then $n \equiv \ell(v) \pmod 2$, as desired. □

Continuing the discussion of Ext groups, we note Corollary 3.6 gives a calculation of the groups $\mathrm{Ext}^n(L(v), L(\tau))$, for $v, \tau \in \Gamma(\lambda)$, once the Kazhdan-Lusztig polynomials $P_{v,\tau}$ in (3.0.5) are related to the Kazhdan-Lusztig polynomials $P_{y,w}$, $y, w \in W$. (We regard these as polynomials in $t = q^{1/2}$.) However, using [So; p. 566], we verify that

$$P_{y \cdot \lambda, w \cdot \lambda} = \sum_{z \in W_\lambda} (-1)^{\ell(z)} P_{yz,w}.$$

(Thus, if λ is regular, we have $P_{y \cdot \lambda, w \cdot \lambda} = P_{y,w}$.) Applying (3.6), we obtain the following result. Recall $\bar{P}_{a,b}(t)=P_{a,b}(t^{-1})$.

(3.8.2) COROLLARY. *Let $\Gamma(\lambda)$ be as above. For $y, w\in W$,*

$$\sum_n \dim_{\mathbf{C}} \operatorname{Ext}^n_{\mathcal{O}}(L(y\cdot\lambda), L(w\cdot\lambda))t^n=\sum_{z\in W^\lambda}\sum_{a,b\in W_\lambda} t^{\ell(y)+\ell(w)-2\ell(z)}(-1)^{\ell(a)+\ell(b)}\bar{P}_{za,y}\bar{P}_{zb,w}\,.$$

In the special case of regular weights, the calculation (3.8.2) has also been obtained by Beilinson-Ginzburg [BG], stated with only a sketch of a very sophisticated proof. Also, in this case, Irving [I1] gives a calculation for Ext^1 between two simple modules.

(3.9) EXAMPLE: Ext groups for algebraic groups. We refer ahead to the context and notation of §5. The following result is immediate from Theorems 3.5 and 5.3.

(3.9.1) THEOREM. *Let G be a semisimple, simply connected algebraic group over an algebraically closed field k of characteristic $p\geq h$. Let $\lambda\in C_{\mathbf{Z}}$ (i.e., λ is a dominant weight in the bottom p-alcove). For dominant weights $w\cdot\lambda$, $y\cdot\lambda$ ($y, w\in W_p$) satisfying the Jantzen condition* (5.0.2), *we have*

$$\sum_n \dim_k \operatorname{Ext}^n_G(L(y\cdot\lambda), L(w\cdot\lambda))t^n=\sum_{z\cdot\lambda\ \text{dominant},\ z\in W_p} t^{\ell(y)+\ell(w)-2\ell(z)}\bar{P}_{zw_0,yw_0}\bar{P}_{zw_0,ww_0}\,.$$

(Here, $P_{a,b}$ denotes the Kazhdan-Lusztig polynomial associated to $a, b\in W_p$.)

We conclude this section with some ways to construct new examples from old ones.

(3.10) PROPOSITION. *Let $\mathscr{C}$ be a highest weight category having a Kazhdan-Lusztig theory relative to a length function $\ell:\Lambda\to\mathbf{Z}$. Let $\Gamma\subset\Lambda$ be an ideal, and let $\Omega\subset\Gamma$ be a coideal. The highest weight category $\mathscr{C}[\Gamma]$ (resp., $\mathscr{C}(\Omega)$ if Γ is finite) has a Kazhdan-Lusztig theory relative to $\ell|_\Gamma$ (resp., $\ell|_\Omega$). For $\lambda, \nu\in\Gamma$ (resp., $\lambda, \nu\in\Omega$) the Poincaré polynomials $p^L_{\nu,\lambda}$ and $p^R_{\nu,\lambda}$ are the same whether computed in $\mathscr{C}$ or $\mathscr{C}[\Gamma]$ (resp., $\mathscr{C}(\Omega)$).*

PROOF. By (1.3) and (1.5), the inclusion functor $i_*:\mathscr{C}[\Gamma]\to\mathscr{C}$ maps the induced and Weyl objects of $\mathscr{C}[\Gamma]$ to the corresponding objects in $\mathscr{C}$, and it induces a full embedding $i_*:D^b(\mathscr{C}[\Gamma])\to D^b(\mathscr{C})$. Therefore, $\mathscr{C}[\Gamma]$ has a Kazhdan-Lusztig theory, and the Poincaré polynomials for $\mathscr{C}[\Gamma]$ can be computed in $\mathscr{C}$.

By (1.5), (3.4) and the definitions in §2, the quotient functor $j^*:D^b(\mathscr{C}[\Gamma])\to D^b(\mathscr{C}(\Omega))$ takes $\mathscr{E}^L(\mathscr{C}[\Gamma])$ and $\mathscr{E}^R(\mathscr{C}[\Gamma])$ to $\mathscr{E}^L(\mathscr{C}(\Omega))$ and $\mathscr{E}^R(\mathscr{C}(\Omega))$, respectively. For $\lambda\in\Omega$, $j^*L(\lambda)$ is the simple object corresponding to λ. If $\gamma\in\Gamma\backslash\Omega$, then $j^*A(\gamma)\cong j^*V(\gamma)\cong 0$. Thus, $\mathscr{C}(\Omega)$ has a Kazhdan-Lusztig theory by (3.4). Also, j^* induces homomorphisms $K^L_0(\mathscr{C}[\Gamma],\ell|_\Gamma)\to K^L_0(\mathscr{C}(\Omega),\ell|_\Omega)$ and $K^R_0(\mathscr{C}[\Gamma],\ell|_\Gamma)\to K^R_0(\mathscr{C}(\Omega),\ell|_\Omega)$ of Grothendieck groups. Hence, the Poincaré polynomials $p^L_{\nu,\lambda}$ for $\lambda, \nu\in\Omega$ can be computed in either $\mathscr{C}[\Gamma]$ (and thus $\mathscr{C}$) or in $\mathscr{C}(\Omega)$ by (3.0.1), (3.0.2). □

The assumption that Γ is finite in proving that $\mathscr{C}(\Omega)$ has a Kazhdan-Lusztig theory can be replaced by the assumption that Ω is a finite coideal, but we omit the details here.

