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AN ALGEBRAIC HALFWAY MODEL FOR THE EVERSION OF THE SPHERE

(WITH AN APPENDIX BY BERNARD MORIN)

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Abstract. In this paper, I construct a new version of the halfway model for the eversion of the sphere, called the closed halfway model, whose image can readily be shown to be the set of zeros of an explicit polynomial of degree eight. For this purpose, a 4-parameter family of halfway models is thoroughly investigated. This family also contains the so-called open halfway model constructed in [A2]. The closed halfway model is chosen among the immersions of this family whose multiple loci contain two circles. Applied to the results of [A1], a similar study leads to notice that there exist Boy surfaces depending on two parameters, each of which intersects a given sphere along four circles (one parallel and three meridians). In the Appendix, Morin gives a coding in differential topological terms, of a sphere eversion which turns out to be minimal in many respects, so that, from now on, we no longer need to refer to pictures in order to present the subject.

Introduction. The present paper is the first step in the program we would eventually like to carry out. Indeed, the task we have in mind, is to construct an eversion of the sphere in terms of a continuous family of immersions of $\boldsymbol{S}^2$ into $\boldsymbol{R}^3$ such that the images of all members of the family be real algebraic surfaces, i.e., the sets of zeros of some polynomials in three variables. Obviously, we are looking for surfaces with minimal complexity (where the word complexity must be understood in a rather vague sense), so that the singular locus of each surface could be controlled in such a way that computations should lead to an easy description of such a homotopy. Assuming that the eversion has some symmetry with respect to time, the first task, in order to solve the problem, is to build a handy central step for the eversion. Here we present an algebraic candidate for this so-called *halfway model* which turns out to be a good one since it minimizes the complexity in many respects.

In fact, there are two differential types of the halfway model, the *open halfway model* and the *closed halfway model*, as B. Morin called them. First, we obtain and examine an algebraic version of the open halfway model, already mentioned in [A2], by modifying the construction given in [A1] for the Boy surface, in such a way that this threefolded symmetric object is now replaced by a fourfold surface. Instead of the

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immersion of $\boldsymbol{P}^2$ given in [A1], we now get an immersion of $\boldsymbol{S}^2$ into $\boldsymbol{R}^3$ which has the differential behavior of the open halfway model described in [M-P] (compare the drawings shown there with the computer picture reproduced on the cover of [A2].

Unfortunately, the computations needed in order to study the singular locus of such an open halfway model turn out to be rather complicated. Therefore, it is easy to guess that, by trying to deform an object of this kind, one would be led into a tremendous mess. In order to avoid a riding into this sort of trouble, we take advantage of the amount of freedom left in the construction, and, thereby, get an algebraic version of B. Morin's differentiable closed halfway model. The set of multiple points in this modified immersion now splits in such a way that the necessary computations, in order to prove the transversality statements, can easily be carried through. Moreover, we are able to show that a certain polynomial of minimal degree (i.e., of degree eight,) vanishes exactly on the image of our immersion.

Of course, there is still quite a lot to be done since it remains to construct a family of rational algebraic surfaces connecting the immersion presented here, to some algebraic embedding of $\boldsymbol{S}^2$ into $\boldsymbol{R}^3$. However, experience shows that such a task becomes much easier when the halfway model has all possible nice properties. The present paper stems out of conversations with B. Morin whose ideas permeate the subject. I also thank the referee for his most pertinent and kind suggestions.

1. Notation and formulation of the problem. Let $\mathscr{M}$, $\mathscr{I}$, $\mathscr{J}$ and $\mathscr{P}$, be respectively the sets of mappings, of immersions, of transverse immersions and of embeddings (of class C^1) of $\boldsymbol{S}^2$ into $\boldsymbol{R}^3$. The C^1-norm on the vector space $\mathscr{M}$ turns it into a Banach space in which $\mathscr{I}$, $\mathscr{J}$ and $\mathscr{P}$ are open subsets such that $\mathscr{P} \subseteq \mathscr{J} \subseteq \mathscr{I} \subseteq \mathscr{M}$. Recall that an immersion is a C^1-mapping everywhere of maximal rank, that a transverse immersion has double and triple points with tangent planes in general position, and that an embedding is a one-to-one immersion. Let us now say what one means by an eversion of the sphere.

