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EINSTEIN-KÄHLER TORIC FANO FOURFOLDS

Dedicated to Professor Hideki Ozeki on his sixtieth birthday

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Abstract. We investigate the relationship between Matsushima's obstruction and the Futaki invariant for the existence of Einstein-Kähler metrics on toric Fano fourfolds. In particular, we determine all toric Fano fourfolds with vanishing Futaki invariant. Moreover, we construct a non-trivial example of an Einstein-Kähler toric Fano fourfold.

Introduction. Let Y be a Fano r-fold, which is by definition, an r-dimensional compact connected non-singular projective algebraic variety, defined over $\boldsymbol{C}$, with ample anti-canonical line bundle. Then one can naturally ask whether Y admits an Einstein-Kähler metric. As to such existence of Einstein-Kähler metrics, two obstructions are known (see Matsushima [9] and Futaki [4]). We here consider the following for toric Fano r-folds (see Definiton 1.1).

PROBLEMS. (I_r) Classify all toric Fano r-folds with vanishing Futaki invariant.

(II_r) For a toric Fano r-fold Y with vanishing Futaki invariant, is its automorphism group $\mathrm{Aut}(Y)$ a reductive algebraic group?

(III_r) Does a toric Fano r-fold with vanishing Futaki invariant always admit an Einstein-Kähler metric?

Note that if (III_r) is true, then (II_r) is also true (see Matsushima [9]). For $r \leqq 3$, (I_r) and (III_r) were settled (see Mabuchi [7], Siu [14], Tian and Yau [15]).

By Batyrev's recent classification of toric Fano fourfolds [2], it is now possible to study the above problems for $r=4$. In this paper, we give a complete classification for (I_4), and answer the question (II_4) (see Theorem 3.5). Moreover, we can solve (III_4) except in one case (see Theorem 4.1).

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1. Toric Fano manifolds.

In this section, we recall some basic notions and facts

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concerning toric Fano manifolds (see [3], [6], [11] or [12] for more details).

Let $\boldsymbol{Z}_{\geqq 0}$ and $\boldsymbol{R}_{\geqq 0}$ be the sets of non-negative integers and non-negative real numbers, respectively. Moreover let r be a positive integer, and $T_r := (\boldsymbol{C}^*)^r$ an r-dimensional algebraic torus. We put $N := \boldsymbol{Z}^r$ and $M := \mathrm{Hom}_{\boldsymbol{Z}}(N, \boldsymbol{Z})$ $(\cong \boldsymbol{Z}^r)$. The natural pairing $\langle\ ,\ \rangle : M \times N \to \boldsymbol{Z}$ is extended to the bilinear form $\langle\ ,\ \rangle : M_{\boldsymbol{R}} \times N_{\boldsymbol{R}} \to \boldsymbol{R}$ where $M_{\boldsymbol{R}} := M \otimes_{\boldsymbol{Z}} \boldsymbol{R}\ (\cong \boldsymbol{R}^r)$ and $N_{\boldsymbol{R}} := N \otimes_{\boldsymbol{Z}} \boldsymbol{R}\ (\cong \boldsymbol{R}^r)$.

DEFINITION 1.1. An r-dimensional compact connected complex manifold X with ample anti-canonical line bundle K_X^{-1} is called a *toric Fano r-fold* if T_r acts biholomorphically on X with an open dense orbit isomorphic to T_r.

DEFINITION 1.2. A convex polytope P in $N_{\boldsymbol{R}}$ is called a *Fano r-polytope* if the following conditions are satisfied:

(1) P is an integral polytope, namely, the set $\mathscr{V}(P)$ of vertices of P is contained in $N = \boldsymbol{Z}^r$;

(2) The origin 0 is contained in the interior of P;

(3) P is a simplicial polytope, that is, each face (which is always assumed to be closed) of P is a simplex;

(4) For an arbitrary codimension one face E of P, let $a_1, a_2, \dots, a_r$ be its vertices. Then $\{a_1, a_2, \dots, a_r\}$ forms a $\boldsymbol{Z}$-basis of N.

Let P be a Fano r-polytope in $N_{\boldsymbol{R}}$. For each $(k-1)$-dimensional face F of P, let $b_1, b_2, \dots, b_k$ be its vertices, and we put

$$\sigma(F) := \boldsymbol{R}_{\geqq 0} b_1 + \boldsymbol{R}_{\geqq 0} b_2 + \cdots + \boldsymbol{R}_{\geqq 0} b_k ,$$

$$\Delta_P(k) := \{\sigma(F) \mid (k-1)\text{-dimensional faces } F \text{ of } P\} , \qquad k = 1, 2, \dots, r ,$$

$$\Delta_P(0) := \{0\} ,$$

$$\Delta_P := \bigcup_{k=0}^{r} \Delta_P(k) .$$

Then $\sigma(F)$ is a *strongly convex rational polyhedral cone* in $N_{\boldsymbol{R}}$ (see [12; p. 1]) and Δ_P is a *fan* of N (see [12; p. 2]). The following theorem is fundamental in the study of toric Fano r-folds.

THEOREM 1.3 (see [12]). (a) *For each Fano r-polytope P in $N_{\boldsymbol{R}}$, there exists a unique toric Fano r-fold X_P satisfying the following*:

(1) *To each $\sigma \in \Delta_P(k)$, $0 \leqq k \leqq r$, there corresponds a unique $(r-k)$-dimensional T_r-orbit, denoted by $\boldsymbol{O}(\sigma)$, such that $X_P = \bigcup_{\sigma \in \Delta_P} \boldsymbol{O}(\sigma)$*;

(2) *For each $\sigma \in \Delta_P(k)$, $0 \leqq k \leqq r$, the closure $V(\sigma)$ of $\boldsymbol{O}(\sigma)$ in X_P is an irreducible normal $(r-k)$-dimensional T_r-invariant subvariety of X_P of the form $V(\sigma) = \bigcup_{\sigma \prec \tau} \boldsymbol{O}(\tau)$, where $\sigma \prec \tau$ means that σ is a* face *of τ (see [12; p. 2]).*

(b) *Every toric Fano r-fold X is T_r-equivariantly isomorphic to X_P for some Fano* *r-polytope P in* $N_{\boldsymbol{R}}$.

We recall results of Batyrev [2] which introduced *primitive collections* and *primitive relations* in the classification of Fano r-polytopes.

