GLOBAL CONTINUA OF PERIODIC SOLUTIONS
TO SOME DIFFERENCE-DIFFERENTIAL
EQUATIONS OF NEUTRAL TYPE

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Abstract. In this paper, we apply a global bifurcation theorem of Alexander-Yorke type to investigate the global continua of periodic solutions of a scalar neutral functional differential equation arising from the study of lossless transmission lines. It will be shown that the considered equation has a slowly oscillating periodic solution and several rapidly oscillating periodic solutions. It will also be demonstrated that this seemingly simple equation raises several unanswered questions.

1. Introduction. In a series of papers [29], [30], [31], Jones proved that Wright's equation

$$(1.1) \qquad \dot{x}(t) = -\alpha x(t-1)[1+x(t)]$$

has nontrivial periodic solutions for $\alpha > \pi/2$. Since then, considerable research has been devoted to the question of the existence and qualitative behaviors of periodic solutions for some classes of scalar autonomous *retarded* functional differential equations. Main technical tools to obtain existence of nontrivial periodic solutions have been (i) some fixed point index and fixed point theorems for cone mappings as well as mappings of a convex set into itself with an ejective fixed point as an extreme point of the convex set (see, [5], [6], [7], [18], [19], [32], [46], [47]); (ii) an extension to retarded equations of the well-known global bifurcation theorem of Alexander-Yorke [1] for periodic solutions obtained by Chow, Ize, Mallet-Paret, Nussbaum and Yorke (see, [8], [9], [10], [27], [28], [48], [49], [50]); and (iii) Kaplan-Yorke's method to yield periodic solutions for a scalar retarded equation from an associated system of ordinary differential equations (see, [8], [34]). For details, we refer to [25], [41], [42] and references therein.

Little progress has been made on the existence of nontrivial periodic solutions of autonomous *neutral* functional differential equations. Although local Hopf bifurcation has been considered by Hale [23], [24], Oliveria [26], [54], and Staffans [61], and the existence of nontrivial periodic solutions to certain classes of difference-differential equations has been studied by Brayton [3], [4], Nussbaum [47] and Preisenberger

[56], the aforementioned main technical tools to obtain existence of periodic solutions have neither been sufficiently generalized to general neutral equations, nor been applied to broad classes of specific examples.

This lack of results on existence of periodic solutions should be filled in. Though neutral equations are more difficult to motivate (as Hale pointed out in his book [25]), more and more neutral equations arise naturally in electrodynamics, economics and mathematical biology. For example, basing on his investigation on laboratory populations of Daphnia magna, Smith [59] argued that the per capita growth rate in the classical logistic single-species population equation

$$(1.2) \qquad \dot{x}(t) = rx(t) \left[1 - \frac{x(t)}{K} \right],$$

where r is the intrinsic growth rate of the species x and K is the environment capacity for x, should be replaced by $r[1 - (x(t) + p\dot{x}(t))/K]$. Furthermore, if we think of x as a species grazing upon vegetation, which takes time τ to recover, then we will be led to the following neutral-delay logistic equation

$$(1.3) \qquad \dot{x}(t) = rx(t) \left[1 - \frac{x(t-\tau) + p\dot{x}(t-\tau)}{K} \right].$$

See [16], [37], [55] and references therein. Recently, it has been observed that the following neutral equation

$$(1.4) \qquad \frac{d}{dt}[x(t) - g(x(t-\tau))] = -h(x(t)) + h(x(t-\tau)),$$

where g and h are continuous functions, describes the mass transmission process in certain compartmental systems where each compart produces or swallows materials (see [17], [63]). Classical examples of neutral equations include also those arising in the study of two or more simple oscillatory systems with some interconnections between them, and in particular, in the study of lossless transmission line problems. For details, we refer to [3], [4], [25], [38] and references therein.

In this paper, we provide some general technical tools which can be used to obtain existence of periodic solutions for neutral equations and illustrate the use of these ideas for a specific example arising from the lossless transmission line problem. More precisely, we will present an extension to neutral equations of Alexander-Yorke's global bifurcation theorem, and then apply such an extension to investigate the global continua of periodic solutions bifurcating from stationary points of the following difference-differential equation of neutral type

$$(1.5) \qquad \frac{d}{dt}[x(t) - qx(t-r)] = -ax(t) - bqx(t-r) - g[x(t) - qx(t-r)],$$

where $q \in (0, 1)$, $a, b, r > 0$ are constants and g is a continuously differentiable function. It will be shown that there exists a monotone increasing sequence $\{q_n\} \subset (0, 1)$ such that for $q \in (q_n, 1)$, the system (1.5) has a slowly oscillating periodic solution and $n - 1$ rapidly oscillating periodic solutions. It will also be demonstrated that this seemingly simple equation raises a number of unanswered questions.

2. A global bifurcation theorem for neutral equations. Let $r \geq 0$ be a given constant. We denote by X the Banach space of continuous functions from $[-r, 0]$ into R^n with the sup norm

$$\|\phi\| = \sup_{-r \leq \theta \leq 0} |\phi(\theta)| , \qquad \phi \in X .$$

In what follows, if $x \in C([-r, \infty); R^n)$ and $t \geq 0$, then $x_t \in X$ is defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$.

We consider the following one-parameter family of neutral equations

$$(2.1) \qquad \frac{d}{dt} [x(t) - B(\alpha, x_t)] = F(\alpha, x_t) , \qquad \alpha \in R ,$$

where B and F are twice continuously differentiable mappings from $R \times X$ into R^n, F is completely continuous and B satisfies the following Lipschitz condition

$$(2.2) \qquad |B(\alpha, \phi) - B(\alpha, \psi)| \leq k \|\phi - \psi\|$$

for $\phi, \psi \in X$, $\alpha \in R$ and for some constant $k \in [0, 1)$.

Clearly, we can identify the subspace X_0 of X consisting of constant mappings with R^n. Let $\hat{F} = F|_{R \times X_0}$ and $\hat{B} = B|_{R \times X_0}$. A point $(\alpha_0, x_0) \in R \times R^n$ is said to be a *stationary point* of (2.1) if $\hat{F}(\alpha_0, x_0) = 0$. A stationary point (α_0, x_0) is *nonsingular* if the mapping $D_x \hat{F}(\alpha_0, x_0) \colon R^n \to R^n$ is an isomorphism.

To a given stationary point (α_0, x_0) of (2.1), we associate a *characteristic equation*

$$(2.3) \qquad \det \Delta_{(\alpha_0, x_0)}(\lambda) = 0 ,$$

where for any given complex number λ, $\Delta_{(\alpha_0, x_0)}(\lambda)$ is an $n \times n$ complex matrix defined by

$$\Delta_{(\alpha_0, x_0)}(\lambda) = \lambda [I - D_\phi B(\alpha_0, \hat{x}_0)(e^{\lambda \cdot} \cdot)] - D_\phi F(\alpha_0, \hat{x}_0)(e^{\lambda \cdot} \cdot)$$

and

$$(e^{\lambda \cdot} \cdot)(\theta, v) = e^{\lambda \theta} v$$

for $(\theta, v) \in [-r, 0] \times C^n$. Here and in what follows, $\hat{x}_0$ denotes the constant mapping from $[-r, 0]$ or R to R^n with the value $x_0 \in R^n$.

