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THE CHOW RINGS AND GKZ-DECOMPOSITIONS FOR $\boldsymbol{Q}$-FACTORIAL TORIC VARIETIES

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Abstract. We define the Chow ring for a $\boldsymbol{Q}$-factorial toric variety as the Stanley-Reisner ring for the corresponding fan modulo the linear equivalence relation. We also define the pull-back homomorphism and the push-forward homomorphism between the Chow rings in terms of the combinatorial structure of fans and a map of fans, and prove the projection formula without using algebro-geometric method. In the second part, we apply the GKZ-decomposition to the $\boldsymbol{Q}$-factorial toric varieties and obtain some information when the corresponding fans are confined to have one-dimensional cones within a fixed set.

Introduction. Let N be a free $\boldsymbol{Z}$-module of rank r and M its dual. An r-dimensional algebraic torus $T_N \cong \boldsymbol{C}^{\times} \times \cdots \times \boldsymbol{C}^{\times}$ (r times) is defined by $T_N := \mathrm{Hom}_{\boldsymbol{Z}}(M, \boldsymbol{C}^{\times})$, where $\boldsymbol{C}^{\times}$ is the multiplicative group of non-zero complex numbers. A toric variety X is a normal algebraic variety containing T_N as a Zariski open dense subset with an algebraic action of T_N on X which is an extension of the group law of T_N. A toric variety X can be described in terms of a certain collection Δ, which is called a fan, of cones in $N_{\boldsymbol{R}} := N \otimes_{\boldsymbol{Z}} \boldsymbol{R}$. From this fact, the properties of a toric variety have strong connection with the combinatorial structure of the corresponding fan and the relations among the generators. One of the purposes of this paper is based on this fact. For the precise definitions of toric varieties, see [4], [18] and [19].

Let $X := T_N \operatorname{emb}(\Delta)$ be the toric variety corresponding to a simplicial fan Δ. Hence X is $\boldsymbol{Q}$-factorial, and has at most quotient singularities. This paper consists of two parts. In Section 1, we first define the Chow ring $A(N, \Delta)$ over the rational number field $\boldsymbol{Q}$ in terms of the simplicial fan Δ. Namely, we define the Chow ring $A(N, \Delta)$ as the Stanley-Reisner ring $\mathrm{SR}(N, \Delta)$ (cf. [27]) of Δ modulo the linear equivalence relation. In Proposition 1.1 we see that for any $0 \le p \le r$ the homogeneous part $A^p(N, \Delta)$ of degree p of the Chow ring $A(N, \Delta)$ is generated over $\boldsymbol{Q}$ by the equivalence classes $v(\sigma)$ of the elements in $\mathrm{SR}(N, \Delta)$ corresponding to $\sigma \in \Delta(p)$. The product satisfies

$$v(\sigma) \cdot v(\sigma') = v(\sigma + \sigma') \qquad \text{whenever} \quad \sigma + \sigma' \in \Delta \quad \text{and} \quad \sigma \cap \sigma' = \{0\} .$$

Note that Danilov [4] and Fulton [7] used different generators $[F_\sigma]$, $\sigma \in \Delta$, which are related to ours by

$$[F_\sigma] = \mathrm{mult}(\sigma) \cdot v(\sigma) ,$$

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where $\mathrm{mult}(\sigma)$ is the *multiplicity* of σ which will be defined later. Hence the product becomes

$$[F_\sigma]\cdot[F_{\sigma'}]=\frac{\mathrm{mult}(\sigma)\cdot\mathrm{mult}(\sigma')}{\mathrm{mult}(\sigma+\sigma')}[F_{\sigma+\sigma'}]$$

for $\sigma, \sigma'\in\Delta$ satisfying $\sigma+\sigma'\in\Delta$ and $\sigma\cap\sigma'=\{0\}$.Thus our generators are more natural in describing the structure of the Chow rings.

We also state some properties of the Chow ring in Section 1, and relate the Chow ring to Ishida's cohomology (cf. [12] and [19]) which is very useful in describing the properties of the Chow ring.

Let N' be a free $\boldsymbol{Z}$-module of rank r', and Δ' a fan for N'. Let $\phi: (N', \Delta')\to(N, \Delta)$ be a map of fans. Then ϕ gives rise to an equivariant holomorphic map $\phi_v: T_{N'}\mathrm{emb}(\Delta')\to T_N\mathrm{emb}(\Delta)$ between toric varieties. If the corresponding map ϕ_v is a *proper* map, then there exists a push-forward homomorphism $\phi_*: A(N', \Delta')\to A(N, \Delta)$ between the Chow rings (cf. [6]). In Section 2, we describe ϕ_* explicitly in terms of the combinatorial structure of the fans Δ, Δ' and a map ϕ of fans, whenever ϕ has *finite cokernel*. We hope to come back to the problem of describing ϕ_* in the interesting case where the cokernel of ϕ is not finite.

Also in Section 2, we describe the pull-back homomorphism $\phi^*: A(N, \Delta)\to A(N', \Delta')$ explicitly for an *arbitrary* map ϕ of fans. If $\phi: (N', \Delta')\to(N, \Delta)$ is a map of fans with finite cokernel, then we can prove directly that the induced homomorphisms ϕ_* and ϕ^* satisfy the *projection formula* (cf. Theorem 2.10), that is,

$$\phi_*(\phi^*(\omega)\cdot\omega')=\omega\cdot\phi_*(\omega') \qquad \text{for} \quad \omega\in A(N, \Delta), \quad \omega'\in A(N', \Delta')\,.$$

In Section 3, we consider equivariant fiber bundles over toric varieties. If $\phi: (N', \Delta')\to(N, \Delta)$ induces an equivariant $\boldsymbol{P}^l(\boldsymbol{C})$- (resp. $\boldsymbol{C}^l$-) bundle over a toric variety, then by [19, Proposition 1.33], Δ' can be described explicitly. From this fact, we get a direct description of the Chow ring $A(N', \Delta')$ in terms of $A(N, \Delta)$.

In the second part, we deal with the GKZ-decompositions for toric varieties, where GKZ stands for Gelfand, Kapranov and Zelevinskij. In [8], [9] and [10] they obtained some decompositions of $\boldsymbol{R}^N$ by using regular triangulations of integral polytopes corresponding to projective toric varieties. We have generalized and reformulated their results in the context of $\boldsymbol{R}$-vector spaces in [24]. Our present purpose is to modify the definition in the context of $\boldsymbol{Q}$-vector spaces and apply it to $\boldsymbol{Q}$-factorial toric varieties.

In Section 4, we define the $\boldsymbol{Q}$-linear Gale transform, and relate it to toric varieties. This concept is very useful in dealing with toric varieties with small Picard numbers. We use this notion in connection with the Chow ring of a toric variety.

Let Ξ be a finite subset of primitive elements in N, such that Ξ spans $N_{\boldsymbol{Q}}:=N\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$ over $\boldsymbol{Q}$. Then, as we show in Theorem 4.1, there exists a simplicial and admissible fan Δ_0 in N, which is *full*, i.e., every $\xi\in\Xi$ gives rise to a one-dimensional cone in Δ_0. Each simplicial fan Δ corresponding to a maximal dimensional GKZ-cone $\mathrm{cpl}(\Delta)$ in the

GKZ-decomposition can be obtained from Δ_0 by a finite succession of *flops* or *star subdivisions* as in [24, Theorem 3.12]. Furthermore, as we show in Theorem 4.5, the union of the $\mathrm{cpl}(\Delta)$'s, with Δ obtained from Δ_0 by finite successions of flops, also is a convex polyhedral cone.

We describe the dual cone of $\mathrm{cpl}(\Delta)$ when Δ is full, simplicial and admissible for a fixed (N, Ξ) in Section 5. It is related to the Mori cone $NE(X)$ of the corresponding toric variety $X := T_N\,\mathrm{emb}(\Delta)$.

In the last section, we apply the GKZ-decomposition to a fan which is a simplicial subdivision of a fixed strongly convex cone π all of whose proper faces are simplicial. We get some information on *small* simplicial subdivisions of π.

Throughout this paper, we fix a free $\boldsymbol{Z}$-module N of rank r over the ring $\boldsymbol{Z}$ of integers, and denote by $M := \mathrm{Hom}_{\boldsymbol{Z}}(N, \boldsymbol{Z})$ its dual $\boldsymbol{Z}$-module with the canonical bilinear pairing

$$\langle\ ,\ \rangle : M \times N \to \boldsymbol{Z}.$$

We denote the scalar extensions of N and M to the field $\boldsymbol{R}$ of real numbers by $N_{\boldsymbol{R}} := N \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ and $M_{\boldsymbol{R}} := M \otimes_{\boldsymbol{Z}} \boldsymbol{R}$, respectively. We follow definitions and notation in [19].

DEFINITION. A finite collection Δ of strongly convex rational polyhedral cones in $N_{\boldsymbol{R}}$ is called a *fan* if it satisfies the following conditions:

(i) Every face of any $\sigma \in \Delta$ is contained in Δ.

(ii) For any $\sigma, \sigma' \in \Delta$, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'.

A fan Δ is said to be *simplicial* if every $\sigma \in \Delta$ is simplicial, i.e., σ can be expressed as

$$\sigma = \boldsymbol{R}_{\geq 0} n_1 + \cdots + \boldsymbol{R}_{\geq 0} n_s$$

for an $\boldsymbol{R}$-linearly independent subset $\{n_1, n_2, \ldots, n_s\}$ of N, where $\boldsymbol{R}_{\geq 0}$ is the set of nonnegative real numbers. A fan Δ is said to be *complete* if $|\Delta| := \bigcup_{\sigma \in \Delta} \sigma = N_{\boldsymbol{R}}$. It is known that the toric variety corresponding to a simplicial fan has at most quotient singularities and is $\boldsymbol{Q}$-factorial. Also, the toric variety is compact if and only if the corresponding fan is complete.

In this paper, we consider only those (finite) fans which are *simplicial* with *r-dimensional convex* support.

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1. The Chow ring. Let Δ be a simplicial fan for $N \cong \boldsymbol{Z}^r$, which may not be complete. In this section, we define the Chow ring $A(N, \Delta)$ in terms of a simplicial fan Δ, describe its generators, and relate it to Ishida's cohomology.

Introduce the polynomial ring S over $\boldsymbol{Q}$ in the variables $\{x(\rho) \mid \rho \in \Delta(1)\}$. We can regard this ring as a graded $\boldsymbol{Q}$-algebra by letting $\deg x(\rho) = 1$ for any $\rho \in \Delta(1)$. Let I be the ideal in S generated by the set

$$\{x(\rho_1)x(\rho_2)\cdots x(\rho_s) \mid \rho_1, \ldots, \rho_s \in \Delta(1) \text{ distinct and } \rho_1 + \cdots + \rho_s \notin \Delta\}\,.$$

Then the residue ring $\mathrm{SR}(N, \Delta) := S/I$ is the *Stanley-Reisner ring* (or *face ring* in [27]) for the fan Δ.

On the other hand, we define another ideal J in S to be the one generated by the set

$$\left\{\theta(m) := \sum_{\rho \in \Delta(1)} \langle m, n(\rho)\rangle x(\rho) \,\middle|\, m \in M\right\},$$

where $n(\rho)$ is the unique primitive element of N contained in $\rho \in \Delta(1)$.

DEFINITION. In the notation above, we define the *Chow ring* over $\boldsymbol{Q}$ for a *simplicial* fan Δ to be the ring

$$A(N, \Delta) := S/(I+J)\,.$$

We simply write $A(\Delta)$ if there is no confusion. $A(\Delta)$ is a finite-dimensional graded $\boldsymbol{Q}$-algebra of the form $A(\Delta) = \bigoplus_{k=0}^{r} A^k(\Delta)$ and is generated by $A^1(\Delta)$ over $A^0(\Delta) = \boldsymbol{Q}$, where $A^k(\Delta)$ is its homogeneous part of degree k. Especially, $A(\Delta)$ is a Gorenstein ring if Δ is complete.

Let us denote by $v(\rho) \in A^1(\Delta)$ the image in $A(\Delta)$ of $x(\rho)$ for $\rho \in \Delta(1)$. By the construction of the Chow ring, we have

$$\sum_{\rho \in \Delta(1)} \langle m, n(\rho)\rangle v(\rho) = 0 \qquad \text{for any} \quad m \in M\,,$$

or more symmetrically, we can write it as a single equality

$$\sum_{\rho \in \Delta(1)} n(\rho) \otimes v(\rho) = 0 \qquad \text{in} \quad N \otimes_{\boldsymbol{Z}} A(\Delta)\,,$$

which we call the *defining relation*.

Since Δ is assumed to be simplicial, each $\sigma \in \Delta(k)$ can be expressed as $\sigma = \rho_1 + \cdots + \rho_k$ for distinct $\rho_1, \ldots, \rho_k \in \Delta(1)$. In this case, we denote by $v(\sigma) \in A^k(\Delta)$ the image in $A(\Delta)$ of the monomial $x(\rho_1)x(\rho_2)\cdots x(\rho_k) \in S$.

For pairs $\sigma, \sigma' \in \Delta$, we have

$$v(\sigma)\cdot v(\sigma') = \begin{cases} 0 & \text{if } \ \sigma+\sigma' \notin \Delta \\ v(\sigma+\sigma') & \text{if } \ \sigma\cap\sigma' = \{0\} \ \text{ and } \ \sigma+\sigma' \in \Delta \,. \end{cases}$$

PROPOSITION 1.1. *Let Δ be a simplicial fan for $N \cong \boldsymbol{Z}^r$. Then we have*

$$A^k(\Delta) = \sum_{\sigma\in\Delta(k)} \boldsymbol{Q}v(\sigma) \qquad \textit{for any} \quad 0 \le k \le r\,.$$

Especially, if $k=r$, then we have

$$A^r(\Delta) \cong \begin{cases} \boldsymbol{Q} & \textit{if } \Delta \textit{ is complete} \\ 0 & \textit{otherwise}\,. \end{cases}$$

PROOF. By induction, it suffices to show that $v(\rho_0)v(\tau)$ for $\tau \in \Delta$ and $\rho_0 \prec \tau$ is expressed as a linear combination of $\{v(\sigma) \mid \sigma \in \Delta, \dim\sigma = \dim\tau + 1\}$.

Since $\{n(\rho) \mid \rho \in \Delta(1), \rho \prec \tau\}$ is a set of linearly independent elements in N, there exists $m_0 \in M$ such that $\langle m_0, n(\rho_0)\rangle = 1$, while $\langle m_0, n(\rho)\rangle = 0$ for all $\rho \in \Delta(1)$, $\rho \neq \rho_0$ with $\rho \prec \tau$. Hence we get

$$v(\rho_0) + \sum_{\rho\in\Delta(1),\, \rho\not\prec\tau} \langle m_0, n(\rho)\rangle v(\rho) = 0\,.$$

Multiplying $v(\tau)$ to this equality, we have

$$v(\rho_0)v(\tau) + \sum_{\rho\in\Delta(1),\, \rho\not\prec\tau} \langle m_0, n(\rho)\rangle v(\rho)v(\tau) = 0\,.$$

Since $v(\rho)v(\tau) = v(\rho+\tau)$ for $\rho \not\prec \tau$ with $\rho+\tau \in \Delta$, while $v(\rho)v(\tau) = 0$ if $\rho+\tau \notin \Delta$, $v(\rho_0)v(\tau)$ can be written as a linear combination of $\{v(\sigma) \mid \sigma \in \Delta, \dim\sigma = \dim\tau + 1\}$.

Now we prove the second statement. If Δ is complete, then every $\tau \in \Delta(r-1)$ is an internal wall. Let us fix $\sigma_0 \in \Delta(r)$. Then for any $\sigma \in \Delta(r)$, there exist $\sigma_1, \ldots, \sigma_k = \sigma \in \Delta(r)$ such that $\sigma_i \cap \sigma_{i+1}$ belongs to $\Delta(r-1)$ for any $i = 0, \ldots, k-1$. From the defining relation, we see that $v(\sigma) = b_\sigma v(\sigma_0)$ for some $b_\sigma \in \boldsymbol{Q}_{>0}$. Hence,

$$A^r(\Delta) = \sum_{\sigma\in\Delta(r)} \boldsymbol{Q}v(\sigma) = \boldsymbol{Q}v(\sigma_0) \cong \boldsymbol{Q}\,,$$

since $A(\Delta)$ is Gorenstein.

If $|\Delta| \neq N_{\boldsymbol{R}}$, we may assume that $\Delta(r) \neq \emptyset$. Then there exists a $\sigma \in \Delta(r)$ having a facet τ which is not an internal wall. Let $\sigma = \tau + \rho_\tau$ for some $\rho_\tau \in \Delta(1)$. From the defining relation, we obtain $v(\sigma) = 0$. Since Δ is not complete, for any $\sigma' \in \Delta(r)$ there exist $\sigma_1, \ldots, \sigma_k = \sigma' \in \Delta(r)$ such that $\sigma_i \cap \sigma_{i+1}$ belongs to $\Delta(r-1)$ for any $i = 1, \ldots, k-1$ and that σ_1 has a facet which is not an internal wall. We use the defining relation again to get $v(\sigma')=b_{\sigma'}v(\sigma_1)=0$ for some $b_{\sigma'}\in \boldsymbol{Q}_{>0}$. Consequently, we have $A^r(\Delta)=0$. q.e.d.

Let $X:=T_N\operatorname{emb}(\Delta)$ be the toric variety corresponding to a simplicial fan Δ. Then

$$H^{\cdot}(X;\boldsymbol{Z})\otimes_{\boldsymbol{Z}}\boldsymbol{Q}\cong A(\Delta)\,.$$

For the proof, see [4], [7], and [14]. Especially, if X is a nonsingular surface, there is an easy proof in [1].