DEFINITION 1.4. For a Fano r-polytope P in $N_{\boldsymbol{R}}$, a non-empty subset $\alpha=\{x_1, x_2, \dots, x_k\}$ of $\mathscr{V}(P)$ is called a *primitive collection*, if the following conditions are satisfied:

(1) For any proper subset $\{x_{i_1}, x_{i_2}, \dots, x_{i_l}\} \subsetneqq \alpha$,

$$\boldsymbol{R}_{\geqq 0}x_{i_1}+\boldsymbol{R}_{\geqq 0}x_{i_2}+\cdots+\boldsymbol{R}_{\geqq 0}x_{i_l}\in \Delta_P\,.$$

(2)
$$\boldsymbol{R}_{\geqq 0}x_1+\boldsymbol{R}_{\geqq 0}x_2+\cdots+\boldsymbol{R}_{\geqq 0}x_k\notin \Delta_P\,.$$

Given a primitive collection $\alpha=\{x_1, x_2, \dots, x_k\}$, we have a face F of P such that $x_1+x_2+\cdots+x_k\in\sigma(F)\in\Delta_P$. For the vertices $y_1, y_2, \dots, y_m$ of F, there exist $c_j\in\boldsymbol{Z}_{\geqq 0}$ such that

$$x_1+x_2+\cdots+x_k=\sum_{j=1}^{m} c_j y_j\,,$$

which is called a *primitive relation.*

The following classification of toric Fano fourfolds is crucial in our study of Einstein-Kähler metrics on such fourfolds.

THEOREM 1.5 (Batyrev [2]). *The Fano r-polytopes can be classified only in terms of the primitive collections and primitive relations. In particular, there exist exactly* 123 *mutually non-isomorphic toric Fano fourfolds.*

REMARK 1.6 (cf. Batyrev [1], K. Watanabe and M. Watanabe [17]). There exist exactly 5 isomorphism classes of toric Fano surfaces and exactly 18 isomorphism classes of toric Fano threefolds.

2. Matsushima's obstruction and the Futaki invariant. In this section, we review the obstructions to the existence of Einstein-Kähler metrics on Fano manifolds due to Matsushima [9] and Futaki [4].

Throughout this section, we fix an r-dimensional compact connected complex manifold Y with ample anti-canonical line bundle K_Y^{-1} and a Kähler form ω on Y representing $2\pi c_1(Y)_{\boldsymbol{R}}$. Let $\mathrm{Aut}(Y)$ be the group of holomorphic automorphisms of Y, and $\mathrm{Aut}^{\circ}(Y)$ its identity component. By $\mathrm{Ric}(\omega)$, we denote the Ricci form corresponding to ω. Since ω and $\mathrm{Ric}(\omega)$ are in the same cohomology class $2\pi c_1(Y)_{\boldsymbol{R}}$, there exists a real-valued C^{∞}-function f_ω on Y such that

$$\mathrm{Ric}(\omega)-\omega=\sqrt{-1}\,\partial\bar{\partial}f_\omega\,,$$

where f_ω is unique up to additive constants.

DEFINITION 2.1. ω is called an *Einstein-Kähler form* if $\mathrm{Ric}(\omega)=\omega$.

The following theorem on automorphism groups is known as Matsushima's obstruction to the existence of Einstein-Kähler forms.

THEOREM 2.2 (Matsushima [9]). *Let X be a compact connected complex manifold. If X admits an Einstein-Kähler form, then* $\mathrm{Aut}(X)$ *is a reductive algebraic group.*

For (Y,ω) as above, let $\mathfrak{X}(Y)$ be the Lie algebra of all holomorphic vector fields on Y. Then $\mathfrak{X}(Y)$ is just the Lie algebra associated to $\mathrm{Aut}(Y)$. We define the *Futaki invariant* $F_Y\colon \mathfrak{X}(Y)\to \boldsymbol{R}$ by

$$F_Y(V):=((2\pi c_1(Y)_{\boldsymbol{R}})^r[Y])^{-1}\,\mathrm{Re}\left(\int_Y (Vf_\omega)\omega^r\right), \qquad V\in\mathfrak{X}(Y),$$

where $\mathrm{Re}(z)$ denotes the real part of z. Recall the following fundamental theorem:

THEOREM 2.3 (Futaki [4]). *For Y as above, the following hold:*
(a) *F_Y does not depend on the choice of ω;*
(b) *F_Y vanishes on the commutator subalgebra $[\mathfrak{X}(Y),\mathfrak{X}(Y)]$ of $\mathfrak{X}(Y)$;*
(c) *If Y admits an Einstein-Kähler form, then F_Y vanishes.*

Let us consider the case where Y is a toric Fano manifold. Let P be a Fano r-polytope in $N_{\boldsymbol{R}}$ and X_P the toric Fano r-fold associated to P. We now put

$$R(P):=\left\{a\in M \;\middle|\; \begin{array}{l}\langle a,b_0\rangle=1 \text{ for some } b_0\in\mathscr{V}(P) \text{ and}\\ \langle a,b\rangle\leqq 0 \text{ for all } b\in\mathscr{V}(P) \text{ with } b\neq b_0\end{array}\right\},$$

$$R_s(P):=R(P)\cap(-R(P)),$$

$$\Sigma_{-K}(P):=\{a\in M_{\boldsymbol{R}}\,|\,\langle a,b\rangle\leqq 1 \text{ for all } b\in\mathscr{V}(P)\}.$$

The following results on the automorphism groups of toric Fano r-folds are important in examining Matsushima's obstruction for toric Fano r-folds.

THEOREM 2.4 (Demazure [3]). (a) $\mathrm{Aut}(X_P)$ *is a reductive algebraic group if and only if $-R(P)$ coincides with $R(P)$.*

(b) *Let G_u be the unipotent radical of* $\mathrm{Aut}^\circ(X_P)$, *and denote by G_s be the reductive algebraic group which has T_r as a maximal algebraic torus and has $R_s(P)$ as the root system. Then*

$$\mathrm{Aut}^\circ(X_P)=G_s\rtimes G_u.$$

Mabuchi's result on Futaki invariants [8; Theorem 0.1] asserts that F_{X_P} vanishes on the Lie algebra of G_u. In view of Theorems 2.3, (b) and 2.4, (b), we can interpret Mabuchi [7; Corollary 5.5] as follows:

THEOREM 2.5. *Let P be a Fano r-polytope in $N_{\boldsymbol{R}}$, and let $\mathfrak{t}_r$ be the Lie algebra of*

T_r ($\subseteqq \mathrm{Aut}(X_P)$). *Then* $F_{X_P}\equiv 0$ *if and only if* $F_{X_P}|_{\mathfrak{t}_r}\equiv 0$.