A solution λ_0 to the characteristic equation (2.3) is called a *characteristic value* of the stationary point (α_0, x_0). Evidently, a stationary point (α_0, x_0) is nonsingular if and only if 0 is not a characteristic value of (α_0, x_0). A nonsingular stationary point (α_0, x_0)

is a *center* if it has a pure imaginary characteristic value. If (α_0, x_0) is the only center in some neighborhood of $(\alpha_0, x_0) \in R \times R^n$, we say that (α_0, x_0) is an *isolated center*.

For a given isolated center (α_0, x_0) of (2.1), by the implicit function theorem there exists $\delta_0 > 0$ and a differentiable mapping $r: [\alpha_0 - \delta_0, \alpha_0 + \delta_0] \to R^n$ such that $(\alpha, r(\alpha))$ is a stationary point for each α and $r(\alpha_0) = x_0$. Define

$$(2.4) \qquad \Delta_\alpha(\lambda) = \Delta_{(\alpha, r(\alpha))}(\lambda), \qquad \alpha \in [\alpha_0 - \delta_0, \alpha_0 + \delta_0] \,.$$

Since (α_0, x_0) is an isolated center of (2.1), there exist $\beta_0 > 0$, $b = b(\alpha_0, \beta_0) > 0$ and $c = c(\alpha_0, \beta_0) > 0$ such that

(i): $\det \Delta_{(\alpha_0, x_0)}(i\beta_0) = 0$;

(ii): if $0 < |\alpha - \alpha_0| \le \delta_0$, then $iR \cap \{\lambda \in C; \det \Delta_\alpha(\lambda) = 0\} = \varnothing$;

(iii): $\det \Delta_{\alpha_0 \pm \delta_0}(\lambda) \ne 0$ for $\lambda = \mu + iv$, $(\mu, v) \in \partial\Omega$, where $\Omega = (0, b) \times (\beta_0 - c, \beta_0 + c) \subset R^2$.

Then we can define the so-called *crossing number* of (α_0, x_0, β_0) as

$$\gamma(\alpha_0, x_0, \beta_0) = \deg_B(\det \Delta_{\alpha_0 - \delta_0}(\cdot), \Omega) - \deg_B(\det \Delta_{\alpha_0 + \delta_0}(\cdot), \Omega)$$

where $\deg_B$ denotes the classical Brouwer degree. By definition, the crossing number $\gamma(\alpha_0, x_0, \beta_0)$ counts the number of characteristic values (with multiplicity) escaping from the region Ω as α crosses α_0 from the left to the right.

THEOREM 2.1. *If there exists an isolated center $(\alpha_0, x_0) \in R \times R^n$ of (2.1) such that $\gamma(\alpha_0, x_0, \beta_0) \ne 0$, then*

(i) *(α_0, x_0) is a bifurcation point, i.e. there exists a sequence $\{(\alpha_n, x_n, \beta_n)\}$ such that $\alpha_n \to \alpha_0$, $\beta_n \to \beta_0$, $x_n(t) \to x_0$ uniformly for $t \in R$ as $n \to \infty$, where $x_n(t)$ is a nonconstant periodic solution of the equation (2.1) with period $2\pi/\beta_n$ and $\alpha = \alpha_n$;*

(ii) *the minimal period of $x_n(t)$ is convergent to the set*

$$\left\{ \frac{2\pi}{m\beta_0}; \pm im\beta_0 \text{ are characteristic values of } (\alpha_0, x_0) \right\} \,.$$

In particular, if $\pm im\beta_0$ are not characteristic value of (α_0, x_0) for any integer $m > 1$, then $2\pi/\beta_n$ is the minimal period of $x_n(t)$ and $2\pi/\beta_n \to 2\pi/\beta_0$ as $n \to \infty$.

THEOREM 2.2. *Suppose that all stationary points of (2.1) are nonsingular and all centers of (2.1) are isolated. Let S denote the closure of the set*

$A = \{(y, \alpha, p) \in BC(R; R^n) \times R \times R_+;$ *y is a nonconstant periodic solution of*

   (2.1) with period $p\}$

in the topology of $BC(R; R^n) \times R \times R_+$, where $BC(R; R^n)$ denotes the Banach space of bounded and continuous functions from R to R^n endowed with the sup norm. If (α_0, x_0) is an isolated center of (2.1) such that $\gamma(\alpha_0, x_0, \beta_0) \ne 0$, then for the connected component $C(\alpha_0, x_0, 2\pi/\beta_0)$ of $(\hat{x}_0, \alpha_0, 2\pi/\beta_0)$ in S, we have either

(i) $C(\alpha_0, x_0, 2\pi/\beta_0)$ is unbounded; or

(ii) there are a finite number of isolated centers $(\alpha_i, x_i) \in R \times R^n$ and constants $\beta_i > 0$, $i = 0, 1, \ldots, q$, such that

$$\sum_{i=0}^{q} \gamma\left(\alpha_i, x_i, \frac{2\pi}{\beta_i}\right) = 0 .$$

Moreover, if $(y, \alpha, p) \in C(\alpha_0, x_0, 2\pi/\beta_0)$ and y is a constant mapping, then $(y, \alpha, p) = (x_i, \alpha_i, 2\pi/\beta_0)$ for some $i \in \{0, 1, \ldots, p\}$.

Theorem 2.1 represents an extension to neutral equations of the local Hopf bifurcation theorem of Krasonsel'skii type ([33]). The proof for (i) of Theorem 2.1 was given in [36] by using the S^1-equivariant degree of [11], the complementary function method of [27], [28] and an equivariant vision of the bijection theorem of [44], [45], [51]. A proof for (ii) of Theorem 2.1 will be given later in this section. Theorem 2.2 provides an extension to neutral equations of the global bifurcation theorem of Alexander-Yorke or Rabinowitz type ([1], [57]). The proof was given in [36]. This result should also be obtained by using the general theory of [2]. For a detailed comparison of this global bifurcation theorem with some existing old ideas, we refer to [12], [36].

To prove (ii) of Theorem 2.1, we need the following result concerning positive lower bounds for the period of some periodic solutions.

LEMMA 2.3 (cf. [36]).   Suppose that $S \subset X$ and there exist constants $L \geq 0$ and $k \in [0, 1)$ such that

$$|F(\alpha, \phi) - F(\alpha, \psi)| \leq L \|\phi - \psi\|$$

and

$$|B(\alpha, \phi) - B(\alpha, \psi)| \leq k \|\phi - \psi\|$$

for $\alpha \in R$ and $\phi, \psi \in S$. If $x(t)$ is a non-constant periodic solution of (2.1) with period $p > 0$ and $x_t \in S$ for $t \in R$, then $p \geq 4(1 - k)/L$.

