Tôhoku Math. J.
44 (1992), 581–595

GLOBAL DENSITY THEOREM FOR A FEDERER MEASURE

HIROSHI SATO

(Received November 18, 1991, revised May 14, 1992)

Abstract. The local and global density theorems for the Lebesgue measure in a Euclidean space play a fundamental role in calculus. On the other hand Federer [5] proved a local density theorem for a measure with a doubling condition on a metric space.

The aim of this paper is to prove a global density theorem for a measure with a doubling condition and a class of integrable functions on a metric space. As a special case this theorem also gives a simple and constructive proof to Federer's local density theorem.

A typical example of the above measures is the Hausdorff measure on a self-similar set.

1. Introduction. Throughout the paper $E=(E, d)$ denotes a metric space, $B(x, r)$ for $x \in E$ and $r>0$ the closed ball $\{y \in E; d(x, y) \leq r\}$ and $U(x, r)$ the open ball $\{y \in E; d(x, y)<r\}$, and λ a measure defined on a σ-algebra $\mathscr{B}_\lambda$ of subsets of E such that $\mathscr{B}_\lambda$ includes the Borel field $\mathscr{B}(E)$ of E,

$$\lambda(A)=\inf\{\lambda(G); A \subset G, G \text{ open}\}, \qquad A \in \mathscr{B}_\lambda,$$

and $\lambda(B(x, r))<\infty$ for any $r>0$ and λ-almost all $x \in E$.

For a real measure μ on $(E, \mathscr{B}(E))$, $(d\mu/d\lambda)(x)$ denotes the Radon-Nikodym derivative *in the sense of the Lebesgue decomposition of* μ *with respect to* λ, that is,

$$d\mu(x)=\frac{d\mu}{d\lambda}(x)d\lambda(x)+d\mu_s(x),$$

where $d\mu_s(x)$ is singular with respect to $d\lambda(x)$.

When λ is the Lebesgue measure, the following density theorems are well-known and fundamental in calculus.

THEOREM 1 (Local density theorem, see for example Dunford and Schwartz [4]). *Let λ be the Lebesgue measure on $E=\boldsymbol{R}^n$. Then we have*

$$\frac{d\mu}{d\lambda}(x)=\lim_{r \downarrow 0} \frac{\mu(B(x, r))}{\lambda(B(x, r))}, \quad a.e.\ (d\lambda),$$

for any real measure μ on $\boldsymbol{R}^n$.

THEOREM 2 (Global density theorem). *Let λ be the Lebesgue measure on $E=\boldsymbol{R}^n$,*

1991 *Mathematics Subject Classification.* Primary 28A15.

φ a λ-integrable function such that $\int_{\boldsymbol{R}^n}\varphi(x)d\lambda(x)=1$ and

$$\sup_{x\in\boldsymbol{R}^n}(1+\|x\|^a)|\varphi(x)|<\infty\,,\qquad \textit{for some}\quad a>n\,,$$

where $\|x\|$ denotes the Euclidean norm, and define

$$\varphi_T(y,x)=T^n\varphi(T(y-x))\,,\qquad x\in\boldsymbol{R}^n\,,\quad T>1\,.$$

Then we have

$$\frac{d\mu}{d\lambda}(x)=\lim_{T\to\infty}\int_{\boldsymbol{R}^n}\varphi_T(y,x)d\mu(y)\,,\quad \textit{a.e. }(d\lambda)\,,$$

for any real measure μ on $\boldsymbol{R}^n$ and the exceptional null set is chosen independently of the choice of φ.

van der Vaart [8], by means of the local density theorem, proved the above theorem in a more general form, and Bourgain and Sato [1] gave a simple and direct proof to Theorem 2.

On the other hand Theorems 2.9.7, 2.9.15 and 2.9.17 of Federer [5] imply the following local density theorem.

THEOREM 3 (Federer [5]). *Assume that*

$$\limsup_{r\downarrow 0}\frac{\lambda(B(x,5r))}{\lambda(B(x,r))}<\infty\,,\quad \textit{a.e. }(d\lambda)\,.$$

Then we have

$$\frac{d\mu}{d\lambda}(x)=\lim_{r\downarrow 0}\frac{\mu(B(x,r))}{\lambda(B(x,r))}\,,\quad \textit{a.e. }(d\lambda)\,,$$

for any real measure μ on E.

The aim of this paper is to prove a global density theorem for a class of measures similar to those defined by Federer and a class of integrable functions.

DEFINITION 1. The *β-function* of λ is the function defined by

$$\beta(x)=\sup_{r>0}\frac{\lambda(B(x,3r))}{\lambda(B(x,r))}\,,$$

with the convention $0/0=1$.

Note that β is lower semi-continuous hence Borel measurable (see Federer [5, 2.9.14]).

DEFINITION 2. λ is called *a Federer measure* if

$$\beta(x) < \infty\ , \quad \text{a.e. } (d\lambda)\ .$$

REMARK. It is easy to show that λ is a Federer measure if and only if there exists a constant $A > 1$ such that

$$\sup_{r>0} \frac{\lambda(B(x, Ar))}{\lambda(B(x, r))} < \infty\ , \quad \text{a.e. } (d\lambda)\ .$$

DEFINITION 3.

$$\gamma_x(r) = \lambda(B(x, r))\ , \qquad r \geq 0\ , \quad x \in E\ .$$

$$H(x) = \log \beta(x)/\log 3\ , \qquad x \in E\ .$$

DEFINITION 4. A family of λ-integrable functions

$$\Phi = \{\varphi_T(\cdot, x);\ x \in E,\ T > 1\}$$

is said to be *admissible* if it satisfies the following conditions.