The next formula of Mabuchi allows us to calculate $F_{X_P}|_{\mathfrak{t}_r}$ explicitly.

THEOREM 2.6 (Mabuchi [7]). *Let P be a Fano r-polytope in $N_{\boldsymbol{R}}$. For $T_r=\{(t_1, t_2, \dots, t_r)\,|\,t_i\in \boldsymbol{C}^*\}$, choose a $\boldsymbol{C}$-basis $\{t_i\partial/\partial t_i\,|\,i=1, 2, \dots, r\}$ for $\mathfrak{t}_r$. We put*

$$\mathfrak{b}_i(P):=\frac{\displaystyle\int_{\Sigma_{-K}(P)} x_i dx_1\wedge dx_2\wedge\cdots\wedge dx_r}{\displaystyle\int_{\Sigma_{-K}(P)} dx_1\wedge dx_2\wedge\cdots\wedge dx_r}\,,\qquad 1\leqq i\leqq r\,,$$

where $(x_1, x_2, \dots, x_r)$ denotes the standard coordinate system for $M_{\boldsymbol{R}}\cong \boldsymbol{R}^r$. Then the barycenter $\mathfrak{B}(P):=(\mathfrak{b}_1(P), \mathfrak{b}_2(P), \dots, \mathfrak{b}_r(P))$ of $\Sigma_{-K}(P)$ is of the form

$$\mathfrak{B}(P)=(F_{X_P}(t_1\partial/\partial t_1), F_{X_P}(t_2\partial/\partial t_2), \dots, F_{X_P}(t_r\partial/\partial t_r))\,.$$

3. Toric Fano fourfolds with vanishing Futaki invariant. In this section, we classify all toric Fano fourfolds with vanishing Futaki invariant. From now on we let $r=4$. We can calculate the Futaki invariant for 123 toric Fano fourfolds in the classification by Batyrev (see Theorem 1.5), thanks to the formula of Mabuchi (see Theorem 2.6). (We carried out our computation of the Futaki invariants by means of Mathematica on a Macintosh computer.) We obtain exactly 11 toric Fano fourfolds with vanishing Futaki invariants. The following 9 toric Fano fourfolds among them are elementary:

$$\begin{aligned}
&\boldsymbol{P}^4(\boldsymbol{C})\,,\qquad \boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^2(\boldsymbol{C})\,,\\
&\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1\times\boldsymbol{P}^1}\oplus\mathcal{O}_{\boldsymbol{P}^1\times\boldsymbol{P}^1}(1,-1))\,,\\
(3.1)\qquad &\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})\times S_3\,,\qquad \boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^3(\boldsymbol{C})\,,\\
&\boldsymbol{P}^2(\boldsymbol{C})\times\boldsymbol{P}^2(\boldsymbol{C})\,,\qquad \boldsymbol{P}^2(\boldsymbol{C})\times S_3\,,\qquad S_3\times S_3\,,\\
&\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})\,,
\end{aligned}$$

where S_3 is a smooth projective algebraic surface obtained from $\boldsymbol{P}^2(\boldsymbol{C})$ by blowing up three points $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$, and $\mathcal{O}_{\boldsymbol{P}^1\times\boldsymbol{P}^1}(1,-1)$ denotes the holomorphic line bundle $p_1^*\mathcal{O}_{\boldsymbol{P}^1}(1)\otimes p_2^*\mathcal{O}_{\boldsymbol{P}^1}(-1)$ over $\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})$ with the projections $p_i\colon \boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})\to\boldsymbol{P}^1(\boldsymbol{C})$, $i=1, 2$, to the i-th factor.

For $\boldsymbol{P}^4(\boldsymbol{C})$ and lower dimensional toric Fano manifolds

$$\boldsymbol{P}^1(\boldsymbol{C})\,,\quad \boldsymbol{P}^2(\boldsymbol{C})\,,\quad S_3\,,\quad \boldsymbol{P}^3(\boldsymbol{C})\,,\quad \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1\times\boldsymbol{P}^1}\oplus\mathcal{O}_{\boldsymbol{P}^1\times\boldsymbol{P}^1}(1,-1))\,,$$

which appear as factors in (3.1), the existence of an Einstein-Kähler form is well-known. In fact, the existence for S_3 is proved by Siu [14] (see also Tian and Yau [15], Nadel [10]) and for $\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1\times\boldsymbol{P}^1}\oplus\mathcal{O}_{\boldsymbol{P}^1\times\boldsymbol{P}^1}(1,-1))$, it is proved by Sakane [13] (see also Mabuchi [7]). Hence the 9 toric Fano fourfolds in (3.1) carry Einstein-Kähler forms, and therefore, the vanishing to their Futaki invariants are obvious.

The two remaining toric Fano fourfolds X_{P_1}, X_{P_2} are non-trivial. Their Futaki invariants turn out to vanish. The corresponding Fano 4-polytopes P_1, P_2 are defined as follows:
P_1 is the convex hull of ten vertices

$$
\begin{aligned}
&e_1:=(1,0,0,0)\,, && e_6:=-e_1=(-1,0,0,0)\,,\\
&e_2:=(0,1,0,0)\,, && e_7:=-e_2=(0,-1,0,0)\,,\\
&e_3:=(0,0,1,0)\,, && e_8:=-e_3=(0,0,-1,0)\,,\\
&e_4:=(0,0,0,1)\,, && e_9:=-e_4=(0,0,0,-1)\,,\\
&e_5:=(-1,-1,-1,-1)\,, && e_{10}:=-e_5=(1,1,1,1)\,,
\end{aligned}
\tag{3.2}
$$

and P_2 is the convex hull of ten vertices

$$
\begin{aligned}
&e'_1:=(1,0,0,0)\,, && e'_6:=(1,-1,0,0)\,,\\
&e'_2:=(0,1,0,0)\,, && e'_7:=(0,0,1,0)\,,\\
&e'_3:=(-1,1,0,0)\,, && e'_8:=(1,0,-1,0)\,,\\
&e'_4:=(-1,0,0,0)\,, && e'_9:=(0,0,0,1)\,,\\
&e'_5:=(0,-1,0,0)\,, && e'_{10}:=(-1,0,0,-1)\,.
\end{aligned}
\tag{3.3}
$$

The Fano fourfolds X_{P_1} and X_{P_2} are obtained by Batyrev as follows.