The following result provides an analog for neutral equations of Corollary 3.2 in [43].

LEMMA 2.4.   Suppose that there exists a sequence of real numbers $\{\alpha_n\}_{n=1}^{\infty}$ such that
(i)   for each n, (2.1) with $\alpha = \alpha_n$ has a non-constant periodic solution $x_n(t)$ with minimal period $T_n > 0$;
(ii) $\lim_{n \to \infty} \alpha_n = \alpha_0 \in R$, $\lim_{n \to \infty} T_n = T_0 < \infty$ and $\lim_{n \to \infty} x_n(t) = x_0 \in R^n$, uniformly for $t \in R$.

Then (α_0, x_0) is a stationary solution of (2.1) and $\pm i2\pi/T_0$ are characteristic values of (2.1) with $\alpha = \alpha_0$.

PROOF. We first show that $T_0 > 0$. Since F is continuously differentiable, there exists $\varepsilon_0 > 0$ such that

$$\| DF(\alpha, \phi) - DF(\alpha, \hat{x}_0) \| < 1$$

if $\| \phi - \hat{x}_0 \| < \varepsilon_0$ and $|\alpha - \alpha_0| < \varepsilon_0$. Consequently, $|F(\alpha, \phi) - F(\alpha, \psi)| \leq L_0 \| \phi - \psi \|$ if $\| \phi - \hat{x}_0 \| < \varepsilon_0$, $\| \psi - \hat{x}_0 \| < \varepsilon_0$ and $|\alpha - \alpha_0| < \varepsilon_0$, where $L_0 = |DF(\alpha, \hat{x}_0)| + 1$. Therefore, under the assumption (ii), we can apply Lemma 2.3 to conclude $T_n \geq 4(1-k)/L_0$ for sufficiently large n, from which it follows that $T_0 \geq 4(1-k)/L_0$.

Next, we show that the following linear equation

$$(2.5) \qquad \frac{d}{dt} [y(t) - DB(\alpha_0, \hat{x}_0)y_t] = DF(\alpha_0, \hat{x}_0)y_t$$

has a periodic solution. For $\tau \in (0, 1)$, define

$$\varepsilon_{n,\tau} = \max_{t \in R} |x_n(t + \tau T_n) - x_n(t)| \,,$$

$$y_n(t) = \varepsilon_{n,\tau}^{-1} [x_n(t + \tau T_n) - x_n(t)] \,, \qquad t \in R \,.$$

$y_n(t)$ satisfies the following equation

$$(2.6) \qquad \frac{d}{dt} [y_n(t) - DB(\alpha_0, \hat{x}_0)y_t - \delta_{1n}(t)] = DF(\alpha_0, \hat{x}_0)y_{nt} + \delta_{2n}(t) \,,$$

where

$$\delta_{1n}(t) = \varepsilon_{n,\tau}^{-1} [B(\alpha_n, x_{nt+\tau T_n}) - B(\alpha_n, x_{nt}) - DB(\alpha_0, \hat{x}_0)(x_{nt+\tau T_n} - x_{nt})] \,,$$

$$\delta_{2n}(t) = \varepsilon_{n,\tau}^{-1} [F(\alpha_n, x_{nt+\tau T_n}) - F(\alpha_n, x_{nt}) - DF(\alpha_0, \hat{x}_0)(x_{nt+\tau T_n} - x_{nt})] \,.$$

Noting that $|y_n(t)| \leq 1$ for $t \in R$, we have

$$|\delta_{1n}(t)| \leq \int_0^1 |DB(\alpha_n, z_{nt}(\theta)) - DB(\alpha_0, \hat{x}_0)| d\theta \to 0 \,,$$

$$|\delta_{2n}(t)| \leq \int_0^1 |DF(\alpha_n, z_{nt}(\theta)) - DF(\alpha_0, \hat{x}_0)| d\theta \to 0$$

as $n \to \infty$ uniformly for $t \in R$, where

$$z_{nt}(\theta) = \theta x_{nt} + (1-\theta)x_{nt+\tau T_n} \,, \qquad \theta \in [0, 1] \,, \quad t \in R \,.$$

For any given $t \geq t'$, integrating (2.1) from t' to t, we get

$$x_n(t) - x_n(t') = B(\alpha_n, x_{nt}) - B(\alpha_n, x_{nt'}) + \int_{t'}^t F(\alpha_n, x_{ns}) ds \,,$$

from which and the Lipschitz condition (2.2) it follows that

$$|x_n(t) - x_n(t')| \leq \frac{L_0}{1-k}(t-t')$$

for sufficiently large n. Consequently,

$$\|z_{nt}(\theta) - z_{nt'}(\theta)\| \leq \frac{L_0}{1-k}(t-t'), \qquad \theta \in [0, 1]$$

for sufficiently large n. On the other hand,

$$|DB(\alpha_n, z_{nt'}(\theta)) - DB(\alpha_0, \hat{x}_0)| \leq \frac{1-k}{2},$$

provided that n is sufficiently large. So

$$|\delta_{1n}(t) - \delta_{1n}(t')| \leq \left| \int_0^1 [DB(\alpha_n, z_{nt}(\theta)) - DB(\alpha_n, z_{nt'}(\theta))]d\theta y_{nt} \right|$$

$$+ \left| \int_0^1 [DB(\alpha_n, z_{nt'}(\theta))d\theta - DB(\alpha_0, \hat{x}_0)](y_{nt} - y_{nt'}) \right|$$

$$\leq [|D^2 B(\alpha_0, \hat{x}_0)| + 1] \frac{L_0}{1-k}(t-t') + \frac{1-k}{2}\|y_{nt} - y_{nt'}\|$$

for sufficiently large n.

We now integrate (2.6) from t' to t to obtain

$$|y_n(t) - y_n(t')| \leq k\|y_{nt} - y_{nt'}\| + [|D^2 B(\alpha_0, \hat{x}_0)| + 1] \frac{L_0}{1-k}(t-t')$$

$$+ \frac{1-k}{2}\|y_{nt} - y_{nt'}\| + (|DF(\alpha_0, \hat{x}_0)| + 1)(t-t')$$

from which it follows that

$$(2.7) \qquad\qquad \max_{t > t'} |y_n(t) - y_n(t')| \leq M(t-t'),$$

where

$$M = \frac{2}{1-k}\left[\frac{|D^2 B(\alpha_0, \hat{x}_0)| + 1}{1-k} L_0 + |DF(\alpha_0, \hat{x}_0)| + 1 \right].$$

Therefore, $\{y_n\}_{n=1}^{\infty}$ has a convergent subsequence, denoted again by y_n for simplicity. Let $y_\tau(t) = \lim_{n\to\infty} y_n(t)$. Then $y_\tau(t)$ is a periodic solution of (2.5) with a period T_0. Since the maximal value of $|y_n(t)|$ is one and the average value of each y_n is 0, the same is true for y_τ. So y_τ is a non-constant periodic solution.