(H.1) $\int_E \varphi_T(y, x) d\lambda(y) = 1$, $x \in E$, $T > 1$.

(H.2) There exists a λ-measurable function $\alpha = \alpha(x) > H(x)$ such that

$$Q_\Phi = Q_\Phi(x) = \sup_{T>1,\, y \in E} |\varphi_T(y, x)|\, \gamma_x\left(\frac{1}{T}\right)[1 + (Td(x, y))^\alpha] < +\infty\ , \qquad \text{if} \quad \beta(x) < \infty\ .$$

The following is our main theorem:

THEOREM 4. *Let λ be a Federer measure on E, $\Phi = \{\varphi_T(\cdot, x);\ x \in E,\ T > 1\}$ be an admissible family, and assume that λ is a Radon measure, that is,*

$$\lambda(A) = \sup\{\lambda(K);\ K \subset A,\ K \text{ compact}\}\ , \quad A \in \mathscr{B}_\lambda\ .$$

Then for any real measure μ on E we have

$$\frac{d\mu}{d\lambda}(x) = \lim_{T \to \infty} \int_E \varphi_T(y, x) d\mu(y)\ , \quad \text{a.e. } (d\lambda)\ ,$$

where the exceptional null set can be chosen independently of the choice of Φ.

2. Examples.

(1) Examples of Federer measures.

EXAMPLE 1 (Lebesgue measure). The Lebesgue measure on $E = \boldsymbol{R}^n$ is a Federer measure with $\beta(x) \equiv 3^n$.

DEFINITION 5. A Borel measure λ on (E, d) is said to be *self-similar* if there exists a positive number H such that

$$0 < c(\lambda) := \inf_{x \in E} \inf_{0 < r < d(E)} \frac{\lambda(B(x, r))}{r^H} \leq C(\lambda) := \sup_{x \in E} \sup_{r > 0} \frac{\lambda(B(x, r))}{r^H} < \infty\ ,$$

where $d(E)$ $(\leq \infty)$ denotes the diameter of E.

Hutchinson [7] showed that the Hausdorff measure on a self-similar set with "the open set condition" is a self-similar measure.

EXAMPLE 2 (Self-similar measure). A self-similar measure on E is a Federer measure.

Indeed, using the notation in Definition 5, we have

$$\beta(x) = \sup_{r>0} \frac{\lambda(B(x, 3r))}{\lambda(B(x, r))} \leq 3^H \frac{C(\lambda)}{c(\lambda)} < \infty .$$

EXAMPLE 3 (Bernoulli measure). Let $S = \{1, 2, 3, \ldots, p\}$ be a finite set and define a metric d on the product space $E = S^\infty$ as follows: For $x = \{x_n\}_{n=1}^\infty$ and $y = \{y_n\}_{n=1}^\infty \in E$ define

$$n(x, y) = \begin{cases} 0\,, & \text{if} \quad x_1 \neq y_1\,, \\ \sup\{n \geq 1;\, x_k = y_k,\, 1 \leq \forall k \leq n\} & \text{if} \quad x_1 = y_1\,, \end{cases}$$

fix a positive number $a > 1$ and define $d(x, y) = a^{-n(x,y)}$, $x, y \in E$.

On the other hand, let λ_0 be a probability measure on S such that $\lambda_0(\{k\}) = 1/p$ for any $1 \leq k \leq p$. Then the product measure $\lambda = (\lambda_0)^\infty$ is a Federer measure.

Indeed, for any $x \in E$ and $r > 0$ such that $a^{-n} \leq r < a^{-(n-1)}$ we have

$$\lambda(B(x, r)) = \lambda(y \in E;\, d(x, y) \leq r) = \lambda(y = \{y_k\} \in E;\, y_k = x_k,\, 1 \leq k \leq n)\,,$$

so that $\lambda(B(x, r)) = 1/p^n$.

Define $H = \log p/\log a$. Then we have

$$a^{-H} = a^{-H}\left(\frac{a^H}{p}\right)^n \leq \frac{\lambda(B(x, r))}{r^H} \leq \left(\frac{a^H}{p}\right)^n = 1\,.$$

EXAMPLE 4. A finite Radon measure λ on E such that

$$\limsup_{r \downarrow 0} \frac{\lambda(B(x, 3r))}{\lambda(B(x, r))} < \infty\,, \quad \text{a.e. } (d\lambda(x))$$

is a Federer measure. The proof is easy.

EXAMPLE 5 (Invariance under the absolute continuity). A finite Borel measure μ on E absolutely continuous with respect to a Federer measure λ is also a Federer measure.

Indeed, we may assume that $d\mu(x) = f(x)d\lambda(x)$ where $f(x)$ is a non-negative λ-integrable function on E. Then by definition we have $\mu(x \in E;\, f(x) = 0) = 0$ and if $f(x) > 0$, by Theorem 4

$$\limsup_{r\downarrow 0}\frac{\mu(B(x,3r))}{\mu(B(x,r))}=\limsup_{r\downarrow 0}\frac{\dfrac{1}{\lambda(B(x,3r))}\displaystyle\int_{B(x,3r)}f(x)d\lambda(x)}{\dfrac{1}{\lambda(B(x,r))}\displaystyle\int_{B(x,r)}f(x)d\lambda(x)}\frac{\lambda(B(x,3r))}{\lambda(B(x,r))}$$

$$\le\sup_{r>0}\frac{\lambda(B(x,3r))}{\lambda(B(x,r))}<\infty\,,\quad \text{a.e. } (d\lambda(x))\,,$$

and then Example 4 shows that μ is a Federer measure.