REMARK 3.4. (i) We consider the product $W_1:=\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})$. Let W_2 be the blowing up of W_1 at the two points $x_0:=([1:0],[1:0],[1:0],[1:0])$ and $x_\infty:=([0:1],[0:1],[0:1],[0:1])$, and define subsets $C_1, C_2, \dots, C_8$ of W_1 by

$$
\begin{aligned}
C_1&:=\boldsymbol{P}^1(\boldsymbol{C})\times\{[1:0]\}\times\{[1:0]\}\times\{[1:0]\}\,,\\
C_2&:=\boldsymbol{P}^1(\boldsymbol{C})\times\{[0:1]\}\times\{[0:1]\}\times\{[0:1]\}\,,\\
C_3&:=\{[1:0]\}\times\boldsymbol{P}^1(\boldsymbol{C})\times\{[1:0]\}\times\{[1:0]\}\,,\\
C_4&:=\{[0:1]\}\times\boldsymbol{P}^1(\boldsymbol{C})\times\{[0:1]\}\times\{[0:1]\}\,,\\
C_5&:=\{[1:0]\}\times\{[1:0]\}\times\boldsymbol{P}^1(\boldsymbol{C})\times\{[1:0]\}\,,\\
C_6&:=\{[0:1]\}\times\{[0:1]\}\times\boldsymbol{P}^1(\boldsymbol{C})\times\{[0:1]\}\,,\\
C_7&:=\{[1:0]\}\times\{[1:0]\}\times\{[1:0]\}\times\boldsymbol{P}^1(\boldsymbol{C})\,,\\
C_8&:=\{[0:1]\}\times\{[0:1]\}\times\{[0:1]\}\times\boldsymbol{P}^1(\boldsymbol{C})\,.
\end{aligned}
$$

Let $\tilde{C}_i$ be the strict transform in W_2 of C_i, and let W_3 be the blowing up of W_2 along these eight curves $\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_8$. Then each exceptional set E_i over $\tilde{C}_i$ is isomorphic to $\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^2(\boldsymbol{C})$. We obtain X_{P_1} from W_3 by contracting all E_i, $1\leqq i\leqq 8$, to the second factor $\boldsymbol{P}^2(\boldsymbol{C})$. Moreover, X_{P_1} is a *symmetric* toric Fano variety in the sense of Voskresenskii and Klyachko [16]. Note that, by $\mathrm{Aut}^\circ(X_{P_1})=T_4$, the toric Fano fourfold X_{P_1} cannot be a homogeneous space.

(ii) Consider the $\boldsymbol{P}^2(\boldsymbol{C})$-bundle $W_4 := \boldsymbol{P}(E)$ over $\boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C})$, where E is the holomorphic vector bundle $\mathcal{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1}(1, -1)$ of rank three over $\boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C})$. W_4 over $\boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C})$ has three natural sections corresponding to the direct summands

$$\{0\} \oplus \{0\} \oplus \mathcal{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1}(1, -1),$$

$$\{0\} \oplus \mathcal{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1} \oplus \{0\},$$

$$\mathcal{O}_{\boldsymbol{P}^1 \times \boldsymbol{P}^1} \oplus \{0\} \oplus \{0\}.$$

We then obtain X_{P_2} from W_4 by blowing up these three sections. Note that X_{P_2} is a fiber bundle over $\boldsymbol{P}^1(\boldsymbol{C}) \times \boldsymbol{P}^1(\boldsymbol{C})$ with fiber S_3.

For these two cases, we can examine the reductivity of the automorphism groups by Theorem 2.4. We then obtain an affirmative answer to the question (II_4) as follows:

THEOREM 3.5. *For a toric Fano r-fold X_P associated to a Fano r-polytope P in $N_{\boldsymbol{R}}$, $1 \leqq r \leqq 4$, the group* $\text{Aut}(X_P)$ *is a reductive algebraic group, provided the Futaki invariant F_{X_P} of X_P vanishes.*

REMARK 3.6. By Theorem 2.4, we can explicitly calculate automorphism groups of toric Fano fourfolds and in particular, the converse of Theorem 3.5 is not true, since 24 isomorphism classes have reductive automorphism groups.

4. Existence of Einstein-Kähler forms on the toric Fano fourfold X_{P_1}.

In this section, we shall prove the following theorem.

THEOREM 4.1. *The toric Fano fourfold X_{P_1} admits an Einstein-Kähler form.*

We now quote the following fact on the existence of Einstein-Kähler forms, which plays an important role in the proof of Theorem 4.1.

THEOREM 4.2 (Nadel [10]). *Let X be an r-dimensional non-singular compact connected complex manifold with ample anti-canonical line bundle. Let G be a compact subgroup of* $\text{Aut}(X)$ *and $G^{\boldsymbol{C}}$ its complexification. Assume that X admits no Einstein-Kähler forms. Then there exists a $G^{\boldsymbol{C}}$-invariant closed analytic subspace $Z \subsetneqq X$, called the "multiplier ideal subscheme" of X, satisfying the following properties*:

(1) $\dim_{\boldsymbol{C}}(H^i(Z, \mathcal{O}_Z)) = 0$, *for* $i > 0$, *and* $\dim_{\boldsymbol{C}}(H^0(Z, \mathcal{O}_Z)) = 1$;

(2) *The complement $X \setminus Z$ has vanishing logarithmic-geometric genus.*

REMARK 4.3. Let Z_{red} be the reduced analytic subspace of X associated to Z, and put $k := \dim_{\boldsymbol{C}} Z$. As stated in Nadel [10], we obtain the following from (1) above in Theorem 4.2:

(4.3.1) $\dim_{\boldsymbol{C}}(H^k(Z_{\text{red}}, \mathcal{O}_{Z_{\text{red}}})) = 0$;

(4.3.2) If $k = 0$, then Z is a single reduced point;

(4.3.3) If $k = 1$, then Z_{red} is a tree of smooth rational curves.