Now, we introduce the multiplicity of a simplicial cone σ in $N_{\boldsymbol{R}}$ (cf. [7] and [15]) to relate the Chow ring to Ishida's cohomology. The multiplicity of a cone is also used in Section 2 to define a push-forward homomorphism $\phi_*\colon A(N',\Delta')\to A(N,\Delta)$ between two Chow rings.

Definition. Let $\sigma:=\rho_1+\cdots+\rho_s$ be an s-dimensional simplicial cone in $N_{\boldsymbol{R}}\cong\boldsymbol{R}^r$. We define the *multiplicity* of σ as the index

$$\operatorname{mult}(\sigma,N):=[N\cap\boldsymbol{R}\sigma:\boldsymbol{Z}n(\rho_1)+\cdots+\boldsymbol{Z}n(\rho_s)]$$

of the submodule $\boldsymbol{Z}n(\rho_1)+\cdots+\boldsymbol{Z}n(\rho_s)$ in $N\cap\boldsymbol{R}\sigma$. We simply denote $\operatorname{mult}(\sigma)$ if there is no confusion.

Let us introduce Ishida's cohomology which is very useful in describing the properties of the Chow rings.

Definition. Let Δ be a simplicial fan for $N\cong\boldsymbol{Z}^r$ and M the dual $\boldsymbol{Z}$-module of N. For any $p,q=0,1,\dots,r$, let

$$C^q(\Delta,\Lambda^p):=\bigoplus_{\sigma\in\Delta(q)}\bigwedge\nolimits^{p-q}(M\cap\sigma^{\perp})$$

and define a coboundary homomorphism

$$\delta\colon C^{q-1}(\Delta,\Lambda^p):=\bigoplus_{\tau\in\Delta(q-1)}\bigwedge\nolimits^{p-q+1}(M\cap\tau^{\perp})\to C^q(\Delta,\Lambda^p)$$

by defining the (τ,σ)-component as follows: For any $\tau\in\Delta(q-1)$ and $\sigma\in\Delta(q)$, define

$$\delta_{\tau/\sigma}\colon\bigwedge\nolimits^{p-q+1}(M\cap\tau^{\perp})\to\bigwedge\nolimits^{p-q}(M\cap\sigma^{\perp})$$

to be $\delta_{\tau/\sigma}=0$ if τ is not a face of σ, while for $\tau\prec\sigma$, there exists a unique $\rho\in\Delta(1)$ such that $\sigma=\rho+\tau$. Moreover, $M\cap\sigma^{\perp}$ is a $\boldsymbol{Z}$-submodule of rank $r-q$ in the $\boldsymbol{Z}$-module $M\cap\tau^{\perp}$ of rank $r-q+1$, hence each element of $\bigwedge^{p-q+1}(M\cap\tau^{\perp})$ is a finite linear combination of elements of the form

$$m_1\wedge m_2\wedge\cdots\wedge m_{p-q+1}\quad\text{with}\quad m_1\in M\cap\tau^{\perp}\quad\text{and}\quad m_2,\dots,m_{p-q+1}\in M\cap\sigma^{\perp}\,.$$

We define $\delta_{\tau/\sigma}$ by

$$\delta_{\tau/\sigma}(m_1\wedge m_2\wedge\cdots\wedge m_{p-q+1}):=\langle m_1,n_{\tau/\sigma}\rangle m_2\wedge\cdots\wedge m_{p-q+1}\,,$$

where $n_{\tau/\sigma}$ is a primitive element of N which is uniquely determined modulo $N\cap\boldsymbol{R}\tau$ so that $\sigma+(-\tau)=\boldsymbol{R}_{\geq 0}n_{\tau/\sigma}+\boldsymbol{R}\tau$. Then $(C^{\bullet}(\Delta, \Lambda^{p}), \delta)$ becomes a complex of $\boldsymbol{Z}$-modules for (N, Δ).

We denote by $H^{\bullet}(\Delta, \Lambda^{p})$ the cohomology group of $C^{\bullet}(\Delta, \Lambda^{p})$ and call it *Ishida's cohomology group* (cf. [12] and [19]).

REMARK. (1) Clearly, we have $H^{q}(\Delta, \Lambda^{p})=0$ unless $0\leq q\leq p$. If $p=q$, then

$$C^{p}(\Delta, \Lambda^{p})=\bigoplus_{\sigma\in\Delta(p)}\bigwedge{}^{0}(M\cap\sigma^{\perp})\cong\bigoplus_{\sigma\in\Delta(p)}\boldsymbol{Z}x(\sigma),$$

where $x(\sigma)$ is the element 1 at the factor corresponding to $\sigma\in\Delta(p)$. If we denote by $y(\sigma)$ the image of $x(\sigma)$ in $H^{p}(\Delta, \Lambda^{p})$, then we have a natural isomorphism

$$H^{p}(\Delta, \Lambda^{p})\otimes_{\boldsymbol{Z}}\boldsymbol{Q}=A^{p}(\Delta)$$

sending $(1/\mathrm{mult}(\sigma))y(\sigma)\in H^{p}(\Delta, \Lambda^{p})\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$ to $v(\sigma)\in A^{p}(\Delta)$.

(2) Oda gave a direct proof for a vanishing theorem for a simplicial and complete convex polyhedral cone decomposition, while Ishida generalized it to a simplicial one which may not be complete (cf. [22, Proposition 4.1 and Theorem 4.2]): Let Δ be a simplicial fan for N, which may not be complete. If there exist a complete simplicial fan $\tilde{\Delta}$ and $\rho\in\tilde{\Delta}(1)$ such that $\Delta=\tilde{\Delta}\setminus\{\sigma\in\tilde{\Delta}\mid\rho\prec\sigma\}$, then

$$H^{q}(\Delta, \Lambda^{p})\otimes_{\boldsymbol{Z}}\boldsymbol{Q}=\begin{cases}A^{p}(\Delta) & \text{if } q=p\\ 0 & \text{otherwise}.\end{cases}$$

(3) If Δ is assumed to be complete and simplicial, then we have a perfect pairing in the Chow ring for Δ (cf. [21] and [22, Proposition 4.1]):

$$A^{r-l}(\Delta)\times A^{l}(\Delta)\longrightarrow A^{r}(\Delta)\xrightarrow[\sim]{[\ ]}\boldsymbol{Q}\qquad\text{for any}\quad 0\leq l\leq r,$$

where $[\ ]: A^{r}(\Delta)\to\boldsymbol{Q}$ is equivalent to the push-forward homomorphism f_{*} induced by the structure morphism

$$f: T_{N}\,\mathrm{emb}(\Delta)\to\mathrm{Spec}(\boldsymbol{C}),$$

as we see in Corollary 2.8.

2. Homomorphisms between the Chow rings. In this section, we define the pull-back homomorphism and the push-forward homomorphism induced by a limited class of maps of fans and prove directly (i.e., without recourse to algebraic geometry) that they satisfy the projection formula.

DEFINITION. Let (N, Δ) and (N', Δ') be two fans for $N\cong\boldsymbol{Z}^{r}$ and $N'\cong\boldsymbol{Z}^{r'}$. A *map of fans* $\phi: (N', \Delta')\to(N, \Delta)$ is a $\boldsymbol{Z}$-linear homomorphism $\phi: N'\to N$ whose scalar extension $\phi: N'_{\boldsymbol{R}}\to N_{\boldsymbol{R}}$ satisfies the following property: For each $\sigma'\in\Delta'$ there exists $\sigma\in\Delta$ such that $\phi(\sigma')\subset\sigma$.

DEFINITION. An $\boldsymbol{R}$-valued function h on $|\Delta|$ is called a *Δ-linear support function* if h is $\boldsymbol{Z}$-valued on $N \cap |\Delta|$ and if h is linear on each cone $\sigma \in \Delta$. We denote by $\mathrm{SF}(N, \Delta)$ the additive group consisting of all Δ-linear support functions (cf. [19]).

Throughout this section, we assume that Δ and Δ' are two *simplicial* fans for $N \cong \boldsymbol{Z}^r$ and $N' \cong \boldsymbol{Z}^{r'}$. We also assume that $\phi: (N', \Delta') \to (N, \Delta)$ is a map of fans.

THEOREM 2.1. *Let Δ and Δ' be simplicial fans for N and N', respectively. Then a map of fans $\phi: (N', \Delta') \to (N, \Delta)$ gives rise to a pull-back homomorphism*

$$\phi^*: A(N, \Delta) \to A(N', \Delta')$$

which is a graded $\boldsymbol{Q}$-algebra homomorphism. Moreover, it is functorial, i.e., if Δ'' is a simplicial fan for N'' and $\psi: (N'', \Delta'') \to (N', \Delta')$ is another map of fans, then we get $\psi^* \circ \phi^* = (\phi \circ \psi)^*$.

PROOF. We use the same notation as that in the definition of the Chow ring $A(N, \Delta)$.

There exists an isomorphism from $\mathrm{SF}(N, \Delta) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ to $\bigoplus_{\rho \in \Delta(1)} \boldsymbol{Q} x(\rho)$ which is defined by

$$\mathrm{SF}(N, \Delta) \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \ni h \otimes q \mapsto \sum_{\rho \in \Delta(1)} q \cdot h(n(\rho)) x(\rho) \in \bigoplus_{\rho \in \Delta(1)} \boldsymbol{Q} x(\rho) .$$

Let us denote by $x'(\rho')$, S', I' and J' those appearing in the definition of the Chow ring $A(N', \Delta')$. Then, similarly, we have an isomorphism $\mathrm{SF}(N', \Delta') \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \xrightarrow{\sim} \bigoplus_{\rho' \in \Delta'(1)} \boldsymbol{Q} x'(\rho')$. Let us define

$$\tilde{\phi}^*: \mathrm{SF}(N, \Delta) \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \to \mathrm{SF}(N', \Delta') \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$$

by sending $h \in \mathrm{SF}(N, \Delta) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ to $\tilde{\phi}^*(h) := h \circ \phi$. Then it induces a homomorphism

$$\tilde{\phi}^*: \bigoplus_{\rho \in \Delta(1)} \boldsymbol{Q} x(\rho) \to \bigoplus_{\rho' \in \Delta'(1)} \boldsymbol{Q} x'(\rho') .$$

By extending it, we get a graded $\boldsymbol{Q}$-algebra homomorphism $\tilde{\phi}^*: S \to S'$.

More precisely, let us denote by ε_ρ the element in $\mathrm{SF}(N, \Delta) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ corresponding to $x(\rho) \in \bigoplus_{\rho \in \Delta(1)} \boldsymbol{Q} x(\rho)$. Hence $\varepsilon_\rho(n(\rho_1)) = \delta_{\rho, \rho_1}$ for $\rho, \rho_1 \in \Delta(1)$ under the above isomorphism, where δ_{ρ, ρ_1} is Kronecker's delta. $\tilde{\phi}^*(\varepsilon_\rho) = \varepsilon_\rho \circ \phi$ is then an element in $\mathrm{SF}(N', \Delta') \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$. For each $\rho' \in \Delta'(1)$, there exists a unique cone

$$\sigma_{\rho'} := \rho_1 + \cdots + \rho_s \in \Delta \qquad \text{for some} \quad \rho_1, \ldots, \rho_s \in \Delta(1) ,$$

which contains $\phi(n'(\rho'))$ in its relative interior, where $n'(\rho') \in N'$ is the unique primitive element contained in $\rho' \in \Delta'(1)$. Thus,

$$\begin{aligned} \phi(n'(\rho')) &= c(\rho', \rho_1) n(\rho_1) + \cdots + c(\rho', \rho_s) n(\rho_s) \\ &= \sum_{\rho \in \Delta(1), \rho \prec \sigma_{\rho'}} c(\rho', \rho) n(\rho) \end{aligned}$$

for some $c(\rho', \rho) > 0$. In this notation, we have

$$(\varepsilon_\rho \circ \phi)(n'(\rho')) = \varepsilon_\rho\left(\sum_{\rho_1 \in \Delta(1),\, \rho_1 \prec \sigma_{\rho'}} c(\rho', \rho_1) n(\rho_1)\right)$$
$$= \begin{cases} c(\rho', \rho) = c(\rho', \rho)\varepsilon'_{\rho'}(n'(\rho')) & \text{if } \rho \prec \sigma_{\rho'} \\ 0 & \text{otherwise}, \end{cases}$$

that is,

$$\varepsilon_\rho \circ \phi = \sum_{\rho' \in \Delta'(1),\, \rho \prec \sigma_{\rho'}} c(\rho', \rho)\varepsilon'_{\rho'},$$

where $\varepsilon'_{\rho'}$ is the element in $\mathrm{SF}(N', \Delta') \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ corresponding to $x'(\rho')$. Hence we get

$$\tilde{\phi}^*(x(\rho)) = \sum_{\rho' \in \Delta'(1),\, \rho \prec \sigma_{\rho'}} c(\rho', \rho) x'(\rho').$$

Now, we show $\phi^*(I) \subset I'$. For any generator $x(\rho_1)x(\rho_2)\cdots x(\rho_s)$ of I,

$$\tilde{\phi}^*(x(\rho_1)\cdots x(\rho_s)) = \tilde{\phi}^*(x(\rho_1))\tilde{\phi}^*(x(\rho_2))\cdots\tilde{\phi}^*(x(\rho_s))$$
$$= \left(\sum_{\rho' \in \Delta'(1),\, \rho_1 \prec \sigma_{\rho'}} c(\rho', \rho_1) x'(\rho')\right)\cdots\left(\sum_{\rho' \in \Delta'(1),\, \rho_s \prec \sigma_{\rho'}} c(\rho', \rho_s) x'(\rho')\right).$$

Suppose that for each ρ_i, $1 \le i \le s$, there exists $\rho'_i \in \Delta'(1)$ with $\rho_i \prec \sigma_{\rho'_i}$ such that $x'(\rho'_1)\cdots x'(\rho'_s) \notin I'$. Recall that a monomial $x'(\rho'_1)x'(\rho'_2)\cdots x'(\rho'_s)$ is an element of I' if and only if $\rho'_1 + \cdots + \rho'_s \notin \Delta'$. Thus we have $\rho'_1 + \cdots + \rho'_s \in \Delta'$. Since ϕ is a map of fans, there exists a cone $\sigma \in \Delta$ which contains $\phi(\rho'_1 + \cdots + \rho'_s)$. Since $\sigma_{\rho'_i}$ is the smallest cone in Δ containing $\phi(n'(\rho'_i))$, we have

$$\rho_i \prec \sigma_{\rho'_i} \prec \sigma, \qquad \text{for any} \quad 1 \le i \le s,$$

and as a result, $\rho_1 + \cdots + \rho_s \prec \sigma \in \Delta$, a contradiction. Consequently,

$$\tilde{\phi}^*(x(\rho_1)x(\rho_2)\cdots x(\rho_s)) \in I'$$

for any generator $x(\rho_1)x(\rho_2)\cdots x(\rho_s)$ of I.

It is clear that $\tilde{\phi}^*(J) \subset J'$. Therefore, ϕ induces a graded $\boldsymbol{Q}$-algebra homomorphism $\phi^*: A(N, \Delta) \to A(N', \Delta')$ and the functoriality is clear. q.e.d.

For a map of fans $\phi: (N', \Delta') \to (N, \Delta)$, let us denote $L(\sigma) := N \cap \boldsymbol{R}\sigma$ and $L'(\sigma') := N' \cap \boldsymbol{R}\sigma'$ for $\sigma \in \Delta$ and $\sigma' \in \Delta'$.

By simple calculation, we obtain the following (cf. (1) and (2) are also found in [7] and [15]):

LEMMA 2.2. *In the same notation as above, we have*:

(1) *$\sigma \subset N_{\boldsymbol{R}} \cong \boldsymbol{R}^r$ is a* nonsingular *cone, that is, there exist a $\boldsymbol{Z}$-basis $\{n_1, \ldots, n_r\}$ of N and $s \le r$ such that $\sigma = \boldsymbol{R}_{\ge 0} n_1 + \cdots + \boldsymbol{R}_{\ge 0} n_s$, if and only if* $\mathrm{mult}(\sigma, N) = 1$.

(2) *Let* $n_0 := a_1 n(\rho_1) + \cdots + a_s n(\rho_s)$ *with positive rational numbers* $a_1, \ldots, a_s \in \boldsymbol{Q}_{>0}$ *be a primitive element contained in* σ. *If we denote*

$$\sigma_i := \rho_0 + \rho_1 + \cdots + \overset{i}{\vee} + \cdots + \rho_s \qquad \text{for} \quad 1 \leq i \leq s,$$

where $\rho_0 := \boldsymbol{R}_{\geq 0} n_0$, *then* $\mathrm{mult}(\sigma_i, N) = a_i \cdot \mathrm{mult}(\sigma, N)$.

(3) *Let* $\phi: (N', \Delta') \to (N, \Delta)$ *be a map of fans with* finite cokernel. *For* $\tau' \in \Delta'(p-1)$, *let us denote by* $\sigma_{\tau'}$ *the smallest cone in* Δ *which contains* $\phi(\tau')$. *Suppose that*

$$\sigma_{\tau'} = \rho_1 + \cdots + \rho_t \in \Delta(t) \qquad \text{with} \quad t := (p-1) - (r' - r),$$

and that there exist $\rho' \in \Delta'(1)$ *and* $\rho \in \Delta(1)$ *which satisfy* $\rho' + \tau' \in \Delta(p)$ *and* $\sigma_{\rho' + \tau'} = \rho + \sigma_{\tau'} \in \Delta(t+1)$. *Then there exist positive integers* a, b *and nonnegative integers* $c_1, \ldots, c_t$ *such that*

$$a \cdot \phi(n(\rho')) = b \cdot n(\rho) + \sum_{j=1}^{t} c_j \cdot n(\rho_j).$$

In this case, we have

$$\begin{aligned} &\frac{\mathrm{mult}(\sigma_{\tau'}, N)}{\mathrm{mult}(\tau', N')} \cdot |\mathrm{coker}(N'/L'(\tau') \to N/L(\sigma_{\tau'}))| \\ &\quad = \frac{b}{a} \cdot \frac{\mathrm{mult}(\rho + \sigma_{\tau'}, N)}{\mathrm{mult}(\rho' + \tau', N')} \cdot |\mathrm{coker}(N'/L'(\rho' + \tau') \to N/L(\rho + \sigma_\tau))|, \end{aligned}$$

where $|G|$ *stands for the order of a finite group* G.