REMARK (Existence of non-Federer measures). Davies [3] and Darst [2] showed that on a compact metric space there exist different probability measures that agree on balls.

(2) Examples of admissible families. Let λ be a Federer measure on E and γ_x, $\beta(x)$ and $H(x)$ be functions given in Definitions 1 and 3.

EXAMPLE 6. For $x, y\in E$ and $T>1$ define

$$\varphi_T(y,x)=\frac{1}{\lambda\left(B\left(x,\dfrac{1}{T}\right)\right)}\boldsymbol{I}_{[0,\frac{1}{T}]}(d(y,x))\,.$$

Then $\Phi=\{\varphi_T(\cdot,x);\,x\in E,\,T>1\}$ is an admissible family. Indeed, for any $\alpha(x)>H(x)$ we have

$$|\varphi_T(y,x)|\gamma_x\left(\frac{1}{T}\right)(1+T^\alpha d(y,x)^\alpha)=\frac{1}{\lambda\left(B\left(x,\dfrac{1}{T}\right)\right)}\boldsymbol{I}_{[0,\frac{1}{T}]}(d(y,x))\gamma_x\left(\frac{1}{T}\right)(1+T^\alpha d(y,x)^\alpha)$$

$$=\boldsymbol{I}_{[0,\frac{1}{T}]}(d(y,x))(1+T^\alpha d(y,x)^\alpha)\le\left(1+T^\alpha\left(\frac{1}{T}\right)^\alpha\right)=2<\infty\,.$$

EXAMPLE 7. For any $\alpha(x)>H(x)$ define

$$\varphi_T(y,x)=\left(\int_E\frac{1}{1+T^\alpha d(y,x)^\alpha}\,d\lambda(y)\right)^{-1}\frac{1}{1+T^\alpha d(y,x)^\alpha}\,,\qquad x,y\in E\,,\quad T>1\,.$$

Then $\Phi=\{\varphi_T(\cdot,x);\,x\in E,\,T>1\}$ is an admissible family. This follows from Lemma 2 below.

Combining Examples 2 and 7, we have the following theorem:

THEOREM 5. *Let λ be a self-similar measure on E given in Definition 5. Then for any real Borel measure μ on E we have for any positive number $\alpha>H$*

$$\frac{d\mu}{d\lambda}(x)=\lim_{T\to\infty}\left(\int_E \frac{1}{1+T^\alpha d(y,x)^\alpha}\,d\lambda(y)\right)^{-1}\int_E \frac{1}{1+T^\alpha d(y,x)^\alpha}\,d\mu(y) \quad a.e.\ (d\lambda),$$

where the exceptional null set can be chosen independently of the choice of α.

3. Preliminaries for the proof of Theorem 4. For the proof of Theorem 4, without loss of generality, we may assume:

(A.1) $0<\gamma_x(r)=\lambda(B(x,r))<\infty$, for any $r>0$ and any $x\in E$.

(A.2) $1\le\beta(x)<\infty$ for any $x\in E$.

LEMMA 1. *For any* $x\in E$, $r>0$ *and* $T>1$, *we have*

(1) $\gamma_x(Tr)\le\beta(x)T^{H(x)}\gamma_x(r)$,

(2) $\gamma_x\left(\frac{r}{T}\right)\ge\beta(x)^{-1}T^{-H(x)}\gamma_x(r)$.

PROOF. First we show (1). By definition we have

$$\gamma_x(3r)\le\beta(x)\gamma_x(r)\,,\quad \text{for any } x\in E\,,\ r>0\,.$$

For any $T>1$, considering $T=3^{\log T/\log 3}\le 3^{[\log T/\log 3]+1}$ ($[t]$ is the largest integer which does not exceed t), we have

$$\gamma_x(Tr)\le\gamma_x(3^{[\log T/\log 3]+1}r)\le\beta(x)^{[\log T/\log 3]+1}\gamma_x(r)\le\beta(x)^{(\log T/\log 3)+1}\gamma_x(r)=\beta(x)T^{H(x)}\gamma_x(r)\,.$$

We obtain (2) by replacing r by r/T in (1).

DEFINITION 6. For any λ-measurable function $\alpha=\alpha(x)>H=H(x)$ define

$$f_T^\alpha(y,x)=\frac{1}{1+T^\alpha d(y,x)^\alpha}\,,\qquad x,y\in E\,,\quad T>1\,,$$

$$c_x^\alpha(T)=\int_E f_T^\alpha(y,x)d\lambda(y)\,,$$

$$F_T^\alpha(y,x)=c_x^\alpha(T)^{-1}f_T^\alpha(y,x)\,,\qquad x,y\in E\,,\quad T>1\,.$$

LEMMA 2. *For any* $x\in E$, $T>1$ *we have*

$$\frac{1}{2}\gamma_x\left(\frac{1}{T}\right)\le c_x^\alpha(T)\le\frac{2\alpha(x)\beta(x)}{\alpha(x)-H(x)}\gamma_x\left(\frac{1}{T}\right)<+\infty\,.$$

