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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR THIRD ORDER NONLINEAR BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we study the existence and uniqueness of the solutions of the general boundary value problems for the third order nonlinear ordinary differential equations. Our results improve some of the known results; moreover, these are very convenient for applications.

1. Introduction. Since 1970, Jackson and several other authors [1]–[12] have made a substantial study for the existence and uniqueness of the solutions for the two-point boundary value problems for third order nonlinear ordinary differential equations.

We discuss in the present paper the existence and uniqueness of the solutions of some general two-point boundary value problems for third order nonlinear ordinary differential equations by making use of third order differential inequalities and by the method of constructing upper and lower solutions.

We consider the third order nonlinear differential equation

$$y'''=f(x, y, y', y'') \tag{1}$$

together with the boundary conditions

$$ay'(0)-by''(0)=A\,,\quad y(1)=B\,,\quad y'(1)=C \tag{2}$$

or

$$y(0)=A\,,\quad y'(0)=B\,,\quad ay'(1)+by''(1)=C\,, \tag{3}$$

where A, B, C are constants, a, b are nonnegative constants, and $a+b>0$.

In Section 2, we state some preliminaries needed in the sequel.

We investigate, in Section 3, the existence and uniqueness of the solutions for the third order nonlinear boundary value problems (1), (2) and (1), (3).

Finally, we give in Section 4 some examples to illustrate the applications of the main results of this paper.

As far as the author knows, the technique of constructing upper and lower solutions

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to study the existence and uniqueness of the solutions for such general third order nonlinear boundary value problems seems to be quite new. The results obtained in this way are convenient for applications, and some known results obtained in [2]–[4], [9]–[11] are improved.

2. Preliminaries. In this section, we state some preliminaries which will be needed in the sequel.

Innes and Jackson [4], and Howes [2] have given two kinds of Nagumo condition for the third order differential equation (1).

We say that $f(x, y, z, w)$ or the equation (1) satisfies the Nagumo condition on $[0, 1] \times R^3$ (R is the real line) if one of the following two conditions holds:

(A) For any $M>0$, there exists $h=h(M)>0$ such that for $(x, y, z, w) \in [0, 1] \times [-M, M] \times R^2$,

$$|f(x, y, z, w)| \leqslant h\Phi_p(|z|)\Phi_q(|w|),$$

where $0 \leqslant p \leqslant 1$, $q \geqslant 0$, $p+2q \leqslant 3$, and for any $r \geqslant 0$,

$$\Phi_r(s) = \max\{1, s^r\} \qquad (0 \leqslant s < \infty).$$

(B) For $(x, y, z) \in [0, 1] \times R^2$,

$$f(x, y, z, w) = O(|w|^2) \qquad \text{as} \quad |w| \to \infty.$$

The Nagumo condition for the equation (1) on $[0, 1] \times R^3$ ensures that its solutions either extend to $[0, 1]$ or become unbounded on their maximal intervals of existence.

As Jackson and Schrader defined in [6], we call $\bar{\omega}(x)$ an upper solution of the equation (1) on $[0, 1]$ if $\bar{\omega}(x) \in C^3[0, 1]$ and for $0 \leqslant x \leqslant 1$,

$$\bar{\omega}'''(x) \leqslant f(x, \bar{\omega}(x), \bar{\omega}'(x), \bar{\omega}''(x));$$

and we call $\underline{\omega}(x)$ a lower solution of the equation (1) on $[0, 1]$ if $\underline{\omega}(x) \in C^3[0, 1]$ and for $0 \leqslant x \leqslant 1$,

$$\underline{\omega}'''(x) \geqslant f(x, \underline{\omega}(x), \underline{\omega}'(x), \underline{\omega}''(x)).$$

LEMMA 1 (cf. [3]). *Assume that $f(x, y, z, w)$ is continuous on $[0, 1] \times R^3$ and that solutions of the equation (1) extend to $[0, 1]$ or become unbounded. Suppose $\{y_n\}$ is a sequence of solutions of $y''' = f_n(x, y, y', y'')$ which together with its derivative sequence $\{y_n'\}$ is uniformly bounded on $[0, 1]$, where f_n are continuous functions on $[0, 1] \times R^3$ which converge uniformly to f on compact subsets of $[0, 1] \times R^3$. Then there is a solution y of the equation (1) on $[0, 1]$ and a subsequence of $\{y_n\}$ which converges uniformly to y on $[0, 1]$.*