Denote by T_τ the minimal period of $y_\tau(t)$. Then $T_0 = kT_\tau$ for some positive integer

k. If $k = 1$, then we are done. If not, we wish to find a collection of solutions of (2.5) having T_0 as the least common multiple of their minimal periods. To do so, we first show that τT_0 is not an integer multiple of T_τ. Indeed, if τT_0 is an integer multiple of T_τ, then $y_n(t + k\tau T_0) = y_n(t)$ implies that

$$0 = [y_n(t + \tau T_n) - y_n(t)]\varepsilon_{n,\tau}^{-1} + [y_n(t + 2\tau T_n) - y_n(t + \tau T_n)]\varepsilon_{n,\tau}^{-1}$$
$$+ \cdots + [y_n(t + k\tau T_n) - y_n(t + (k-1)\tau T_n)]\varepsilon_{n,\tau}^{-1}$$
$$\to y_\tau(t) + y_\tau(t + \tau T_0) + \cdots + y_\tau(t + (k-1)\tau T_0) = k y_\tau(t),$$

a contradiction to the fact that $y_\tau(t)$ is non-constant.

Since for every rational $\tau \in (0, 1)$, there is a period solution y_τ of (2.5) whose minimal period divides T_0, but does not divide τT_0, we may choose some collection $\{z_j\}_{j=0}^{k-1}$ of solutions of (2.5) such that T_0 is the smallest number which is a multiple of their minimal periods. It then follows that for almost any choice of real numbers $\{c_j\}_{j=0}^{k-1}$, $\sum_{j=0}^{k-1} c_j z_j(t)$ is a periodic solution of (2.5) with the minimal period T_0. This completes the proof.

Before giving the proof of Theorem 2.1, we should emphasize that the idea in the above argument is due to Mallet-Paret and Yorke [43]. We only make some technical modifications on their proof in order to obtain an analog, for neutral equations, of Corollary 3.2 in [43].

Now we are in the position to give:

PROOF OF (ii) OF THEOREM 2.1. Let T_n denote the minimal period of $x_n(t)$. Then there exists positive integer m_n such that $2\pi/\beta_n = m_n T_n$. Since $T_n \le 2\pi/\beta_n \to 2\pi/\beta_0$ as $n \to \infty$. $\{T_n\}_{n=1}^\infty$ has a convergent subsequence $\{T_{n_k}\}_{k=1}^\infty$. Let $T_0 = \lim_{k \to \infty} T_{n_k}$. By Lemma 2.4, $\pm i2\pi/T_0$ are characteristic values of (α_0, x_0). On the other hand, since $2\pi/\beta_{n_k} \to 2\pi/\beta_0$, $T_{n_k} \to T_0$ as $k \to \infty$, m_{n_k} is identical to a constant m for sufficiently large k. Consequently, $2\pi/\beta_0 = m T_0$. Thus, $T_{n_k} \to 2\pi/m\beta_0$ and $\pm im\beta_0$ are characteristic values of (α_0, x_0). This completes the proof.

3. Existence of multiple periodic solutions.

Consider the following nonlinear difference-differential equation of neutral type:

$$(3.1) \qquad \frac{d}{dt}[x(t) - qx(t-r)] = -ax(t) - bqx(t-r) - g[x(t) - qx(t-r)],$$

where $a, b, r > 0$ and $q \in (0, 1)$ are given constants, $g : R \to R$ is a continuously differentiable function satisfying the following condition

$$(3.2) \qquad \inf_{x \ne 0} \frac{g(x)}{x} > -a.$$

Equation (3.1) arises from the problem of lossless transmission lines. Usually, the electric networks in a lossless transmission line can be described by a system of linear

hyperbolic partial differential equations subject to certain nonlinear boundary conditions, where the connection by a travelling wave through the partial differential equations can be replaced by connections with delays. Therefore, a series of transformations can relate such a system of partial differential equations to a neutral equation of the type (3.1). For details, we refer to [3], [4], [25], [38], [39], [58] and references therein.

Under the hypothesis (3.2), $g(0) = 0$ and $(q, 0)$ is the only stationary point of (3.1) for any given $q \in (0, 1)$. In fact, if (q, z) is a stationary point of (3.1), then $g((1-q)z) = -(a+bq)z$, and hence $g(y) = -(a+bq)y/(1-q)$, where $y = (1-q)z$. Since $(a+bq)/(1-q)$ is increasing in $q \in (0, 1)$, if $y \neq 0$ then $g(y)/y = (a+bq)/(1-q) < -a$, contradicting (3.2).

Let $g'(0) = c$. Then the linearization of (3.1) at 0 leads to

$$\frac{d}{dt}[X(t) - qX(t-r)] = -(a+c)X(t) - (b-c)qX(t-r),$$

and hence we obtain the following characteristic equation

$$(3.3) \qquad (\lambda + a + c)e^{\lambda r} - q(\lambda - b + c) = 0.$$

In order to locate local Hopf bifurcation points, we let $\lambda = i\beta$ in (3.3) and express the obtained equation in terms of its real and imaginary parts as

$$\begin{cases} -(a+c)\cos\beta r + \beta\sin\beta r = q(b-c), \\ \beta\cos\beta r + (a+c)\sin\beta r = q\beta, \end{cases}$$

which is equivalent to

$$(3.4) \qquad \begin{cases} \tan\beta r = \beta\, \dfrac{a+b}{\beta^2 - (a+c)(b-c)}, \\ \beta^2 = \dfrac{q^2(b-c)^2 - (a+c)^2}{1-q^2}. \end{cases}$$

The following result, summarizing the information about characteristic equation (3.3), is our starting point.

LEMMA 3.1. *Suppose that*

$$(3.5) \qquad -a < g'(0) = c < \frac{b-a}{2}.$$

Then we have the following conclusions:

   (i) *The first equation of (3.4) has infinitely many positive solutions $0 < \beta_1 < \beta_2 < \cdots < \beta_n < \cdots$ such that $\lim_{n \to \infty} \beta_n = \infty$, and*

   (a) *if $\sqrt{(a+c)(b-c)} < \pi/2r$, then $2\pi/\beta_1 > 4r$ and $2\pi/\beta_n \in (2r/n, 2r/(n-1))$ for $n \geq 2$;*

(b) if $\sqrt{(a+c)(b-c)} \in (\pi/2r + m\pi/r, \pi/2r + (m+1)\pi/r)$ for some nonnegative integer m, then $2\pi/\beta_1 > 2r$, $2\pi/\beta_n \in (2r/n, 2r/(n-1))$ for $n \geq 2$;

(c) if $\sqrt{(a+c)(b-c)} = \pi/2r$, then $2\pi/\beta_n \in (2r/(n+1), 2r/n)$ for $n \geq 1$;

(d) if $\sqrt{(a+c)(b-c)} = \pi/2r + m\pi/r$ for some positive integer m, then $2\pi/\beta_1 > 2r$, $2\pi/\beta_n \in (2r/n, 2r/(n-1))$ for $n = 2, \ldots, m$ (if $m \geq 2$) and $2\pi/\beta_n \in (2r/(n+1), 2r/n)$ for $n \geq m+1$;

(ii) $\pm i\beta_n$ are characteristic values of the stationary point $(q_n, 0)$, where

$$q_n = \sqrt{\frac{\beta_n^2 + (a+c)^2}{\beta_n^2 + (b-c)^2}}.$$

Moreover, if $q \neq q_n$, $n = 1, 2, \ldots$, then the stationary point $(q, 0)$ has no pure imaginary characteristic value;

(iii) Let $\lambda_n(q) = u_n(q) + iv_n(q)$ be the root of (3.3) for q close to q_n, such that $u_n(q_n) + iv_n(q_n) = i\beta_n$. Then $(d/dq)u_n(q)|_{q=q_n} > 0$.