PROOF. For $\gamma(r)=\gamma_x(r)$ we have

$$c_x^\alpha(T)=\int_E f_T^\alpha(y,x)d\lambda(y)=\int_E\frac{1}{1+T^\alpha d(y,x)^\alpha}\,d\lambda(y)=\lim_{R\to\infty}\int_{(0,R]}\frac{d\gamma(r)}{1+T^\alpha r^\alpha}+\gamma(0)$$

$$= \lim_{R\to\infty} \left\{ \frac{\gamma(R)}{1+T^\alpha R^\alpha} + \alpha T^\alpha \int_0^R \frac{r^{\alpha-1}\gamma(r)}{(1+T^\alpha r^\alpha)^2}\, dr \right\}.$$

By Lemma 1 and the monotonicity of γ, the right hand side

$$\leq \lim_{R\to\infty} \left\{ \frac{\beta R^H \gamma(1)}{1+T^\alpha R^\alpha} + \alpha \int_0^R \frac{T^\alpha r^{\alpha-1}\gamma(r)}{(1+T^\alpha r^\alpha)^2}\, dr \right\}$$

$$= \alpha \int_0^{1/T} \frac{T^\alpha r^{\alpha-1}\gamma(r)}{(1+T^\alpha r^\alpha)^2}\, dr + \alpha \int_{1/T}^{\infty} \frac{T^\alpha r^{\alpha-1}\gamma(r)}{(1+T^\alpha r^\alpha)^2}\, dr = I_1 + I_2\,,$$

where

$$I_2 = \alpha \int_{1/T}^{\infty} \frac{T^\alpha r^{\alpha-1}\gamma(r)}{(1+T^\alpha r^\alpha)^2}\, dr$$

$$\leq \alpha\beta \int_{1/T}^{\infty} \frac{T^\alpha r^{\alpha-1} T^H r^H}{(1+T^\alpha r^\alpha)^2}\, dr\, \gamma\left(\frac{1}{T}\right)$$

$$\leq \alpha\beta \int_1^{\infty} \frac{r^{\alpha+H-1}}{(1+r^\alpha)^2}\, dr\, \gamma\left(\frac{1}{T}\right) \leq \alpha\beta \int_1^{\infty} \frac{dr}{r^{\alpha-H+1}}\, dr\, \gamma\left(\frac{1}{T}\right)$$

$$= \frac{\alpha\beta}{\alpha - H}\, \gamma\left(\frac{1}{T}\right).$$

$$I_1 = \alpha \int_0^{1/T} \frac{T^\alpha r^{\alpha-1}\gamma(r)}{(1+T^\alpha r^\alpha)^2}\, dr$$

$$\leq \alpha \int_0^{1/T} \frac{T^\alpha r^{\alpha-1}}{(1+T^\alpha r^\alpha)^2}\, dr\, \gamma\left(\frac{1}{T}\right)$$

$$= \alpha \int_0^1 \frac{r^{\alpha-1}}{(1+r^\alpha)^2} dr\, \gamma\left(\frac{1}{T}\right) = \frac{1}{2}\, \gamma\left(\frac{1}{T}\right) \leq \frac{\alpha\beta}{\alpha-H}\, \gamma\left(\frac{1}{T}\right).$$

On the other hand we have

$$c_x^\alpha(T) = \alpha \int_0^{\infty} \frac{T^\alpha r^{\alpha-1}\gamma(r)}{(1+T^\alpha r^\alpha)^2}\, dr \geq \alpha \int_{1/T}^{\infty} \frac{T^\alpha r^{\alpha-1}\gamma(r)}{(1+T^\alpha r^\alpha)^2}\, dr$$

$$= \alpha \int_1^{\infty} \frac{r^{\alpha-1}\gamma(r/T)}{(1+r^\alpha)^2}\, dr \geq \alpha \int_1^{\infty} \frac{r^{\alpha-1}}{(1+r^\alpha)^2}\, dr\, \gamma\left(\frac{1}{T}\right) = \frac{1}{2}\, \gamma\left(\frac{1}{T}\right).$$

LEMMA 3 (see Rudin [6, Lemma 7.3]). *Let A be a measurable subset of E such that $\lambda(A) < +\infty$ and $\mathcal{S} = \{U(x_\iota, r_\iota);\ \iota \in \Lambda\}$ an open covering of A, that is, $A \subset \bigcup_{\iota\in\Lambda} U(x_\iota, r_\iota)$. Then, if $M = \sup_\iota \beta(x_\iota) < +\infty$, there exist $U_1, U_2, \ldots, U_n \in \mathcal{S}$ such that $U_k \cap U_j = \varnothing$, $(k \neq j)$ and $\sum_{k=1}^n \lambda(U_k) > (2M)^{-1}\lambda(A)$.*