3. Appendix. Graded Grothendieck group. We sketch how some of the theory works for *graded* highest weight categories. We define the latter here, at least for the case of a finite weight poset, to mean a category $\mathscr{C}_{\mathrm{gr}}$ of finite dimensional graded modules for a graded quasi-hereditary algebra A; see [CPS3]. (Thus, the algebra A is *positively* graded.) For a weight λ, we assume that the induced (resp., Weyl) module $A(\lambda)$ (resp., $V(\lambda)$) is graded with socle (resp., head) $L(\lambda)$ homogeneous of degree 0. Fix a length function $\ell : \Lambda \to \boldsymbol{Z}$.

If $X \in \mathrm{Ob}(\mathscr{C}_{\mathrm{gr}})$ and $i \in \boldsymbol{Z}$, let $X(i)$ denote the upward *shift in grade* (or *twist*) of X by i steps: $X(i)_n = X_{n-i}$. This induces a similar functor $X \mapsto X(i)$ on the bounded derived category $D^b(\mathscr{C}_{\mathrm{gr}})$. It will be convenient to write $X\{i\}$ for $X[i](i) = X(i)[i]$, $X \in \mathrm{Ob}(D^b(\mathscr{C}_{\mathrm{gr}}))$. Define a graded version $\mathscr{E}^L_{\mathrm{gr}}$ of $\mathscr{E}^L$ by replacing (in the definitions of §2) $V(\lambda)[-\ell(\lambda)]$ everywhere by $V(\lambda)\{-\ell(\lambda)\}$. Let $K^L_0(\mathscr{C}_{\mathrm{gr}}, \ell)$ be the associated left Grothendieck group constructed using $\mathscr{E}^L_{\mathrm{gr}} \oplus \mathscr{E}^L_{\mathrm{gr}}\{1\}$.

By construction, the evident forgetful functors induce a commutative diagram

$$\text{(3A.1)} \qquad \begin{array}{ccc} K^L_0(\mathscr{C}_{\mathrm{gr}}, \ell) & \xrightarrow{\sim} & K_0(\mathscr{C}_{\mathrm{gr}}) \\ \wr\!\downarrow & & \downarrow \\ K^L_0(\mathscr{C}, \ell) & \longrightarrow & K_0(\mathscr{C}) \end{array}$$

of the various Grothendieck groups. Observe that each Grothendieck group in (3A.1) is naturally a $\boldsymbol{Z}[t, t^{-1}]$-module: on $K^L_0(\mathscr{C}_{\mathrm{gr}}, \ell)$ (resp., $K^L(\mathscr{C}, \ell)$, $K_0(\mathscr{C}_{\mathrm{gr}})$, $K_0(\mathscr{C})$) t acts as the operator $[X] \mapsto [X\{-1\}]$ (resp., $[X] \mapsto [X[-1]]$, $[X] \mapsto -[X(-1)]$, $[X] \mapsto -[X]$). Clearly, the morphisms in (3A.1) are $\boldsymbol{Z}[t, t^{-1}]$-module homomorphisms, while both the top horizontal morphism and the left vertical morphism are, in fact, *isomorphisms*. (As usual, $\mathscr{C} = \text{mod-}A$).

Similarly, the category $\mathscr{E}^R_{\mathrm{gr}}$ and the associated right Grothendieck group $K^R_0(\mathscr{C}_{\mathrm{gr}}, \ell)$ are defined using the $A(\lambda)\{-\ell(\lambda)\}$. Also, there is a right analog (3A.1)′ of (3A.1).

We say $\mathscr{C}_{\mathrm{gr}}$ has a *graded Kazhdan-Lusztig theory* if $L(\lambda)\{-\ell(\lambda)\} \in \mathrm{Ob}(\mathscr{E}^L_{\mathrm{gr}} \cap \mathscr{E}^R_{\mathrm{gr}})$ for each λ. If this condition holds, a left Poincaré polynomial $p_{\nu,\lambda} \in \boldsymbol{Z}[t, t^{-1}]$ is defined by

$$\text{(3A.2)} \qquad [L(\lambda)] = \sum_{\nu} \bar{p}_{\nu,\lambda}[V(\nu)]\,.$$

This right Poincaré polynomial $p^R_{\nu,\lambda}$ is similarly defined, and we put $P_{\nu,\lambda} = t^{\ell(\lambda)-\ell(\nu)}\bar{p}_{\nu,\lambda}$ for the corresponding Kazhdan-Lusztig polynomial. These polynomials can also be defined by analogy with (3.0.1) and (3.0.2). By (3A.1), $P_{\nu,\lambda}$ is also a Kazhdan-Lusztig polynomial for the highest weight category $\mathscr{C}$.

Assume that $\mathscr{C}_{\mathrm{gr}}$ has a graded Kazhdan-Lusztig theory. An expression (3A.2), valid in any one of the three isomorphic Grothendieck groups $K^L_0(\mathscr{C}_{\mathrm{gr}}, \ell)$, $K^L_0(\mathscr{C}, \ell)$, or $K_0(\mathscr{C}_{\mathrm{gr}})$ in (3A.1) leads to valid expressions in the other two Grothendieck groups. (Clearly, our assumption implies that $\mathscr{C}$ has a Kazhdan-Lusztig theory.)