For $\tau \in \Delta(q)$, denote $\bar{N} := N/L(\tau)$. For $\sigma \in \Delta$ with $\tau \prec \sigma$, let $\bar{\sigma} := (\sigma + \boldsymbol{R}\tau)/\boldsymbol{R}\tau$ be the image of σ in the quotient vector space $\bar{N}_{\boldsymbol{R}} := N_{\boldsymbol{R}}/\boldsymbol{R}\tau$. Then

$$\bar{\Delta} := \{\bar{\sigma} \mid \sigma \in \Delta, \tau \prec \sigma\}$$

is a fan for $\bar{N}$ and the toric variety $T_{\bar{N}}\,\mathrm{emb}(\bar{\Delta})$ coincides with the closure $V(\tau)$ of $\mathrm{orb}(\tau)$ in $X := T_N\,\mathrm{emb}(\Delta)$ (cf. [19, Corollary 1.7]). Let us denote by $\bar{v}(\bar{\sigma})$ the generator of $A^p(\bar{\Delta})$ corresponding to $\bar{\sigma} \in \bar{\Delta}$. Then we can prove the following:

Proposition 2.3. *In the same notation as above, define a* $\boldsymbol{Q}$*-linear map*

$$\iota_*: A^p(\bar{\Delta}) = \sum_{\bar{\alpha} \in \bar{\Delta}(p)} \boldsymbol{Q}\bar{v}(\bar{\alpha}) \to A^{p+q}(\Delta) = \sum_{\alpha \in \Delta(p+q)} \boldsymbol{Q}v(\alpha)$$

for $0 \leq p \leq r - q$ *by*

$$\bar{v}(\bar{\alpha}) \mapsto (\mathrm{mult}(\alpha, N)/\mathrm{mult}(\bar{\alpha}, \bar{N}))v(\alpha).$$

Then ι_* *is well-defined.*

Let $\phi: (N', \Delta') \to (N, \Delta)$ be a map of fans. Then it is known that the equivariant holomorphic map $\phi_*: T_{N'}\,\mathrm{emb}(\Delta') \to T_N\,\mathrm{emb}(\Delta)$ is proper if and only if for each $\sigma \in \Delta$, the set $\Delta'_\sigma := \{\sigma' \in \Delta' \mid \phi(\sigma') \subset \sigma\}$ is finite and

$$\phi^{-1}(\sigma) = |\Delta'_\sigma| := \bigcup_{\sigma' \in \Delta'_\sigma} \sigma'$$

(cf. [19, Theorem 1.15]). We also say that a map ϕ of fans is *proper* if the corresponding equivariant holomorphic map ϕ_v is proper.

THEOREM 2.4. *Let Δ and Δ' be simplicial fans and ϕ: $(N', \Delta') \to (N, \Delta)$ a proper map of fans which has* finite cokernel. *Then ϕ gives rise to the push-forward **Q**-linear map*

$$\phi_*: A^p(N', \Delta') \to A^{p-(r'-r)}(N, \Delta)$$

for all p. Moreover, the push-forward homomorphism is functorial. Namely, let

$$(N'', \Delta'') \xrightarrow{\psi} (N', \Delta') \xrightarrow{\phi} (N, \Delta)$$

be maps of fans which are proper maps between simplicial fans having finite cokernel. Then the induced homomorphisms satisfy $\phi_* \circ \psi_* = (\phi \circ \psi)_*$.

PROOF. Let $\alpha' \in \Delta'(p)$. Denote by $\sigma_{\alpha'}$ the smallest cone in Δ which contains $\phi(\alpha')$. Define

$$\phi_*(v'(\alpha')) := \frac{\mathrm{mult}(\sigma_{\alpha'}, N)}{\mathrm{mult}(\alpha', N')} \cdot |\mathrm{coker}(N'/L'(\alpha') \to N/L(\sigma_{\alpha'}))| \cdot v(\sigma_{\alpha'})$$

if $\dim \sigma_{\alpha'} = p-(r'-r)$, while $\phi_*(v'(\alpha')) := 0$ otherwise.

$$\phi_*: A^p(N', \Delta') \to A^{p-(r'-r)}(N, \Delta)$$

is then the $\boldsymbol{Q}$-linear extension.

We can prove the functoriality easily because the smallest cone in Δ containing $\phi(\tau')$ is equal to the smallest cone in Δ containing $(\phi \circ \psi)(\tau'')$ for any $\tau'' \in \Delta''$, where τ' is the smallest cone in Δ' containing $\psi(\tau'')$.

It remains to show that this ϕ_* is well-defined. Let us fix a cone $\tau' \in \Delta'(p-1)$. If $\rho' \in \Delta'(1)$ satisfies $\rho' + \tau' \in \Delta'(p)$, then $\sigma_{\tau'} \prec \sigma_{\rho'+\tau'}$. If $\dim \sigma_{\rho'+\tau'} \neq p-(r'-r)$, then $\phi_*(v'(\rho'+\tau')) = 0$. Hence we may restrict ourselves to the case where $\dim \sigma_{\rho'+\tau'} = p-(r'-r)$. Hence, $\dim \sigma_{\tau'} \leq p-(r'-r)$. If $\dim \sigma_{\tau'} \leq (p-2)-(r'-r)$, then τ' cannot be a $(p-1)$-dimensional cone, a contradiction. Thus $\dim \sigma_{\tau'} = (p-1)-(r'-r)$ or $p-(r'-r)$. The rest of the theorem is a consequence of the following three lemmas. q.e.d.

For $\tau' \in \Delta'(p-1)$ and $\sigma_{\tau'}$ which is the smallest cone in Δ containing $\phi(\tau')$, let us denote

$$\begin{aligned}
&\bar{N}' := N'/L'(\tau') \\
&\bar{\Delta}' := \{\bar{\sigma}' := (\sigma' + \boldsymbol{R}\tau')/\boldsymbol{R}\tau' \subset \bar{N}'_{\boldsymbol{R}} \mid \sigma' \in \Delta', \tau' \prec \sigma'\} \\
&\bar{N} := N/L(\sigma_{\tau'}) \\
&\bar{\Delta} := \{\bar{\sigma} := (\sigma + \boldsymbol{R}\sigma_{\tau'})/\boldsymbol{R}\sigma_{\tau'} \subset \bar{N}_{\boldsymbol{R}} \mid \sigma \in \Delta, \sigma_{\tau'} \prec \sigma\},
\end{aligned}$$

where $\bar{N}'_{\boldsymbol{R}} := \bar{N}' \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ and $\bar{N}_{\boldsymbol{R}} := \bar{N} \otimes_{\boldsymbol{Z}} \boldsymbol{R}$. Then $V'(\tau') = T_{\bar{N}'} \operatorname{emb}(\bar{\Delta}')$ and $V(\sigma_{\tau'}) = T_{\bar{N}} \operatorname{emb}(\bar{\Delta})$ are toric varieties of dimensions $r'-(p-1)$ and $r - \dim \sigma_{\tau'}$, respectively. For any $\bar{\rho}' \in \bar{\Delta}'(1)$ (resp. $\bar{\rho} \in \bar{\Delta}(1)$), we denote by $\bar{v}'(\bar{\rho}')$ (resp. $\bar{v}(\bar{\rho})$) the generators of $A^1(\bar{N}', \bar{\Delta}')$ (resp. $A^1(\bar{N}, \bar{\Delta})$), by $\bar{n}'(\bar{\rho}')$ (resp. $\bar{n}(\bar{\rho})$) the primitive element of $\bar{N}'$ (resp. $\bar{N}$) contained in $\bar{\rho}'$ (resp. $\bar{\rho}$), and

$$\bar{L}'(\bar{\rho}') := \bar{N}' \cap \boldsymbol{R}\bar{\rho}' \text{ (resp. } \bar{L}(\bar{\rho}) := \bar{N} \cap \boldsymbol{R}\bar{\rho}) .$$

LEMMA 2.5. *If* $\dim \sigma_{\tau'} = (p-1)-(r'-r)$, *then* $\operatorname{rank} \bar{N}' = \operatorname{rank} \bar{N}$. *Hence the two toric varieties* $V'(\tau')$ *and* $V(\sigma_{\tau'})$ *have the same dimension, and the induced map*

$$\bar{\phi} : (\bar{N}', \bar{\Delta}') \to (\bar{N}, \bar{\Delta})$$

is also a proper map with finite cokernel. In this case, define

$$\phi_* : \bar{v}'(\bar{\rho}') \mapsto \begin{cases} |\operatorname{coker}(\bar{N}'/\bar{L}'(\bar{\rho}') \to \bar{N}/\bar{L}(\bar{\rho}))| \cdot \bar{v}(\bar{\rho}) & \textit{if } \ \bar{\phi}(\bar{\rho}') = \bar{\rho} \textit{ for some } \ \bar{\rho} \in \bar{\Delta}(1) \\ 0 & \textit{otherwise} \end{cases}$$

for $\bar{\rho}' \in \bar{\Delta}'(1)$. *Then we get a well-defined* $\boldsymbol{Q}$*-linear map*

$$\phi_* : A^1(\bar{N}', \bar{\Delta}') \to A^1(\bar{N}, \bar{\Delta}) .$$

PROOF. Note that if $\bar{\phi}(\bar{\rho}') = \bar{\rho}$ for some $\bar{\rho} \in \bar{\Delta}(1)$, then

$$|\operatorname{coker}(\bar{N}'/\bar{L}'(\bar{\rho}') \to \bar{N}/\bar{L}(\bar{\rho}))| = \frac{|\operatorname{coker}(M \cap (\sigma_{\tau'})^{\perp} \to M' \cap (\tau')^{\perp})|}{[\boldsymbol{Z}\bar{n}(\bar{\rho}) : \boldsymbol{Z}\bar{\phi}(\bar{n}'(\bar{\rho}'))]} .$$

If we denote by $\bar{\phi}^*$ the map dual to $\bar{\phi}$, then for any $m'_0 \in M' \cap (\tau')^{\perp}$ there exists an $m_0 \in M \cap (\sigma_{\tau'})^{\perp}$ such that

$$|\operatorname{coker}(M \cap (\sigma_{\tau'})^{\perp} \to M' \cap (\tau')^{\perp})| \cdot m'_0 = \bar{\phi}^*(m_0) .$$

Hence, for $m'_0 \in M' \cap (\tau')^{\perp}$, we have

$$\langle m'_0, \bar{n}'(\bar{\rho}') \rangle \cdot |\operatorname{coker}(\bar{N}'/\bar{L}'(\bar{\rho}')) \to \bar{N}/\bar{L}(\bar{\rho}))| = \langle m_0, \bar{n}(\bar{\rho}) \rangle .$$

Since $\bar{\phi}$ is proper, for any $\bar{\rho} \in \bar{\Delta}(1)$ there exists $\bar{\rho}' \in \Delta'(1)$ satisfying $\bar{\phi}(\bar{\rho}') = \bar{\rho}$. In particular, such a $\bar{\rho}'$ is unique because $\bar{\phi}$ has finite cokernel. Hence, the set $\bar{\Delta}(1)$ is in ono-to-one correspondence with the subset $\{\bar{\rho}' \in \Delta'(1) \mid \bar{\phi}(\bar{\rho}') \in \bar{\Delta}(1)\}$ of $\bar{\Delta}'(1)$.

Combining these, we have for any $m'_0 \in M' \cap (\tau')^{\perp}$,

$$\phi_* \left(\sum_{\bar{\rho}' \in \bar{\Delta}'(1)} \langle m'_0, \bar{n}'(\bar{\rho}') \rangle \cdot \bar{v}'(\bar{\rho}') \right) = \sum_{\bar{\rho} \in \bar{\Delta}(1)} \langle m_0, \bar{n}(\bar{\rho}) \rangle \cdot \bar{v}(\bar{\rho}) = 0 ,$$

since $m_0 \in M \cap (\sigma_{\tau'})^{\perp}$. q.e.d.

LEMMA 2.6. *If* $\dim \sigma_{\tau'} = (p-1)-(r'-r)$, *then the following diagram is commutative*:

$$\begin{array}{ccc} A^p(N', \Delta') & \xrightarrow{\phi_*} & A^{p-(r'-r)}(N, \Delta) \\ \iota_* \uparrow & & \uparrow \iota_* \\ A^1(\bar{N}', \bar{\Delta}') & \xrightarrow{\phi_*} & A^1(\bar{N}, \bar{\Delta}), \end{array}$$

where ι_* *is the map defined in Proposition* 2.3.

PROOF. It is a consequence of the following easy equality:

$$|\operatorname{coker}(\bar{N}'/\bar{L}'(\bar{\rho}') \to \bar{N}/\bar{L}(\bar{\rho}))| = |\operatorname{coker}(N'/L'(\rho' + \tau') \to N/L(\rho + \sigma_{\tau'}))|$$

for any $\bar{\rho}' \in \bar{\Delta}'(1)$ satisfying $\bar{\phi}(\bar{\rho}') = \bar{\rho} \in \bar{\Delta}(1)$. q.e.d.

LEMMA 2.7. *If* $\dim \sigma_{\tau'} = p - (r' - r)$, *then* $V'(\tau')$ (*resp.* $V(\sigma_{\tau'})$) *is a toric variety of dimension* $r' - (p-1)$ (*resp.* $r' - p$). *In this case, define*

$$\phi_*(\bar{v}'(\bar{\rho}')) := \begin{cases} |\operatorname{coker}(\bar{N}' \to \bar{N})| \cdot 1 & \text{if } \bar{\phi}(\bar{\rho}') = \bar{0} \\ 0 & \text{otherwise} \end{cases}$$

for any $\bar{\rho}' \in \bar{\Delta}'(1)$. *Then we get a well-defined* $\boldsymbol{Q}$*-linear map*

$$\phi_* : A^1(\bar{N}', \bar{\Delta}') \to A^0(\bar{N}, \bar{\Delta}).$$

Moreover, the following diagram is commutative:

$$\begin{array}{ccc} A^p(N', \Delta') & \xrightarrow{\phi_*} & A^{p-(r'-r)}(N, \Delta) \\ \iota_* \uparrow & & \uparrow \iota_* \\ A^1(\bar{N}', \bar{\Delta}') & \xrightarrow{\phi_*} & A^0(\bar{N}, \bar{\Delta}), \end{array}$$

where ι_* *is the map defined in Proposition* 2.3.

PROOF. Since $\bar{\phi} : (\bar{N}', \bar{\Delta}') \to (\bar{N}, \bar{\Delta})$ is a proper map with $\operatorname{rank}(\ker \bar{\phi}) = 1$, there exist exactly two $\bar{\rho}'_1$ and $\bar{\rho}'_2$ in $\bar{\Delta}'(1)$ which are mapped to $\bar{0}$ by $\bar{\phi}$. In fact, $\bar{\rho}'_1 = -\bar{\rho}'_2$.

Consequently, for any $m' \in M' \cap (\tau')^\perp$, we have

$$\begin{aligned} \phi_*\left(\sum_{\bar{\rho}' \in \bar{\Delta}'(1)} \langle m', \bar{n}'(\bar{\rho}')\rangle \cdot \bar{v}'(\bar{\rho}')\right) &= \{\langle m', \bar{n}'(\bar{\rho}'_1)\rangle + \langle m', \bar{n}'(\bar{\rho}'_2)\rangle\} \cdot |\operatorname{coker}(\bar{N}' \to \bar{N})| \\ &= 0. \end{aligned}$$

The commutativity of the diagram is similar to that in Lemma 2.6. q.e.d

As important special cases of Theorem 2.4, we have the following two corollaries:

COROLLARY 2.8. *Let* $X := T_N \operatorname{emb}(\Delta)$ *be a* compact *toric variety which has at most quotient singularities. Then the structure morphism* $f : X \to \operatorname{Spec}(\boldsymbol{C})$ *gives rise to the push-forward homomorphism* $[\ ] := f_* : A^r(\Delta) \to \boldsymbol{Q}$ *which is defined by*

$$A^r(\Delta) \ni v(\sigma) \mapsto [v(\sigma)] := 1/\mathrm{mult}(\sigma) \in \boldsymbol{Q}\,.$$

COROLLARY 2.9. *Let Δ' be a simplicial* subdivision *of a simplicial fan Δ. Then the inclusion $\iota\colon (N, \Delta') \to (N, \Delta)$ becomes a proper map and gives rise to the push-forward homomorphism*

$$\iota_*\colon A^p(\Delta') \to A^p(\Delta)$$

such that for $\alpha' \in \Delta'$,

$$\iota_*(v'(\alpha')) := \begin{cases} \dfrac{\mathrm{mult}(\sigma_{\alpha'}, N)}{\mathrm{mult}(\alpha', N)} \cdot v(\sigma_{\alpha'}) & \text{if} \quad \dim \sigma_{\alpha'} = \dim \alpha' \\ 0 & \text{otherwise}\,, \end{cases}$$

where $\sigma_{\alpha'}$ is the smallest cone in Δ containing α'.

THEOREM 2.10. *Let Δ and Δ' be simplicial fans and $\phi\colon (N', \Delta') \to (N, \Delta)$ a* proper *map of fans with* finite cokernel. *Then the induced homomorphisms ϕ^* and ϕ_* defined in the above theorems satisfy the* projection formula, *that is,*

$$\phi_*(\phi^*(\omega) \cdot \omega') = \omega \cdot \phi_*(\omega')$$

for any $\omega \in A(\Delta)$ and $\omega' \in A(\Delta')$.