The open subset $\mathscr{P}$ of $\mathscr{M}$ has two connected components, one of which contains the standard embedding, and the other, the antipodal embedding. Therefore, there exists no path (i.e., no continuous mapping $[-1, 1] \to \mathscr{P}$) connecting these two embeddings. A result due to Smale [S] states that the open subset $\mathscr{I}$ is connected, and hence, that there exist paths in $\mathscr{I}$ starting at the standard embedding and ending at the antipodal embedding. Such paths are called *eversions of the sphere.*

The story telling how various people tried to illustrate the paradoxical discovery of S. Smale, can be found in [M-P]. However, the first explicit parametrization was constructed by Morin [M] in 1978. The main trouble with Morin's parametrization lies in the fact that his formulae are so complicated, that they allow no control on the multiple locus of the evolving immersion. A full control on this multiple locus is indeed a key point for a thorough understanding of the all procedure. The present paper intends to be a step toward a more tractable answer to the problem.

The open subset $\mathscr{J}\subseteq\mathscr{I}$ of transverse immersions, has infinitely many connected components; nervertheless, any eversion of the sphere can be approximated by an eversion $f\colon[-1,1]\to\mathscr{I}$ such that $f^{-1}(\mathscr{I}\setminus\mathscr{J})$ has finitely many points, and therefore, such that it intersects only finitely many of these connected components. Now let $f\colon[-1,1]\to\mathscr{J}$ be a path $t\mapsto f_t$ in $\mathscr{J}$. There exist an ambient isotopy $\psi\colon[-1,1]\to\mathrm{Diff}_+(\boldsymbol{R}^3)$ and an isotopy $\varphi\colon[-1,1]\to\mathrm{Diff}_+(\boldsymbol{S}^2)$ such that $\psi_0=1_{\boldsymbol{R}^3}$ and $\varphi_0=1_{\boldsymbol{S}^2}$, and such that for all $t\in[-1,1]$, one has $f_t=\psi_t\circ f_0\circ\varphi_t$ (where $\mathrm{Diff}_+(M)$ denotes the set of orientation-preserving C^1-diffeomorphisms of the orientable C^1-manifold M, with the C^1-topology). For any eversion of the sphere $f\colon[-1,1]\to\mathscr{I}$ such that $f^{-1}(\mathscr{I}\setminus\mathscr{J})$ is a finite subset of $[-1,1]$, one therefore readily sees that the differentiable type of f_t, and consequently its topological type vary only at $f^{-1}(\mathscr{I}\setminus\mathscr{J})$ (i.e., when the deformation f passes from one connected component of $\mathscr{J}$ to another). For any so-called *generic eversion of the sphere* (as well as for any generic regular homotopy of a surface into $\boldsymbol{R}^3$) there are only six types of such transitions, the description of which can be found in [P] and [M-P] (see also Appendix below). Let us recall what are the local models for these six types.

2. The six generic types of transitions between connected components of the space of transverse immersions. Let $f\colon[-1,1]\to\mathscr{I}$ be such that $f_t\in\mathscr{I}\setminus\mathscr{J}$ for some value of $t\in]-1,1[$, so that there exists at least one point $m\in\boldsymbol{R}^3$ such that f_t is not transverse at m. By this, we mean that the set $f_t^{-1}(m)$ contains at least p elements $x_1,\dots,x_p$, where $p\geq 2$, such that the tangent planes at $f_t(x_1),\dots,f_t(x_p)$, are not in general position. Supposing that for such an m, the inverse image $f_t^{-1}(m)$ contains only the points $x_1,\dots,x_p$, let $g_1,\dots,g_p$ be continuous mappings of $[-1,1]$ into $C^1(\boldsymbol{R}^2,\boldsymbol{R}^3)$:

DEFINITION. One says that $g_1,\dots,g_p$ is a *local model of f in neighbourhoods of $t_0\in]-1,1[$ and of $x_1,\dots,x_p\in\boldsymbol{S}^2$*, when there exist $\varepsilon>0$ such that $]t_0-\varepsilon,t_0+\varepsilon[\subseteq]-1,1[$, open neighbourhoods $V_1,\dots,V_p$ of O in $\boldsymbol{R}^2$, p continuous mappings $\varphi_i\colon]t_0-\varepsilon,t_0+\varepsilon[\to\mathrm{Diff}(U_i,x_i;V_i,O)$ (where $\mathrm{Diff}(U,x;V,O)$ denotes the set of C^1-diffeomorphisms with the C^1-topology from an open subset U of $\boldsymbol{S}^2$ onto an open subset V of $\boldsymbol{R}^2$ mapping x to O), as well as a continuous mapping $\psi\colon]t_0-\varepsilon,t_0+\varepsilon[\to\mathrm{Diff}(\boldsymbol{R}^3)$ such that for all $t\in]t_0-\varepsilon,t_0+\varepsilon[$, the following diagram commutes:

$$\begin{tikzcd} U_i \arrow[r, "f_t"] \arrow[d, "\varphi_{i,t}"'] & \boldsymbol{R}^3 \arrow[d, "\psi_t"] \\ V_i \arrow[r, "g_{i,t}"'] & \boldsymbol{R}^3 . \end{tikzcd}$$