LEMMA 2. *Assume that $f(x, y, z, w)$ is nondecreasing in y and continuous on $[0, 1] \times R^3$ and satisfies the Nagumo condition on $[0, 1] \times R^3$. If there exist an upper* *solution* $\bar{\omega}(x)$ *and a lower solution* $\underline{\omega}(x)$ *of the equation* (1) *on* $[0, 1]$ *such that*

$$\bar{\omega}(x) \leqslant \underline{\omega}(x)\,, \qquad \underline{\omega}'(x) \leqslant \bar{\omega}'(x)$$

for $0 \leqslant x \leqslant 1$, *and*

$$a\underline{\omega}'(0) - b\underline{\omega}''(0) \leqslant A \leqslant a\bar{\omega}'(0) - b\bar{\omega}''(0)\,,$$
$$\bar{\omega}(1) \leqslant B \leqslant \underline{\omega}(1)\,, \quad \underline{\omega}'(1) \leqslant C \leqslant \bar{\omega}'(1)\,,$$

then the boundary value problem (1), (2) *has a solution.*

PROOF. For $n = 1, 2, \ldots$, and $(x, y, z, w) \in [0, 1] \times R^3$, we define

$$g_n(x, y, z, w) = \begin{cases} f(x, y, z, w)\,, & |w| \leqslant n\,, \\ f(x, y, z, n \cdot \operatorname{sgn} w)\,, & |w| > n\,, \end{cases}$$

$$h_n(x, y, z, w) = \begin{cases} g_n(x, y, \underline{\omega}'(x), w) - \dfrac{\underline{\omega}'(x) - z}{1 + \underline{\omega}'(x) - z}\,, & z < \underline{\omega}'(x)\,, \\ g_n(x, y, z, w)\,, & \underline{\omega}'(x) \leqslant z \leqslant \bar{\omega}'(x)\,, \\ g_n(x, y, \bar{\omega}'(x), w) + \dfrac{z - \bar{\omega}'(x)}{1 + z - \bar{\omega}'(x)}\,, & z > \bar{\omega}'(x)\,, \end{cases}$$

$$f_n(x, y, z, w) = \begin{cases} h_n(x, \bar{\omega}(x), z, w)\,, & y < \bar{\omega}(x)\,, \\ h_n(x, y, z, w)\,, & \bar{\omega}(x) \leqslant y \leqslant \underline{\omega}(x)\,, \\ h_n(x, \underline{\omega}(x), z, w)\,, & y > \underline{\omega}(x)\,. \end{cases}$$

Then for each natural number n, $f_n(x, y, z, w)$ is continuous and bounded on $[0, 1] \times R^3$.

It is easily seen that Green's function of the boundary value problem

$$y''' = 0$$
$$ay'(0) - by''(0) = 0\,, \qquad y(1) = 0\,, \qquad y'(1) = 0$$

is

$$G(x, s) = \begin{cases} \dfrac{1}{2}\left[(x-1)(s-1)\left(1 + \dfrac{ax+b}{a+b}\right) - (1-s)^2\right], & 0 \leqslant x \leqslant s \leqslant 1\,, \\ \dfrac{1}{2}\left[(x-1)(s-1)\left(1 + \dfrac{ax+b}{a+b}\right) - (1-s)^2 + (x-s)^2\right], & 0 \leqslant s \leqslant x \leqslant 1\,. \end{cases}$$

Thus, on applying Schauder's fixed point theorem to the operator

$$T_n[y(x)] = \int_0^1 G(x, s) f_n(s, y(s), y'(s), y''(s))\,ds + \frac{aC - A}{2(a+b)}(x^2 + 1) + \frac{bC + A}{a+b}x + B - C\,,$$

it follows that the boundary value problem for the equation

$$y'''=f_n(x,y,y',y'')$$

satisfying the conditions (2) has a solution $y=y_n(x)$.

Let

$$L=\max\left\{\max_{0\leqslant x\leqslant 1}|\bar{\omega}''(x)|,\ \max_{0\leqslant x\leqslant 1}|\underline{\omega}''(x)|\right\}.$$

We now prove that for $n>L$,

$$\underline{\omega}'(x)\leqslant y_n'(x)\leqslant\bar{\omega}'(x)\qquad(0\leqslant x\leqslant 1)\,. \tag{4}$$