There are conceptual advantages to attaching different meanings to the (3A.2) equation for the simple module $L(\lambda)$. As a simple application, consider the question of "inverting" the Kazhdan-Lusztig polynomials theoretically. By (3A.1), it suffices to invert (3A.2) when read as an equation in $K_0(\mathscr{C}_{\mathrm{gr}})$. Clearly, we can write $[V(\nu)]=\sum_{\lambda'\leq\nu}\bar{q}_{\nu,\lambda'}[L(\lambda')]$ in $K_0(\mathscr{C}_{\mathrm{gr}})$, where $q_{\nu,\lambda'}$ is a polynomial in $-t$, acting as (-1), with positive coefficients. That is, $q_{\nu,\lambda'}=\sum_n(-1)^n a_n t^n$, $a_n\geq 0$. Substituting this expression into the image of (3A.2) in $K_0(\mathscr{C}_{\mathrm{gr}})$ yields the identity $\sum_{\lambda'\leq\nu\leq\lambda}\bar{p}_{\nu,\lambda}\bar{q}_{\nu,\lambda'}=\delta_{\lambda,\lambda'}$. Putting $Q_{\nu,\lambda'}=(-t)^{\ell(\nu)-\ell(\lambda')}\bar{q}_{\nu,\lambda'}$, we obtain the equivalent expression

(3A.3) $$\sum_{\lambda'\leq\nu\leq\lambda}(-1)^{\ell(\lambda)-\ell(\nu)}P_{\nu,\lambda}Q_{\nu,\lambda'}=\delta_{\lambda,\lambda'}\,.$$

Observe (3A.3) implies that the $Q_{\nu,\lambda'}$ are polynomials in t^2. These polynomials thus have positive coefficients by their construction. Of course, for elementary reasons an inverse to the matrix $[P_{\nu,\lambda}]$ must exist, but our discussion supplies an explicit interpretation (as well as the positivity) of that inverse. Compare also [I1], [L2] and [A2].

Using an analog of Theorem 3.5, it can be shown, when $\mathscr{C}_{\mathrm{gr}}$ has a graded Kazhdan-Lusztig theory, that the graded quasi-hereditary algebra A is *formal* (Koszul) in the sense of Beilinson-Ginzburg [BG]. It can further be shown that its homological dual $A^!$ is also a graded quasi-hereditary algebra. The corresponding graded highest weight category $A^!$-mod (with weight poset Λ^{op}) has a graded Kazhdan-Lusztig theory in which the $Q_{\nu,\lambda}$ are the Kazhdan-Lusztig polynomials. Further details will appear elsewhere.

4. A parity-based decomposition theorem. We begin with the following important complete reducibility criterion. In a limited way, the result plays a role analogous to Gabber's purity theorem [BBD; (5.3.8)]; see the remarks after (4.2) below.

(4.1) THEOREM. *Suppose $\mathscr{C}$ is a highest weight category with weight poset Λ, and $X\in\mathrm{Ob}(\mathscr{C})$. For each $\lambda,\nu\in\Lambda$ with* $\mathrm{Hom}(X,A(\lambda))\neq 0\neq\mathrm{Hom}(X,A(\nu))$ *assume*

$$\mathrm{Ext}^1(V(\lambda),L(\nu))=0\,,$$

and dually, if $\mathrm{Hom}(V(\lambda),X)\neq 0\neq\mathrm{Hom}(V(\nu),X)$, *assume*

$$\mathrm{Ext}^1(L(\nu),A(\lambda))=0\,.$$

Then X is completely reducible.

PROOF. Let $\lambda\in\Lambda$ be maximal with the property that $L(\lambda)$ is a composition factor of X. By (1.3), we can replace Λ by an ideal Γ and $\mathscr{C}$ by the highest weight category $\mathscr{C}[\Gamma]$ to assume that the weight λ is maximal, and, thus, $V(\lambda)$ is projective and $A(\lambda)$ is injective.

We claim that all composition factors of X isomorphic to $L(\lambda)$ occur in the head of X. If not, there is a subobject Y of $\mathrm{rad}(X)$ with $L(\lambda)$ as a quotient. Since $A(\lambda)$ is injective, the composite morphism $Y \to L(\lambda) \subseteq A(\lambda)$ extends to a morphism $X \to A(\lambda)$ with image Q properly containing $L(\lambda)$.

Let ν be maximal among the weights of composition factors of $Q/L(\lambda)$. Then there is a nonzero map $Q/L(\lambda) \to A(\nu)$, and thus a nonzero morphism $X \to A(\nu)$. By hypothesis, $\mathrm{Ext}^1(V(\nu), L(\lambda)) = 0$. However, $\nu \neq \lambda$, so $\mathrm{Hom}(V(\nu), A(\lambda)) = 0$ by Lemma 2.2. Hence, $\mathrm{Hom}(V(\nu), A(\lambda)/L(\lambda))$ injects into $\mathrm{Ext}^1(V(\nu), L(\lambda))$. But $\mathrm{Hom}(V(\nu), A(\lambda)/L(\lambda))$ contains $\mathrm{Hom}(V(\nu), Q/L(\lambda))$, which is nonzero. This contradiction establishes the claim.

Dually, all composition factors of X isomorphic to $L(\lambda)$ occur in the socle of X. This implies that X is the direct sum of an object X' and a nonzero number of copies of $L(\lambda)$. An evident induction on the length of X completes the proof. □

In the presence of a Kazhdan-Lusztig theory in the sense of Definition 3.3, we have the following consequence of the above theorem.

(4.2) COROLLARY. *Let $\mathscr{C}$ be a highest weight category having a Kazhdan-Lusztig theory with respect to a length function $\ell : \Lambda \to \mathbf{Z}$. Let X be an object in $\mathscr{C}$ having a left parity and a right parity. Then X is completely reducible.*

PROOF. If $\lambda, \nu \in \Lambda$ satisfy

$$\mathrm{Hom}(X, A(\lambda)) \neq 0 \neq \mathrm{Hom}(X, A(\nu)),$$

then $\ell(\lambda) \equiv 0$ and $\ell(\nu) \equiv 0 \pmod 2$ if X has an even left parity, while $\ell(\lambda) \equiv 1$ and $\ell(\mu) \equiv 1 \pmod 2$ if X has an odd left parity. In either case, $\ell(\lambda) \equiv \ell(\nu) \pmod 2$. Therefore, since $\mathscr{C}$ has a Kazhdan-Lusztig theory, $\mathrm{Ext}^1(V(\lambda), L(\nu)) = 0$.