PROOF. The proof for (i) is straight forward as the first equation of (3.4) can be easily analysed graphically. For example, the following figure provides graphic solutions for the first equation of (3.4) in the case where $\sqrt{(a+c)(b-c)} < \pi/2r$.

Conclusion (ii) follows immediately from the second equation of (3.4).

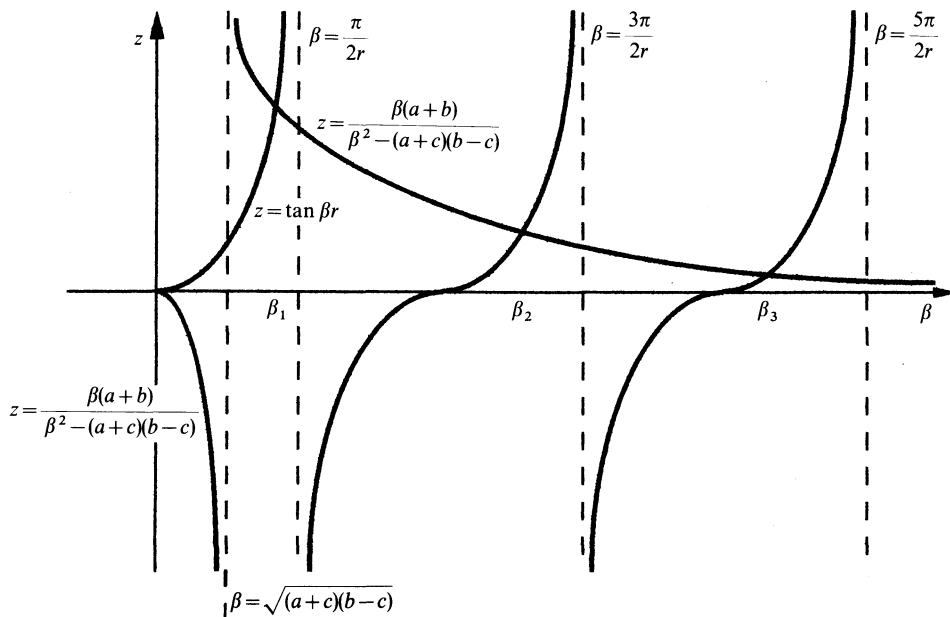To prove (iii), we rewrite (3.3) in terms of its real part and imaginary part to obtain

FIGURE.

$$\begin{cases} (u_n(q)+a+c)e^{u_n(q)r}\cos(v_n(q)r)-v_n(q)e^{u_n(q)r}\sin(v_n(q)r)=q[u_n(q)-b+c] \,, \\ (u_n(q)+a+c)e^{u_n(q)r}\sin(v_n(q)r)+v_n(q)e^{u_n(q)r}\cos(v_n(q)r)=qv_n(q) \,. \end{cases}$$

Differentiating both sides of the above system and substituting $u_n(q_n)+iv_n(q_n)=i\beta_n$, we get

$$\begin{cases} A_n u'(q_n)-B_n v'(q_n)=-b+c \,, \\ B_n u'(q_n)+A_n v'(q_n)=\beta_n \,, \end{cases}$$

where

$$\begin{cases} A_n=\cos(\beta_n r)+r(a+c)\cos(\beta_n r)-\beta_n r\sin(\beta_n r)-q_n \,, \\ B_n=r(a+c)\sin(\beta_n r)+\sin(\beta_n r)+r\beta_n\cos(\beta_n r) \,. \end{cases}$$

From the system above (3.4), we have

$$A_n=\cos(\beta_n r)-rq_n(b-c)-q_n \,,$$

$$B_n=rq_n\beta_n+\sin(\beta_n r) \,,$$

$$\cos(\beta_n r)=\frac{q_n[\beta_n^2-(b-c)(a+c)]}{(a+c)^2+\beta_n^2} \,,$$

$$\sin(\beta_n r)=\frac{q_n\beta_n(a+b)}{(a+c)^2+\beta_n^2} \,.$$

So

$$\begin{cases} A_n=\dfrac{-q_n}{(a+c)^2+\beta_n^2}\left[(b-c)(a+c)+r(b-c)(a+c)^2+r(b-c)\beta_n^2+(a+c)^2\right]<0 \,, \\ B_n=\dfrac{q_n\beta_n}{(a+c)^2+\beta_n^2}\left[a+c+r(a+c)^2+r\beta_n^2\right]>0 \,. \end{cases}$$

Therefore,

$$u'_n(q_n)=\frac{1}{A_n^2+B_n^2}\left[-(b-c)A_n+\beta_n B_n\right]>0 \,.$$

This completes the proof.

We are now in a position to employ Theorem 2.1 to obtain a local Hopf bifurcation of periodic solutions of small amplitudes. However, in order to obtain information about the continua of these periodic solutions emanating from the point $(0, q_n)$, we need further information to exclude the existence of nontrivial solutions of certain periods.

The following result represents an extension to neutral equations of a well known result for retarded equations due to Chow and Mallet-Paret [8].

LEMMA 3.2.   (3.1) *has no nonconstant periodic solution of period $2r/m$ for any positive integer m.*

PROOF.   Clearly, it suffices to prove that (3.1) has no nonconstant periodic solution of period $2r$. By way of contradiction, we assume that there is a nonconstant periodic solution $x(t)$ of period $2r$. Let $y(t) = x(t-r)$. Then $(x(t), y(t))$ satisfies the following system of ordinary differential equations

$$\begin{cases} \dfrac{d}{dt}\,[x(t)-qy(t)] = F(q, x(t), y(t)), \\[2mm] \dfrac{d}{dt}\,[y(t)-qx(t)] = F(q, y(t), x(t)), \end{cases}$$

where

$$F(q, x, y) = -ax - bqy - g(x-qy).$$

Set

$$u(t) = x(t) - qy(t), \quad v(t) = y(t) - qx(t),$$

that is,

$$x(t) = \frac{u(t)+qv(t)}{1-q^2}, \quad y(t) = \frac{qu(t)+v(t)}{1-q^2}.$$

We can show that $(u(t), v(t))$ satisfies the following system of ordinary differential equations