Suppose to the contrary that (4) is not valid (without loss of generality, we may suppose the right side inequality does not hold). It then follows from $y_n'(1)\leqslant\bar{\omega}'(1)$ $(n=1,2,\ldots)$ that there exists $n_0>L$ such that the positive maximum value of the function $y_{n_0}'(x)-\bar{\omega}'(x)$ on $[0,1]$ must be attained at some $x_0\in[0,1)$. If $x_0\neq 0$, then it is evident that

$$y_{n_0}''(x_0)-\bar{\omega}''(x_0)=0 \tag{5}$$

and

$$y_{n_0}'''(x_0)-\bar{\omega}'''(x_0)\leqslant 0\,. \tag{6}$$

If $x_0=0$, then, by

$$ay_{n_0}'(x_0)-by_{n_0}''(x_0)\leqslant a\bar{\omega}'(x_0)-b\bar{\omega}''(x_0)\,,$$

we see that $b\neq 0$, and hence $y_{n_0}''(x_0)\geqslant\bar{\omega}''(x_0)$. Then again, because $x_0=0$ is a maximum point of $y_{n_0}'(x)-\bar{\omega}'(x)$ on $[0,1]$, we know that (5) holds, and so does (6).

On the other hand, by $y_{n_0}'(x_0)>\bar{\omega}'(x_0)$ and the monotonicity of $f(x,y,z,w)$ in y, we have

$$y_{n_0}'''(x_0)\geqslant f(x_0,\bar{\omega}(x_0),\bar{\omega}'(x_0),\bar{\omega}''(x_0)+\frac{y_{n_0}'(x_0)-\bar{\omega}'(x_0)}{1+y_{n_0}'(x_0)-\bar{\omega}'(x_0)}>\bar{\omega}'''(x_0)\,.$$

This contradicts (6), and (4) is established. It then follows from $\bar{\omega}(1)\leqslant y_n(1)\leqslant\underline{\omega}(1)$ $(n=1,2,\ldots)$ that for $n>L$,

$$\bar{\omega}(x)\leqslant y_n(x)\leqslant\underline{\omega}(x)\qquad(0\leqslant x\leqslant 1)\,,$$

and hence for $n>L$, $y=y_n(x)$ is a solution of the boundary value problem for the equation

$$y'''=g_n(x,y,y',y'')$$

satisfying the conditions (2). The proof of Lemma 2 now follows as an application of Lemma 1.

3. Main Results. In this section, we discuss the existence and uniqueness of the solutions for the third order nonlinear boundary value problems (1), (2) and (1), (3). For this, we need the following hypotheses:

H_1. The function $f(x, y, z, w)$ and its first order partial derivatives with respect to y, z and w are continuous on $[0, 1]\times R^3$, and $f(x, y, z, w)$ satisfies the Nagumo condition on $[0, 1]\times R^3$.

H_2. For $0\leqslant x\leqslant 1$, $f(x, 0, 0, 0)\equiv 0$.

THEOREM 1. *Assume that* H_1 *and* H_2 *hold. If there exist* μ, $\sigma>0$ *such that*

$$0\leqslant f_y(x, y, z, w)<\mu\,, \quad f_z(x, y, z, w)\geqslant 0\,, \quad f_w(x, y, z, w)\leqslant -\sigma$$

for $(x, y, z, w)\in[0, 1]\times R^3$, *and*

$$\frac{\mu}{4}-\frac{\sigma^3}{27}<0\,, \tag{7}$$

then the boundary value problem (1), (2) *has a unique solution.*

PROOF. We prove first the existence. Let

$$\bar{\omega}(x)=\frac{a\lambda-b}{\lambda^2(a^2+b^2)}|A|e^{\lambda x}+\left(\frac{|C|}{\lambda}-|B|\right)e^{\lambda(x-1)}\,,$$

$$\underline{\omega}(x)=\frac{b-a\lambda}{\lambda^2(a^2+b^2)}|A|e^{\lambda x}+\left(|B|-\frac{|C|}{\lambda}\right)e^{\lambda(x-1)} \qquad (0\leqslant x\leqslant 1)\,,$$

where

$$\lambda=\frac{\sigma}{3}\left(2\cos\frac{\varphi+2\pi}{3}-1\right), \tag{8}$$

$$\varphi=\arccos\left(\frac{27\mu}{2\sigma^3}-1\right).$$

It then follows from (7) that $\lambda<0$ satisfies

$$\lambda^3+\sigma\lambda^2-\mu=0\,. \tag{9}$$

Thus, by the assumptions and

$$\bar{\omega}(x)\leqslant 0\leqslant\underline{\omega}(x)\,, \quad \underline{\omega}'(x)\leqslant 0\leqslant\bar{\omega}'(x)\,, \quad \bar{\omega}''(x)\leqslant 0\leqslant\underline{\omega}''(x)$$