PROOF. Since λ is a Radon measure, there exists a compact subset K such that $K \subset A$ and $\lambda(K) > \lambda(A)/2$. Since K is compact, there exist $S_1, S_2, \ldots, S_p \in \mathscr{S}$ such that $\bigcup_{k=1}^{p} S_k \supset K$. Without loss of generality we may assume that $S_k = U(x_k, r_k)$, and $r_1 \geq r_2 \geq \cdots \geq r_p > 0$. Define $U_1, U_2, \ldots, U_n$ by

$$U_1 = S_1,\ k(2) = \min\{k > 1;\ S_k \cap U_1 = \varnothing\},$$

$$U_2 = S_{k(2)},\ k(3) = \min\{k > k(2);\ S_k \cap (U_1 \cup U_2) = \varnothing\},$$

$$U_3 = S_{k(3)},\ k(4) = \min\{k > k(2);\ S_k \cap (U_1 \cup U_2 \cup U_3) = \varnothing\},$$

$$\vdots$$

For any $1 \leq i \leq p$ there exists $1 \leq j \leq i$ such that $U_j \cap S_i \neq \varnothing$, $r_j \geq r_i$. Therefore we have $S_i \subset U(x_{k(j)}, 3r_{k(j)})$ so that

$$K \subset \bigcup_{j=1}^{p} S_j \subset \bigcup_{j=1}^{n} U(x_{k(j)}, 3r_{k(j)}).$$

Consequently we have

$$\frac{1}{2}\lambda(A) < \lambda(K) \leq \sum_{j=1}^{n} \lambda(U(x_{k(j)}, 3r_{k(j)})) \leq \sum_{j=1}^{n} \beta(x_{k(j)})\lambda(U(x_{k(j)}, r_{k(j)}))$$

$$\leq M \sum_{j=1}^{n} \lambda(U(x_{k(j)}, r_{k(j)})) = M \sum_{j=1}^{n} \lambda(U_j).$$

Let $\alpha = \alpha(x)$ be a λ-measurable function such that $\alpha(x) > H(x)$, and define for any finite Borel measure μ on E

$$D_\alpha\mu(x) = \limsup_{T \to \infty} \int_E F_T^\alpha(y, x) d\mu(y).$$

Let $\boldsymbol{Q}$ be the set of all rational numbers. Then we have

$$D_\alpha\mu(x) = \inf_{N \in \boldsymbol{N}} \sup_{T \in \boldsymbol{Q}, T > N} \int_E F_T^\alpha(y, x) d\mu(y),$$

so that $D_\alpha\mu(x)$ is a λ-measurable function.

The following lemma is fundamental in the proof of Theorem 4.

LEMMA 4. *Let λ be a Federer Radon measure, μ a finite Borel measure on E, and A a Borel subset of E such that $\mu(A) = 0$. Then there exists a Borel subset A_0 of E such that $\lambda(A \setminus A_0) = 0$ and*

$$D_\alpha\mu(x) = 0, \qquad x \in A_0.$$

PROOF. Without loss of generality we may assume that μ is a probability measure and, since λ is σ-finite, $\lambda(A) < \infty$.

In order to prove $\lambda(\{x\in A;\ D_\alpha\mu(x)>0\})=0$, it is enough to show $\lambda(A_p)=0$ for any p, where we define

$$A_p=\left\{x\in A;\ D_\alpha\mu(x)>\frac{1}{p},\ \beta(x)\le p,\ \alpha(x)>\left(1+\frac{1}{p}\right)H(x)\right\},\quad p\in \boldsymbol{N}.$$

Note that, by definition, for any $p\in \boldsymbol{N}$, $x\in A_p$ there exists a sequence $T_k=T_k(x)\nearrow+\infty$ such that

$$\int_E F^\alpha_{T_k}(y, x)d\mu(y)>\frac{1}{p},\qquad k\in \boldsymbol{N}.$$

For $x\in A_p$ and $T=T(x)>1$ assume

$$\int_E F^\alpha_T(y, x)d\mu(y)>\frac{1}{p},\qquad k\in \boldsymbol{N},$$

and define

$$\Gamma_k=\Gamma_k(x)=\{y\in E;\ e^{-k}\ge f^\alpha_T(y, x)>e^{-(k+1)}\},\qquad k=0, 1, 2, \cdots.$$

Then, since $0<f^\alpha_T(y, x)\le 1$, $x, y\in E$, we have $E=\bigcup_{k=0}^\infty \Gamma_k$ and by Lemma 2

$$\frac{1}{2p}\gamma_x\left(\frac{1}{T}\right)\le\frac{1}{p}c^\alpha_x(T)<\int_E f^\alpha_T(y, x)d\mu(y)\le\sum_{k=0}^\infty e^{-k}\mu(\Gamma_k).$$

Let $l=l(T)$ be the minimal integer that exceeds $\log(4pe/(e-1))-\log\gamma_x(1/T)$. Then we have

$$\frac{1}{2p}\gamma_x\left(\frac{1}{T}\right)\le\sum_{k=0}^\infty e^{-k}\mu(\Gamma_k)\le\sum_{k<l}e^{-k}\mu(\Gamma_k)+\sum_{k\ge l}e^{-k}$$
$$\le\sum_{k<l}e^{-k}\mu(\Gamma_k)+\frac{e^{-l}}{1-e^{-1}}\le\sum_{k<l}e^{-k}\mu(\Gamma_k)+\frac{1}{4p}\gamma_x\left(\frac{1}{T}\right),$$

so that

$$\sum_{k<l}e^{-k}\mu(\Gamma_k)\ge\frac{1}{4p}\gamma_x\left(\frac{1}{T}\right).$$

Define $b=(1-(H/\alpha))/2>0$ and $L=(1-e^{-b})/4p$. Then there exists a natural number $k(T)<l=l(T)$ such that $\mu(\Gamma_{k(T)})\ge L\gamma_x(1/T)e^{(1-b)k(T)}$. Otherwise we have

$$\sum_{k<l}e^{-k}\mu(\Gamma_k)<\sum_{k<l}L\gamma_x\left(\frac{1}{T}\right)e^{-bk}\le\frac{1}{4p}\gamma_x\left(\frac{1}{T}\right),$$

which is a contradiction.