One says that the path $f\colon[-1,1]\to\mathscr{I}$ is *generic* when f_{-1} and f_1 are transverse,

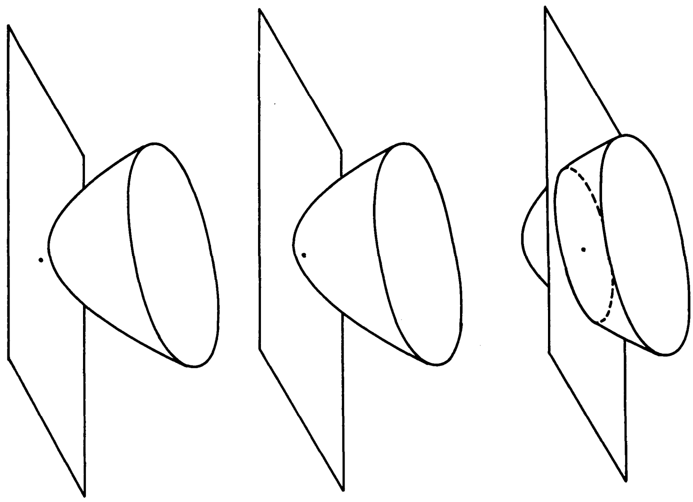

FIGURE 1. Transition of type D_0 or D_2.

and when, for any couple (t_0, m) such that f_{t_0} is not transverse at m, there exists a local model $g_1, \ldots, g_p$ of f in a neighbourhood of t_0 and in a neighbourhood of $f_{t_0}^{-1}(m)$, of one of the six following types:

Birth or death of a closed curve of double points: $p=2$,

$$g_{1,t}(u, v)=(u, v, t) \qquad g_{2,t}(u, v)=(u, v, u^2+v^2)$$

or

$$g_{1,t}(u, v)=(u, v, -t) \qquad g_{2,t}(u, v)=(u, v, u^2+v^2)\,.$$

The first transition is called a *birth* and is said to be of *type* $\boldsymbol{D}_0$. The second is called a *death* and is said to be of *type* $\boldsymbol{D}_2$ (see Figure 1).

Surgery on the curve of double points: $p=2$,

$$g_{1,t}(u, v)=(u, v, t) \qquad g_{2,t}(u, v)=(u, v, uv)\,.$$

The transition called a *surgery* is said to be of *type* $\boldsymbol{D}_1$ (see Figure 2).

Birth or death of a pair of triple points: $p=3$,

$$g_{1,t}(u, v)=(u, v, u+t) \qquad g_{2,t}(u, v)=(u, v, -u+t) \qquad g_{3,t}(u, v)=(u, v, u^2+v^2)$$

or

$$g_{1,t}(u, v)=(u, v, u-t) \qquad g_{2,t}(u, v)=(u, v, -u, -t) \qquad g_{3,t}(u, v)=(u, v, u^2+v^2)\,.$$

The first transition is called a *birth of two triple points* and is said to be of *type* $\boldsymbol{T}^+$, the second, a *death of two triple points*, is said to be of *type* $\boldsymbol{T}^-$ (see Figure 3).

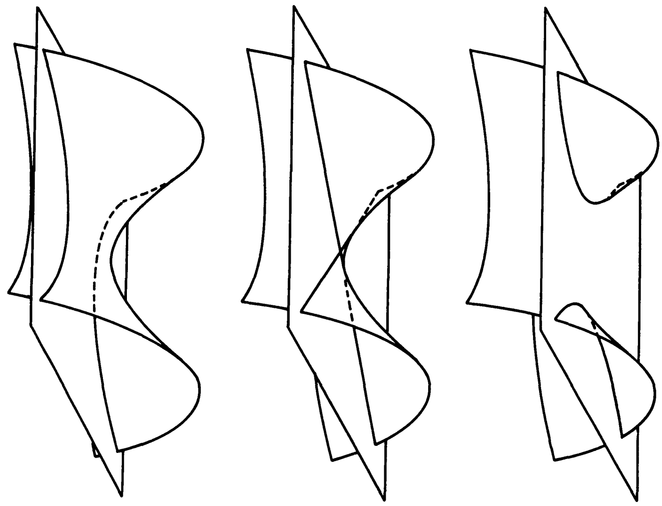

FIGURE 2. Transition of type $\boldsymbol{D}_1$.