for $0\leqslant x\leqslant 1$, we get

$$f(x, \bar{\omega}(x), \bar{\omega}'(x), \bar{\omega}''(x))\geqslant\mu\bar{\omega}(x)-\sigma\bar{\omega}''(x)=\bar{\omega}'''(x)\,,$$
$$f(x, \underline{\omega}(x), \underline{\omega}'(x), \underline{\omega}''(x))\leqslant\mu\underline{\omega}(x)-\sigma\underline{\omega}''(x)=\underline{\omega}'''(x)$$

for $0\leqslant x\leqslant 1$. Moreover,

$$a\underline{\omega}'(0)-b\underline{\omega}''(0)\leqslant A\leqslant a\bar{\omega}'(0)-b\bar{\omega}''(0)\,,$$
$$\bar{\omega}(1)\leqslant B\leqslant\underline{\omega}(1)\,, \quad \underline{\omega}'(1)\leqslant C\leqslant\bar{\omega}'(1)\,.$$

Therefore, the existence of the solutions for the boundary value problem (1), (2) follows by Lemma 2.

Now, we prove the uniqueness. If the assertion were false, then there would exist two different solutions $y=y_1(x)$ and $y=y_2(x)$ for the boundary value problem (1), (2). Let $y_0(x)=y_2(x)-y_1(x)$. Then it will be seen by (2) that $y_0'(x)\not\equiv 0$ for $0\leqslant x<1$. Without loss of generality, we may assume that $y_0'(x_0)<0$ at some $x_0\in[0, 1)$.

It is not difficult to show that for any constant c, $y=cy_0(x)$ is a solution of the boundary value problem

$$\begin{aligned} &y'''=N(x)y''+P(x)y'+Q(x)y\,, \\ &ay'(0)-by''(0)=0\,, \quad y(1)=0\,, \quad y'(1)=0\,, \end{aligned} \tag{10}$$

where

$$N(x)=\int_0^1 f_w(x, y_1(x), y_1'(x), y_1''(x)+\theta y_0''(x))d\theta\,,$$

$$P(x)=\int_0^1 f_z(x, y_1(x), y_1'(x)+\theta y_0'(x), y_2''(x))d\theta\,,$$

$$Q(x)=\int_0^1 f_y(x, y_1(x)+\theta y_0(x), y_2'(x), y_2''(x))d\theta\,.$$

Let

$$\omega(x)=e^{\lambda x}\,,$$

where $\lambda<0$ is as in (8). Then, the set $E=\{c\,|\,cy_0'(x)>\omega'(x),\ 0\leqslant x\leqslant 1\}$ is nonempty and bounded from above. Let $c_0=\sup E$. Then

$$c_0 y_0'(x)\geqslant\omega'(x) \qquad (0\leqslant x\leqslant 1)\,. \tag{11}$$

Evidently, $c_0\notin E$, and hence there exists $x_1\in[0, 1)$ such that

$$c_0 y_0'(x_1)=\omega'(x_1)\,.$$

We maintain $x_1>0$. In fact, if $x_1=0$, then, by (11),

$$c_0 y_0''(0)\geqslant\omega''(0)$$

and so

$$a(c_0 y_0'(0))-b(c_0 y_0''(0))<0\,.$$

This contradicts (10), and hence $x_1\in(0, 1)$. Therefore,

$$c_0 y_0''(x_1)=\omega''(x_1)\,.$$

Moreover, $\omega(1)>c_0 y_0(1)$ and (11) imply

$$c_0 y_0(x) < \omega(x) \qquad (0 \leqslant x \leqslant 1) .$$

Then again, by the assumptions and (9),

$$\omega'''(x) > N(x)\omega'''(x) + P(x)\omega'(x) + Q(x)\omega(x) \qquad (0 \leqslant x \leqslant 1) ,$$

and so

$$\omega'''(x_1) > c_0 y_0'''(x_1) .$$

Therefore, there exists $\delta \in (0, 1-x_1)$ such that for $x_1 < x \leqslant x_1 + \delta$,

$$c_0 y_0'(x) < \omega'(x) .$$

This contradicts (11). Theorem 1 is thus proved.