By a dual argument, if $\mathrm{Hom}(V(\lambda), X) \neq 0 \neq \mathrm{Hom}(V(\nu), X)$, then $\mathrm{Ext}^1(L(\nu), A(\lambda)) = 0$. By the theorem, we conclude X is completely reducible. □

MacPherson's work in [Sp], which uses Gabber's decomposition theorem [BBD], can instead by carried out with (4.2) and the fact, proved by Kazhdan-Lusztig [KL2], that intersection cohomology complexes associated to Schubert varieties have cohomology only in even degrees. Related parity considerations made in [Sp] helped guide the definition of the enriched Grothendieck groups in §2. See also remarks following (5.6) below.

Our original proof of a complete reducibility criterion, like that given in Corollary 4.2, was based on a variation of the following result, interesting in its own right.

(4.3) THEOREM. *Let $\mathscr{C}$ be a highest weight category with finite weight poset Λ and length function $\ell : \Lambda \to \mathbf{Z}$. Let $X \in \mathrm{Ob}(D^b(\mathscr{C}))$ have a left parity, and let $\nu \in \Lambda$ be such that $L(\nu)$ has a right parity. Then for each integer n, the natural map*

$$\mathrm{Hom}^n(X, L(\nu)) \to \mathrm{Hom}^n(X, A(\nu)) \tag{4.3.1}$$

is surjective. Dually, let $X \in \mathrm{Ob}(D^b(\mathscr{C}))$ have a right parity, and let $\nu \in \Lambda$ be such that $L(\nu)$ has a left parity. Then for each integer n, the natural map

$$\mathrm{Hom}^n(L(\nu), X) \to \mathrm{Hom}^n(V(\nu), X) \tag{4.3.2}$$

is surjective.

PROOF. It suffices to consider the first case (4.3.1), which is dual to (4.3.2).

We can clearly assume that X has even left parity, i.e., $X \in \mathrm{Ob}(\mathscr{E}^L)$ by Theorem 2.4. Certainly, the result holds if X has the form $V(\tau)[-\ell(\tau)+2k]$. Thus, by induction and the construction of $\mathscr{E}^L$, there is a distinguished triangle $Y \xrightarrow{y} X \xrightarrow{x} Z \xrightarrow{z}$ where $Y, Z \in \mathrm{Ob}(\mathscr{E}^L)$ and (4.3.1) is surjective with X replaced by Y or Z. Consider the corresponding commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}^n(Z, L(\nu)) & \longrightarrow & \mathrm{Hom}^n(X, L(\nu)) & \longrightarrow & \mathrm{Hom}^n(Y, L(\nu)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}^n(Z, A(\nu)) & \longrightarrow & \mathrm{Hom}^n(X, A(\nu)) & \longrightarrow & \mathrm{Hom}^n(Y, A(\nu)) & \longrightarrow & 0\,. \end{array} \tag{4.3.3}$$

To show that (4.3.1) is surjective, we can assume that $\mathrm{Hom}^n(X, A(\nu)) \neq 0$, so that, $n \equiv \ell(\nu)$ (mod 2). Assuming $L(\nu)$ has a left parity, it follows both horizontal rows in (4.3.3) are exact. Now the surjectivity of (4.3.1) for Y, Z and the snake lemma imply that (4.3.1) surjective for X. □

5. Applications to the Lusztig conjecture. Early work of Vogan [V1] established that the complete reducibility of certain modules implies the Kazhdan-Lusztig conjecture for the category $\mathcal{O}$. These and other observations of Vogan were subsequently adapted by Andersen [A2] to the modular representation theory of reductive algebraic groups. In this section, we begin by showing how the results of §4 yield, in the context of this earlier work, a significant simplification of the Lusztig conjecture.

Let G be a semisimple, simply connected algebraic group over an algebraically closed field k of positive characteristic $p \geq h$, the Coxeter number of G. We generally adhere to the notation in [J]. Thus, T is a fixed maximal torus contained in a Borel subgroup B. Regarding the opposite Borel subgroup as positive, let R^+ (resp., S) be the positive (resp., simple) roots for the root system R of T in G. Let $X(T)$ (resp., $X(T)^+$) be the set of weights (resp., dominant weights) on T. Let $\rho \in X(T)^+$ be the half-sum of the positive roots, and let $\alpha_0 \in R^+$ be the maximal short root.

The affine Weyl group W_p is generated by affine reflections $s_{\alpha,rp}$, $\alpha \in R$, $r \in \boldsymbol{Z}$ in $X(T) \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ [J; §II.6.1], and, as a Coxeter group, $\Sigma \equiv \{s_{\alpha,0} \mid \alpha \in S\} \cup \{s_{\alpha_0,p}\}$ is a set of simple reflections. The "dot" action of W_p is defined by $w \cdot \lambda$ for $w(\lambda+\rho)-\rho$ for $w \in W_p$, $\lambda \in X(T) \otimes_{\boldsymbol{Z}} \boldsymbol{R}$.

For $\lambda, \mu \in X(T)$, write $\lambda \leq \mu \Leftrightarrow \mu - \lambda \in \boldsymbol{Z}^{\geq 0}R^+$. Also, put $\lambda \uparrow \nu$ if there is a sequence $\lambda = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_t = \nu$ in $X(T)$ such that, for $0 \leq i \leq t$, there exists a positive root α_i and a positive integer n_i with $s_{\alpha_i, n_i p} \cdot \lambda_i = \lambda_{i+1}$. Both $\leq$ and $\uparrow$ induce partial orderings on $X(T)$. For $\lambda, \mu \in X(T)^+ - \rho$, if $\lambda \uparrow \mu$, then $\lambda \uparrow \mu$ in the sense of [J; §II.6.4]. (The two notions agree on $X(T)^+ - \rho$ by [Y], [W], although we will not use this fact).