(3.6)
$$\begin{cases} \dot{u}(t) = G(q, u(t), v(t)), \\ \dot{v}(t) = G(q, v(t), u(t)), \end{cases}$$

where

$$G(q, u, v) = F(q, (1-q^2)^{-1}(u+qv), (1-q^2)^{-1}(v+qu)).$$

The diagonal $\Delta = \{(u, v) \in R^2; u = v\} \cong R$ is invariant for (3.6) due to the symmetry of u and v in (3.6). Moreover, any vector field on Δ cannot have nonconstant periodic solutions. So $(u(t), v(t)) \notin \Delta$ for all t. Without loss of generality, we may assume that $u(t) < v(t)$ for all $t \in R$. This implies that $u(t-r) < v(t-r)$ for $t \in R$. However, we have

$$v(t-r) = y(t-r) - qx(t-r) = x(t-2r) - qx(t-r) = x(t) - qy(t) = u(t)$$

and

$$u(t-r) = x(t-r) - qy(t-r) = x(t-r) - qx(t-2r) = y(t) - qx(t) = v(t).$$

This contradicts the assumption $u(t) < v(t)$ for $t \in R$, and thus the proof is complete.

REMARK 3.3. The same argument can be applied to exclude the existence of nontrivial periodic solutions of period $2r$ for the following neutral equation

(3.7)
$$\frac{d}{dt}[x(t) - qx(t-r)] = h(x(t - N_1 r), x(t - N_2 r), \dots, x(t - N_k r)),$$

where $h: R^k \to R$ is a continuously differentiable function, and N_i are integers for $i = 1, \dots, k$. In this case,

$$G(q, u, v) = h(f_1(u, v), f_2(u, v), \dots, f_k(u, v)),$$

$$f_j(u, v) = \begin{cases} (u + qv)/(1 - q^2) & \text{if } N_j \text{ is even}, \\ (v + qu)/(1 - q^2) & \text{if } N_j \text{ is odd}, \end{cases}$$

and $j = 1, \dots, k$. The general idea that periodic solutions of retarded equations of period $2r$ satisfy an associated system of ordinary differential equations has been employed before, see [8], [34] and references therein.

We now consider the existence of *a priori* bounds for periodic solutions of (3.1).

LEMMA 3.4. *Suppose that* $\lim_{z \to \pm\infty} g(z)/z = \infty$. *Then for any* $\delta \in (0, 1)$ *there exists* $M = M(\delta) > 0$ *such that if* $x(t)$ *is a periodic solution of* (3.1) *with* $q \in (0, \delta)$, *then* $|x(t)| \le M(\delta)$ *for* $t \in R$.

PROOF. Since $\lim_{z \to \pm\infty} g(z)/z = \infty$, for any $\delta \in (0, 1)$ there exists $M = M(\delta) > 0$ such that if $|z| > (1 - \delta)M(\delta)$, then

(3.8)
$$a + \frac{g(z)}{z} > \frac{(a+b)\delta}{1 - \delta}.$$

Suppose that x is a periodic solution of (3.1). Let $y(t) = x(t) - qx(t-r)$ for $t \in R$. Then we can easily prove that $x(t) = \sum_{i=0}^{\infty} q^i y(t - ir)$ for $t \in R$. So by (3.1), $y(t)$ satisfies the following scalar retarded equation with unbounded delay

$$\dot{y}(t) = -ay(t) - g(y(t)) - (a+b)\sum_{i=1}^{\infty} q^i y(t - ir).$$

Let $\tau \in R$ be given so that $y^2(\tau) = \max_{s \in R} y^2(s)$ and assume that y is not identical to zero. Then $2\dot{y}(\tau)y(\tau) = 0$. Therefore

$$a + \frac{g(y(\tau))}{y(\tau)} = -(a+b)\sum_{i=1}^{\infty} q^i \frac{y(\tau - ir)}{y(\tau)}.$$

From $|y(\tau - ir)| \le |y(\tau)|$, it follows that

$$a + \frac{g(y(\tau))}{y(\tau)} \le (a+b)\sum_{i=1}^{\infty} q^i = \frac{(a+b)q}{1 - q} \le \frac{(a+b)\delta}{1 - \delta}.$$

So by the choice of $M(\delta)$, $|y(\tau)| \le (1 - \delta)M(\delta)$. Hence, for $t \in R$,

$$|x(t)| = \left| \sum_{i=0}^{\infty} q^i y(t - ir) \right| \leq \sum_{i=0}^{\infty} q^i (1 - \delta) M(\delta) \leq \frac{1 - \delta}{1 - q} M(\delta) \leq M(\delta) .$$

This completes the proof.

REMARK 3.5.   The technique employed in the above proof, to associate a neutral equation with a retarded equation of unbounded delay in the consideration of stability and boundedness of solutions to the neutral equation, is motivated by the work of Staffans [60] and has been used before in [21], [22]. Moreover, a result in a forthcoming paper by Haddcok, Krisztin, Terjeki and Wu [20] can also be applied to obtain *a priori* bounds for periodic solutions of (3.1).

Now we are in a position to state our main result.

THEOREM 3.6.   *Suppose* $\lim_{z \to \pm \infty} g(z)/z = \infty$, *(3.2) and (3.5) are satisfied. Let* q_n *be as given in* (ii) *of Lemma 3.1. Then for any* $n \geq 2$ *and* $q \in (q_n, 1)$, *system (3.1) has* $n - 1$ *periodic solutions* $x_{k,q}$, $k = 2, \ldots, n$, *with periods* $p_{k,q}$ *satisfying*

(i) $p_{k,q} \in (2r/k, 2r/(k-1))$ *for* $k = 2, \ldots, n$, *if* $\sqrt{(a+c)(b-c)} < \pi/2r$;

(ii) $p_{k,q} \in (2r/k, 2r/(k-1))$ *for* $k = 2, \ldots, n$, *if* $\sqrt{(a+c)(b-c)} \in (\pi/2r + m\pi/r, \pi/2r + (m+1)\pi/r)$ *for some nonnegative integer* m;

(iii) $p_{k,q} \in (2r/(k+1), 2r/k)$ *for* $k = 2, \ldots, n$, *if* $\sqrt{(a+c)(b-c)} = \pi/2r$;

(iv) $p_{k,q} \in (2r/k, 2r/(k-1))$ *for* $k = 2, \ldots, m$ *(if* $m \geq 2$*) and* $p_{k,q} \in (2r/(k+1), 2r/k)$ *for* $k = m+1, \ldots, n$, *if* $\sqrt{(a+c)(b-c)} = \pi/2r + m\pi/r$ *for some positive integer* m.