Before proving Theorem 4.1, we introduce some notation and prove a crucial technical lemma (see Lemma 4.4 below). Let $e_1, e_2, \dots, e_{10}$ be the same as in (3.2), and we now put

$$I := \{(i, j, k, l) \in \boldsymbol{Z}^4 \mid 1 \leqq i < j \leqq 5 < k < l \leqq 10,\ \{i, j\} \cap \{k-5, l-5\} = \emptyset\},$$

$$J := \left\{(i, j, k) \in \boldsymbol{Z}^3 \,\middle|\, \begin{array}{l} \text{either } 1 \leqq i < j \leqq 5 < k \leqq 10,\ k-5 \notin \{i, j\} \\ \text{or } 1 \leqq i \leqq 5 < j < k \leqq 10,\ i+5 \notin \{j, k\} \end{array}\right\},$$

$$K_1 := \{(i, j) \in \boldsymbol{Z}^2 \mid 1 \leqq i < j \leqq 5 \text{ or } 6 \leqq i < j \leqq 10\},$$

$$K_2 := \{(i, j) \in \boldsymbol{Z}^2 \mid 1 \leqq i \leqq 5 < j \leqq 10,\ i \neq j-5\},$$

$$\sigma_{i,j,k,l} := \boldsymbol{R}_{\geqq 0} e_i + \boldsymbol{R}_{\geqq 0} e_j + \boldsymbol{R}_{\geqq 0} e_k + \boldsymbol{R}_{\geqq 0} e_l, \qquad (i, j, k, l) \in I,$$

$$\tau_{i,j,k} := \boldsymbol{R}_{\geqq 0} e_i + \boldsymbol{R}_{\geqq 0} e_j + \boldsymbol{R}_{\geqq 0} e_k, \qquad (i, j, k) \in J,$$

$$\rho_{i,j} := \boldsymbol{R}_{\geqq 0} e_i + \boldsymbol{R}_{\geqq 0} e_j, \qquad (i, j) \in K_1,$$

$$\eta_{i,j} := \boldsymbol{R}_{\geqq 0} e_i + \boldsymbol{R}_{\geqq 0} e_j, \qquad (i, j) \in K_2,$$

$$\varepsilon_i := \boldsymbol{R}_{\geqq 0} e_i, \qquad i = 1, 2, \dots, 10.$$

Then $\Delta_{P_1}(4)$, $\Delta_{P_1}(3)$, $\Delta_{P_1}(2)$ and $\Delta_{P_1}(1)$ consist of 30, 60, 40 and 10 strongly convex rational polyhedral cones, respectively, and are of the form

$$\Delta_{P_1}(4) = \{\sigma_{i,j,k,l} \mid (i, j, k, l) \in I\},$$

$$\Delta_{P_1}(3) = \{\tau_{i,j,k} \mid (i, j, k) \in J\},$$

$$\Delta_{P_1}(2) = \{\rho_{i,j} \mid (i, j) \in K_1\} \cup \{\eta_{i,j} \mid (i, j) \in K_2\},$$

$$\Delta_{P_1}(1) = \{\varepsilon_i \mid i = 1, 2, \dots, 10\}.$$

To specify our compact subgroup G of $\mathrm{Aut}(X_{P_1})$, we introduce the following matrices in $GL(4, \boldsymbol{Z})$:

$$A_1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_2 := \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad A_3 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$A_4 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_5 := \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad A_6 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$A_7:=\begin{pmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{pmatrix},\quad A_8:=\begin{pmatrix}1&0&-1&0\\0&1&-1&0\\0&0&-1&0\\0&0&-1&1\end{pmatrix},\quad A_9:=\begin{pmatrix}1&0&0&-1\\0&1&0&-1\\0&0&1&-1\\0&0&0&-1\end{pmatrix},$$

$$A_{10}:=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix},\quad A_{11}:=\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}.$$

For each $1\leqq i\leqq 11$, let φ_{i*} be the T_4-equivariant automorphism of X_{P_1} associated to the automorphism φ_i of the fan Δ_{P_1} induced by the matrix $A_i\in GL(4, \boldsymbol{Z})$ (see [12; p. 19]). The elements φ_i, $i=1, 2, \dots, 10$, generate the full permutation group on the set $\{e_1, e_2, e_3, e_4, e_5\}$. Therefore in $\mathrm{Aut}(X_{P_1})$, the corresponding φ_{i*}, $i=1, 2, \dots, 11$, generate a finite subgroup G_1 isomorphic to the product $\mathfrak{S}_5\times \boldsymbol{Z}_2$ of the symmetric group $\mathfrak{S}_5$ of degree 5 and the cyclic group $\boldsymbol{Z}_2$ of order 2. Let G be the compact subgroup of $\mathrm{Aut}(X_{P_1})$ generated by the 4-dimensional compact real torus $U(1)^4$ ($\subset T_4$) and G_1, where $U(1):=\{t\in \boldsymbol{C} \mid |t|=1\}$. Using the same notation as in Section 1, Theorem 1.3 allows us to determine all $G^{\boldsymbol{C}}$-invariant closed subvarieties of X_{P_1}.

(i) The only zero-dimensional $G^{\boldsymbol{C}}$-invariant closed subvariety Ξ in X_{P_1} is of the form

$$\Xi=\bigcup_{(i,j,k,l)\in I} V(\sigma_{i,j,k,l}),$$

where each component $V(\sigma_{i,j,k,l})$ in Ξ is a single reduced point. In particular, Ξ is a set of 30 distinct points.

(ii) The only one-dimensional $G^{\boldsymbol{C}}$-invariant closed subvariety Γ in X_{P_1} is of the form

$$\Gamma=\bigcup_{(i,j,k)\in J} V(\tau_{i,j,k}).$$

Note that each $V(\tau_{i,j,k})$ with $(i, j, k)\in J$ is isomorphic to $\boldsymbol{P}^1(\boldsymbol{C})$. Moreover, for any two distinct elements $(i, j, k), (i', j', k')\in J$, we have $\sharp(V(\tau_{i,j,k})\cap V(\tau_{i',j',k'}))\leqq 1$, where $\sharp S$ denotes the cardinality of a set S. Therefore, Γ is the union of sixty $\boldsymbol{P}^1(\boldsymbol{C})$'s and contains cycles $V(\tau_{3,4,6})\cup V(\tau_{3,4,7})\cup V(\tau_{3,4,10})$, $V(\tau_{3,6,7})\cup V(\tau_{4,6,7})\cup V(\tau_{5,6,7})$ of $\boldsymbol{P}^1(\boldsymbol{C})$'s, which therefore do not form trees of $\boldsymbol{P}^1(\boldsymbol{C})$'s.