THEOREM 2. *Assume that* H_1 *and* H_2 *hold. If there exist* $\mu, \sigma > 0$ *such that*

$$0 \leqslant f_y(x, y, z, w) < \mu , \quad f_z(x, y, z, w) \geqslant \sigma , \quad f_w(x, y, z, w) \leqslant 0$$

for $(x, y, z, w) \in [0, 1] \times R^3$, *and*

$$\frac{\mu^2}{4} - \frac{\sigma^3}{27} < 0 , \tag{12}$$

then there exists a unique solution for the boundary value problem (1), (2).

PROOF. Let

$$\bar{\omega}(x) = \frac{a\rho - b}{\rho^2(a^2+b^2)} |A| e^{\rho x} + \left(\frac{|C|}{\rho} - |B|\right) e^{\rho(x-1)} ,$$

$$\underline{\omega}(x) = \frac{b - a\rho}{\rho^2(a^2+b^2)} |A| e^{\rho x} + \left(|B| - \frac{|C|}{\rho}\right) e^{\rho(x-1)} \qquad (0 \leqslant x \leqslant 1) ,$$

where

$$\rho = \frac{2\sqrt{3\sigma}}{3} \cos\frac{\theta + 2\pi}{3} , \qquad \theta = \arccos\frac{3\mu\sqrt{3\sigma}}{2\sigma^2} .$$

Then we can show by (12) that $\rho < 0$ satisfies

$$\rho^3 - \sigma\rho - \mu = 0 .$$

The rest of the proof of Theorem 2 is similar to that of Theorem 1.

THEOREM 3. *Assume that* H_1 *and* H_2 *hold. If there exist* $\mu, \sigma > 0$ *such that*

$$-\mu < f_y(x, y, z, w) \leqslant 0 , \quad f_z(x, y, z, w) \geqslant 0 , \quad f_w(x, y, z, w) \geqslant \sigma$$

for $(x, y, z, w) \in [0, 1] \times R^3$, *and* $\mu/4 - \sigma^3/27 < 0$, *then the boundary value problem* (1), (3) *has a unique solution.*

PROOF. Let $t=1-x$, and let

$$F\left(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}\right)=-f\left(1-t, y, -\frac{dy}{dt}, \frac{d^2y}{dt^2}\right).$$

Then the boundary value problem (1), (3) is equivalent to the boundary value problem

$$\frac{d^3y}{dt^3}=F\left(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}\right), \tag{13}$$

$$\left(a\frac{dy}{dt}-b\frac{d^2y}{dt^2}\right)\bigg|_{t=0}=-C, \quad y|_{t=1}=A, \quad \frac{dy}{dt}\bigg|_{t=1}=-B. \tag{14}$$

In addition, the function $F(t, y, z, w)$ satisfies all the conditions of Theorem 1 on $[0, 1]\times R^3$. Thus, there exists a unique solution $y=\tilde{y}(t)$ for the boundary value problem (13), (14). Therefore, the boundary value problem (1), (3) has a unique solution $y=\tilde{y}(1-x)$, and the theorem is proved.

By reasoning similar to the proof of Theorem 3 and by Theorem 2, we can deduce the following:

THEOREM 4. *Assume that* H_1 *and* H_2 *hold. If there exist* $\mu, \sigma>0$ *such that*

$$-\mu<f_y(x, y, z, w)\leqslant 0, \quad f_z(x, y, z, w)\geqslant\sigma, \quad f_w(x, y, z, w)\geqslant 0$$

for $(x, y, z, w)\in[0, 1]\times R^3$, *and* $\mu^2/4-\sigma^3/27<0$, *then the boundary value problem* (1), (3) *has a unique solution.*

4. Examples. The following examples with certain generality illustrate the applications of the main results of this paper.

EXAMPLE 1. Consider the equation

$$\begin{aligned} y''' = {} & \exp\{\mathrm{arctg}[e^{-\pi/2}y+F_1(x)(y')^{2m+1}]\} \\ & -[2y''+F_2(x)(y'')^{(4n+1)/(2n+1)}]\log[e+(y'')^{2p}]-1, \end{aligned} \tag{15}$$

where m, n and p are nonnegative integers and $F_i(x)$ $(i=1, 2)$ are nonnegative and continuous on $[0, 1]$. Let

$$\begin{aligned} f(x, y, z, w) = {} & \exp\{\mathrm{arctg}(e^{-\pi/2}y+F_1(x)z^{2m+1})\} \\ & -(2w+F_2(x)w^{(4n+1)/(2n+1)})\log(e+w^{2p})-1. \end{aligned}$$

Then, $f(x, y, z, w)$ and its first order partial derivatives with respect to y, z and w are continuous on $[0, 1]\times R^3$, and for $(x, y, z, w)\in[0, 1]\times R^3$,

$$0<f_y(x, y, z, w)<1, \quad f_z(x, y, z, w)\geqslant 0, \quad f_w(x, y, z, w)\leqslant -2,$$

and

$$f(x, y, z, w)=O(|w|^2) \qquad \text{as} \quad |w| \to \infty ,$$

and so $f(x, y, z, w)$ satisfies the Nagumo condition. Moreover,

$$f(x, 0, 0, 0) \equiv 0$$

for $0 \leqslant x \leqslant 1$. Therefore, the boundary value problem (15), (2) has a unique solution by Theorem 1.