PROOF. Since ϕ_* and ϕ^* are $\boldsymbol{Q}$-linear maps, it is enough to prove

$$(*) \qquad \phi_*(\phi^*(v(\sigma)) \cdot v'(\tau')) = v(\sigma) \cdot \phi_*(v'(\tau')) \qquad \text{for any} \quad \sigma \in \Delta \quad \text{and} \quad \tau' \in \Delta'\,.$$

We prove $(*)$ by induction on the dimension of $\sigma \in \Delta$.

If $\dim \sigma = 0$, then the formula $(*)$ is obviously true.

Assume that the equality holds for any $\sigma \in \Delta$ with $\dim \sigma \leq k-1$.

Let $\dim \sigma = k$. Then $\sigma = \rho + \tau$ for some $\rho \in \Delta(1)$ and $\tau \in \Delta(k-1)$. Let $\tau' \in \Delta'(p)$. We may assume that $\sigma \cap \sigma_{\tau'} = \{0\}$. Recall that $v(\sigma) = v(\rho)v(\tau)$. Hence we have

$$\begin{aligned} \phi_*(\phi^*(v(\sigma)) \cdot v'(\tau')) &= \phi_*(\phi^*(v(\tau)) \cdot \phi^*(v(\rho)) \cdot v'(\tau')) \\ &= v(\tau) \cdot \phi_*(\phi^*(v(\rho)) \cdot v'(\tau'))\,, \end{aligned}$$

by the induction hypothesis. If the formula $(*)$ holds for $\sigma = \rho \in \Delta(1)$, that is,

$$(**) \qquad \phi_*(\phi^*(v(\rho)) \cdot v'(\tau')) = v(\rho) \cdot \phi_*(v'(\tau')) \qquad \text{for any} \quad \rho \in \Delta(1) \quad \text{and} \quad \tau' \in \Delta'$$

holds, then

$$\begin{aligned} \phi_*(\phi^*(v(\sigma)) \cdot v'(\tau')) &= v(\tau) \cdot \phi_*(\phi^*(v(\rho)) \cdot v'(\tau')) \\ &= v(\tau) \cdot (v(\rho) \cdot \phi_*(v'(\tau'))) \\ &= v(\sigma) \cdot \phi_*(v'(\tau'))\,. \end{aligned}$$

Thus it is enough to prove the equality $(**)$.

Note that $\dim \sigma_{\tau'} \geq p - (r' - r)$. If $\dim \sigma_{\tau'} \geq (p+2) - (r' - r)$, then we have $\phi_*(v'(\tau')) = 0$

and for any $\rho' \in \Delta'(1)$, we have $\dim \sigma_{\rho'+\tau'} \geq (p+2)-(r'-r)$. Hence $\phi_*(v'(\rho')v'(\tau'))=0$, which implies that $\phi_*(\phi^*(v(\rho))\cdot v'(\tau'))=0$. Thus, the equality (**) holds.

If $\dim \sigma_{\tau'}=(p+1)-(r'-r)$, then $\phi_*(v'(\tau'))=0$. Now consider the commutative diagram in Lemma 2.7:

$$\begin{array}{ccc} A^{p+1}(\Delta') & \longrightarrow & A^{(p+1)-(r'-r)}(\Delta) \\ \uparrow & & \uparrow \\ A^1(\bar{\Delta}') & \longrightarrow & A^0(\bar{\Delta}) . \end{array}$$

Then we have for any $\bar{\rho} \in \bar{\Delta}(1)$

$$\phi_*(\phi^*(\bar{v}(\bar{\rho})))=\phi_*\left(\sum_{\bar{\rho}' \in \bar{\Delta}'(1),\ \bar{\rho} \prec \bar{\sigma}_{\bar{\rho}'}} c(\bar{\rho}', \bar{\rho})\bar{v}'(\bar{\rho}')\right),$$

where $\bar{\sigma}_{\bar{\rho}'}$ is the smallest cone in $\bar{\Delta}$ containing $\bar{\phi}(\bar{\rho}')$. For any $\bar{\rho}' \in \bar{\Delta}'(1)$ with $\bar{\rho} \prec \bar{\sigma}_{\bar{\rho}'}$, the image of $\bar{v}'(\bar{\rho}')$ under ϕ_* is not 0 only if $\dim \bar{\sigma}_{\bar{\rho}'}=0$. However, there is no cone $\bar{\rho}' \in \bar{\Delta}'(1)$ satisfying $\bar{\rho} \prec \bar{\sigma}_{\bar{\rho}'}$ and $\dim \bar{\sigma}_{\bar{\rho}'}=0$. Hence we have for $\rho \in \Delta(1)$

$$\phi_*(\phi^*(v(\rho))\cdot v'(\tau'))=0=v(\rho)\cdot\phi_*(v'(\tau')) .$$

If $\dim \sigma_{\tau'}=p-(r'-r)$, then we consider the diagram in Lemma 2.6:

$$\begin{array}{ccc} A^{p+1}(\Delta') & \longrightarrow & A^{(p+1)-(r'-r)}(\Delta) \\ \uparrow & & \uparrow \\ A^1(\bar{\Delta}') & \longrightarrow & A^1(\bar{\Delta}) . \end{array}$$

From the above diagram, we have for $\rho \in \Delta(1)$ with $\rho+\sigma_{\tau'} \in \Delta((p+1)-(r'-r))$,

$$\begin{aligned} &\phi_*(\phi^*(v(\rho))\cdot v'(\tau')) \\ &=c(\rho', \rho)\cdot\frac{\operatorname{mult}(\rho+\sigma_{\tau'}, N)}{\operatorname{mult}(\rho'+\tau', N')}\cdot|\operatorname{coker}(N'/L'(\rho'+\tau')\to N/L(\rho+\sigma_{\tau'}))|\cdot v(\rho+\sigma_{\tau'}) , \end{aligned}$$

where $\rho' \in \Delta'(1)$ satisfying $\rho'+\tau' \in \Delta'(p+1)$ and $\sigma_{\rho'+\tau'}=\rho+\sigma_{\tau'} \in \Delta((p+1)-(r'-r))$. Note that

$$v(\rho)\cdot\phi_*(v'(\tau'))=\frac{\operatorname{mult}(\sigma_{\tau'}, N)}{\operatorname{mult}(\tau', N')}\cdot|\operatorname{coker}(N'/L'(\tau')\to N/L(\sigma_{\tau'}))|\cdot v(\rho+\sigma_{\tau'}) .$$

Thus, the projection formula (**) holds in this case by Lemma 2.2, (3). q.e.d.

3. Fibrations and the Chow rings. In this section, we calculate the Chow rings of equivariant fiber bundles over toric varieties. Our calculation does not resort to algebraic geometry. We rather interpret, in terms of fans, the standard algebro-geometric proof found in Fulton [6]. We consider equivariant $\boldsymbol{C}^l$-bundles and $\boldsymbol{P}^l(\boldsymbol{C})$-bundles over toric varieties, and compare their Chow rings.

Throughout this section, we denote by Δ (resp. Δ') a *simplicial* fan for a lattice $N \cong \boldsymbol{Z}^r$(resp. $N' \cong \boldsymbol{Z}^{r'}$).

The key point in considering the Chow rings for equivariant fiber bundles is the following:

PROPOSITION 3.1. (cf. [19, Proposition 1.33]). *Consider a map of fans $\phi: (N', \Delta') \to (N, \Delta)$ and the corresponding equivariant holomorphic map*

$$\phi_v: X' := T_{N'}\,\mathrm{emb}(\Delta') \to X := T_N\,\mathrm{emb}(\Delta)$$

of toric varieties. Denote by N'' the kernel of the $\boldsymbol{Z}$-homomorphism $\phi: N' \to N$ and let Δ'' be a fan for N''. Then $\phi_v: X' \to X$ is an equivariant fiber bundle with $X'' := T_{N''}\,\mathrm{emb}(\Delta'')$ as typical fiber if and only if the following is satisfied: $\phi: N' \to N$ is surjective and there exists a subfan $\tilde{\Delta} \subset \Delta'$ such that

(i) *ϕ induces a homeomorphism $|\tilde{\Delta}| \xrightarrow{\sim} |\Delta|$*

(ii) *$\Delta = \{\tilde{\sigma} + \sigma'' \mid \tilde{\sigma} \in \tilde{\Delta}, \sigma'' \in \Delta''\}$ and*

(iii) *for each $\tilde{\sigma} \in \tilde{\Delta}$, ϕ induces a $\boldsymbol{Z}$-isomorphism $N' \cap \boldsymbol{R}\tilde{\sigma} \xrightarrow{\sim} N \cap \boldsymbol{R}\phi(\tilde{\sigma})$.*

REMARK. (1) We need to add (iii), which is missing in [19, Proposition 1.33].

(2) In particular, if $X' := T_{N'}\,\mathrm{emb}(\Delta')$ is a $\boldsymbol{P}^l(\boldsymbol{C})$-bundle over X, then there exist equivariant line bundles $L_0, \dots, L_l$ such that

$$X' := T_{N'}\,\mathrm{emb}(\Delta') = \boldsymbol{P}(L_0 \oplus \cdots \oplus L_l)\,.$$

Let h_i be the support function corresponding to the line bundle L_i, $0 \le i \le l$. Denote by $\{n_1'', \dots, n_l''\}$ a $\boldsymbol{Z}$-basis for N'', and let $n_0'' := -(n_1'' + \cdots + n_l'')$. For each $\sigma \in \Delta$, we denote by $\tilde{\sigma}$ the image of σ under the $\boldsymbol{R}$-linear map $N_{\boldsymbol{R}} \to N'_{\boldsymbol{R}}$

$$N_{\boldsymbol{R}} \ni x \mapsto \left(x, \sum_{0 \le i \le l} h_i(x) n_i''\right) \in N'_{\boldsymbol{R}}\,.$$

(cf. the minus sign in [19, p. 59] needs to be deleted.) Let $\tilde{\Delta} := \{\tilde{\sigma} \mid \sigma \in \Delta\}$, $\rho_i'' := \boldsymbol{R}_{\ge 0} n_i''$, and

$$\sigma_i'' := \rho_0'' + \cdots + \overset{i}{\vee} + \cdots + \rho_l''$$

for $0 \le i \le l$. If we denote $\Delta'' := \{\text{the faces of } \sigma_i'' \mid 0 \le i \le l\}$, then $T_{N''}\,\mathrm{emb}(\Delta'') = \boldsymbol{P}^l(\boldsymbol{C})$ and we have

$$\Delta' = \{\tilde{\sigma} + \sigma'' \mid \tilde{\sigma} \in \tilde{\Delta}, \sigma'' \in \Delta''\}\,.$$

Let $\{m_1'', \dots, m_l''\}$ be the dual basis for the dual $\boldsymbol{Z}$-module M'' of N''. By applying m_i'' to the defining relation $\sum_{\rho' \in \Delta'(1)} n'(\rho') \otimes v'(\rho') = 0$, we get

$$v'(\rho_0'') + \sum_{\tilde{\rho} \in \tilde{\Delta}(1)} h_0(n(\rho)) v'(\tilde{\rho}) = v'(\rho_i'') + \sum_{\tilde{\rho} \in \tilde{\Delta}(1)} h_i(n(\rho)) v'(\tilde{\rho}) \qquad \text{for all} \quad 1 \le i \le l\,.$$

For any $0 \le i \le l$, let us denote

$$\eta'_i := \sum_{\tilde{\rho} \in \tilde{\Delta}(1)} h_i(n(\rho)) v'(\tilde{\rho}) \in A^1(\Delta')$$

and

$$\xi := v'(\rho''_0) + \sum_{\tilde{\rho} \in \tilde{\Delta}(1)} h_0(n(\rho)) v'(\tilde{\rho}) \in A^1(\Delta') .$$

Then we have

$$\xi = v'(\rho''_0) + \eta'_0 = v'(\rho''_1) + \eta'_1 = \cdots = v'(\rho''_l) + \eta'_l .$$

We also denote

$$\eta_i := \sum_{\rho \in \Delta(1)} h_0(n(\rho)) v(\rho) \in A^1(\Delta) \qquad \text{for} \quad 0 \le i \le l .$$

(3) Similarly, we can express an equivariant $\boldsymbol{C}^l$-bundle $X'_0 := T_{N'} \operatorname{emb}(\Delta'_0)$ over a toric variety $X := T_N \operatorname{emb}(\Delta)$ as a direct sum of equivariant line bundles and obtain an expression for the fan Δ'_0 in terms of the cones in Δ and support functions corresponding to the line bundles. Indeed, let X'_0 be an equivariant $\boldsymbol{C}^l$-bundle over X. Then there exist equivariant line bundles $L_1, \ldots, L_l$ over X such that $X'_0 = L_1 \oplus \cdots \oplus L_l$. Let L_0 be an equivariant line bundle over X. Then $\boldsymbol{P}(L_0 \oplus \cdots \oplus L_l)$ is an equivariant $\boldsymbol{P}^l(\boldsymbol{C})$-bundle over X. Let us use the same notation as above. Then we have

$$\Delta'_0 = \{\tilde{\sigma} + \sigma'' \mid \tilde{\sigma} \in \tilde{\Delta},\ \sigma'' \in \Delta''_0\} ,$$

where we denote by Δ''_0 the collection of all the faces of $\sigma''_0 \in \Delta''$.

Let $\phi_v \colon T_{N'} \operatorname{emb}(\Delta') \to T_N \operatorname{emb}(\Delta)$ be an equivariant $\boldsymbol{P}^l(\boldsymbol{C})$-bundle. From now on, we use the same notation as above.

Note that the corresponding map $\phi \colon (N', \Delta') \to (N, \Delta)$ is proper and surjective. Hence we have the pull-back homomorphism ϕ^* and the push-forward homomorphism ϕ_* induced by ϕ. By construction, we easily see that

$$\phi^*(v(\sigma)) = v'(\tilde{\sigma}) \qquad \text{for any} \quad \sigma \in \Delta .$$

Note that $\phi(\tilde{\sigma} + \sigma'') = \sigma$ and $\dim(\tilde{\sigma} + \sigma'') = \dim \sigma + \dim \sigma''$ for any $\tilde{\sigma} \in \tilde{\Delta}$, $\sigma'' \in \Delta''$. Hence $\phi_*(v'(\tilde{\sigma} + \sigma'')) = 0$ whenever $\dim \sigma'' \ne r' - r = l$. On the other hand,

$$\operatorname{mult}(\sigma, N) = \operatorname{mult}(\tilde{\sigma} + \sigma''_i, N') \qquad \text{for any} \quad 0 \le i \le l ,$$

since ϕ induces a homeomorphism $|\tilde{\Delta}| \xrightarrow{\sim} |\Delta|$ and a $\boldsymbol{Z}$-isomorphism $N' \cap \boldsymbol{R}\tilde{\sigma} \xrightarrow{\sim} N \cap \boldsymbol{R}\sigma$. Thus, it follows that

$$\phi_*(v'(\tilde{\sigma} + \sigma'')) = \begin{cases} v(\sigma) & \text{if} \quad \sigma'' = \sigma''_i \quad \text{for some} \quad 0 \le i \le l \\ 0 & \text{otherwise} \end{cases}$$

for any $\tilde{\sigma} \in \tilde{\Delta}$, $\sigma'' \in \Delta''$.

Now following Fulton [6], let us define the *Segre class operation*

$$S_i(\Delta'): A^p(\Delta) \to A^{p+i}(\Delta)$$

in the same notation as above, to be the homomorphism which sends $\alpha \in A^p(\Delta)$ to

$$S_i(\Delta') \cdot \alpha := \phi_*(\xi^{l+i} \cdot \phi^*(\alpha)) \in A^{p+i}(\Delta)$$

for any i, where $\xi := v'(\rho''_0) + \sum_{\tilde{\rho} \in \tilde{\Delta}(1)} h_0(n(\rho)) v'(\tilde{\rho}) \in A^1(\Delta')$.

By the projection formula, we can easily prove the following:

LEMMA 3.2 (cf. [6, Proposition 3.1, (a)]). *For any* $\alpha \in A^p(\Delta)$,

(1) $S_i(\Delta') \cdot \alpha = 0$ *for* $i < 0$;

(2) $S_0(\Delta') \cdot \alpha = \alpha$.

THEOREM 3.3. *Let* $\phi_v: T_{N'} \operatorname{emb}(\Delta') \to T_N \operatorname{emb}(\Delta)$ *be an equivariant* $\boldsymbol{P}^l(\boldsymbol{C})$*-bundle. Then we have the following*:

(1) *The induced pull-back homomorphism*

$$0 \longrightarrow A^p(\Delta) \xrightarrow{\phi^*} A^p(\Delta')$$

is a split monomorphism for all p.

(2) *The induced push-forward homomorphism*

$$A^p(\Delta') \xrightarrow{\phi_*} A^{p-l}(\Delta) \longrightarrow 0$$

is a split epimorphism for all p with $\phi_* \circ \phi^* = 0$.

(3) *In the same notation as the one in the remark above, we have a canonical isomorphism*

$A(\Delta') \cong A(\Delta)[\xi]$ *with the defining relation* $(\xi - \eta_0)(\xi - \eta_1) \cdots (\xi - \eta_l) = 0$.

PROOF. (1) By construction, we have

$$\phi^*(v(\sigma)) = v'(\tilde{\sigma}) \qquad \text{for any} \quad \sigma \in \Delta .$$

Define a homomorphism $\zeta: A^p(\Delta') \to A^p(\Delta)$ by sending $v'(\sigma')$ for any $\sigma' \in \Delta'(p)$ to

$$\zeta(v'(\sigma')) := \phi_*(v'(\sigma') \cdot v'(\sigma''_0)) .$$

Then ζ is a well-defined homomorphism and from the projection formula (cf. Theorem 2.10), we also see that $\zeta \circ \phi^* = \mathrm{id}_{A^p(\Delta)}$ and ϕ^* is injective.