(iii) All two-dimensional $G^{\boldsymbol{C}}$-invariant closed subvarieties are Ψ_1, Ψ_2 and Ψ_3 in X_{P_1}, written in the form,

$$\Psi_1=\bigcup_{(i,j)\in K_1} V(\rho_{i,j}),$$

$$\Psi_2 = \bigcup_{(i,j)\in K_2} V(\eta_{i,j}) ,$$
$$\Psi_3 = \Psi_1 \cup \Psi_2 .$$

Each component $V(\rho_{i,j})$ in Ψ_1 is isomorphic to $\boldsymbol{P}^2(\boldsymbol{C})$, and therefore, Ψ_1 is a union of twenty $\boldsymbol{P}^2(\boldsymbol{C})$'s. For any three distinct elements $(i,j), (i',j'), (i'',j'')$ in K_1, we have $\sharp(V(\rho_{i,j}) \cap V(\rho_{i',j'})) \leqq 1$ and $\sharp(V(\rho_{i,j}) \cap V(\rho_{i',j'}) \cap V(\rho_{i'',j''})) = 0$. Furthermore,

$$\Xi = \bigcup_{(i,j)\neq(i',j')} (V(\rho_{i,j}) \cap V(\rho_{i',j'})) ,$$

where the union is taken over all pairs of distinct elements $(i,j), (i',j')$ in K_1. On the other hand, each $V(\eta_{i,j})$ with $(i,j)\in K_2$ is isomorphic to S_3, and therefore, Ψ_2 is the union of twenty S_3's. Moreover, $\Psi_1 \cap \Psi_2 = \Gamma$ and in particular, $\dim_{\boldsymbol{C}}(\Psi_1 \cap \Psi_2) = 1$.

(iv) The only three-dimensional $G^{\boldsymbol{C}}$-invariant closed subvariety Φ in X_{P_1} is of the form

$$\Phi = \bigcup_{i=1}^{10} V(\varepsilon_i) .$$

Its complement $X_{P_1} \setminus \Phi$ in X_{P_1} is nothing but $T_4 = (\boldsymbol{C}^*)^4$.

We need the following lemma for the proof of Theorem 4.1.

LEMMA 4.4. $H^2(\Psi_2, \mathcal{O}_{\Psi_2}) \cong \boldsymbol{C}$.

PROOF. We put $\square_{\Psi_2} := \{\sigma \in \Delta_{P_1} \mid V(\sigma) \subseteqq \Psi_2\}$. Then $\square_{\Psi_2}$ is expressible as

$$\square_{\Psi_2} = \{\sigma_{i,j,k,l} \mid (i,j,k,l) \in I\} \cup \{\tau_{i,j,k} \mid (i,j,k) \in J\} \cup \{\eta_{i,j} \mid (i,j) \in K_2\} .$$

For each $\kappa \in \square_{\Psi_2}$, we put $\square^{\kappa}_{\Psi_2} := \{\sigma \in \square_{\Psi_2} \mid \sigma \prec \kappa\}$. We now consider Ishida's fourth complex of $\boldsymbol{Z}$-modules for $\square^{\kappa}_{\Psi_2}$ as in [5]:

$$C^{\cdot}(\square^{\kappa}_{\Psi_2}; 4) = \left(\{0\} \longrightarrow C^0(\square^{\kappa}_{\Psi_2}; 4) \xrightarrow{\delta^{\kappa}_0} \cdots \xrightarrow{\delta^{\kappa}_3} C^4(\square^{\kappa}_{\Psi_2}; 4) \longrightarrow \{0\} \right) .$$

We first consider the case $\dim \kappa = 2$. Let $\kappa = \eta_{4,7}$ for instance. In this case, we have $\square^{\eta_{4,7}}_{\Psi_2} = \{\eta_{4,7}\}$ and

$$C^i(\square^{\eta_{4,7}}_{\Psi_2}; 4) = \begin{cases} \boldsymbol{Z} e^1 \wedge e^3 , & \text{for } i=2 ; \\ \{0\} , & \text{otherwise} , \end{cases}$$

$$\delta^{\eta_{4,7}}_i = 0 , \qquad 0 \leqq i \leqq 3 ,$$

where $\{e^1, e^2, e^3, e^4\}$ is the $\boldsymbol{Z}$-basis for M dual to the standard $\boldsymbol{Z}$-basis $\{e_1, e_2, e_3, e_4\}$ for N. The cohomology groups of the complex $\boldsymbol{C} \otimes_{\boldsymbol{Z}} C^{\cdot}(\square^{\eta_{4,7}}_{\Psi_2}; 4)$ turn out to be

$$H^i(\boldsymbol{C} \otimes_{\boldsymbol{Z}} C^{\cdot}(\square^{\eta_{4,7}}_{\Psi_2}; 4)) \cong \begin{cases} \boldsymbol{C} , & \text{for } i=2 ; \\ \{0\} , & \text{otherwise} . \end{cases}$$

For an arbitrary $\kappa \in \square_{\Psi_2} \cap \Delta_{P_1}(2)$ in general, the same calculation as above yields

$$H^i(\boldsymbol{C} \otimes_{\boldsymbol{Z}} C^{\boldsymbol{\cdot}}(\square^{\kappa}_{\Psi_2}; 4)) \cong \begin{cases} \boldsymbol{C}, & \text{for } i=2; \\ \{0\}, & \text{otherwise}. \end{cases}$$

We next consider the case $\dim \kappa = 3$. Let $\kappa = \tau_{3,4,7}$ for instance. In this case, we have $\square^{\tau_{3,4,7}}_{\Psi_2} = \{\eta_{3,7}, \eta_{4,7}, \tau_{3,4,7}\}$ and

$$C^i(\square^{\tau_{3,4,7}}_{\Psi_2}; 4) = \begin{cases} \boldsymbol{Z}e^1 \wedge e^3 \oplus \boldsymbol{Z}e^1 \wedge e^4, & \text{for } i=2; \\ \boldsymbol{Z}e^1, & \text{for } i=3; \\ \{0\}, & \text{otherwise}, \end{cases}$$

$$\begin{cases} \delta_2^{\tau_{3,4,7}}(e^1 \wedge e^3) = -e^1, \quad \delta_2^{\tau_{3,4,7}}(e^1 \wedge e^4) = -e^1, \\ \delta_i^{\tau_{3,4,7}} = 0, \quad \text{for } i \neq 2. \end{cases}$$

The cohomology groups of the complex $\boldsymbol{C} \otimes_{\boldsymbol{Z}} C^{\boldsymbol{\cdot}}(\square^{\tau_{3,4,7}}_{\Psi_2}; 4)$ turn out to be

$$H^i(\boldsymbol{C} \otimes_{\boldsymbol{Z}} C^{\boldsymbol{\cdot}}(\square^{\tau_{3,4,7}}_{\Psi_2}; 4)) \cong \begin{cases} \boldsymbol{C}, & \text{for } i=2; \\ \{0\}, & \text{otherwise}. \end{cases}$$

For an arbitrary $\kappa \in \square_{\Psi_2} \cap \Delta_{P_1}(3)$ in general, we similarly have

$$H^i(\boldsymbol{C} \otimes_{\boldsymbol{Z}} C^{\boldsymbol{\cdot}}(\square^{\kappa}_{\Psi_2}; 4)) \cong \begin{cases} \boldsymbol{C}, & \text{for } i=2; \\ \{0\}, & \text{otherwise}. \end{cases}$$