PROOF.   We consider (i) only. Other cases can be dealt with analogously. First of all, we choose $\alpha^* \in (0, 1)$ such that

(3.9) $$\inf_{x \neq 0} \frac{g(x)}{x} > -\frac{a - \alpha^* b}{1 + \alpha^*} ,$$

where the existence of such α^* is guaranteed by (3.2). For any given integer $n \geq 2$, we consider the following neutral equation

(3.10) $$\frac{d}{dt} [x(t) - Q_n(\alpha) x(t - r)] = -ax(t) - bQ_n(\alpha) x(t - r) - g[x(t) - Q_n(\alpha) x(t - r)] ,$$

where

(3.11) $$Q_n(\alpha) = \frac{q_{n+1} + \alpha^*}{\pi} \left(\arctan \alpha + \frac{\pi}{2} \right) - \alpha^* .$$

Clearly, $Q_n : (-\infty, \infty) \to (-\alpha^*, q_{n+1})$ is an increasing and continuous bijection. Therefore, if we let $B(\alpha, \phi) = Q_n(\alpha) \phi(-r)$ for $\phi \in C([-r, 0]; R)$, then B satisfies the Lipschitz condition (2.2) with $k = \max\{\alpha^*, q_{n+1}\} < 1$. By (3.9) we can easily show that for any $\alpha \in (-\infty, +\infty)$, $(\alpha, 0)$ is the only stationary point of (3.10) and 0 is never a characteristic

value, that is, all stationary points of (3.10) are nonsingular.

Let $\alpha_k = Q_n^{-1}(q_k)$ for $k = 1, \ldots, n$. Then $(\alpha_k, 0)$ are isolated centers of (3.10). Except at these isolated centers, there are no pure imaginary characteristic values of (3.10). Moreover, since $Q_n'(\alpha) = q_{n+1} + \alpha^*/\pi(1 + \alpha^2) > 0$, by (iii) of Lemma 3.1, we can show that if $u_k^*(\alpha) + iv_k^*(\alpha)$ is the characteristic root of (3.10), α close to α_k, such that $u_k^*(\alpha_k) + iv^*(\alpha_k) = i\beta_k$, then $(d/d\alpha)u_k^*(\alpha)|_{\alpha = \alpha_k} > 0$. So the crossing number of $(\alpha_k, 0, \beta_k)$ satisfies $\gamma(\alpha_k, 0, \beta_k) = -1$ for $k = 1, 2, \ldots, n$.

We now consider the connected component $C(\alpha_2, 0, 2\pi/\beta_2)$. By Theorem 2.2, $C(\alpha_2, 0, 2\pi/\beta_2)$ is unbounded. Let $z_n \in R$ denote the unique number such that $Q_n(z_n) = 0$. Then at $\alpha = z_n$, equation (3.10) degenerates to the following ordinary differential equation

$$\dot{x}(t) = -ax(t) - g(x(t)),$$

which has no nonconstnat periodic solutions since the origin is globally asymptotically stable under the assumption (3.2). So we can concluded that

$$C\left(\alpha_2, 0, \frac{2\pi}{\beta_2}\right) \subset BC(R; R) \times (z_n, \infty) \times [0, \infty).$$

Since $2\pi/\beta_2 \in (r, 2r)$ and for $\alpha \in (z_n, \infty)$, $Q_n(\alpha) \in (0, q_{n+1})$, by Lemmas 3.2 and 3.4, there exist $M_n = M(q_{n+1}) > 0$ such that

$$C\left(\alpha_2, 0, \frac{2\pi}{\beta_2}\right) \subset BC(M_n) \times (z_n, \infty) \times [r, 2r],$$

where

$$BC(M_n) = \left\{ y \in BC(R; R);\ \sup_{t \in R} |y(t)| < M_n \right\}.$$

Therefore, the projection of $C(\alpha_2, 0, 2\pi/\beta_2)$ onto the parameter (α) space must be an unbounded interval containing $[\alpha_2, \infty)$. Note that $Q_n(\alpha_2, \infty) = (q_2, q_{n+1})$. So for all $q \in (q_2, q_{n+1})$, system (3.1) has a nonstant periodic solution $x_{2,q}$ with period $p_{2,q}$ in $(r, 2r)$.

Applying a similar argument to $C(\alpha_k, 0, 2\pi/\beta_k)$, $k = 3, \ldots, n$, we can show the existence of periodic solutions $x_{k,q}$ with periods $p_{k,q} \in (2r/k, 2r/(k-1))$. This completes the proof.

To obtain existence of periodic solutions of periods greater than $2r$, we recall a result about circulant matrices. Let $a_1, a_2, \ldots, a_n$ be given numbers. A square matrix of order n

$$
A = \begin{pmatrix}
a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\
a_n & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\
a_{n-1} & a_n & a_1 & \cdots & a_{n-3} & a_{n-2} \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
a_3 & a_4 & a_5 & \cdots & a_1 & a_2 \\
a_2 & a_3 & a_4 & \cdots & a_n & a_1
\end{pmatrix}
$$

is called a circulant matrix and will be denoted by $A = \text{circ}(a_1, a_2, \ldots, a_n)$. For a circulant matrix $A = \text{circ}(a_1, a_2, \ldots, a_n)$, Nussbaum [52] proved the following result

$$
(3.12) \qquad \inf\left\{ \langle Ay, y \rangle;\ y \in R^n,\ \sum_{i=1}^{n} y_i^2 = 1 \right\} = \min\left\{ \text{Re}\left(\sum_{j=1}^{n} a_j z^{j-1} \right);\ z \in C,\ z^n = 1 \right\}.
$$

We now introduce the following quantities

$$
c_n(a, b, q) = \inf\left\{ \text{Re}\left(\frac{a + b\alpha^n}{1 - \alpha^n} + \frac{a + b}{1 - \alpha^n} \frac{1 - (\alpha z)^{n-1}}{1 - \alpha z} \alpha z \right);\ z \in C,\ z^n = 1,\ 0 < \alpha < q \right\},
$$

where $q \in (0, 1)$ and n is any natural number.

LEMMA 3.7. *If there exists $q^* \in (0, 1)$ and $n \geq 1$ such that*

$$
(3.13) \qquad\qquad \inf_{y \neq 0} \frac{g(y)}{y} + c_n(a, b, q^*) > 0,
$$

then (3.1) *has no nonconstant periodic solutions of period nr for all $q \in (0, q^*)$.*

PROOF. Suppose that $x(t)$ is a nr-periodic solution of (3.1). Let

$$
(3.14) \qquad
\begin{cases}
y_1(t) = x(t) - q x(t - r), \\
y_2(t) = y_1(t - r) = x(t - r) - q x(t - 2r), \\
y_3(t) = y_1(t - 2r) = x(t - 2r) - q x(t - 3r), \\
\qquad\vdots \\
y_n(t) = y_1(t - (n-1)r) = x(t - nr + r) - q x(t).
\end{cases}
$$

Then

$$
\begin{aligned}
x(t) &= y_1(t) + q x(t - r) \\
&= y_1(t) + q y_2(t) + q^2 x(t - 2r) \\
&\ \ \vdots \\
&= y_1(t) + q y_2(t) + q^2 y_3(t) + \cdots + q^{n-1} y_n(t) + q^n x(t).
\end{aligned}
$$