For any $k\in \boldsymbol{N}$ we have

$$\Gamma_k(x) \subset \{y \in E; f_T^\alpha(y, x) > e^{-(k+1)}\} = \left\{y \in E; \frac{1}{1+T^\alpha d(y, x)^\alpha} > e^{-(k+1)}\right\}$$

$$\subset \{y \in E; d(y, x) < T^{-1} e^{(k+1)/\alpha}\} =: S_k(x)$$

and, since $e^{(k+1)/\alpha} > 1$, by Lemma 1

$$\lambda(S_k(x)) \leq \gamma_x\left(\frac{1}{T} e^{(k+1)/\alpha}\right) \leq \beta\gamma_x\left(\frac{1}{T}\right) e^{H(k+1)/\alpha},$$

so that

$$\mu(S_{k(T)}(x)) \geq \mu(\Gamma_{k(T)}) \geq L\gamma_x\left(\frac{1}{T}\right) e^{(1-b)k(T)} \geq \frac{L}{\beta} e^{(1-b)k(T) - H(k(T)+1)/\alpha} \lambda(S_{k(T)}(x)).$$

On the other hand since $x \in A_p$ we have $\beta \leq p$ and $\alpha(x) > (p+1)H(x)/p$, so that

$$(1-b)k - \frac{H}{\alpha}(k+1) = \frac{1}{2}\left(1 - \frac{H}{\alpha}\right)k - \frac{H}{\alpha} \geq -\frac{H}{\alpha} > -\frac{p}{p+1},$$

$$L \geq \frac{1}{4p}(1 - e^{-1/2(p+1)}),$$

and there exists a positive number

$$\delta = \frac{1}{4p^2}(1 - e^{-1/2(p+1)}) e^{-p/(p+1)}$$

independent of $x \in A_p$ and $k(T)$ such that

$$\mu(S_{k(T)}(x)) \geq \delta\lambda(S_{k(T)}(x)).$$

On the radius of $S_{k(T)}$ we have by Lemma 1(2)

$$\text{radius}(S_{k(T)}) = \frac{1}{T}\exp\left[\frac{k(T)+1}{\alpha}\right] \leq \frac{1}{T}\exp\left[\frac{l(T)+1}{\alpha}\right]$$

$$\leq \frac{1}{T}\exp\left[\frac{1}{\alpha}\left(\log\frac{4pe^3}{e-1} - \log\gamma_x\left(\frac{1}{T}\right)\right)\right] \leq \frac{1}{T}\left\{\frac{4pe^3}{e-1}\gamma_x\left(\frac{1}{T}\right)^{-1}\right\}^{1/\alpha}$$

$$\leq \frac{1}{T}\left(\frac{\beta e^3 T^H}{(e-1)\gamma_x(1)}\right)^{1/\alpha} \to 0 \quad \text{as} \quad T \to \infty.$$

Define a collection of open balls

$$\mathcal{S} = \left\{S_{k(T)}(x); x \in A_p, T > 1, \int_E F_T^\alpha(y, x) d\mu(y) > \frac{1}{p}\right\}.$$

Then for any $x \in A_p$ and any $\varepsilon > 0$ there exists $S \in \mathcal{S}$ with $x \in S$ such that radius$(S) < \varepsilon$

and $\mu(S) \geq \delta\lambda(S)$.

Since μ is a finite Borel measure and $\mu(A_p)=0$, for any $\varepsilon>0$ there exists an open subset $G \supset A_p$ such that $\mu(G)<\varepsilon$. Furthermore for any $x \in A_p$ there exists $S_x \in \mathscr{S}$ with $x \in S_x$ such that $S_x \subset G$. Then $\mathscr{W}=\{S_x; x \in A_p\}$ is an open covering of A_p and, since $\sup_{x \in A_p} \beta(x) \leq p$, by Lemma 3 there exists a mutually disjoint finite subcovering $U_1, U_2, \ldots, U_n$ such that

$$\sum_{k=1}^{n} \lambda(U_k) > \frac{1}{2p} \lambda(A_p) .$$

Consequently we have

$$\lambda(A_p) < 2p \sum_{k=1}^{n} \lambda(U_k) \leq \frac{2p}{\delta} \sum_{k=1}^{n} \mu(U_k) = \frac{2p}{\delta} \mu\left(\bigcup_{k=1}^{n} U_k\right) \leq \frac{2p}{\delta} \mu(G) < \frac{2p}{\delta} \varepsilon$$

and since ε is arbitrary we have $\lambda(A_p)=0$.

4. Proof of Theorem 4. Recall that we made the assumptions (A.1) and (A.2).

Let $\Phi=\{\varphi_T(y, x); y, x \in E, T>1\}$ be an admissible family which satisfies (H.1) and (H.2) for $\alpha=\alpha(x)>H(x)$ and μ a probability measure on E.

First we prove the theorem when $\varphi_T(y, x) \geq 0$, $y, x \in E$, $T>1$.