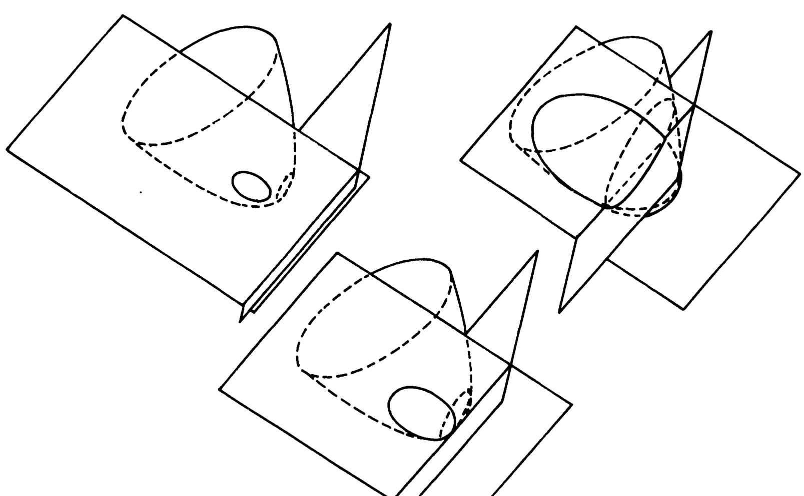

FIGURE 3. Transition of type $\boldsymbol{T}^+$ or $\boldsymbol{T}^-$.

The quadruple point: $p=4$,

$$g_{1,t}(u,v)=(u,v,\ -u-v+t) \qquad g_{2,t}(u,v)=(u,v,u+v)$$

and

$$g_{3,t}(u,v)=(u,v,u-v) \qquad g_{4,t}(u,v)=(u,v,v-u)\,.$$

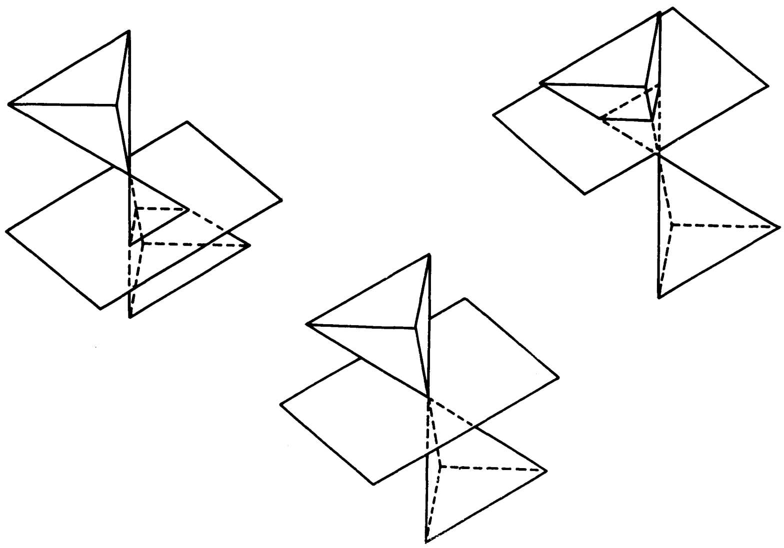

FIGURE 4. Transition of type Q.

This transition, *the eversion of a tetrahedron*, is said to be of *type* Q (see Figure 4). Notice that, in particular, a generic regular homotopy

$$S^2 \times [-1, 1] \to R^3 \times [-1, 1]$$

in the sense of Poenaru [P], determines a generic path $[-1, 1] \to \mathscr{I}$ in the present sense.

3. The open halfway model. The eversion of the sphere $f: [-1, 1] \to \mathscr{I}$ is said to be *generic*, when the path f is generic in the sense given in §2. Notice that, as well as in the case of nice functions in the sense of S. Smale, for a given value of the parameter t such that $f_t \in \mathscr{I} \setminus \mathscr{J}$, we allow more than one point $m \in R^3$ at which f_t is not transverse. Such a generic eversion of the sphere is described in [M-P] (see also §A2 below), up to parametrization of the source sphere S^2 with the help of drawings. Notice, however, that the eversion of [M-P] can easily be coded up to ambient isotopies both of the source and of the target space, in purely differential topological terms, without the aid of pictures. To such a coding, is associated that sequence of types used during the deformation, but this sequence does not characterize completely the coding (see Appendix below). In such sequences, parenthesis indicate those elementary moves that may appear simultaneously. For instance, the main example of eversion of the sphere described in [M-P], gives rise to the sequence

$$D_0 D_0 T^+ T^+ D_1 D_1 (D_1 Q) D_1 D_1 T^- T^- D_2 D_2 .$$

Recall that the eversion f of [M-P] satisfies the following equivariant condition:

$$f_{-t} = \rho_{\pi/2} \circ f_t \circ \rho' \qquad \forall t \in [-1, 1] ,$$

where $\rho_{\pi/2}$ denotes the rotation of $\boldsymbol{R}^3$ of ninety degrees around the vertical axis, and ρ', the restriction to $\boldsymbol{S}^2$ of the composition of the rotation $\rho_{\pi/2}$ by the orthogonal symmetry with respect to the horizontal coordinate plane. Notice that, with this requirement, if f_{-1} is assumed to be the standard embedding of $\boldsymbol{S}^2$ into $\boldsymbol{R}^3$, then f_1 has to be the antipodal embedding, and that