Finally, we consider the case $\dim \kappa = 4$. Let $\kappa = \sigma_{3,4,6,7}$ for instance. Ishida's complex $C^{\boldsymbol{\cdot}}(\square^{\sigma_{3,4,6,7}}_{\Psi_2}; 4)$ is explicitly written as follows:

$$\square^{\sigma_{3,4,6,7}}_{\Psi_2} = \{\eta_{3,6}, \eta_{4,6}, \eta_{3,7}, \eta_{4,7}, \tau_{3,4,7}, \tau_{3,4,6}, \tau_{3,6,7}, \tau_{4,6,7}, \sigma_{3,4,6,7}\},$$

$$C^i(\square^{\sigma_{3,4,6,7}}_{\Psi_2}; 4) = \begin{cases} \boldsymbol{Z}e^1 \wedge e^3 \oplus \boldsymbol{Z}e^1 \wedge e^4 \oplus \boldsymbol{Z}e^2 \wedge e^3 \oplus \boldsymbol{Z}e^2 \wedge e^4, & \text{for } i=2; \\ \boldsymbol{Z}e^1 \oplus \boldsymbol{Z}e^2 \oplus \boldsymbol{Z}e^3 \oplus \boldsymbol{Z}e^4, & \text{for } i=3; \\ \boldsymbol{Z}, & \text{for } i=4; \\ \{0\}, & \text{for } i=0, 1, \end{cases}$$

$$\begin{cases} \delta_2^{\sigma_{3,4,6,7}}(e^1 \wedge e^3) = -e^1 - e^3, \quad \delta_2^{\sigma_{3,4,6,7}}(e^1 \wedge e^4) = -e^1 - e^4, \\ \delta_2^{\sigma_{3,4,6,7}}(e^2 \wedge e^3) = -e^2 - e^3, \quad \delta_2^{\sigma_{3,4,6,7}}(e^2 \wedge e^4) = -e^2 - e^4, \\ \delta_3^{\sigma_{3,4,6,7}}(e^1) = -1, \quad \delta_3^{\sigma_{3,4,6,7}}(e^2) = -1, \quad \delta_3^{\sigma_{3,4,6,7}}(e^3) = 1, \quad \delta_3^{\sigma_{3,4,6,7}}(e^4) = 1, \\ \delta_i^{\sigma_{3,4,6,7}} = 0, \quad \text{for } i=0, 1. \end{cases}$$

The cohomology groups of the complex $\boldsymbol{C} \otimes_{\boldsymbol{Z}} C^{\boldsymbol{\cdot}}(\square^{\sigma_{3,4,6,7}}_{\Psi_2}; 4)$ turn out to be

$$H^i(\boldsymbol{C} \otimes_{\boldsymbol{Z}} C^{\boldsymbol{\cdot}}(\square^{\sigma_{3,4,6,7}}_{\Psi_2}; 4)) \cong \begin{cases} \boldsymbol{C}, & \text{for } i=2; \\ \{0\}, & \text{otherwise}. \end{cases}$$

For an arbitrary $\kappa \in \square_{\Psi_2} \cap \Delta_{P_1}(4)$ in general, we again have

$$H^i(\boldsymbol{C}\otimes_{\boldsymbol{Z}}C^{\bullet}(\square^{\kappa}_{\Psi_2};4))\cong\begin{cases}\boldsymbol{C}, & \text{for } i=2;\\ \{0\}, & \text{otherwise}.\end{cases}$$

Therefore, for any $\kappa\in\square_{\Psi_2}$ of an arbitrary dimension, we obtain

$$H^i(\boldsymbol{C}\otimes_{\boldsymbol{Z}}C^{\bullet}(\square^{\kappa}_{\Psi_2};4))\cong\begin{cases}\boldsymbol{C}, & \text{for } i=2;\\ \{0\}, & \text{otherwise}.\end{cases}$$

Then by applying Ishida's criterion [5; Theorem 5.10] (see also [12; p. 126]) to Ψ_2, we see that Ψ_2 is a Gorenstein variety with the dualizing sheaf isomorphic to $\mathcal{O}_{\Psi_2}$. From Serre-Grothendieck's duality theorem, we conclude that

$$H^2(\Psi_2,\mathcal{O}_{\Psi_2})\cong H^0(\Psi_2,\mathcal{O}_{\Psi_2})\cong\boldsymbol{C},$$

as required. ∎

It is now possible to prove our main result.

PROOF OF THEOREM 4.1. Suppose, for contradictions, that X_{P_1} admits no Einstein-Kähler forms. Then X_{P_1} has a multiplier ideal subscheme Z by Theorem 4.2. Since Z is $G^{\boldsymbol{C}}$-invariant, Z_{red} is one of the six varieties Ξ, Γ, Ψ_1, Ψ_2, Ψ_3, Φ. We first observe that Z_{red} cannot be Ξ, since Ξ is a set of thirty distinct points in contradicion to (4.3.2). Secondly, Z_{red} cannot be Γ, since Γ is not a tree of $\boldsymbol{P}^1(\boldsymbol{C})$'s in contradiction to (4.3.3). Thirdly, Z_{red} cannot be Φ, since $(\boldsymbol{C}^*)^4\cong X_{P_1}\setminus\Phi$ has positive logarithmic-geometric genus, contradicting Theorem 4.2, (2). Fourthly, we do not have $Z_{\text{red}}=\Psi_2$ either, in view of Lemma 4.4 and (4.3.1).

We next consider the case $Z_{\text{red}}=\Psi_3=\Psi_1\cup\Psi_2$. Let $Z':=\Psi_1\coprod\Psi_2$ be the disjoint union of Ψ_1 and Ψ_2, and let $\varpi\colon Z'\to Z_{\text{red}}$ be the natural projection. Then we have a short exact sequence

$$0\to\mathcal{O}_{Z_{\text{red}}}\to\varpi_*\mathcal{O}_{Z'}\to\mathcal{F}_1:=(\varpi_*\mathcal{O}_{Z'})/\mathcal{O}_{Z_{\text{red}}}\to 0, \tag{4.5}$$

where the support $\text{Supp}(\mathcal{F}_1)$ of $\mathcal{F}_1$ is just the one-dimensional variety $\Gamma=\Psi_1\cap\Psi_2$. From (4.5), we obtain a long exact sequence

$$\cdots\to H^2(Z_{\text{red}},\mathcal{O}_{Z_{\text{red}}})\to H^2(Z',\mathcal{O}_{Z'})\to H^2(Z_{\text{red}},\mathcal{F}_1)\to\cdots. \tag{4.6}$$

Since $\dim_{\boldsymbol{C}}(\text{Supp}(\mathcal{F}_1))=1$, we have $H^2(Z_{\text{red}},\mathcal{F}_1)\cong\{0\}$. By (4.3.1), we also have $H^2(Z_{\text{red}},\mathcal{O}_{Z_{\text{red}}})\cong\{0\}$. Hence, $H^2(\Psi_1,\mathcal{O}_{\Psi_1})\oplus H^2(\Psi_2,\mathcal{O}_{\Psi_2})\cong H^2(Z',\mathcal{O}_{Z'})\cong\{0\}$ by (4.6), contradicting Lemma 4.4.