(First step) Define

$$(D_\Phi \mu)(x) = \lim_{T \to \infty} \int_E \varphi_T(y, x) d\mu(y) , \qquad x \in E ,$$

if the limit exists. By (H.2) and Lemma 1 we have

$$\varphi_T(y, x) \leq \frac{Q_\Phi(x)}{\gamma_x(1/T)(1+T^\alpha d(y, x)^\alpha)} \leq \frac{2\alpha\beta Q_\Phi}{\alpha-H} F_T^\alpha(y, x) , \qquad y, x \in E , \quad T>1 ,$$

so that

$$\limsup_{T \to \infty} \int_E \varphi_T(y, x) d\mu(y) \leq \frac{2\alpha\beta Q_\Phi}{\alpha-H} (D_\alpha \mu)(x) , \qquad x \in E .$$

Let A be a λ-measurable subset of E such that $\mu(A)=0$. Then by Lemma 4 there exists a Borel subset $A_0 \subset A$, which is determined only by $\{F_T^\alpha(y, x)\}$, such that $\lambda(A \setminus A_0)=0$ and

$$(D_\Phi \mu)(x)=0 , \qquad x \in A_0 .$$

(Second step) Denote the Lebesgue decomposition of μ with respect to λ by

$$d\mu(x) = f(x) d\lambda(x) + d\mu_s(x) ,$$

where μ_s is singular with respect to λ so that there exists a λ-measurable subset $J \subset E$ such that $\mu_s(J^c)=\lambda(J)=0$. From the first step there exists a λ-null Borel subset N_0 such that

$$(D_\Phi \mu_s)(x)=0\,, \qquad x\in J^c\setminus N_0\,.$$

Therefore in order to prove the theorem it is enough to show

$$(D_\Phi \mu)(x)=f(x)\,, \quad \text{a.e.}\,(d\lambda)\,,$$

if $d\mu(x)=f(x)d\lambda(x)$.

(Third step) Fix any $x_0\in E$. For any $m\in \boldsymbol{N}$, any $r\in \boldsymbol{Q}$ and any λ-measurable subset A define

$$\mu_r^m(A)=\int_{A\cap B(x_0,m)\cap\{y;f(y)>r\}}(f(y)-r)d\lambda(y)\,.$$

Then μ_r^m is a finite measure on E.

Put $E_r=\{y;f(y)\le r\}$. Then we have $\mu_r^m(E_r)=0$ so that by Lemma 4 there exists a λ-measurable subset $C_r^m\subset E_r$ such that $\lambda(E_r\setminus C_r^m)=0$ and

$$(D_\Phi \mu_r^m)(x)=\lim_{T\to\infty}\int_E \varphi_T(y,x)d\mu_r^m(y)=0\,, \qquad x\in C_r^m\,.$$

Define $C_r=\bigcap_{n\in \boldsymbol{N}}C_r^m$. We have $\lambda(E\setminus C_r)=0$. Furthermore define $N=\bigcup_{r\in \boldsymbol{Q}}(E_r\setminus C_r)$. Then we have $\lambda(N)=0$.

(Fourth step) For any $x\notin N\cup J\cup N_0$ fix any $r\in \boldsymbol{Q}$ such that $f(x)\le r$. By definition we have $x\in E_r$ and $x\notin N$ so that $x\in C_r$. By Lemma 1(2) we have for any $m>d(x,x_0)$

$$\begin{aligned}\int_{\{y;d(y,x_0)>m\}}\varphi_T(y,x)d\mu(y)&\le\frac{Q_\Phi}{\gamma_x(1/T)}\int_{\{y;d(y,x_0)>m\}}\frac{1}{1+T^\alpha d(y,x)^\alpha}d\mu(y)\\&\le\frac{Q_\Phi\beta}{\gamma_x(1)}\int_{\{y;d(y,x_0)>m\}}\frac{T^H}{1+T^\alpha d(y,x)^\alpha}d\mu(y)\\&\le\frac{Q_\Phi\beta}{\gamma_x(1)(m-d(x,x_0))^H}\sup_{u>0}\frac{u^H}{1+u^\alpha}\mu(y;d(y,x)>m)\,,\end{aligned}$$

so that for any $\varepsilon>0$ there exists $m_0=m(x,\varepsilon)\in \boldsymbol{N}$ such that

$$\inf_{T>1}\int_{B(x_0,m_0)}\varphi_T(y,x)d\mu(y)\ge\int_E\varphi_T(y,x)d\mu(y)-\varepsilon\,.$$

Considering for any λ-measurable subset A

$$\mu(A\cap B(x_0,m_0))-r\lambda(A\cap B(x_0,m_0))=\int_{A\cap B(x_0,m_0)}(f(y)-r)d\lambda(y)\,,$$

we have

$$\int_{B(x_0,m_0)} \varphi_T(y,x)d\mu(y) - r\int_{B(x_0,m_0)} \varphi_T(y,x)d\lambda(y) = \int_{B(x_0,m_0)} (f(y)-r)\varphi_T(y,x)d\lambda(y)$$

$$\leq \int_{B(x_0,m_0)\cap\{y;f(y)>r\}} (f(y)-r)\varphi_T(y,x)d\lambda(y) \leq \int_E \varphi_T(y,x)d\mu_r^{m_0}(y),$$

so that

$$\int_E \varphi_T(y,x)d\mu(y) - \varepsilon \leq \int_{B(x_0,m_0)} \varphi_T(y,x)d\mu(y) \leq \int_E \varphi_T(y,x)d\mu_r^{m_0}(y) + r\int_E \varphi_T(y,x)d\lambda(y),$$