$$f_t = \rho_{\pi/2}^2 \circ f_t \circ \rho'^2 \qquad \forall t \in [-1, 1],$$

so that each model f_t has a symmetry of order two. Moreover, when $t=0$, one gets

$$f_0 = \rho_{\pi/2} \circ f_0 \circ \rho',$$

so that the halfway model f_0 admits a fourfold symmetry. The halfway model f_0, thoroughly described in [M-P], is the so-called *open halfway model*. Of course, in [M-P], this fourfold object is determined up to a parametrization of the source sphere $\boldsymbol{S}^2$, and up to ambient isotopy of the target space $\boldsymbol{R}^3$.

More generally, one says that the eversion of the sphere $f: [-1, 1] \to \mathscr{I}$ is *symmetric*, when there exist an isotopy $\psi: [0, 1] \to \mathrm{Diff}_+(\boldsymbol{R}^3)$ and an isotopy $\varphi: [0, 1] \to \mathrm{Diff}_-(\boldsymbol{S}^2)$, such that

$$f_t = \psi_t \circ f_{-t} \circ \varphi_t \qquad \forall t \geqslant 0.$$

Then f_0 is the so-called *halfway model for the eversion f.*

DEFINITION. The point $m \in \boldsymbol{R}^3$ is said to be *of type* $\boldsymbol{D}_{0,2}$ (resp. $\boldsymbol{D}_1$, $\boldsymbol{T}$, $\boldsymbol{Q}$) *with respect to the map* $f_0 \in \mathscr{M}$, when $m \in f_0(\boldsymbol{S}^2)$ is such that $f_0^{-1}(m)$ contains only finitely many points $x_1, \ldots, x_p \in \boldsymbol{S}^2$, and when there exist diffeomorphisms $\varphi_i \in \mathrm{Diff}(U_i, x_i; V_i, O)$, where U_i is an open neighbourhood of $x_i \in \boldsymbol{S}^2$ and V_i an open neighbourhood of $0 \in \boldsymbol{R}^3$, as well as a diffeomorphism $\psi \in \mathrm{Diff}(\boldsymbol{R}^3)$, such that the following diagram commutes:

$$\begin{array}{ccc} U_i & \xrightarrow{f_0} & \boldsymbol{R}^3 \\ {\scriptstyle \varphi_i}\big\downarrow & & \big\downarrow{\scriptstyle \psi} \\ V_i & \xrightarrow[g_{i,0}]{} & \boldsymbol{R}^3, \end{array}$$

where $g_{1,0}, \ldots, g_{p,0}$ are deduced from the g_i's of §2 of the corresponding types $\boldsymbol{D}_0$ or $\boldsymbol{D}_2$ (resp. $\boldsymbol{D}_1$, $\boldsymbol{T}^+$ or $\boldsymbol{T}^-$, $\boldsymbol{Q}$).

Notice that, it is not possible to distinguish between points of types $\boldsymbol{D}_0$ and $\boldsymbol{D}_2$, as well as between points of types $\boldsymbol{T}^+$ and $\boldsymbol{T}^-$. It is the reason for which we have defined points of type $\boldsymbol{D}_{0,2}$ and $\boldsymbol{T}$.

A result due to Banchoff and Max [B-M], unfortunately proved by referring to pictures, implies that a generic halfway model, i.e., the halfway model of a generic symmetric eversion of the sphere, must have an odd number of quadruple points, i.e., of points of type $\boldsymbol{Q}$. In order to construct a halfway model as simple as possible, this fact leads us to look for examples of immersions of $\boldsymbol{S}^2$ into $\boldsymbol{R}^2$ admitting exactly one quadruple point. In [A2, p. 104], an immersion f_O of $\boldsymbol{S}^2$ into $\boldsymbol{R}^3$, with a fourfold symmetry axis containing a point of type $\boldsymbol{Q}$ and a point of type $\boldsymbol{D}_1$, is defined by the formula:

$$f_O(\vartheta, \eta) = (1 - \sqrt{2}\sin\eta\cos\eta\sin 2\vartheta)^{-1} \cdot \cos\eta \begin{pmatrix} (1/2)(\sqrt{2}\cos\eta + \sin\eta)\cos\vartheta \\ (1/2)(\sqrt{2}\cos\eta - \sin\eta)\sin\vartheta \\ \cos\eta \end{pmatrix},$$