Let $\mathscr{C}$ be the category of finite dimensional rational G-modules. For $\lambda \in X(T)^+$, $L(\lambda)$ is the irreducible G-module of highest weight λ, and

$$(5.0.1) \qquad A(\lambda)=H^0(\lambda) \qquad (\text{resp., } V(\lambda)=H^0(-w_0\lambda)^*)$$

is the induced module (resp., Weyl module) of highest weight λ [J; §II.5]. The following observation will be used in the sequel.

Let Γ be a finite ideal in the poset $(X(T)^+, \uparrow)$. The full subcategory $\mathscr{C}[\Gamma]$ of $\mathscr{C}$ consisting of G-modules with composition factors $L(\gamma)$, $\gamma \in \Gamma$, is a highest weight category (satisfying (1.1)–(1.2)) *with weight poset $(\Gamma, \uparrow)$. For $\lambda \in \Gamma$, the induced (resp., Weyl) module $A(\lambda)$ (resp., $V(\lambda)$) is defined by* (5.0.1). *Also, $\mathscr{C}[\Gamma]$ has a natural duality D.*

To see this, let $\tilde{\Gamma}$ be the ideal in the poset $(X(T)^+, \leq)$ generated by Γ. Then $\mathscr{C}[\tilde{\Gamma}]$ is a highest weight category with induced objects $A(\lambda)$ and Weyl objects $V(\lambda)$ as in (5.0.1). (Let $\tilde{\mathscr{C}}$ be the category of all rational G-modules. It follows from [CPS1; 3.3d, 3.5] that the subcategory $\tilde{\mathscr{C}}[\tilde{\Gamma}]$ consisting of rational G-modules having composition factors $L(\lambda)$ with $\lambda \in \tilde{\Gamma}$ is a highest weight category having the indicated induced and Weyl objects. However, since $\tilde{\Gamma}$ is finite, the injective objects in $\tilde{\mathscr{C}}[\tilde{\Gamma}]$ are finite dimensional. It follows that $\mathscr{C}[\tilde{\Gamma}]$ has enough injective objects and is a highest weight category as described. See also [PS; §6].)

If $\tau \in \Gamma$, the composition factors $L(\nu)$ of $A(\tau)$ and $V(\tau)$ satisfy $\nu \uparrow \tau$, by strong linkage [A1] (see also [J; §II.6.13]). For $\lambda \in \Gamma$, the injective envelope $I(\lambda)$ of $L(\lambda)$ in $\mathscr{C}[\tilde{\Gamma}]$ has a filtration with sections $A(\sigma)$ such that $\lambda \uparrow \sigma$ by reciprocity [J; §II.4.18] and strong linkage again. Thus, $I(\lambda)$ is the injective hull of $L(\lambda)$ in $\mathscr{C}[\Gamma]$. This proves $\mathscr{C}[\Gamma]$ is a highest weight category relative to $(\Gamma, \uparrow)$. Also, the duality of [CPS2; (3.3)] induces a duality D on $\mathscr{C}[\Gamma]$. This establishes the observation.

Let $C_{\mathbf{Z}}=\{\lambda \in X(T) \mid 0<(\lambda+\rho, \alpha^\vee)<p,\ \forall \alpha \in R^+\}$ [J; §II.5.5]. Since $p \geq h$, $C_{\mathbf{Z}} \neq \varnothing$. For $\lambda \in C_{\mathbf{Z}}$, let $O_\lambda^+=W_p \cdot \lambda \cap X(T)^+$, and define a length function ℓ on O_λ^+ by putting $\ell(w \cdot \lambda)=\ell(w)$. Let $\Gamma(\lambda)$ be the ideal of $(O_\lambda^+, \uparrow)$ consisting of weights τ satisfying the Jantzen condition

$$(5.0.2) \qquad (\tau+\rho, \alpha_0^\vee) \leq p(p-h+2)\,.$$

Let $s \in \Sigma$, and fix $\nu \in \bar{C}_{\mathbf{Z}}$ lying on the face defined by s. Using the Jantzen translation operators T_ν^λ, $T_\lambda^\nu : \mathscr{C} \to \mathscr{C}$, define $\Theta_s=T_\nu^\lambda \circ T_\lambda^\nu : \mathscr{C} \to \mathscr{C}$, an exact functor commuting with duality, which is (left and right) adjoint to itself. (For details concerning these operators, see [J; §II.7].) Thus, for $M \in \operatorname{Ob}(\mathscr{C})$, the adjunction morphisms associated to the adjoint pairs $(T_\nu^\lambda, T_\lambda^\nu)$ and $(T_\lambda^\nu, T_\nu^\lambda)$ define morphisms

$$(5.0.3) \qquad M \xrightarrow{\delta(M)} \Theta_s M \xrightarrow{\varepsilon(M)} M\,.$$

As noted by Vogan [V2] (in a different context), for some M (e.g., $L(\tau)$, $A(\tau)$, or $V(\tau)$, $\tau \in O_\lambda^+$), the diagram (5.0.3) is a complex. In this case, Vogan sets $U_s M$ equal to the homology of the complex. We will use this notation also, and, in addition, let $\beta_s M$

denote the complex itself concentrated in degrees $-1, 0, 1$ (and viewed as an object in $D^b(\mathscr{C})$). We will return to β_s in (5.6), (5.7).

The following result quickly reveals the strength of Theorem 4.1. For $\tau = w\cdot\lambda$ and $s\in\Sigma$, we write τs for $ws\cdot\lambda$.