(2) Since

$$\phi_*(v'(\tilde{\sigma} + \sigma'')) = \begin{cases} v(\sigma) & \text{if } \sigma'' = \sigma''_i \text{ for some } 0 \le i \le l \\ 0 & \text{otherwise} \end{cases}$$

for any $\tilde{\sigma} \in \tilde{\Delta}$, $\sigma'' \in \Delta''$, ϕ_* must be surjective and $\phi_* \circ \phi^* = 0$.

Define a homomorphism $\gamma: A^{p-l}(\Delta) \to A^p(\Delta')$ by sending $v(\sigma)$ for any $\sigma \in \Delta(p-l)$ to

$$\gamma(v(\sigma)) := \phi^*(v(\sigma)) \cdot v'(\sigma_0'') = v'(\tilde{\sigma} + \sigma_0'') .$$

Then γ is a well-defined homomorphism and from the projection formula, we also see that $\phi_* \circ \gamma = \mathrm{id}_{A^{p-l}(\Delta)}$.

(3) Let $q := \min\{p, l\}$. Define a homomorphism

$$\theta_{\Delta'}: \bigoplus_{i=0}^{q} A^{p-i}(\Delta) \to A^p(\Delta')$$

by sending $\bigoplus \alpha_i$, with $\alpha_i \in A^{p-i}(\Delta)$, to

$$\theta_{\Delta'}(\bigoplus \alpha_i) := \sum_{i=0}^{q} \xi^i \cdot \phi^*(\alpha_i) .$$

For any $\sigma' \in \Delta'$, we let $\sigma' = \tilde{\sigma} + \sigma''$ for some $\tilde{\sigma} \in \tilde{\Delta}$ and $\sigma'' \in \Delta''$. Since $v'(\rho_i'') = \xi - \eta_i' = \xi - \phi^*(\eta_i)$ for any $1 \le i \le l$ and $v'(\tilde{\sigma}) = \phi^*(\sigma)$, we have

$$v'(\sigma') = v'(\tilde{\sigma}) \cdot v'(\sigma'') = \phi^*(\sigma) \cdot v'(\sigma'') = \sum_{i=0}^{q} \xi^i \cdot \phi^*(\alpha_i)$$

for some $\alpha_i \in A^{p-i}(\Delta)$. Hence $\theta_{\Delta'}$ is surjective.

To show that $\theta_{\Delta'}$ is injective, let us assume that there exists a nontrivial relation $\beta = \sum_{i=0}^{q} \xi^i \cdot \phi^*(\alpha_i) = 0$. Let k be the largest index such that $\alpha_k \neq 0$. Then by Lemma 3.2, we get

$$0 = \phi_*(\xi^{l-k} \cdot \beta) = \sum_{i=0}^{q} \phi_*(\xi^{l-k+i} \cdot \phi^*(\alpha_i)) = \alpha_k ,$$

which is a contradiction.

Consequently, we have isomorphisms

$$\begin{aligned} A^j(\Delta') &\cong A^j(\Delta) \oplus (A^{j-1}(\Delta) \cdot \xi) \oplus \cdots \oplus (A^0(\Delta) \cdot \xi^j) && \text{for} \quad 0 \le j \le l \\ A^k(\Delta') &\cong A^k(\Delta) \oplus (A^{k-1}(\Delta) \cdot \xi) \oplus \cdots \oplus (A^{k-l}(\Delta) \cdot \xi^l) && \text{for} \quad l < k \le r' = r + l . \end{aligned}$$

Furthermore, since $\rho_0'' + \rho_1'' + \cdots + \rho_l'' \notin \Delta'$, we have $v'(\rho_0'')v'(\rho_1'') \cdots v'(\rho_l'') = 0$, that is

$$(\xi - \eta_0')(\xi - \eta_1') \cdots (\xi - \eta_l') = 0 \qquad \text{in} \quad A(\Delta') .$$

Thus we obtained the desired isomorphism

$$A(\Delta') \cong A(\Delta)[\xi]/((\xi - \eta_0)(\xi - \eta_1) \cdots (\xi - \eta_l)) .$$

q.e.d.

Let us recall Ishida's cohomology. We have seen that $A^p(\Delta) \cong H^p(\Delta, \Lambda^p) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ in the remark at the end of Section 1. Even though we define the Chow ring only for simplicial fans, Ishida's cohomology can be defined for a *locally star closed* subset Δ'

of a fan Δ, that is, a subcollection of Δ which is required to contain σ whenever $\tau \prec \sigma \prec \pi$ for $\tau, \pi \in \Delta'$. In this case, $C^{\cdot}(\Delta', \Lambda^p)$ with a coboundary map δ defined in Section 1 becomes a complex (cf. [12] and [13]). From this fact, Ishida's cohomology is a useful tool in considering the Chow rings for simplicial fans.

Furthermore, if Δ_0 is a subfan of a fan Δ, then we get a cohomology long exact sequence

$$\cdots \to H^{p-1}(\Delta_0, \Lambda^p) \to H^p(\Delta \setminus \Delta_0, \Lambda^p) \to H^p(\Delta, \Lambda^p) \to H^p(\Delta_0, \Lambda^p) \to 0 .$$

From this fact and the remark at the end of Section 1, we have the following:

PROPOSITION 3.4. *Let Δ be a simplicial fan for N and Δ_0 a subfan of Δ. We denote the corresponding inclusions by $i\colon (N, \Delta \setminus \Delta_0) \to (N, \Delta)$ and $j\colon (N, \Delta_0) \to (N, \Delta)$, respectively. Then we have an exact sequence*

$$H^p(\Delta \setminus \Delta_0, \Lambda^p) \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \xrightarrow{i_\sharp} A^p(\Delta) \xrightarrow{j^*} A^p(\Delta_0) \longrightarrow 0 ,$$

where $i_\sharp$ is the induced homomorphism between Ishida's cohomology groups and j^* is the induced pull-back homomorphism. Namely, $\ker(j^*)$ *is the set of linear combinations of $v(\sigma)$ with $\sigma \in \Delta \setminus \Delta_0$.*

As we saw in the remark after Proposition 3.1, for any equivariant $\boldsymbol{C}^l$-bundle $X'_0 := T_{N'}\,\mathrm{emb}(\Delta'_0)$, there exist equivariant line bundles $L_1, \dots, L_l$ over $X := T_N\,\mathrm{emb}(\Delta)$ such that $X'_0 = L_1 \oplus \cdots \oplus L_l$. Let L_0 be an equivariant line bundle over X. Then $\boldsymbol{P}(L_0 \oplus \cdots \oplus L_l)$ is an equivariant $\boldsymbol{P}^l(\boldsymbol{C})$-bundle over X. Let us use the same notation as the one in Theorem 3.3. Then we have

$$\Delta'_0 = \{\tilde{\sigma} + \sigma'' \mid \tilde{\sigma} \in \tilde{\Delta}, \sigma'' \in \Delta''_0\} ,$$

where we denote by Δ''_0 the collection of all the faces of $\sigma''_0 \in \Delta''$. Let us denote

$$\Delta' \setminus \Delta'_0 = \mathrm{Star}_{\rho''_0}(\Delta') := \{\sigma \in \Delta' \mid \rho''_0 \prec \sigma\} .$$

Then there exists a cohomology long exact sequence

$$\cdots \to H^{p-1}(\Delta'_0, \Lambda^p) \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \to H^p(\mathrm{Star}_{\rho''_0}(\Delta'), \Lambda^p) \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \to A^p(\Delta') \to A^p(\Delta'_0) \to 0$$

as we have seen in Proposition 3.4.

THEOREM 3.5. *If $\phi_v\colon T_{N'}\,\mathrm{emb}(\Delta'_0) \to T_N\,\mathrm{emb}(\Delta)$ is an equivariant $\boldsymbol{C}^l$-bundle, then the induced full-back homomorphism $\phi^*\colon A(N, \Delta) \to A(N', \Delta'_0)$ is an isomorphism.*

PROOF. In the same notation as above, we have an exact sequence

$$H^p(\mathrm{Star}_{\rho''_0}(\Delta'), \Lambda^p) \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \xrightarrow{i_\sharp} A^p(\Delta') \xrightarrow{j^*} A^p(\Delta'_0) \longrightarrow 0 .$$

Hence

$$A^p(\Delta'_0) \cong A^p(\Delta')/\ker(j^*) .$$

Notice that

$$A^p(\Delta') = \phi^*(A^p(\Delta)) \oplus (v'(\rho''_0) \cdot A^{p-1}(\Delta')) .$$

From the above exact sequence, we see that

$$v'(\rho''_0) \cdot A^{p-1}(\Delta') \subset \ker(j^*) = \operatorname{im}(i_\sharp) \subset v'(\rho''_0) \cdot A^{p-1}(\Delta') .$$

Thus, $\ker(j^*) = v'(\rho''_0) \cdot A^{p-1}(\Delta')$. This implies that

$$\begin{aligned} A^p(\Delta'_0) &\cong \{\phi^*(A^p(\Delta)) \oplus (v'(\rho''_0) \cdot A^{p-1}(\Delta'))\}/\{v'(\rho''_0) \cdot A^{p-1}(\Delta')\} \\ &= \phi^*(A^p(\Delta)) \cong A^p(\Delta) , \end{aligned}$$

since ϕ^* is injective. q.e.d.

REMARK. We can apply these facts to equivariant $\boldsymbol{P}^1$- (resp. $\boldsymbol{C}^1$-, resp. $\boldsymbol{C}^\times$-) bundles over a toric variety. We obtain more special facts, and relate them to the strong Lefschetz theorem (cf. [21]). Indeed, let us fix a simplicial fan Δ for $N \cong \boldsymbol{Z}^r$ and a support function η for Δ with $\eta(n(\rho)) > 0$ for any $\rho \in \Delta(1)$. Then

$$\bar{\eta} := \sum_{\rho \in \Delta(1)} \eta(n(\rho)) v(\rho)$$

is an element in $A^1(\Delta)$, and we have a map

$$\bar{\eta}: A^{p-1}(\Delta) \to A^p(\Delta) \qquad \text{for} \quad p = 1, \dots, r ,$$

sending $v(\tau)$, $\tau \in \Delta(p-1)$, to

$$\bar{\eta} \cdot v(\tau) = \sum_{\substack{\rho \in \Delta(1) \\ \rho + \tau \in \Delta(p)}} (\eta(n(\rho)) - \langle z_\tau, n(\rho) \rangle) v(\rho + \tau) .$$

For each $\sigma \in \Delta$, we define

$$\tilde{\sigma} := \{(x, \eta(x)) \mid x \in \sigma\} \subset N_{\boldsymbol{R}} \times \boldsymbol{R} .$$

Since η is linear on σ, we see that $\tilde{\sigma}$ is a strongly convex rational polyhedral cone with $\dim \tilde{\sigma} = \dim \sigma$. Furthermore, we fix

$$\rho_0 := \boldsymbol{R}_{\geq 0}(0, 1) \subset N_{\boldsymbol{R}} \times \boldsymbol{R}$$

and define the following fans for $N' := N \times \boldsymbol{Z}$ associated with η:

$$\begin{aligned} \Phi^\flat &:= \{\tilde{\sigma} \mid \sigma \in \Delta\} \\ \Phi &:= \Phi^\flat \coprod \{\tilde{\sigma} + \rho_0 \mid \tilde{\sigma} \in \Phi^\flat\} \\ \tilde{\Phi} &:= \Phi \coprod \{\tilde{\sigma} + (-\rho_0) \mid \tilde{\sigma} \in \Phi^\flat\} , \end{aligned}$$

where $-\rho_0 := \boldsymbol{R}_{\geq 0}(0, -1) \subset N_{\boldsymbol{R}} \times \boldsymbol{R}$. Then $T_{N'}\operatorname{emb}(\tilde{\Phi})$ (resp. $T_{N'}\operatorname{emb}(\Phi)$, resp.

$T_{N'}\,\mathrm{emb}(\Phi^b))$ is an equivariant $\boldsymbol{P}^1(\boldsymbol{C})$- (resp. $\boldsymbol{C}$-, resp. $\boldsymbol{C}^\times$-) bundle over $T_N\,\mathrm{emb}(\Delta)$. We have the following:

(1) The projection pr gives rise to a canonical isomorphism

$$\mathrm{pr}^*: A^p(\Delta) \xrightarrow{\sim} A^p(\Phi)$$

for any $p=0, 1, \ldots, r$.

(2) As we have seen in Theorem 3.3,

$$A^p(\tilde{\Phi}) \cong A^p(\Delta) \oplus (\xi \cdot A^{p-1}(\Delta)) \qquad \text{for any} \quad 1 \le p \le l\,.$$

where $\xi \in A^1(\tilde{\Phi})$ with the defining relation $\xi(\xi - \bar{\eta}) = 0$.

(3) For any $p=1, \ldots, r$, we have an exact sequence

$$A^{p-1}(\Delta) \xrightarrow{\bar{\eta}} A^p(\Delta) \xrightarrow{\mathrm{pr}^*} A^p(\Phi^b) \longrightarrow 0\,,$$

where the first map is the multiplication by $\bar{\eta} \in A^1(\Delta)$. Hence, we have

$$A^p(\Phi^b) \cong A^p(\Delta)/\bar{\eta}A^{p-1}(\Delta)\,.$$

If Δ is simplicial and complete, then the above (3) is closely related to the strong Lefschetz theorem. Oda showed that $\bar{\eta}: A^{p-1}(\Delta) \to A^p(\Delta)$ is surjective for $r/2 < p$ if and only if (1′) $H^p(\Phi^b, \Lambda^p) = 0$ for $r/2 < p$ (cf. [21, Corollary 4.5]).

4. The GKZ-decomposition. We have defined the linear Gale transform in the context of $\boldsymbol{R}$-vector spaces and stated some properties of it in [24]. In this second part of the paper, we modify the definition in the context of $\boldsymbol{Q}$-vector spaces and apply it to $\boldsymbol{Q}$-factorial toric varieties.

Let Ξ be a finite subset of *primitive* elements in N, such that Ξ spans $N_{\boldsymbol{Q}} := N \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ over the field $\boldsymbol{Q}$ of rational numbers. Let Z be the $\boldsymbol{Q}$-vector space with a basis $\{e_\xi \mid \xi \in \Xi\}$, which is in one-to-one correspondence with Ξ. By sending e_ξ to $\xi \in \Xi$, we get a surjective linear map $Z \to N_{\boldsymbol{Q}}$. Let $Z^* := \mathrm{Hom}_{\boldsymbol{Q}}(Z, \boldsymbol{Q})$ be the dual space with the dual basis $\{e_\xi^* \mid \xi \in \Xi\}$. Then we have the dual injective linear map $M_{\boldsymbol{Q}} := M \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \to Z^*$ which sends $m \in M_{\boldsymbol{Q}}$ to $\sum_{\xi \in \Xi} \langle m, \xi \rangle e_\xi^*$. The cokernel $G^{\boldsymbol{Q}} := Z^*/M_{\boldsymbol{Q}}$ of the injective map is a $\boldsymbol{Q}$-vector space of dimension $\sharp\Xi - r$, where $\sharp\Xi$ is the cardinality of Ξ. For each $\xi \in \Xi$, we denote by $g(\xi) \in G^{\boldsymbol{Q}}$ the image of $e_\xi^* \in Z^*$. Then by definition, the defining relations among the elements in $g(\Xi) := \{g(\xi) \mid \xi \in \Xi\}$ are

$$\sum_{\xi \in \Xi} \langle m, \xi \rangle g(\xi) = 0 \qquad \text{for all} \quad m \in M_{\boldsymbol{Q}}\,.$$

More symmetrically, they can be written as a single equality

$$\sum_{\xi \in \Xi} \xi \otimes g(\xi) = 0 \qquad \text{in} \quad N_{\boldsymbol{Q}} \otimes_{\boldsymbol{Q}} G^{\boldsymbol{Q}}\,,$$

which we call the *defining relation*. We call the pair $(G^{\boldsymbol{Q}}, g(\Xi))$ the ***Q**-linear Gale transform* of $(N_{\boldsymbol{Q}}, \Xi)$.

We regard $G^{\boldsymbol{Q}}$ as a subset of its scalar extension $G := G^{\boldsymbol{Q}} \otimes_{\boldsymbol{Q}} \boldsymbol{R}$. Hence $(G, g(\Xi))$ is the linear Gale transform of $(N_{\boldsymbol{R}}, g(\Xi))$ in the sense of [24]. We define a cone $G_{\geq 0}$ in G by

$$G_{\geq 0} := \sum_{\xi \in \Xi} \boldsymbol{R}_{\geq 0} g(\xi) .$$

If Ξ positively spans $N_{\boldsymbol{R}}$ over $\boldsymbol{R}$, that is, $N_{\boldsymbol{R}} = \sum_{\xi \in \Xi} \boldsymbol{R}_{\geq 0} \xi$, then we easily see that $G_{\geq 0}$ becomes a strongly convex cone (cf. [24, Proposition 1.3]).