where $\vartheta \in \boldsymbol{R}/2\pi\boldsymbol{Z}$ and $\eta \in [-\pi/2, \pi/2]$ respectively denote the longitude and the latitude on $\boldsymbol{S}^2$ with respect to an arbitrarily chosen reference meridian (a Greenwich meridian) of $\boldsymbol{S}^2$. This model is a bouquet of ellipses entirely contained in (and actually equal to) the set of zeros of the following polynomial P_O of degree eight:

$$\begin{aligned} P_O(X, Y, Z) = {} & (A-D)^4 + 2AD(B^2 - 2C^2) + 2BD(A^2 + 4AD - 3D^2 + 5B^2 - 11C^2) \\ & + D^2(2B^2 + 11C^2) + B(-A^3 + 5A^2B - 4AB^2 + 4B^3) + 2CDE(2B - 11A - D), \end{aligned}$$

where $A = Z(1-Z)$, $B = 2(X^2 + Y^2)$, $C = 2(Y^2 - X^2)$, $D = 2Z^2$ and $E = 4XY$.

This immersion is a reasonable candidate for what should be called the canonical open halfway model, since it satisfies all the properties imposed on such a model in [M-P]. In particular, its multiple locus is of the required type. Indeed, this multiple locus geometrically splits into two non symmetric threebladed propellers having their triple points at the quadruple point of the model, and intersecting once more at the

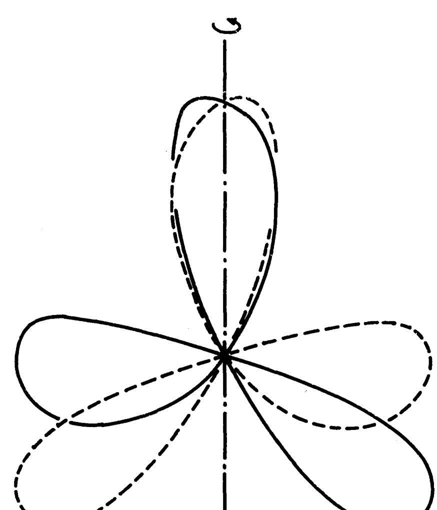

FIGURE 5. Multiple locus of the open halfway model.

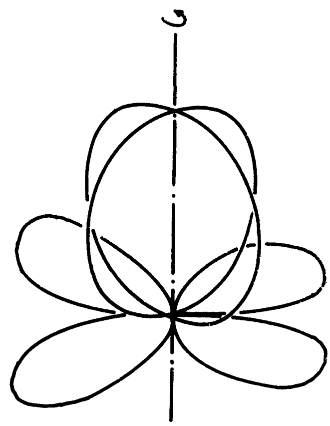

FIGURE 6. Multiple locus of the closed halfway model.

point of type $\boldsymbol{D}_1$ of the model, one of them being obtained by applying to the other the rotation $\rho_{\pi/2}$ (see Figure 5).

Unfortunately, this splitting is not algebraic, and therefore, the complication of such an irreducible singular locus leads to theoretically feasible, but very messy, transversality calculations. On the other hand, the proof that $f_O(\boldsymbol{S}^2) \subseteq \boldsymbol{R}^3$ is exactly the set of zeros of the polynomial P_O, which can be obtained by sharpening the methods used in §6 below, is too long to be inserted here, and would appear uselessly tedious.

In order to avoid this kind of trouble, we construct in §5 an explicit homotopy which deforms f_O into a new immersion f_F of $\boldsymbol{S}^2$, called the closed halfway model. The transversality calculations will become very easy, since the multiple locus of f_F splits into two ellipses, and into an algebraic fourbladed propeller, whose symmetry axis coincides with the symmetry axis of the model (see Figure 6).

Let us first describe the 4-dimensional submanifold of $\mathscr{I}$ in which the homotopy leading from f_O to f_F can be easily defined (see §4). Next, we will be in position to study in all details, f_F itself (see §5).

4. The space of halfway models. In [A2], one can find a geometric construction for a sequence of immersions with an arbitrarly high order of symmetry. In the quoted book (see also §5 below), the construction is thoroughly described in the case where the order of symmetry is equal to 3 (Boy immersion). Let us now recall how this construction works in the case we are presently studying, i.e., in the case of the halfway model, where the symmetry order is equal to 4. Actually, the construction depends on four parameters, here called γ, α, ρ, β, giving rise to an all family of immersions $f_{\alpha,\rho,\beta,\gamma}$, where $(\alpha, \rho, \beta, \gamma)$ ranges in a contractible open subset Δ of $\boldsymbol{S}^1 \times \boldsymbol{R}^3$ (see Proposition 2 below). As noticed in [A2], halfway models can be generated by the images $\mathscr{E}_\vartheta$ of the meridians of the sphere $\boldsymbol{S}^2$, where ϑ is the longitude, and $\mathscr{E}_\vartheta$ an ellipse for all $\vartheta \in \boldsymbol{R}/2\pi\boldsymbol{Z}$.