So

$$x(t) = \frac{1}{1-q^n} \left[y_1(t) + qy_2(t) + q^2 y_3(t) + \cdots + q^{n-1} y_n(t) \right].$$

Using the symmetry, we get

$$\begin{cases} x(t-r) = \dfrac{1}{1-q^n} \left[y_2(t) + qy_3(t) + q^2 y_4(t) + \cdots + q^{n-2} y_n(t) + q^{n-1} y_1(t) \right], \\ \quad\vdots \\ x(t-nr+r) = \dfrac{1}{1-q^n} \left[y_n(t) + qy_1(t) + q^2 y_2(t) + \cdots + q^{n-2} y_{n-2}(t) + q^{n-1} y_{n-1}(t) \right], \end{cases}$$

that is,

$$(3.15) \qquad \begin{pmatrix} x(t) \\ x(t-r) \\ \vdots \\ x(t-nr+r) \end{pmatrix} = \frac{1}{1-q^n} \operatorname{circ}(1, q, \ldots, q^{n-1}) \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}.$$

Notice that (3.1) can be rewritten as

$$\frac{d}{dt} \left[x(t) - qx(t-r) \right] = -g[x(t) - qx(t-r)] + b[x(t) - qx(t-r)] - (a+b)x(t).$$

Therefore, replacing t by t, $t-r$, $t-2r$, ..., $t-nr+r$, respectively in (3.1) and using (3.14), we can show that $(y_1(t), y_2(t), \ldots, y_n(t))$ satisfies the following cyclic system of ordinary differential equations

$$(3.16) \qquad \frac{d}{dt} \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix} = \begin{pmatrix} -g(y_1) + by_1 \\ -g(y_2) + by_2 \\ \vdots \\ -g(y_n) + by_n \end{pmatrix} - \frac{a+b}{1-q^n} \operatorname{circ}(1, q, \ldots, q^{n-1}) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

Let

$$H(y) = \begin{pmatrix} g(y_1) \\ g(y_2) \\ \vdots \\ g(y_n) \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad A = \operatorname{circ}(a_1, a_2, \ldots, a_n),$$

where

$$\begin{cases} a_1 = \dfrac{a + bq^n}{1-q^n}, \\ a_j = \dfrac{(a+b)q^{j-1}}{1-q^n}, \qquad 2 \le j \le n. \end{cases}$$

Then (3.16) can be rewritten as

$$\dot{y}(t) = -H(y) - Ay \, .$$

Let $V(y) = \langle y, y \rangle / 2$. Then

$$\frac{d}{dt} V(y(t)) = -\langle H(y), y \rangle - \langle Ay, y \rangle$$

$$\leq -\langle H(y), y \rangle - c(A) \langle y, y \rangle$$

$$= -\sum_{i=1}^{n} g(y_i) y_i - c(A) \sum_{i=1}^{n} y_i^2 \, ,$$

where by the result of Nussbaum, we have

$$c(A) = \min \left\{ \mathrm{Re} \left(\sum_{j=1}^{n} a_j z^{j-1} \right); \ z \in C, \ z^n = 1 \right\} .$$

Observe that

$$\sum_{j=1}^{n} a_j z^{j-1} = \frac{a + bq^n}{1 - q^n} + \sum_{j=2}^{n} \frac{(a+b)(qz)^{j-1}}{1 - q^n}$$

$$= \frac{a + bq^n}{1 - q^n} + \frac{a+b}{1 - q^n} \frac{1 - (qz)^{n-1}}{1 - qz} \, qz \, .$$

Therefore $c(A) \geq c_n(a, b, q^*)$. Consequently,

$$\frac{d}{dt} V(y(t)) \leq -2 \left(\inf_{y \neq 0} \frac{g(y)}{y} + c_n(a, b, q^*) \right) V(y(t)) \, ,$$

which implies that

$$V(y(t)) \leq V(y(0)) \exp \left(-2 \left(\inf_{y \neq 0} g(y)/y + c_n(a, b, q^*) \right) t \right) \to 0$$

as $t \to \infty$, that is, the only nr-periodic solution of (3.1) is the trivial solution. This completes the proof.

REMARK 3.8. The ideas, to associate a periodic solution of large periods to a certain *retarded* equation with a cyclic system of ordinary differential equations and then to use results about circulant matrices in conjunction with appropriate Liapunov functions to exclude the existence of nontrivial periodic solutions of the associated cyclic system, have been used before by Nussbaum and Potter [52], [53].

THEOREM 3.9. *Suppose* $\lim_{z \to \pm \infty} g(z)/z = \infty$, (3.2) *and* (3.5) *are satisfied.*
(i) *In the case where* $\sqrt{(a+c)(b-c)} < \pi/2r$, *let* $n_0 > 5$ *be the integer such that* $(n_0 - 1)r < 2\pi/\sqrt{(a+c)(b-c)} \leq n_0 r$. *If there exist an integer* $n \geq n_0$ *and a real*

       number $q^* \in (q_1, 1)$ *such that* (3.13) *holds, then* (3.1) *has a periodic solution* $x_{1,q}$
       of period $p_{1,q} \in (4r, nr)$ *for all* $q \in (q_1, q^*)$;

(ii)   *In the case where* $\pi/2r + m\pi/r < \sqrt{(a+c)(b-c)} \le \pi/2r + (m+1)\pi/r$ *for some*
       nonnegative integer m, *if there exists an integer* $n \ge 4$ *and a number* $q^* \in (q_1, 1)$
       such that (3.13) *holds, then* (3.1) *has a periodic solution* $x_{1,q}$ *of period* $p_{1,q} \in (2r, nr)$
       for all $q \in (q_1, q^*)$.

PROOF.   One can easily show that in the case (i),

$$4r < \frac{2\pi}{\beta_1} < \frac{2\pi}{\sqrt{(a+c)(b-c)}} \le n_0 r \le nr ,$$

and in the case (ii),

$$2r < \frac{2\pi}{\beta_1} < 4r .$$

So employing the same argument to $C(\alpha_1, 0, 2\pi/\beta_1)$ as that for Theorem 3.6, we can prove the conclusions.

REMARK 3.10.   In cases (i) and (ii) of Theorem 3.1, we proved the existence of periodic solutions of period greater than $2r$ for equation (3.1) with $q \in (q_1, q^*)$. We borrow the terminology *slowly oscillating* for these periodic solutions, though whether the separation between consecutive zeros of these periodic solutions is greater than r is still an interesting problem.

REMARK 3.11.   The existence of periodic solutions of period less than $2r$ for equation (3.1) with $q \in (q_2, 1)$ has been guaranteed in all cases. We say these solutions to be *rapidly oscillating*. It has been observed, both numerically and theoretically, that slowly oscillating periodic solutions are stable and rapidly oscillating periodic solutions seem to be unstable for many retarded equations. It would be interesting to find out whether the same phenomenon happens to neutral equations.

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