EXAMPLE 2. We consider the equation

$$\begin{aligned} y''' &= \frac{1}{\pi}\left[\left(\frac{\pi}{2}+\operatorname{arctg} y\right)y-\frac{1}{2}\log(1+y^2)\right] \\ &\quad +\frac{[2\sqrt{1+(y')^{4m}}+F_1(x)(y')^{2m}]y'}{\sqrt{1+(y')^{4m}}}-\frac{F_2(x)(y'')^{2n+1}}{1+(y'')^{2n}} , \end{aligned} \tag{16}$$

where m and n are nonnegative integers and $F_i(x)$ $(i=1, 2)$ are nonnegative and continuous on $[0, 1]$. Let

$$\begin{aligned} f(x, y, z, w) &= \frac{1}{\pi}\left[\left(\frac{\pi}{2}+\operatorname{arctg} y\right)y-\frac{1}{2}\log(1+y^2)\right] \\ &\quad +\frac{(2\sqrt{1+z^{4m}}+F_1(x)z^{2m})z}{\sqrt{1+z^{4m}}}-\frac{F_2(x)w^{2n+1}}{1+w^{2n}} . \end{aligned}$$

Then $f(x, y, z, w)$ and its first order partial derivatives with respect to y, z and w are continuous on $[0, 1]\times R^3$, and for $(x, y, z, w)\in[0, 1]\times R^3$,

$$0<f_y(x, y, z, w)<1 , \quad f_z(x, y, z, w)\geqslant 2 , \quad f_w(x, y, z, w)\leqslant 0 .$$

Let

$$K=\max_{0\leqslant x\leqslant 1}\{|F_1(x)|+|F_2(x)|\} .$$

Then for any $M>0$, choosing

$$h=2+K+M(M+1) ,$$

we have for $(x, y, z, w)\in[0, 1]\times[-M, M]\times R^2$,

$$|f(x, y, z, w)|\leqslant h\Phi_1(|z|)\Phi_1(|w|) ,$$

where

$$\Phi_1(s)=\max\{1, s\} \qquad (0\leqslant s<\infty) ,$$

and hence $f(x, y, z, w)$ satisfies the Nagumo condition. Moreover, for $0\leqslant x\leqslant 1$,

$$f(x, 0, 0, 0)\equiv 0 .$$

It follows from Theorem 2 that there exists a unique solution for the boundary value problem (16), (2).

It is not difficult to give some examples for the applications of Theorems 3 and 4 by making a few obvious modifications in Examples 1 and 2. However, for illustrating that the results can be applied in wide range, we shall give more examples here.

EXAMPLE 3. Let us consider the equation

$$y''' = 3y''\sqrt{1+F_1(x)(y'')^2} + F_2(x)[1+(y'')^2] \operatorname{arctg} y'' + \operatorname{arctg}[F_3(x)(y')^{2m+1} - y], \tag{17}$$

where $F_i(x)$ $(i=1, 2, 3)$ are nonnegative and continuous on $[0, 1]$ and m is a nonnegative integer. By Theorem 3, we conclude that the boundary value problem (17), (3) has a unique solution.

EXAMPLE 4. We now consider the equation

$$\begin{aligned} y''' = {} & \left[\frac{F_1(x)(y'')^{2m}}{\sqrt{1+(y'')^{4m}}} + F_2(x) \operatorname{arctg}(y'')^{2n}\right] y'' \\ & + \left[2 + \frac{F_3(x)(y')^{2p}}{1+(y')^{2p}}\right] y' - \frac{y}{4}\sqrt{1+\operatorname{arctg} y^2}, \end{aligned} \tag{18}$$

where m, n and p are nonnegative integers and $F_i(x)$ $(i=1, 2, 3)$ are nonnegative and continuous on $[0, 1]$. From Theorem 4, we deduce easily that there exists a unique solution for the boundary value problem (18), (3).

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