(5.1) LEMMA. *Let $\lambda\in C_{\mathbf{Z}}$ and let $\Gamma(\lambda)$ be the ideal in $(O_\lambda^+, \uparrow)$ of weights satisfying* (5.0.2). *Assume that $\mathscr{C}[\Gamma(\lambda)]$ has a Kazhdan-Lusztig theory relative to the length function ℓ defined above. Let $\tau<\tau s$ be weights in $\Gamma(\lambda)$. Then $U_sL(\tau)$ is completely reducible.*

PROOF. Let $\sigma\in\Gamma(\lambda)$. We will show that $U_sL(\tau)$ satisfies the condition

$$\text{(5.1.1)} \qquad \operatorname{Ext}^i(U_sL(\tau), A(\sigma))\neq 0 \Rightarrow i\equiv \ell(\tau s)-\ell(\sigma) \pmod 2 .$$

If $\sigma\in\Gamma(\lambda)$ is maximal, then (5.1.1) is automatic if $i>1$ since $A(\sigma)$ is injective in $\mathscr{C}[\Gamma(\lambda)]$. Since all the composition factors $L(\gamma)$ of $U_sL(\tau)$ satisfy $\gamma\uparrow\tau s$ by [J; §II.7.8], the $i=0$ case of (5.1.1) is also clear for σ maximal (since only $\sigma=\tau s$ need be considered).

Using [J; §II.7.13], we see that Θ_s induces an operator $[\Theta_s]$ on the Grothendieck group $K_0(\mathscr{C})$ satisfying the identity $[\Theta_s]^2=2[\Theta_s]$. Thus, $[\Theta_sL(\tau)]=2[L(\tau)]+[U_sL(\tau)]$, so that $\Theta_sU_sL(\tau)=0$. Hence,

$$\text{(5.1.2)} \qquad 0=\operatorname{Ext}^i(\Theta_sU_sL(\tau), A(\sigma))\cong \operatorname{Ext}^i(U_sL(\tau), \Theta_sA(\sigma)), \quad \forall i .$$

Assume $\sigma<\sigma s\in X(T)^+$. The short exact sequence $0\to A(\sigma)\to\Theta_sA(\sigma)\to A(\sigma s)\to 0$ [J; §II.7.19] applied to (5.1.2) thus yields

$$\text{(5.1.3)} \qquad \operatorname{Ext}^i(U_sL(\tau), A(\sigma))\cong \operatorname{Ext}^{i-1}(U_sL(\tau), A(\sigma s)), \quad \forall i .$$

By the previous two paragraphs, we must show that (5.1.1) holds for $\sigma>\sigma s$ (with σs possibly not dominant). Using the exact sequence [A2; (2.9)] (obtained from decomposing (5.0.3) into two short exact sequences) and [A2; (2.6)] (an application of [GJ; (5.18)]), the assumption that $\mathscr{C}[\Gamma(\lambda)]$ has a Kazhdan-Lusztig theory implies statement (5.1.1) in this case. (For a more conceptual point of view, see (5.7) below.)

Since $U_sL(\tau)$ is self-dual, the statement dual to (5.1.1)

$$\operatorname{Ext}^i(V(\sigma), U_sL(\tau))\neq 0 \Rightarrow i\equiv \ell(\tau s)-\ell(\sigma) \pmod 2$$

holds as well. It follows easily that $X=U_sL(\tau)$ satisfies the hypothesis of Theorem 4.1 (or Corollary 4.2). Thus, $U_sL(\tau)$ is completely reducible. □

The Lusztig conjecture [L1; §3] asserts, for $w\cdot\lambda\in\Gamma(\lambda)$, that

$$\text{(5.2)} \qquad \operatorname{ch} L(w\cdot\lambda)=\sum_{y\cdot\lambda\in X(T)^+,\, yw_0\leq ww_0} (-1)^{\ell(y)-\ell(w)}P_{yw_0,ww_0}(-1)\operatorname{ch} V(y\cdot\lambda) .$$

Here the $P_{u,v}$, $u, v\in W_p$, are the Kazhdan-Lusztig polynomials [KL1] for W_p, and $w_0\in W$ is the long word. We follow the formulation in [J; §II.7.20], but differ in that, because we view the $P_{u,v}$ as polynomials in t (rather than $q=t^2$), we evaluate them at $t=-1$.

The following result provides an attractive reduction of the above conjecture, relating it to the more flexible notion of a Kazhdan-Lusztig theory. Vogan knew [GJ; p. 285], [V1; I, 3.9; II, 3.5] in the category $\mathcal{O}$ context, that the substance of (a) was implied by (b) as well as by (c), and that complete reducibility of $U_s L(\tau)$ implied (b). The key new ingredient is our Lemma 5.1 (which in turn relies on Theorem 4.1).

(5.3) THEOREM. *Let G be a semisimple, simply connected algebraic group over k, as above. Fix $\lambda \in C_{\mathbf{Z}}$. The following statements are equivalent*:

(a) *The highest weight category $\mathscr{C}[\Gamma(\lambda)]$ has a Kazhdan-Lusztig theory relative to the length function ℓ on $(\Gamma(\lambda), \uparrow)$ defined by $\ell(w \cdot \lambda) = \ell(w)$.*

(b) *The character formula* (5.2) *is valid whenever $w \cdot \lambda \in \Gamma(\lambda)$.*

(c) *Whenever $\tau, \sigma \in \Gamma(\lambda)$ lie in adjacent p-alcoves, then* $\mathrm{Ext}^1_G(L(\tau), L(\sigma)) \neq 0$.

If these conditions hold, the Kazhdan-Lusztig polynomials (3.0.3) *identify with the Kazhdan-Lusztig polynomials for W_p as follows*:

$$P_{y \cdot \lambda, w \cdot \lambda} = P_{w_0 y, w_0 w}(w \cdot \lambda,\ y \cdot \lambda \in \Gamma(\lambda)) .$$

PROOF. The equivalence of the three conditions follows from Lemma 5.1 together with the results of [A2; §2]. Assuming this, the identification of the polynomials (3.0.3) follows from [A2; (2.12)], using the fact that "$w \cdot \lambda$" in the notation of that paper is "$w_0 w \cdot \lambda$" in the notation of the present paper. (We mention that the proof of [A2; (2.12)] is more transparent using the results of Deodhar [D; (3.4)].) $\square$

Notice in (5.3c) the order of weights is irrelevant, since $L(\tau)$, $L(\sigma)$ are fixed by the duality D. We now establish other new equivalent forms of the Lusztig conjecture.