EXAMPLE. Let Δ be a *complete* and simplicial fan with $\{n(\rho) \mid \rho \in \Delta(1)\} = \Xi$ and $X := T_N \operatorname{emb}(\Delta)$ the corresponding compact toric variety. We use the same notation as that in the definition of the Chow ring $A(N, \Delta)$. If we denote by $T_N \operatorname{Div}(X)_{\boldsymbol{Q}}$ the scalar extension to $\boldsymbol{Q}$ of the group of T_N-invariant divisors and by $V(\rho)$ the closure of the codimension-one T_N-orbit $\operatorname{orb}(\rho)$ corresponding to each cone $\rho \in \Delta(1)$, then from [19, Proposition 2.1 and Corollary 2.5] we have

$$T_N \operatorname{Div}(X)_{\boldsymbol{Q}} = \bigoplus_{\rho \in \Delta(1)} \boldsymbol{Q} V(\rho) \quad \text{and} \quad \operatorname{Pic}(X)_{\boldsymbol{Q}} = A^1(\Delta) .$$

Note that we have a natural isomorphism

$$T_N \operatorname{Div}(X)_{\boldsymbol{Q}} \cong S^1 := \bigoplus_{\rho \in \Delta(1)} \boldsymbol{Q} x(\rho) .$$

As we have seen in Proposition 1.1, $A^1(\Delta)$ is generated over $\boldsymbol{Q}$ by the set $\{v(\rho) \mid \rho \in \Delta(1)\}$. By the definition of $A^1(\Delta)$ and the natural isomorphism $T_N \operatorname{Div}(X)_{\boldsymbol{Q}} \cong S^1$ above, we see that $v(\rho)$ is the linear equivalence class of the T_N-invariant divisor $V(\rho)$.

On the other hand, since Δ is assumed to be complete and simplicial, we have a perfect pairing in the Chow ring for Δ (cf. Corollary 2.8 and [22, Proposition 4.1])

$$A^{r-1}(\Delta) \times A^1(\Delta) \longrightarrow A^r(\Delta) \xrightarrow[\sim]{[\ ]} \boldsymbol{Q} ,$$

which enables us to identify $A^{r-1}(\Delta)$ with the dual space of $A^1(\Delta)$. Hence, we have mutually dual short exact sequences of $\boldsymbol{Q}$-vector spaces

$$\begin{array}{l} 0 \longleftarrow N_{\boldsymbol{Q}} \longleftarrow (T_N \operatorname{Div}(X)_{\boldsymbol{Q}})^* \longleftarrow A^{r-1}(\Delta) \longleftarrow 0 \\ 0 \longrightarrow M_{\boldsymbol{Q}} \longrightarrow \quad T_N \operatorname{Div}(X)_{\boldsymbol{Q}} \quad \longrightarrow \quad A^1(\Delta) \quad \longrightarrow 0 , \end{array}$$

where $(T_N \operatorname{Div}(X)_{\boldsymbol{Q}})^*$ denotes the dual space of $T_N \operatorname{Div}(X)_{\boldsymbol{Q}}$ (cf. Example in Section 5). Thus by the definition of the $\boldsymbol{Q}$-linear Gale transform, the pair

$$(A^1(\Delta), \{v(\rho) \mid \rho \in \Delta(1)\})$$

becomes the $\boldsymbol{Q}$-linear Gale transform of $(N_{\boldsymbol{Q}}, \{n(\rho) \mid \rho \in \Delta(1)\})$, and the defining relation for the $\boldsymbol{Q}$-linear Gale transform coincides with the defining relation for the Chow ring:

$$\sum_{\rho \in \Delta(1)} n(\rho) \otimes v(\rho) = 0 \qquad \text{in} \quad N_{\boldsymbol{Q}} \otimes_{\boldsymbol{Q}} A^1(\Delta)\,.$$

If a subset Ξ of N is given, then it is known that there exists a convex polyhedral cone decomposition of G, called the GKZ-*decomposition* (cf. [24]), with support $G_{\geq 0}$. To describe it, let us introduce necessary concepts.

For the time being, we assume that Δ is a simplicial fan for N such that the support $|\Delta|$ is convex and spans $N_{\boldsymbol{R}}$ over $\boldsymbol{R}$. (Note that Δ may not be complete.)

Let us denote $\mathrm{PL}(\Delta) := \mathrm{SF}(N, \Delta) \otimes_{\boldsymbol{Z}} \boldsymbol{R}$. A function η in $\mathrm{PL}(\Delta)$ is said to be *convex* if

$$\eta(w+w') \leq \eta(w) + \eta(w') \qquad \text{for all} \quad w, w' \in |\Delta|\,.$$

A function $\eta \in \mathrm{PL}(\Delta)$ is said to be *strictly convex with respect to* Δ if there exists an $m_\sigma \in M_{\boldsymbol{R}}$ for each $\sigma \in \Delta$ such that

$$\begin{aligned} \eta(w) &= \langle m_\sigma, w \rangle \qquad && \text{if} \quad w \in \sigma \\ \eta(w) &> \langle m_\sigma, w \rangle \qquad && \text{otherwise}\,. \end{aligned}$$

A fan Δ is said to be *quasi-projective* if there exists an $\eta \in \mathrm{PL}(\Delta)$ which is strictly convex with respect to Δ. If a fan Δ is complete and quasi-projective, then Δ is said to be *projective*.

We denote by $\mathrm{CPL}(\Delta)$ the cone consisting of all convex functions in $\mathrm{PL}(\Delta)$.

By using the toric Kleiman-Nakai criterion (cf. [24, Theorem 2.3]), we see that a fan Δ is quasi-projective if and only if $\mathrm{CPL}(\Delta)$ spans $\mathrm{PL}(\Delta)$ over $\boldsymbol{R}$.

From now on, we fix a finite subset Ξ of primitive elements in N such that Ξ spans $N_{\boldsymbol{R}}$ over $\boldsymbol{R}$.

DEFINITION. A fan Δ for N is said to be *admissible* for (N, Ξ) if

(i) Δ is quasi-projective,

(ii) $|\Delta| = \sum_{\xi \in \Xi} \boldsymbol{R}_{\geq 0}\xi$ and

(iii) $\Delta(1) \subset \{\boldsymbol{R}_{\geq 0}\xi \mid \xi \in \Xi\}$.

We denote by $\Xi(\Delta)$ the subset consisting of those elements in Ξ which are of the form $n(\rho)$ for some $\rho \in \Delta(1)$. Note that $\Xi(\Delta) \neq \Xi$ may happen. For any given Ξ, however, there always exists a simplicial fan Δ such that Δ is admissible for (N, Ξ) with $\Xi(\Delta) = \Xi$, as we now prove by using the concept of *pulling* (cf. [11]).

DEFINITION. Let P be a convex polytope in $\boldsymbol{R}^r$ with the vertex set $\mathrm{ver}(P) = \Xi$. For $\xi \in \Xi$ and $c > 1$, the convex hull $P_* := \mathrm{conv}((\mathrm{ver}(P) \setminus \{\xi\}) \cup \{c\xi\})$ is said to be *obtained from P by pulling ξ to $c\xi$* if $(\xi, c\xi] \cap H = \varnothing$ for the hyperplane H determined by any facet of P, where $(\xi, c\xi] := \{a\xi \mid 1 < a \leq c\}$.

Eggleston, Grübaum and Klee [5] described all the faces of P_* explicitly. Using a similar concept of *pushing* instead of *pulling* of vertices, Klee [16] constructed a simplicial convex polytope P_* from a given convex polytope P.

THEOREM 4.1. *Let Ξ be a finite subset of primitive elements in N such that Ξ spans $N_{\boldsymbol{R}}$ over $\boldsymbol{R}$. Then there exists a simplicial and admissible fan Δ for N which is* full, *that is,* $\Xi(\Delta)=\Xi$. *In the two-dimensional case, such a fan Δ is unique.*

In order to prove this theorem, we use the following lemma:

LEMMA 4.2. *Suppose that Δ is a simplicial fan with r-dimensional convex support. Then Δ is quasi-projective if and only if there exists $c_\xi>0$ for each $\xi\in\Xi(\Delta)$ such that the convex hull* $\operatorname{conv}(\{c_\xi\cdot\xi \mid \xi\in\Xi(\Delta)\}\cup\{0\})$ *gives rise to the same fan as Δ by projection from* 0.

PROOF. Suppose that Δ is quasi-projective. Then there exists an $\eta\in\mathrm{PL}(\Delta)$ which is strictly convex with respect to Δ. Replacing η by $\eta+m$ for a suitable $m\in M$, we may assume that $\eta(\xi)>0$ for any $\xi\in\Xi$. Put $c_\xi:=1/\eta(\xi)$. Let us denote $P:=\operatorname{conv}(\{c_\xi\xi \mid \xi\in\Xi(\Delta)\}\cup\{0\})$. Let $\mathscr{F}$ be the set of all facets of P which do not contain 0. For any $\sigma\in\Delta(r)$, let $\sigma=\boldsymbol{R}_{\geq 0}\xi_1+\cdots+\boldsymbol{R}_{\geq 0}\xi_r$ for $\xi_1,\dots,\xi_r\in\Xi(\Delta)$. We denote by H_σ the hyperplane passing through the points $c_{\xi_1}\xi_1,\dots,c_{\xi_r}\xi_r$. Since η is strictly convex with respect to Δ, $P\cap H_\sigma$ becomes a facet of P satisfying $\sum_{x\in P\cap H_\sigma}\boldsymbol{R}_{\geq 0}x=\sigma$ and $0\notin P\cap H_\sigma$. So we can find a facet $P\cap H_\sigma\in\mathscr{F}$ corresponding to each $\sigma\in\Delta(r)$.

On the other hand, for any facet $F\in\mathscr{F}$, let H_F be the hyperplane containing F. Then there exists a linearly independent subset $\{\xi_1,\dots,\xi_r\}\subset\Xi$ such that $H_F\cap P=\operatorname{conv}\{c_{\xi_1}\xi_1,\dots,c_{\xi_r}\xi_r\}$ and that $\boldsymbol{R}_{\geq 0}\xi_1+\cdots+\boldsymbol{R}_{\geq 0}\xi_r\in\Delta(r)$, because η is strictly convex with respect to Δ. Thus there exists a one-to-one correspondence between the subset $\mathscr{F}$ of facets of P and the set of r-dimensional cones in Δ.

For the converse, we define a map η by $\eta(\xi)=1/c_\xi$ for any $\xi\in\Xi$ and extend it to $|\Delta|$ in such a way that η becomes piecewise linear with respect to Δ. This is possible, because Δ is assumed to be simplicial. Obviously η is strictly convex with respect to Δ by assumption. q.e.d.

THE PROOF OF THEOREM 4.1. Let us denote $P_0:=\operatorname{conv}(\Xi\cup\{0\})$. If $\operatorname{ver}(P_0)\neq\Xi$ (or $\Xi\cup\{0\}$, if Δ is not complete), then we can find $x_\xi>0$ for each $\xi\in\Xi\setminus\operatorname{ver}(P_0)$ such that

$$P:=\operatorname{conv}(\operatorname{ver}(P_0)\cup\{x_\xi\xi \mid \xi\in\Xi\setminus\operatorname{ver}(P_0)\}\cup\{0\})$$

becomes a convex polytope with $\operatorname{ver}(P)=\Xi$ (or $\Xi\cup\{0\}$, if Δ is not complete).

Note that this convex polytope P may have a facet which is not an $(r-1)$-simplex. But if we use a method similar to that in [5, Theorem 2.1] and [16, Corollary 2.5], we can find a $c_\xi>0$ for each $\xi\in\Xi$ such that every facet of the new convex polytope P_*, which is obtained from P by pulling ξ to $c_\xi\xi$ for any $\xi\in\Xi$, is an $(r-1)$-simplex. Let us define

$$\sigma_F:=\bigcup_{x\in F}\boldsymbol{R}_{\geq 0}x$$

for any facet F of P_* with $0\notin F$. Then it is clear that σ_F is an r-dimensional cone. Now we define

$$\Delta := \{\text{the faces of } \sigma_F \,|\, F\text{: a facet of } P_* \text{ with } 0 \notin F\}\,.$$

Then Δ becomes a simplicial fan with $\Xi(\Delta) = \Xi$. It is clear that Δ is quasi-projective by Lemma 4.2.

The uniqueness in the two-dimensional case is clear. q.e.d.

Recall the exact sequence of $\boldsymbol{Q}$-vector spaces

$$0 \to M_{\boldsymbol{Q}} \to Z^* = \bigoplus_{\xi \in \Xi} \boldsymbol{Q} e_\xi^* \to G^{\boldsymbol{Q}} \to 0\,.$$

For any simplicial and admissible fan Δ, we define the cone $\mathrm{CPL}^{\sim}(\Delta)$ in $Z_{\boldsymbol{R}}^* := Z^* \otimes_{\boldsymbol{Q}} \boldsymbol{R}$ to be the set of all elements $x = \sum_{\xi \in \Xi} x_\xi e_\xi^* \in Z_{\boldsymbol{R}}^*$ satisfying the following: There exists an $\eta \in \mathrm{CPL}(\Delta)$ such that

$$x_\xi \geq \eta(\xi) \qquad \text{for all} \quad \xi \in \Xi \quad \text{and that} \quad x_\xi = \eta(\xi) \qquad \text{for all} \quad \xi \in \Xi(\Delta)\,.$$

$\mathrm{CPL}^{\sim}(\Delta)$ contains the nontrivial vector subspace $M_{\boldsymbol{R}}$. We denote by $\mathrm{cpl}(\Delta)$ the image of $\mathrm{CPL}^{\sim}(\Delta)$ in $G := G^{\boldsymbol{Q}} \otimes_{\boldsymbol{Q}} \boldsymbol{R}$. Then $\mathrm{cpl}(\Delta)$ is a maximal-dimensional strongly convex cone, that is,

$$\mathrm{cpl}(\Delta) \cap (-\mathrm{cpl}(\Delta)) = \{0\}$$

and

$$\dim \mathrm{cpl}(\Delta) = \dim G = \#\Xi - r\,,$$

since Δ is assumed to be simplicial and quasi-projective.

REMARK. (1) We have proved the following in [24, Proposition 3.3 and Theorem 3.5]: Let Ξ be a finite subset of primitive elements in N. Assume that Ξ spans $N_{\boldsymbol{R}}$ over $\boldsymbol{R}$. Then we get

$$\bigcup_{\Delta} \mathrm{CPL}^{\sim}(\Delta) = M_{\boldsymbol{R}} + \sum_{\xi \in \Xi} \boldsymbol{R}_{\geq 0} e_\xi^*$$

and

$$(***) \qquad \bigcup_{\Delta} \mathrm{cpl}(\Delta) = \sum_{\xi \in \Xi} \boldsymbol{R}_{\geq 0}\, g(\xi) = G_{\geq 0}\,,$$

where Δ runs through all the simplicial fans admissible for (N, Ξ).

(2) V. Batyrev pointed out that the above $(***)$ can be regared as one on the existence and uniqueness of the Zariski decomposition of effective divisors, and suggests a possible nice formulation of the problem for general higher-dimensional algebraic varieties and arithmetic varieties.

(3) In fact, the collection of all faces of $\mathrm{cpl}(\Delta)$'s for all simplicial and admissible fans Δ becomes a cone decomposition with support equal to $G_{\geq 0}$. We call this decomposition the GKZ-*decomposition* for $(N_{\boldsymbol{R}}, \Xi)$ and call $\mathrm{cpl}(\Delta)$ the GKZ-*cone* for

Δ. Furthermore, we can describe all the elements in this collection explicitly. Indeed, by defining the GKZ-cones for any admissible convex polyhedral cone decompositions, we see that GKZ-cones corresponding to nonsimplicial fans become faces of GKZ-cones corresponding to some simplicial fans.

By [24, Theorem 3.12], we can describe a relation among the GKZ-cones in the GKZ-decomposition as a relation among the corresponding fans. Namely, the cone $\mathrm{cpl}(\Delta)\cap\mathrm{cpl}(\Delta')$ is a facet of both $\mathrm{cpl}(\Delta)$ and $\mathrm{cpl}(\Delta')$ if and only if one of Δ and Δ' is a *star subdivision* or a *flop* of the other. For the definition of a star subdivision and a flop, see [24].

It Δ is simplicial, by the definition of $\mathrm{cpl}(\Delta)$ we have

$$\mathrm{cpl}(\Delta)=\bigcap_{\sigma\in\Delta(r)}\left(\sum_{\xi\in\Xi\setminus(\Xi(\Delta)\cap\sigma)}\boldsymbol{R}_{\geq 0}g(\xi)\right).$$

If Δ is simplicial with $\Xi=\Xi(\Delta)$, then the above expression is related with [26]. As we have seen in [24, Corollary 2.4], η is strictly convex with respect to Δ if and only if

$$\bar{\eta}:=\sum_{\xi\in\Xi}\eta(\xi)g(\xi)\in\bigcap_{\sigma\in\Delta(r)}\left(\sum_{\xi\in\Xi\setminus(\Xi\cap\sigma)}\boldsymbol{R}_{>0}g(\xi)\right),$$

where $\boldsymbol{R}_{>0}:=\{x\in\boldsymbol{R}\mid x>0\}$. This is the same result as [26, (3) Theorem].

By the property of the linear Gale transform, the set $\Lambda\subset\Xi$ is an $\boldsymbol{R}$-basis of $N_{\boldsymbol{R}}$ if and only if $g(\Xi\setminus\Lambda):=\{g(\xi)\mid\xi\in\Xi\setminus\Lambda\}$ is an $\boldsymbol{R}$-basis of G. Hence we see that every GKZ-cone $\mathrm{cpl}(\Delta)$ can be written as an intersection of cones which are generated by some $\boldsymbol{R}$-bases for G. Moreover, we get the converse correspondence as follows:

THEOREM 4.3 (cf. [3]). *For an $\boldsymbol{R}$-basis $\Omega\subset g(\Xi)$ for G, we denote*

$$C_\Omega:=\sum_{g(\xi)\in\Omega}\boldsymbol{R}_{\geq 0}g(\xi),$$

which is a maximal dimensional cone, that is, $\dim C_\Omega=\#\Xi-r$. Let A be a $(\#\Xi-r)$-dimensional cone in $G_{\geq 0}$ of the form $A=\bigcap_\Omega C_\Omega$, where $\Omega\subset g(\Xi)$ runs through some $\boldsymbol{R}$-bases for G. Suppose that for any $\boldsymbol{R}$-basis $\Omega'\subset g(\Xi)$ for G, $C_{\Omega'}$ contains A whenever $C_{\Omega'}$ meets the interior of A. Then there exists a unique simplicial and admissible fan Δ satisfying $\mathrm{cpl}(\Delta)=A$.