We finally consider the case $Z_{\text{red}}=\Psi_1$. Then Z is expressible in the form

$$Z=\bigcup_{(i,j)\in K_1}\tilde{V}(\rho_{i,j}),$$

where $\tilde{V}(\rho_{i,j})$ is an analytic subspace of X_{P_1} such that $\tilde{V}(\rho_{i,j})_{\text{red}}=V(\rho_{i,j})$. Note that $\tilde{V}(\rho_{i,j})$, $(i,j)\in K_1$, are all $G^{\boldsymbol{C}}$-congruent. Let

$$Z'' := \coprod_{(i,j)\in K_1} \tilde{V}(\rho_{i,j})$$

be the disjoint union of $\tilde{V}(\rho_{i,j})$, $(i,j)\in K_1$, and let $\varpi' : Z'' \to Z$ be the natural projection. Then we have a short exact sequence

$$0 \to \mathcal{O}_Z \to \varpi'_* \mathcal{O}_{Z''} \to \mathcal{F}_2 := (\varpi'_* \mathcal{O}_{Z''})/\mathcal{O}_Z \to 0 . \tag{4.7}$$

Note that $\mathrm{Supp}(\mathcal{F}_2)$ is just Ξ consisting of thirty $G^{\boldsymbol{C}}$-congruent points. Moreover, $\mathcal{F}_2$ is $G^{\boldsymbol{C}}$-invariant. Now by (4.7), we have

$$\{0\} \to H^0(Z, \mathcal{O}_Z) \to H^0(Z'', \mathcal{O}_{Z''}) \to H^0(Z, \mathcal{F}_2) \to H^1(Z, \mathcal{O}_Z) \to \cdots . \tag{4.8}$$

Since all $\tilde{V}(\rho_{i,j})$'s in Z'' and all $V(\sigma_{i,j,k,l})$'s in Ξ are $G^{\boldsymbol{C}}$-congruent, respectively, and since $\mathcal{F}_2$ is $G^{\boldsymbol{C}}$-invariant, there exist some p, q in $\boldsymbol{Z}_{\geqq 0}$ such that

$$\dim_{\boldsymbol{C}}(H^0(Z'', \mathcal{O}_{Z''})) = 20p \quad \text{and} \quad \dim_{\boldsymbol{C}}(H^0(Z, \mathcal{F}_2)) = 30q .$$

Since $\dim_{\boldsymbol{C}}(H^0(Z, \mathcal{O}_Z)) = 1$ and $\dim_{\boldsymbol{C}}(H^1(Z, \mathcal{O}_Z)) = 0$ by Theorem 4.2, (1), the long exact sequence (4.8) above yields $20p - 1 = 30q$ in contradiction. Thus, we can conclude that X_{P_1} admits an Einstein-Kähler form. ■

REFERENCES

[1] V. V. BATYREV, Toroidal Fano 3-folds, Math. USSR-Izv. 19 (1982), 13–25; Izv. Akad. Nauk SSSR 45 (1981), 704–717.

[2] V. V. BATYREV, On the classification of toric Fano 4-folds, preprint.

[3] M. DEMAZURE, Sous-groupes algébriques de rang maximum du groupe de Cremona, Ann. Sci. École Norm. Sup. 3 (1970), 507–588.

[4] A. FUTAKI, An obstruction to the existence of Kähler Einstein metrics, Invent. Math. 73 (1983), 437–443.

[5] M.-N. ISHIDA, Torus embeddings and dualizing complexes, Tôhoku Math. J. 32 (1980), 111–146.

[6] G. KEMPF, F. KNUDSEN, D. MUMFOLD AND B. SAINT-DONAT, Toroidal embeddings I, Lecture Notes in Mathematics 339, Springer-Verlag, Heidelberg, 1973.

[7] T. MABUCHI, Einstein-Kähler forms, Futaki invariants and convex geometry on toric Fano varieties, Osaka J. Math. 24 (1987), 705–737.

[8] T. MABUCHI, An algebraic character associated with the Poisson brackets, in Recent Topics in Differential and Analytic Geometry, Adv. Stud. in Pure Math. 18-I, Kinokuniya and Academic Press, Tokyo and Boston, 1990, pp. 339–358.

[9] Y. MATSUSHIMA, Sur la structure du groupe d'homomorphismes analitique d'une certaine variété kählerienne, Nagoya Math. J. 46 (1957), 145–150.

[10] A. M. NADEL, Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature, Ann. of Math. 132 (1990), 549–596.

[11] T. ODA, Lecture on torus embeddings and applications (Based on joint works with Katsuya Miyake), Tata Inst. of Fund. Research 58, Springer-Verlag, Heidelberg, 1978.

[12] T. ODA, Convex bodies and algebraic geometry: An introduction to the theory of toric varieties, Ergebnisse der Math. (3) 15, Springer-Verlag, Heidelberg, 1988.

[13] Y. SAKANE, Example of compact Einstein Kähler manifolds with positive Ricci tensor, Osaka J. Math. 23 (1986), 585–616.

[14] Y.-T. SIU, The existence of Kähler-Einstein metrics on manifolds with positive anticanonical line bundle and a suitable finite symmetry group, Ann. of Math. 127 (1988), 585–627.

[15] G. TIAN AND S.-T. YAU, Kähler-Einstein metrics on complex surfaces with $C_1>0$, Commun. Math. Phys. 112 (1987), 175–203.

[16] V. E. VOSKRESENSKII AND A. A. KLYACHKO, Toroidal Fano varieties and root systems, Math. USSR-Izv. 24 (1985), 221–244; Izv. Akad. Nauk SSSR 48 (1984), 237–263.

[17] K. WATANABE AND M. WATANABE, The classification of Fano 3-folds with torus embeddings, Tokyo J. Math. 5 (1982), 37–48.

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