(5.4) COROLLARY. *Let G, etc. be as above. The validity of the character formula* (5.2) *for all $w \cdot \lambda \in \Gamma(\lambda)$ is equivalent to each of the following assertions*:

(a) *Whenever $s \in \Sigma$ and $\tau < \tau s$ belong to $\Gamma(\lambda)$, there is a quotient $X(\tau s)$ of the Weyl module $V(\tau s)$ with exactly two composition factors $L(\tau)$ and $L(\tau s)$.* (*The quotient $X(\tau s)$ is unique and can be explicitly described.*)

(b) *For $\tau, \nu \in \Gamma(\lambda)$ satisfying $\ell(\tau) \equiv \ell(\nu)$* (mod 2), *we have*

$$\mathrm{Ext}^1(V(\nu), L(\tau)) = 0 .$$

(c) *The natural map*

$$\mathrm{Ext}^1(L(\nu), L(\tau)) \to \mathrm{Ext}^1(V(\nu), L(\tau))$$

is surjective for all weights $\nu, \tau \in \Gamma(\lambda)$.

(d) *Any quotient M of a Weyl module in $\mathscr{C}[\Gamma(\lambda)]$ having exactly two B-stable lines has exactly two composition factors.*

PROOF. Clearly, (a) $\Rightarrow$ (5.3c), while (5.3c) $\Rightarrow$ (a) by the universal mapping property of Weyl modules. By [J; §II.7.18], $V(\tau s)$ has a *unique* quotient $X(\tau s)$ with simple head $L(\tau s)$, simple socle $L(\tau)$, and such that any other composition factor $L(\nu)$ satisfies $\tau \neq \nu \neq \tau s$. We can explicitly describe $X(\tau s)$ as follows: $V(\tau s)$ contains unique B-fixed lines

kv and kv^+ of weights τ and τs, respectively. Then $X(\tau s)=V(\tau s)/M$, where M is the unique G (or B)-submodule of $V(\tau s)$ maximal with respect to not containing v or v^+.

Suppose $X(\tau s)$ has more than two composition factors. Let ν be maximal $\neq \tau s$ with $L(\nu)$ a composition factor of $X(\tau s)/L(\tau)$. Thus, there is a nonzero homomorphism $f: V(\nu)\to \operatorname{rad} X(\tau s)/L(\tau)\to A(\tau)/L(\tau)$, giving $\operatorname{Ext}^1(V(\nu), L(\tau))\neq 0$. However, one can similarly argue that $\operatorname{Ext}^1(V(\nu), L(\tau s))\neq 0$ using $DX(\tau s)$. This contradicts the hypothesis of (b), so (b) $\Rightarrow$ (a). Conversely, it is clear that (5.3a) $\Rightarrow$ (b).

Next, (5.3a) $\Rightarrow$ (c) by Theorem 4.3, while (c) $\Rightarrow$ (5.3c) by [J; §II.7.18].

Clearly, (d) $\Rightarrow$ (a). Finally, let $V(\sigma)\to M$ be a surjective morphism in $\mathscr{C}[\Gamma(\lambda)]$ in which M has exactly two B-stable lines, having weights $\sigma>\tau$. Thus, τ is the maximal weight of $\operatorname{rad} M$, giving an embedding $\operatorname{rad} M\subset A(\tau)$. We show (b) implies $\operatorname{rad} M\cong L(\tau)$. If not, let $L(\xi)$ be a composition factor of $\operatorname{rad} M/L(\tau)$ with ξ maximal. As above, this implies $\operatorname{Ext}^1(V(\xi), L(\tau))\neq 0$. A dual argument gives $\operatorname{Ext}^1(V(\xi), L(\sigma))\neq 0$. Similarly, $\operatorname{Ext}^1(V(\tau), L(\sigma))\neq 0$. This contradicts the parity conditions of (b), so (b) $\Rightarrow$ (d). □

(5.5) REMARK. By Andersen [A3; (6.13)], Jantzen's Weyl module filtration conjecture implies the Lusztig conjecture. *We observe that the $i=1$ case of the Jantzen conjecture implies the Lusztig conjecture for G*: In the notation of [J; §II.8], for $\tau<\tau s$ in $\Gamma(\lambda)$, there is a $\boldsymbol{Z}$-module homomorphism $f_{\boldsymbol{Z}}: V(\tau)_{\boldsymbol{Z}}\to V(\tau s)_{\boldsymbol{Z}}$ inducing a nonzero G-homomorphism $f: V(\tau)\to V(\tau s)$. (Since $\operatorname{Hom}(V(\tau), V(\tau s))$ has the same dimension as $\operatorname{Hom}(V(\tau), A(\tau s)/L(\tau s))\cong \operatorname{Ext}^1(V(\tau), L(\tau s))\neq 0$, f exists.) Then the module $(pV(\tau s)_{\boldsymbol{Z}}+f_{\boldsymbol{Z}}(V(\tau)_{\boldsymbol{Z}})/V(\tau s)^2_{\boldsymbol{Z}}\otimes k$ defines a nontrivial extension of $L(\tau)$ by $L(\tau s)$ provided $fV(\tau)^1\subset V(\tau s)^2$. (Making use of [A3; (6.4i)], this latter inclusion is equivalent to the non-vanishing of $\operatorname{Ext}^1(L(\tau), L(\tau s))$.)

Let $\mathscr{H}_a$ be the affine Hecke algebra over $\boldsymbol{Z}[t, t^{-1}]$ associated with W_p. Let $\{T_y\}_{y\in W_p}$ be the standard basis of $\mathscr{H}_a$. (Thus, $T_s^2=(t^2-1)T_s+t^2T_1$, $s\in\Sigma$.) Let $x\equiv x_W=\sum_{w\in W}T_w$, so that the right ideal $x\mathscr{H}_a$ is the natural "$q=t^2$-analog" of the right W_p-permutation module $W\backslash W_p$.