PROOF. Let Θ be the set of all $\boldsymbol{R}$-bases $\Omega\subset g(\Xi)$ for G satisfying $C_\Omega\supset A$. Choose an element y from the interior of A. Let x be the pre-image of y in $Z^*_{\boldsymbol{R}}$ under the map $Z^*_{\boldsymbol{R}}\to G$. Then x is contained in the set

$$M_{\boldsymbol{R}}+\sum_{\xi\in\Xi}\boldsymbol{R}_{\geq 0}e^*_\xi=\bigcup\{\mathrm{CPL}^\sim(\Delta)\mid\Delta:\text{simplicial and admissible}\}.$$

Thus there exists a simplicial and admissible fan Δ satisfying $x\in\mathrm{CPL}^\sim(\Delta)$. Namely, there exists an $m_\sigma\in M_{\boldsymbol{R}}$ for any $\sigma\in\Delta(r)$ such that $x_\xi\geq\langle m_\sigma,\xi\rangle$ for $\xi\in\Xi$ and that

$x_\xi = \langle m_\sigma, \xi \rangle$ for $\xi \in \Xi(\Delta) \cap \sigma$. We claim that x is contained in the interior of $\mathrm{CPL}^\sim(\Delta)$. To show this, suppose that x is contained in the boundary of $\mathrm{CPL}^\sim(\Delta)$. Then there exist $\sigma_0 \in \Delta(r)$ and $\xi_0 \in \Xi \setminus (\Xi(\Delta) \cap \sigma_0)$ such that $x_{\xi_0} = \langle m_{\sigma_0}, \xi_0 \rangle$. Let $\sigma_0 = \boldsymbol{R}_{\geq 0}\xi_1 + \cdots + \boldsymbol{R}_{\geq 0}\xi_r$ for an $\boldsymbol{R}$-basis $\{\xi_1, \ldots, \xi_r\} \subset \Xi(\Delta)$. Then the set $\Omega := \{g(\xi) \mid \xi \in \Xi,\ \xi \neq \xi_1, \ldots, \xi_r\} \subset g(\Xi)$ becomes an $\boldsymbol{R}$-basis for G. We have

$$y \in \sum_{\substack{\xi \in \Xi \\ \xi \neq \xi_0, \xi_1, \ldots, \xi_r}} \boldsymbol{R}_{\geq 0} g(\xi) \subset \sum_{\substack{\xi \in \Xi \\ \xi \neq \xi_1, \ldots, \xi_r}} \boldsymbol{R}_{\geq 0} g(\xi) = C_\Omega .$$

By assumption, we have $C_\Omega \supset A$. Hence y is contained in the interior of C_Ω, a contradiction to the assumption $x_{\xi_0} = \langle m_{\sigma_0}, \xi_0 \rangle$. Hence x is contained in the interior of $\mathrm{CPL}^\sim(\Delta)$. Hence Δ is the unique fan satisfying $x \in \mathrm{CPL}^\sim(\Delta)$.

As we have seen above, any r-dimensional cone $\sigma \in \Delta(r)$ gives rise to an $\boldsymbol{R}$-basis

$$\Omega := \{g(\xi) \mid \xi \in \Xi,\ \boldsymbol{R}_{\geq 0}\xi \not\prec \sigma\} \subset g(\Xi)$$

for G, satisfying $C_\Omega \supset A$. Conversely, for any $\Omega \in \Theta$, the set

$$\sigma := \sum_{\substack{\xi \in \Xi \\ g(\xi) \notin \Omega}} \boldsymbol{R}_{\geq 0}\xi$$

becomes an r-dimensional cone in Δ. Consequently, we have

$$\mathrm{cpl}(\Delta) = \bigcap_{\sigma \in \Delta(r)} \left(\sum_{\xi \in \Xi \setminus (\Xi(\Delta) \cap \sigma)} \boldsymbol{R}_{\geq 0} g(\xi) \right) = \bigcap_{\Omega \in \Theta} \left(\sum_{g(\xi) \in \Omega} \boldsymbol{R}_{\geq 0} g(\xi) \right) = A .$$

q.e.d.

COROLLARY 4.4. *There exists a one-to-one correspondence between the set of simplicial and admissible fans and the set of maximal dimensional cones* $\bigcap_{\Omega \in \Theta} C_\Omega$ *which are not separated by* $C_{\Omega'}$ *for any* $\boldsymbol{R}$*-basis* $\Omega' \subset g(\Xi)$ *for* G, *where* Θ *runs through all the possible subsets of all the* $\boldsymbol{R}$*-bases* $\Omega \subset g(\Xi)$ *for* G.

PROOF. By what we stated before Theorem 4.3, a simplicial and admissible fan gives rise to a cone of the form $\bigcap_\Omega C_\Omega$. We get the converse correspondence by Theorem 4.3. q.e.d.

EXAMPLE. Let $\Xi := \{n, n', -n, -n-n', n-n'\} \subset N \cong \boldsymbol{Z}^2$, where $\{n, n'\}$ is a $\boldsymbol{Z}$-basis for N. Then there exist eight different simplicial admissible fans. Among those fans, there is a unique fan Δ_0 which is full (cf. Theorem 4.1). Let Δ be a fan consisting of all the faces of the following three cones:

$$\begin{aligned}
\sigma_1 &:= \boldsymbol{R}_{\geq 0} n' + \boldsymbol{R}_{\geq 0}(n - n') \\
\sigma_2 &:= \boldsymbol{R}_{\geq 0} n' + \boldsymbol{R}_{\geq 0}(-n - n') \\
\sigma_3 &:= \boldsymbol{R}_{\geq 0}(n - n') + \boldsymbol{R}_{\geq 0}(-n - n') .
\end{aligned}$$

Then the corresponding toric variety $S := T_N \mathrm{emb}(\Delta)$ becomes the weighted projective plane $\boldsymbol{P}(1, 1, 2)$ (cf. [20]). The toric variety $X_0 := T_N \operatorname{emb}(\Delta_0)$ corresponding to Δ_0 is obtained from S by blowing-up at the following two T_N-fixed points of S:

$$p_1 := V(\sigma_1) \quad \text{and} \quad p_2 := V(\sigma_2) .$$

$G_{\geq 0}$ is a three-dimensional strongly convex cone spanned by the set

$$\{v(\boldsymbol{R}_{\geq 0} n) ,\ v(\boldsymbol{R}_{\geq 0}(n-n')) ,\ v(\boldsymbol{R}_{\geq 0}(-n)) ,\ v(\boldsymbol{R}_{\geq 0}(-n-n'))\}$$

in $A^1(\Delta_0)_{\boldsymbol{R}} := A^1(\Delta_0) \otimes_{\boldsymbol{Q}} \boldsymbol{R}$. By choosing all the $\boldsymbol{R}$-bases for $G = A^1(\Delta_0)_{\boldsymbol{R}}$ from the set $g(\Xi) = \{v(\rho) \mid \rho \in \Delta_0(1)\}$, we get the GKZ-decomposition consisting of eight different three-dimensional cones. Using Theorem 4.3, we can express the corresponding fans immediately. The corresponding $\boldsymbol{Q}$-factorial toric varieties are

(i) $S = \boldsymbol{P}(1, 1, 2)$,

(ii) (resp. (iii)) the equivariant blowing-up X_1 (resp. X_2) of S at the T_N-fixed point p_1 (resp. p_2),

(iv) X_0,

(v) (resp. (vi)) the Hirzebruch surface $F_1 =: Y_1$ (resp. Y_2) obtained from X_0 by contracting $V(\boldsymbol{R}_{\geq 0}(n-n'))$ (resp. $V(\boldsymbol{R}_{\geq 0}(-n-n'))$), and

(vii) (resp. (viii)) the projective plane $\boldsymbol{P}_2(\boldsymbol{C}) =: Z_1$ (resp. Z_2) obtained from Y_1 (resp. Y_2) by contracting $V(\boldsymbol{R}_{\geq 0}(-n))$ (resp. $V(\boldsymbol{R}_{\geq 0}(n))$) in Y_1 (resp. Y_2).

It is clear that the GKZ-decomposition of G is uniquely determined by the given set Ξ. From this, we obtain all *possible* fans and get information on the relations among these fans.

Suppose that Δ is a complete fan for N. Then by the property of the linear Gale transform, $G_{\geq 0}$ becomes a strongly convex cone. As the example above suggests, the GKZ-decomposition of G has some *core* which is the union of the GKZ-cones corresponding to fans which are full, simplicial and admissible. Δ becomes coarser as $\operatorname{cpl}(\Delta)$ goes to the boundary of $G_{\geq 0}$. In fact, the *core* in the above sense also becomes a cone in $G_{\geq 0}$, even if Δ is not complete, as we now show.

THEOREM 4.5. *Let Ξ be a finite subset of primitive elements in N such that Ξ spans $N_{\boldsymbol{R}}$ over $\boldsymbol{R}$. We denote by $\mathscr{C}$ the union of* CPL$^{\sim}(\Delta)$*'s corresponding to all fans Δ which are* full, simplicial *and* admissible *for (N, Ξ). Then $\mathscr{C}$ is equal to the set of those elements*

$$x = \sum_{\xi \in \Xi} x_\xi e_\xi^* \in M_{\boldsymbol{R}} + \sum_{\xi \in \Xi} \boldsymbol{R}_{\geq 0} e_\xi^*$$

which satisfy

$$a_1 x_{\xi_1} + \cdots + a_p x_{\xi_p} \geq x_\xi ,$$

whenever

$$\xi_1, \ldots, \xi_p, \xi \in \Xi \quad \textit{and} \quad a_1\xi_1 + \cdots + a_p\xi_p = \xi \qquad \textit{for some} \quad a_1, \ldots, a_p \geq 0 .$$

So the image $\mathscr{C}$ *of* $\tilde{\mathscr{C}}$ *in* G *becomes a convex polyhedral cone contained in* $G_{\geq 0}$. *If both* $\mathrm{CPL}^{\sim}(\Delta)$ *and* $\mathrm{CPL}^{\sim}(\Delta')$ *are contained in* $\tilde{\mathscr{C}}$, *then* Δ *can be obtained from* Δ' *by a finite succession of flops.*

PROOF. Suppose that $x=\sum_{\xi\in\Xi}x_\xi e_\xi^*$ is contained in $\tilde{\mathscr{C}}$, hence x is in $M_{\boldsymbol{R}}+\sum_{\xi\in\Xi}\boldsymbol{R}_{\geq 0}e_\xi^*$. By assumption, there exists a fan Δ which is full, simplicial, admissible, and satisfying $x\in\mathrm{CPL}^{\sim}(\Delta)$. Hence, there exists an $\eta\in\mathrm{CLP}(\Delta)$ such that $x_\xi=\eta(\xi)$ for any $\xi\in\Xi$. If $a_1\xi_1+\cdots+a_p\xi_p=\xi$ holds for $\xi_1,\ldots,\xi_p,\xi\in\Xi$ and for some $a_1,\ldots,a_p>0$, then

$$x_\xi=\eta(\xi)=\eta(a_1\xi_1+\cdots+a_p\xi_p)\leq a_1\eta(\xi_1)+\cdots+a_p\eta(\xi_p)=a_1x_{\xi_1}+\cdots+a_px_{\xi_p}\,,$$

because η is convex.

Conversely, suppose that $x=\sum_{\xi\in\Xi}x_\xi e_\xi^*\in M_{\boldsymbol{R}}+\sum_{\xi\in\Xi}\boldsymbol{R}_{\geq 0}e_\xi^*$ satisfies the assumption. Recall that

$$M_{\boldsymbol{R}}+\sum_{\xi\in\Xi}\boldsymbol{R}_{\geq 0}e_\xi^*=\bigcup_{\Delta}\mathrm{CPL}^{\sim}(\Delta)\,,$$

where Δ runs through all the simplicial and admissible fans. Thus there exists a simplicial and admissible fan Δ satisfying $x\in\mathrm{CPL}^{\sim}(\Delta)$. Namely, there exists an $\eta\in\mathrm{CPL}(\Delta)$ such that $x_\xi\geq\eta(\xi)$ for any $\xi\in\Xi$ and that the equality holds if $\xi\in\Xi(\Delta)$. For any $\xi\in\Xi\setminus\Xi(\Delta)$, we can find an r-dimensional cone $\sigma:=\boldsymbol{R}_{\geq 0}\xi_1+\cdots+\boldsymbol{R}_{\geq 0}\xi_r\in\Delta(r)$ containing ξ. Thus,

$$\xi=a_1\xi_1+\cdots+a_r\xi_r\qquad\text{for some}\quad a_1,\ldots,a_r\geq 0\,.$$

Hence we have

$$x_\xi\geq\eta(\xi)=\eta(a_1\xi_1+\cdots+a_r\xi_r)=a_1\eta(\xi_1)+\cdots+a_r\eta(\xi_r)=a_1x_{\xi_1}+\cdots+a_rx_{\xi_r}\geq x_\xi\,,$$

by assumption. This implies that $x_\xi=\eta(\xi)$ for all $\xi\in\Xi$. We can find a subdivision Δ' of Δ such that Δ' is full, simplicial and admissible as in Theorem 4.1. It is clear that $x\in\mathrm{CPL}^{\sim}(\Delta')$.

As for the last statement of the theorem, we just note that Δ and Δ' are full. So $\Delta(1)=\Delta'(1)$ and the case of a star subdivision in [24, Theorem 3.12] cannot occur in the present situation. q.e.d.

5. Full and simplicial fans. In this section we consider only those fans which are full, simplicial and admissible for a fixed (N,Ξ).

Recall that $\dim N_{\boldsymbol{R}}=r$. An $(r-1)$-dimensional cone $\tau\in\Delta(r-1)$ is called an *internal wall* if there exist σ and σ' in $\Delta(r)$ such that $\tau=\sigma\cap\sigma'$. It is clear that every $(r-1)$-dimensional cone is an internal wall when Δ is complete. We have described the dual cone of the GKZ-cone $\mathrm{cpl}(\Delta)$ in [24] for a convex polyhedral cone decomposition Δ having convex r-dimensional support. If Δ is full and simplicial, we can describe it more explicitly as follows:

THEOREM 5.1 (cf. [24, Theorem 2.3]). *Let Δ be a full and simplicial fan for N with r-dimensional convex support. Then for each internal wall $\tau \in \Delta(r-1)$, there exists a nonzero element $l_\tau \in G^*$ uniquely determined up to positive scalar multiple such that*

$$\mathrm{cpl}(\Delta)^\vee = \sum_{\tau:\ \text{internal wall}} \boldsymbol{R}_{\geq 0} l_\tau .$$

PROOF. Let $\tau \in \Delta(r-1)$ be an internal wall. Then there exist $\xi_1, \dots, \xi_{r+1} \in \Xi$ and $\sigma_1(\tau), \sigma_2(\tau) \in \Delta(r)$ which depend on τ and satisfy $\tau = \sigma_1(\tau) \cap \sigma_2(\tau)$ with

$$\begin{aligned}\sigma_1(\tau) &= \tau + \rho_1(\tau)\\ \sigma_2(\tau) &= \tau + \rho_2(\tau)\\ \tau &= \rho_3(\tau) + \rho_4(\tau) + \cdots + \rho_{r+1}(\tau)\,,\end{aligned}$$

where we denote $\rho_i(\tau) := \boldsymbol{R}_{\geq 0} \xi_i(\tau) \in \Delta(1)$ for $i = 1, \dots, r+1$. By renumbering the indices if necessary, we have a relation

$$\sum_{i=1}^{p} a_i \xi_i(\tau) = \sum_{j=1}^{q} (-a_{p+j}) \xi_{p+j}(\tau) \qquad \text{for some} \quad a_1, \dots, a_p, (-a_{p+1}), \dots, (-a_{p+q}) > 0$$

among the elements in a minimal linearly dependent subset of $\{\xi_1(\tau), \dots, \xi_{r+1}(\tau)\}$, where p, q are intergers with $p \geq 2$, $q \geq 0$ and $p+q \leq r+1$. If we put

$$l_\tau := \sum_{i=1}^{p} a_i e_{\xi_i(\tau)} - \sum_{j=1}^{q} (-a_{p+j}) e_{\xi_{p+j}}(\tau)\,,$$

then by the definition of $\mathrm{cpl}(\Delta)$, we can deduce that $\mathrm{cpl}(\Delta)^\vee = \sum_{\tau:\ \text{internal wall}} \boldsymbol{R}_{\geq 0} l_\tau$.

q.e.d.

Let Δ be a *complete* simplicial fan which is full. Recall that there exists a perfect pairing

$$A^{r-1}(\Delta) \times A^1(\Delta) \longrightarrow A^r(\Delta) \xrightarrow[\sim]{[\ ]} \boldsymbol{Q}$$

in this case (cf. Example in Section 4), and that $A^{r-1}(\Delta) = \sum_{\tau \in \Delta(r-1)} \boldsymbol{Q} v(\tau)$ by Proposition 1.1.

PROPOSITION 5.2. *Let Δ be a complete simplicial fan which is full. Then*

$$A^{r-1}(\Delta) \ni \gamma \mapsto \sum_{\rho \in \Delta(1)} [\gamma \cdot v(\rho)] e_\rho \in Z = \bigoplus_{\rho \in \Delta(1)} \boldsymbol{Q} e_\rho$$

induces an isomorphism

$$A^{r-1}(\Delta)_{\boldsymbol{R}} := A^{r-1}(\Delta) \otimes_{\boldsymbol{Q}} \boldsymbol{R} \xrightarrow{\sim} G^* := (G^{\boldsymbol{Q}})^* \otimes_{\boldsymbol{Q}} \boldsymbol{R}$$

which sends $\mathrm{cpl}(\Delta)^\vee$ *onto* $(A^{r-1}(\Delta)_{\boldsymbol{R}})_{\geq 0} := \sum_{\tau \in \Delta(r-1)} \boldsymbol{R}_{\geq 0} v(\tau)$.