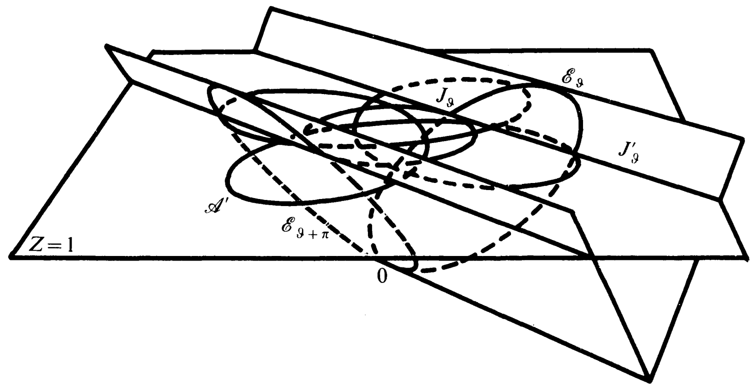

FIGURE 7. Generation of a general halfway model.

All these ellipses pass through the origin O (which is the image of the poles of S^2), and are tangent to the plane $Z=0$ at O. The plane $\mathscr{P}_\vartheta$ containing the ellipse $\mathscr{E}_\vartheta$ is tangent to a cone with vertex O whose intersection with the plane $Z=1$ is a hypocycloid with four cusps denoted by $\mathscr{A}$. (A *fourcuspidal hypocycloid* is defined to be the curve generated by a point K attached to a circle of radius, say $\gamma>0$, with moving center ω, rolling without gliding inside a circle of radius 4γ centered at a fixed point, say Ω.) The ellipse $\mathscr{E}_\vartheta$ intersects the plane $Z=1$ at two points J_ϑ and J'_ϑ. The first point J_ϑ of these two points, runs along the circie of radius 2γ inscribed inside $\mathscr{A}$. Meanwhile, the second point, J'_ϑ, generates an elongated hypocycloid $\mathscr{A}'$ (see Figure 7).

Recall what an elongated hypocycloid is assumed to be. With the previous notation, let K' be any point attached to the moving line (ω, K). For each choice of the ratio $\varepsilon=\|\overrightarrow{\omega K}\|/\gamma$, the point K' generates an *elongated hypocycloid of elongation* ε:

for $\varepsilon=0$, one gets the circle of radius 3γ centered at Ω,
for $0<\varepsilon<1$, the elongated hypocycloid is a smooth simple curve,
for $\varepsilon=1$, one gets the fourcuspidal hypocycloid,
for $1<\varepsilon<3$, the elongated hypocycloid is a smooth curve with four double points,
for $\varepsilon=3$, the smooth curve admits a quadruple point, and
for $\varepsilon>3$, the elongated hypocycloid is a smooth curve with eight double points (see Figure 8).

By identifying the plane containing $\mathscr{A}$ to the complex line, so that Ω becomes 0 and m the complex number z_m, one gets

$$z_{K'}=3\gamma e^{i\vartheta}+\varepsilon\gamma e^{-3i\vartheta},$$

where $\vartheta\in \boldsymbol{R}/2\pi\boldsymbol{Z}$ parametrizes the elongated hypocycloid (see Lemma 1 below). In order to describe the motion of J'_ϑ on its elongated hypocycloid, let us proceed as follow.

Since the line $(J_\vartheta, J'_\vartheta)=\mathscr{P}_\vartheta\cap[Z=1]$ envelopes $\mathscr{A}$, by setting

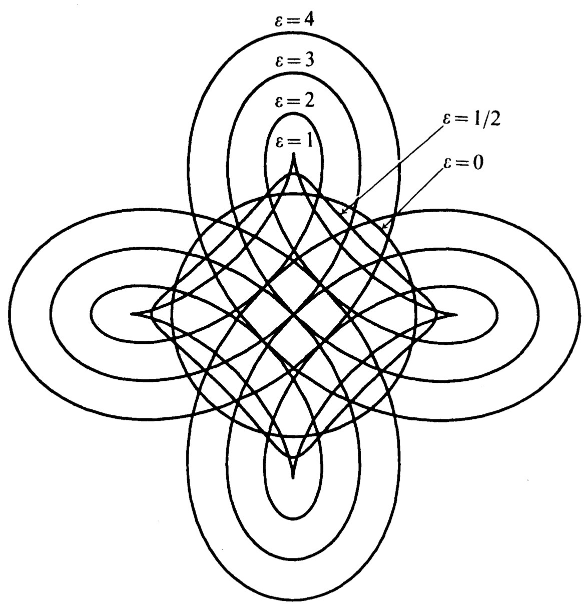

FIGURE 8. Elongated hypocycloids.