The following result and remarks provide a conceptual link between this section and §2. Write $K_0^L(\mathscr{C}_\lambda)=\varinjlim K_0^L(\mathscr{C}[\Gamma], \ell)$, the directed union over all finite ideals Γ of O_λ^+.

(5.6) PROPOSITION. *There is a natural action of $\mathscr{H}_a$ on $K_0^L(\mathscr{C}_\lambda)$, isomorphic to the action of $\mathscr{H}_a$ on $x\mathscr{H}_a$. In this isomorphism, $t^{\ell(y)}[V(y\cdot\lambda)]$ corresponds to xT_y for $y\cdot\lambda\in X(T)^+$. The element $b_s=t^{-1}(T_s+1)$ in $\mathscr{H}_a$ sends $[V(y\cdot\lambda)]$ to $[\beta_sV(y\cdot\lambda)]$.*

PROOF. Let $\Omega\subset W_p$ be the set of distinguished right coset representatives for W in W_p (i.e., $y\in\Omega\Leftrightarrow \ell(wy)=\ell(w)+\ell(y)$ $\forall w\in W\Leftrightarrow y\cdot\lambda\in X(T)^+$). Thus, $\{[V(y\cdot\lambda)]\}_{y\in\Omega}$ (resp., $\{xT_y\}_{y\in\Omega}$) is a basis for the free $\boldsymbol{Z}[t, t^{-1}]$-module $K_0^L(\mathscr{C}_\lambda)$ (resp., $x\mathscr{H}_a$), and the map $\iota: K_0^L(\mathscr{C}_\lambda)\to x\mathscr{H}_a$ defined by $\iota[V(y\cdot\lambda)]=t^{-\ell(y)}xT_y$ $(y\in\Omega)$ is an isomorphism of $\boldsymbol{Z}[t, t^{-1}]$-modules.

The description of translation functors [J; §II.7] shows that $\Theta_sV(y\cdot\lambda)=0$ unless

$ys\cdot\lambda\in X(T)^+$, in which case $\Theta_s V(y\cdot\lambda)$ has two sections $V(ys\cdot\lambda)$ and $V(y\cdot\lambda)$ (with the Weyl module of smaller highest weight appearing as a quotient). The map $\varepsilon(V(y\cdot\lambda))$ of (5.0.3) is surjective if $y\cdot\lambda<ys\cdot\lambda$, while $\delta(V(y\cdot\lambda))$ is injective if $y\cdot\lambda>ys\cdot\lambda$. If follows, for $y\in\Omega$, that $[\beta_s V(y\cdot\lambda)]=t^{-1}[V(y\cdot\lambda)]+[V(ys\cdot\lambda)]$ (resp., $t[V(y\cdot\lambda)]+[V(ys\cdot\lambda)]$, $(t+t^{-1})[V(y\cdot\lambda)]$) if $y<ys\in\Omega$ (resp., $ys<y$, $y<ys\notin\Omega$). A direct computation shows that $\iota[\beta_s V(y\cdot\lambda)]=(\iota[V(y\cdot\lambda)])b_s$. (Observe that $xT_w=t^{2\ell(w)}x$, $w\in W$.) $\square$

This result parallels corresponding geometric Hecke actions of Lusztig-Vogan [LV] and of MacPherson [Sp]. Similar results hold for the category $\mathcal{O}$, using the Hecke algebra for W in place of $\mathscr{H}_q$ and taking $x=1$, and for perverse sheaves, agreeing precisely with MacPherson's work. Through [PS; §5], MacPherson's work inspired (5.6) and led to the results of this section. The existence of a Kazhdan-Lusztig theory, i.e., "even-odd vanishing", allows irreducible objects to be represented in $K_0^L(\mathscr{C}[\Gamma(\lambda)])\subset K_0^L(\mathscr{C}_\lambda)$. The computations of [A.2] on the "Vogan conjecture", i.e., the complete reducibility in (5.1), can be recovered from this action, the combinatorics of [D], and the identity

$$(5.7)\qquad \langle[\beta_s L(\nu)],[A(\omega)]\rangle=\langle[L(\nu)],[\beta_s A(\omega)]\rangle\,,$$

for $L(\nu)\in\mathrm{Ob}(\hat{\mathscr{E}}^R)$. (This identity essentially follows from the exact sequences proved in [A2]; we will publish another proof in a future paper.) Casian makes a related assertion [C; (8.9)], with different hypotheses, and (5.7) was inspired by his version of β_s (called U_s there) for the geometric "push-forward/pull-back" operator on perverse sheaves.

The Hecke algebra calculations given in [LV] and [Sp] relying on "weights" and cohomology, respectively, can be compared via the formalism of the appendix to §3.

(5.8) Remark. There are analogs of (5.3), (5.4) in other contexts. For example, these reductions hold for Lusztig's quantum conjecture [L3; (8.2)] (announced as a theorem in [KL3], depending on other work) at an l-th root of unity (l odd), at least for a root system of type A in the comodule set-up of [PW]. The proofs are the same, using [PW] to supply the analogs of characteristic p results used in this section. For any type, one may use [APW] for $l>3$ a prime power or [AW] for more general l with certain restrictions. The validity of the quantum conjecture would also give calculations of Ext groups between simple modules having regular highest weights.

Our proofs apply in the category $\mathcal{O}$ case, giving algebraic reductions of the original Kazhdan-Lusztig conjecture. In particular, the analog of (5.1) settles a question regarding "even-odd vanishing" apparently left open by Vogan (see [V2; Prob. 7, p. 739] for the Harish-Chandra module case), despite a more optimistic communication in [KL1].

As already remarked above (4.3), our results apply to perverse sheaves on Schubert varieties, providing a partial replacement for the decomposition theorem.

These results can all be treated simultaneously within an appropriate axiomatic setting, including perverse sheaves as well as potential examples arising from finite dimensional algebras. Details will be presented elsewhere.

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