PROOF. Since Δ is complete, every wall $\tau \in \Delta(r-1)$ is an internal wall, that is, there exist $\rho_1, \rho_2 \in \Delta(1)$ such that

$$\sigma_1 := \tau + \rho_1 \in \Delta(r), \qquad \sigma_2 := \tau + \rho_2 \in \Delta(r),$$

satisfying a relation

$$a_1 n(\rho_1) + a_2 n(\rho_2) + \sum_{\rho \in \Delta(1),\, \rho \prec \tau} a_\rho n(\rho) = 0$$

for some $a_1, a_2 > 0$ and $a_\rho \in \boldsymbol{Q}$. Recall that (cf. Theorem 5.1) $\mathrm{cpl}(\Delta)^\vee = \sum_{\tau \in \Delta(r-1)} \boldsymbol{R}_{\geq 0} l_\tau$ and that we can put

$$l_\tau = a_1 e_{\rho_1} + a_2 e_{\rho_2} + \sum_{\rho \in \Delta(1),\, \rho \prec \tau} a_\rho e_\rho$$

in this case. If we regard $\mathrm{cpl}(\Delta)^\vee \subset G^*$ as a subset of $Z_{\boldsymbol{R}} := \bigoplus_{\rho \in \Delta(1)} \boldsymbol{R} e_\rho$, then we have

$$\langle l_\tau, e_\rho^* \rangle = \begin{cases} a_1\,(\text{resp.}\, a_2) & \text{if} \quad \rho = \rho_1\,(\text{resp.}\, \rho_2) \\ a_\rho & \text{if} \quad \rho \prec \tau \\ 0 & \text{otherwise}. \end{cases}$$

On the other hand, we have an isomorphism

$$A^{r-1}(\Delta) \xrightarrow{\sim} (A^1(\Delta))^* := \mathrm{Hom}_{\boldsymbol{Q}}(A^1(\Delta), \boldsymbol{Q})$$

by identifying $\gamma \in A^{r-1}(\Delta)$ with a map which sends $v(\rho) \in A^1(\Delta)$ to $[\gamma \cdot v(\rho)] \in \boldsymbol{Q}$, in view of the perfect pairing

$$A^{r-1}(\Delta) \times A^1(\Delta) \longrightarrow A^r(\Delta) \underset{\sim}{\xrightarrow{[\ ]}} \boldsymbol{Q}.$$

Hence by the map

$$A^{r-1}(\Delta) \subsetneq Z = \bigoplus_{\rho \in \Delta(1)} \boldsymbol{Q} e_\rho, \qquad \gamma \mapsto \sum_{\rho \in \Delta(1)} [\gamma \cdot v(\rho)] e_\rho,$$

we have isomorphisms $A^{r-1}(\Delta)_{\boldsymbol{R}} \xrightarrow{\sim} G^*$ as well as

$$\boldsymbol{R}_{\geq 0} v(\tau) \xrightarrow{\sim} \boldsymbol{R}_{\geq 0} l_\tau$$

for any $\tau \in \Delta(r-1)$ by Corollary 2.8. q.e.d.

EXAMPLE. Let $X := T_N \mathrm{emb}(\Delta)$ be the toric variety corresponding to a complete simplicial fan Δ which is full. The above isomorphism $A^{r-1}(\Delta)_{\boldsymbol{R}} \xrightarrow{\sim} G^*$ induces the mutually dual short exact sequences (cf. Example in Section 4)

$$0 \longleftarrow N_{\boldsymbol{R}} \longleftarrow (T_N \mathrm{Div}(X))^*_{\boldsymbol{R}} = \bigoplus_{\rho \in \Delta(1)} \boldsymbol{R} e_\rho \longleftarrow A^{r-1}(\Delta)_{\boldsymbol{R}} \longleftarrow 0$$

$$0 \longrightarrow M_{\boldsymbol{R}} \longrightarrow T_N \mathrm{Div}(X)_{\boldsymbol{R}} = \bigoplus_{\rho \in \Delta(1)} \boldsymbol{R} V(\rho) \longrightarrow A^1(\Delta)_{\boldsymbol{R}} \longrightarrow 0.$$

$G_{\geq 0}=(A^1(\Delta)_{\boldsymbol{R}})_{\geq 0}$ is equal to the cone spanned by the linear equivalence classes of T_N-stable effective divisors, and $\mathrm{cpl}(\Delta)\subset(A^1(\Delta)_{\boldsymbol{R}})_{\geq 0}$ becomes the cone spanned by the linear equivalence classes of numerically effective divisors. By the isomorphism $A^{r-1}(\Delta)_{\boldsymbol{R}}\xrightarrow{\sim} G^*$ in Proposition 5.2, we have the identification of the dual cone

$$\mathrm{cpl}(\Delta)^\vee\cong(A^{r-1}(\Delta)_{\boldsymbol{R}})_{\geq 0}\subset A^{r-1}(\Delta)_{\boldsymbol{R}}.$$

Thus $\mathrm{cpl}(\Delta)^\vee$ becomes the cone of effective one-cycles modulo linear equivalence, that is, the *Mori cone* $NE(X):=\sum_{\tau\in\Delta(r-1)}\boldsymbol{R}_{\geq 0}v(\tau)$ (cf. [19] and [25]).

REMARK. Batyrev [2, Theorem 2.15] expressed the Mori cone in a different way, when Δ is complete and nonsingular. He used a new concept of *primitive collections.* If Δ is complete and nonsingular, then $\mathrm{cpl}(\Delta)^\vee=\sum_{\tau\in\Delta(r-1)}\boldsymbol{R}_{\geq 0}l_\tau$. If $\boldsymbol{R}_{\geq 0}l_\tau$ is an extremal ray (i.e., a one-dimensional face) of $\mathrm{cpl}(\Delta)^\vee$, then τ gives rise to a primitive collection. We see that not all of the primitive collections come from the extremal rays of $\mathrm{cpl}(\Delta)^\vee$ in this way. The total cone $\mathrm{cpl}(\Delta)^\vee$ itself, however, is equal to the cone $\mathrm{Pr}(X)$ generated by the primitive relations for primitive collections, and coincides with the Mori cone $NE(X)$.

Coming back to the general case where Δ may not be complete but has r-dimensional convex support, we have another proof of a result in Reid [25].

PROPOSITION 5.3 (cf. [25, Corollary 2.10]). *Let* $\dim N_{\boldsymbol{R}}=r\geq 3$, *and let* Δ *be full, simplicial and admissible for* (N,Ξ). *Let us denote*

$$\begin{aligned}
\sigma_1 &:=\tau_0+\rho_1\in\Delta(r)\\
\sigma_2 &:=\tau_0+\rho_2\in\Delta(r)\\
\tau_0 &=\sigma_1\cap\sigma_2:=\rho_3+\cdots+\rho_{r+1}\in\Delta(r-1)\\
\tau^k &:=\rho_3+\cdots+\overset{k}{\vee}+\cdots+\rho_{r+1}\in\Delta(r-2) \qquad \textit{for any}\quad k=3,\dots,r+1\,.
\end{aligned}$$

Furthermore, let

$$\sum_{i=1}^{p}a_in(\rho_i)=\sum_{j=1}^{q}a_{p+j}n(\rho_{p+j})\,,$$

for some $a_1,\dots,a_{p+q}>0$, *be a relation among the elements in a minimal linearly dependent subset of* $\{n(\rho_1),\dots,n(\rho_{r+1})\}$. *Suppose that* $\boldsymbol{R}_{\geq 0}l_{\tau_0}$ *is an extremal ray of* $\mathrm{cpl}(\Delta)^\vee$. *Then we have the following*:

(1) *For any* $k=3,\dots,p$, *let*

$$\begin{aligned}
\tau_1 &:=\tau^k+\rho_1\in\Delta(r-1)\\
\tau_2 &:=\tau^k+\rho_2\in\Delta(r-1)\,.
\end{aligned}$$

Then $l_{\tau_1},l_{\tau_2}\in\boldsymbol{R}_{\geq 0}l_{\tau_0}$ *and* $\tau^k+\rho_1+\rho_2\in\Delta(r)$.

(2) *For any* $k=p+q+1,\dots,r$, *if there exists a* $n(\rho')\in\Xi=\{n(\rho)\mid\rho\in\Delta(1)\}$ *distinct*

from $n(\rho_1), \dots, n(\rho_{r+1})$ *such that* $\tau' := \rho' + \tau^k \in \Delta(r-1)$, *then* $l_{\tau'} \in \boldsymbol{R}_{\geq 0} l_{\tau_0}$ *and* $\rho_1 + \tau'$, $\rho_2 + \tau' \in \Delta(r)$.

PROOF. For any $k = 3, \dots, r+1$, it is clear that $\tau^k + \rho_k, \tau^k + \rho_1, \tau^k + \rho_2 \in \Delta(r-1)$. Suppose there exists a $\rho_0 \in \Delta(1) \setminus \{\rho_1, \dots, \rho_{r+1}\}$ such that $\rho_0 + \tau^k \in \Delta(r-1)$. Since $|\Delta|$ is convex, we may assume that $\sigma := \tau^k + \rho_0 + \rho_1 \in \Delta(r)$. For any $x \in \mathrm{CPL}^{\sim}(\Delta) \subset \bigoplus_{\rho \in \Delta(1)} \boldsymbol{R} e_\rho^*$, we can express x as

$$x = z_\sigma + \sum_{\rho \in \Delta(1),\, \rho \not\prec \sigma} x_\rho e_\rho^* \qquad \text{for some} \quad z_\sigma \in M_{\boldsymbol{R}} \quad \text{and} \quad x_\rho \geq 0 .$$

Recall that $\boldsymbol{R}_{\geq 0} l_{\tau_0}$ is an extremal ray of $\mathrm{cpl}(\Delta)^\vee$ if and only if

$$\tilde{F} := \{x \in \mathrm{CPL}^{\sim}(\Delta) \mid \langle x, l_{\tau_0} \rangle = 0\}$$

is a facet of $\mathrm{CPL}^{\sim}(\Delta)$. We may take $l_{\tau_0} := \sum_{i=1}^p a_i e_{\rho_i} - \sum_{j=1}^q a_{p+j} e_{\rho_{p+j}}$ (cf. Theorem 5.1). Hence we have for $x \in \tilde{F}$,

$$\begin{aligned} 0 = \langle x, l_{\tau_0} \rangle &= \left\langle z_\sigma + \sum_{\rho \in \Delta(1),\, \rho \not\prec \sigma} x_\rho e_\rho^* \,,\; \sum_{i=1}^p a_i e_{\rho_i} - \sum_{j=1}^q a_{p+j} e_{\rho_{p+j}} \right\rangle \\ &= \left\langle z_\sigma \,,\; \sum_{i=1}^p a_i n(\rho_i) - \sum_{j=1}^q a_{p+j} n(\rho_{p+j}) \right\rangle + x_{\rho_2} a_2 + x_{\rho_k} a_k \\ &= x_{\rho_2} a_2 + x_{\rho_k} a_k \,. \end{aligned}$$

(1) If $3 \leq k \leq p$, then $a_2, a_k > 0$. So $\tilde{F}$ cannot be a facet of $\mathrm{CPL}^{\sim}(\Delta)$. Hence for any $\rho \in \Delta(1)$ distinct from $\rho_1, \dots, \rho_{r+1}$, we have $\rho + \tau^k \notin \Delta(r-1)$. Since $|\Delta|$ is convex, we have $\tau^k + \rho_1 + \rho_2 \in \Delta(r)$, and clearly $l_{\tau_1}, l_{\tau_2} \in \boldsymbol{R}_{\geq 0} l_{\tau_0}$.

(2) If $p+q+1 \leq k \leq r+1$, then $\langle x, l_{\tau_0} \rangle = x_{\rho_2} a_2$. So $x \in \tilde{F}$ implies $x_{\rho_2} = 0$. If there is another $\rho \in \Delta(1)$ distinct from $\rho_0, \dots, \rho_{r+1}$ such that $\rho + \tau^k \in \Delta(r-1)$, then $x_\rho = 0$ for the same reason, which contradicts the fact that $\tilde{F}$ is a facet of $\mathrm{CPL}^{\sim}(\Delta)$. So we have

$$\{\rho \in \Delta(1) \mid \rho + \tau^k \in \Delta(r-1)\} = \{\rho_0, \rho_1, \rho_2, \rho_k\} .$$

Hence we are done, if we take $\rho' := \rho_0$. q.e.d.

6. Application to isolated singularities. As another application of the GKZ-decomposition, we consider a toric variety which has at most one bad isolated singularity and possibly some quotient singularities.

Choose and fix an r-dimensional strongly convex rational polyhedral cone π in $N_{\boldsymbol{R}}$ such that any proper face of π is simplicial.

Let Δ_0 be the fan consisting of all the faces of π. Then the corresponding toric variety X_0 has one possible bad isolated singularity at the point $\mathrm{orb}(\pi)$ and $X_0 \setminus \mathrm{orb}(\pi)$ has at most quotient singularities.

Let $\Xi := \{n(\rho) \mid \rho \in \Delta_0(1)\}$ and consider the $\boldsymbol{Q}$-linear Gale transform of (N, Ξ). Since π is strongly convex, $G_{\geq 0}$ becomes the whole space G (cf. [24, Proposition 1.4]). For any maximal dimensional GKZ-cone $\mathrm{cpl}(\Delta)$ in the GKZ-decomposition, the corresponding fan Δ is a quasi-projective simplicial subdivision of Δ_0 with $\Delta(1)=\Delta_0(1)$ (cf. [24, Corollary 3.8]). The corresponding toric variety $X=T_N\,\mathrm{emb}(\Delta)$ has at most quotient singularities. We also see that for any pair of maximal dimensional GKZ-cones, the corresponding fans can be obtained from each other by a finite succession of flops.

DEFINITION. A fan Δ is called a *small* simplicial subdivision of π if it satisfies the following:

(i) Δ is simplicial.

(ii) $|\Delta|=\pi$.

(iii) Any proper face of π is contained in Δ.

(iv) $\dim\sigma$ for any cone $\rho\in\Delta$ is greater than $r/2$ whenever σ meets the interior $\mathrm{int}(\pi)$ of π.

Such a small simplicial subdivision may not exist and may not be unique. In fact, we have examples of π which do not have any small simplicial subdivision.

PROPOSITION 6.1. *Let π be an even-dimensional strongly convex cone and $\Xi=\{n(\rho)\,|\,\rho\prec\pi,\ \dim\rho=1\}$. Suppose that there exists a small and quasi-projective simplicial subdivision Δ of π. If* $\mathrm{cpl}(\Delta')$ *is a* GKZ-*cone such that $F:=\mathrm{cpl}(\Delta)\cap\mathrm{cpl}(\Delta')$ is a facet of both* $\mathrm{cpl}(\Delta)$ *and* $\mathrm{cpl}(\Delta')$, *then Δ' cannot be small.*

PROOF. Let $R:=\boldsymbol{R}_{\geq 0}l_\tau$ be the extremal ray (cf. Theorem 5.1) of $\mathrm{cpl}(\Delta)^\vee$ corresponding to the facet F. Then there exist a minimal linearly dependent set $\{n(\rho_1),\dots,n(\rho_{p+q})\}$ and a relation

$$\sum_{i=1}^{p}a_in(\rho_i)=\sum_{j=1}^{q}a_{p+j}n(\rho_{p+j})\qquad\text{for some}\quad a_i,\,a_{p+j}>0\,,$$

where $\rho_i,\rho_{p+j}\in\Delta(1)$ are one-dimensional faces of $\sigma_1,\sigma_2\in\Delta(r)$ satisfying $\sigma_1\cap\sigma_2=\tau$, while p and q are intergers such that $p+q\leq r+1$ and $p,q\geq 2$.

Without loss of generality, we may assume that $\rho_1+\tau=\sigma_1$ and $\rho_2+\tau=\sigma_2$. Then, by the construction of a flop Δ' of Δ, we see that

$$\begin{aligned}&\rho_1+\cdots+\rho_p\notin\Delta\,, &&\rho_{p+1}+\cdots+\rho_{p+q}\in\Delta\,,\\ &\rho_1+\cdots+\rho_p\in\Delta'\,, &&\rho_{p+1}+\cdots+\rho_{p+q}\notin\Delta'\,.\end{aligned}$$

Thus, $\rho_1+\cdots+\rho_p$ and $\rho_{p+1}+\cdots+\rho_{p+q}$ are not proper faces of π, and these cones intersect the interior of π. Since Δ is small, we have $q>r/2$, hence $p+r/2<p+q\leq r+1$. Thus $p\leq r/2$, which implies that Δ' cannot be small. q.e.d.

If we cut this cone π by a hyperplane not passing through the origin, then the intersection becomes an $(r-1)$-dimensional simplicial convex polytope. Thus, by considering combinatorial types of simplicial convex polytopes (cf. [19, Appendix]), we have some information in lower dimensional cases.

Recall that a simplicial subdivision Δ of π is said to be *non-divisorial* if it does not introduce any one-dimensional cones other than the one-dimensional faces of π (cf. [23]). As we have seen in [24, Corollary 3.8], for any π there always exists a non-divisorial quasi-projective simplicial subdivision Δ of π.

PROPOSITION 6.2. (1) *If* $r=3$, *then every non-divisorial simplicial subdivision of* π *is small.*

(2) *If* π *with* $r=4$ *has a small simplicial subdivision, then it is unique.*

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