and by (H.1), the extreme right hand side

$$= \int_E \varphi_T(y,x)d\mu_r^{m_0}(y) + r.$$

Since $x \in C_r$,

$$\limsup_{T\to\infty} \int_E \varphi_T(y,x)d\mu(y) \leq r + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\limsup_{T\to\infty} \int_E \varphi_T(y,x)d\mu(y) \leq r,$$

and since $r \geq f(x)$ is arbitrary, we have

$$\limsup_{T\to\infty} \int_E \varphi_T(y,x)d\mu(y) \leq f(x).$$

(Fifth step) For any $m \in \boldsymbol{N}$, any $r \in \boldsymbol{Q}$ and any λ-measurable subset A define a finite measure ν_r^m on E by

$$\nu_r^m(A) = \int_{A\cap B(x_0,m)\cap\{y;f(y)<r\}} (r - f(y))d\lambda(y).$$

By discussions similar to those as in the third and fourth steps we have

$$\liminf_{T\to\infty} \int_E \varphi_T(y,x)d\mu(y) \geq f(x).$$

Summing up the above we have thus proved

$$f(x) = \lim_{T\to\infty} \int_E \varphi_T(y,x)d\mu(y), \quad \text{a.e.}\ (d\lambda(x)).$$

Next we shall prove the theorem when φ_T need not be non-negative.

(Sixth step) Define

$$\varphi_T^+(y, x) = \max(\varphi_T(y, x), 0), \quad \varphi_T^-(y, x) = \max(-\varphi_T(y, x), 0).$$

Then by (H.2) we have

$$|\varphi_T^+(y, x)| \le \frac{2\alpha\beta Q_\Phi}{\alpha - H} F_T^\alpha(y, x), \quad |\varphi_T^-(y, x)| \le \frac{2\alpha\beta Q_\Phi}{\alpha - H} F_T^\alpha(y, x).$$

Applying the preceding discussions in the second to fifth steps, for λ-almost all $x \in E$ we have for any $r \in \boldsymbol{Q}$ such that $f(x) \le r$ and any $\varepsilon > 0$

$$\limsup_{T\to\infty} \int_E \varphi_T(y, x) d\mu(y) = \limsup_{T\to\infty} \int_E \varphi_T^+(y, x) - \varphi_T^-(y, x) d\mu(y)$$

$$\le \limsup_{T\to\infty} r \left\{ \int_E \varphi_T^+(y, x) d\lambda(y) - \int_E \varphi_T^-(y, x) d\lambda(y) \right\} + \varepsilon = r + \varepsilon,$$

so that

$$\limsup_{T\to\infty} \int_E \varphi_T(y, x) d\mu(y) \le f(x).$$

Similarly we have

$$\liminf_{T\to\infty} \int_E \varphi_T(y, x) d\mu(y) \ge f(x),$$

and hence the theorem.

5. Further generalizations. Theorem 4 can be extended to a non-Radon measure. In fact one of the following three conditions (M.1)–(M.3) implies

$$\lambda(\textstyle\bigcup_\iota U(x_\iota, r_\iota)) = 0 \quad \text{provided} \quad \lambda(U(x_\iota, r_\iota)) = 0 \qquad \text{for all } \iota.$$

(M1) The metric space E is separable.

(M.2) $\lambda(B(x, r)) > 0$ for any $r > 0$ and λ-almost all $x \in E$.

(M.3) λ is τ-regular, that is, for every directed family $\{O_\iota\}$ of open subsets of E, $\lambda(\bigcup_\iota O_\iota) = \sup_\iota \mu(O_\iota)$.

Therefore we have the following theorem:

THEOREM 6. *Assume that*

$$\beta_1(x) = \sup_{r>0} \frac{\lambda(B(x, 5r))}{\lambda(B(x, r))} < \infty, \quad a.e. (d\lambda)$$

and let $H_1(x) = \log \beta_1(x)/\log 5$. *Define the admissible family* Φ *by Definition* 4 *for* $\beta_1(x)$ *and* $H_1(x)$, *instead of* $\beta(x)$ *and* $H(x)$, *respectively. If one of* (M.1)–(M.3) *are satisfied, then we have the same conclusion as that in Theorem* 4.

References

[1] J. Bourgain and H. Sato, A direct proof to van der Vaart's theorem, Studia Math. 84 (1986), 125–131.

[2] R. B. Darst, Two singular measures can agree on balls, Mathematika 20 (1973), 224–225.

[3] R. O. Davies, Measures not approximable or not specifiable by means of balls, Mathematika, 18 (1971), 157–160.

[4] N. Dunford and J. T. Schwartz, Linear operators, Part I. Interscience, New York, 1957.

[5] H. Federer, Geometric measure theory. Grundlehren der Math. Wiss. 153, Springer-Verlag, Berlin, Heidelberg and New York, 1969.

[6] W. Rudin, Real and complex analysis, McGraw-Hill, New York, 1966.

[7] J. E. Hutchinson, Fractals and self similarity, Indiana Univ. Math. J. 30 (1981), 713–747.

[8] H. R. van der Vaart, Determining the absolutely continuous component of a probability distribution from its Fourier-Stieltjes transform, Ark. Mat. 7 (1967), 331–342.

Department of Mathematics
Faculty of Science
Kyushu University-33
Fukuoka 812
Japan