$$J_\vartheta = \begin{pmatrix} 2\gamma\cos\vartheta \\ 2\gamma\sin\vartheta \\ 1 \end{pmatrix},$$

the parametrization of $\mathscr{A}$ reads

$$K_\vartheta = \begin{pmatrix} 4\gamma\cos^3\vartheta \\ 4\gamma\sin^3\vartheta \\ 1 \end{pmatrix}.$$

(For the halfway model f_O, $\gamma = 1/2\sqrt{2}$.) Notice that, at K_ϑ, up to sign, the unit tangent vector to $\mathscr{A}$ is equal to

$$\begin{pmatrix} \cos\vartheta \\ -\sin\vartheta \\ 0 \end{pmatrix}.$$

Choose

$$J'_\vartheta = J_\vartheta + 2\gamma\rho\cos(2\theta+\alpha)\cdot\begin{pmatrix} \cos\vartheta \\ -\sin\vartheta \\ 0 \end{pmatrix},$$

where $\rho \in \boldsymbol{R}$ and $\alpha \in \boldsymbol{R}/2\pi\boldsymbol{Z}$, are respectively our second and first parameter. When $\rho = 1$

and $\alpha=0$, one gets $J'_\vartheta=K_\vartheta$. In order to obtain the open halfway model, set $\rho=1$ and $\alpha=-\pi/2$.

LEMMA 1. *The point J'_ϑ generates an elongated hypocycloid of elongation*

$$\varepsilon=3\rho(4+4\rho\cos\alpha+\rho^2)^{-1/2}.$$

PROOF. By identifying the plane $Z=1$ to the complex line, as above, one has

$$Z_{J'_\vartheta}=2\gamma e^{i\vartheta}+2\gamma\rho\cos(2\theta+\alpha)e^{-i\vartheta}=\gamma(2+\rho e^{i\alpha})e^{i\vartheta}+\gamma\rho e^{-i(3\vartheta+\alpha)}.$$

By setting $2+\rho e^{i\alpha}=\rho' e^{i\alpha'}$ where $\rho'\geq 0$ and $\alpha'\in\boldsymbol{R}/2\pi\boldsymbol{Z}$, and $\vartheta_0=-(\alpha+\alpha')/4$, one gets

$$Z_{J'_\vartheta}=\gamma e^{i(3\alpha'-\alpha)/4}(\rho' e^{i(\vartheta-\vartheta_0)}+\rho e^{-3i(\vartheta-\vartheta_0)}).$$

This relation shows that J'_ϑ generates an elongated hypocycloid of elongation $\varepsilon=3\rho/\rho'$. ■

In order to parametrize the general halfway model, for the second variable, one uses $\tau=\tan\eta$ where $\eta\in[-\pi/2,\pi/2]$ is the latitude on $\boldsymbol{S}^2$. Since the ellipse $\mathscr{E}_\vartheta$ has to satisfy four linearly independent conditions in the plane $\mathscr{P}_\vartheta$, there remains only one degree of freedom, say β, at our disposal (the third parameter in $f_{\alpha,\rho,\beta,\gamma}$). With these conventions, the parametrization of $\mathscr{E}_\vartheta$ reads:

$$\beta(\beta-2\gamma\tau\rho\cos(2\vartheta+\alpha)+\beta\tau^2)^{-1}\cdot\begin{pmatrix}(2\gamma+\beta\tau)\cos\vartheta\\(2\gamma-\beta\tau)\sin\vartheta\\1\end{pmatrix}.$$

Notice that, for the halfway model of §3, one has $\beta=1$. Our construction yields a 4-parameter family of mappings $f_{\alpha,\rho,\beta,\gamma}$ from $\boldsymbol{S}^1\times[-\pi/2,\pi/2]$ into $\boldsymbol{R}^3$ given by

$$f_{\alpha,\rho,\beta,\gamma}(\vartheta,\eta)=\beta(\beta-\gamma\rho\sin(2\eta)\cos(2\vartheta+\alpha))^{-1}\cdot\cos\eta\begin{pmatrix}(2\gamma\cos\eta+\beta\sin\eta)\cos\vartheta\\(2\gamma\cos\eta-\beta\sin\eta)\sin\vartheta\\\cos\eta\end{pmatrix},$$

where $\alpha\in\boldsymbol{R}/2\pi\boldsymbol{Z}$ and $\rho,\beta,\gamma>0$.

REMARK 1. **Symmetry properties of the $f_{\alpha,\rho,\beta,\gamma}$'s.** The family of mappings $f_{\alpha,\rho,\beta,\gamma}$ satisfy the following relations:

(i) $f_{\alpha,\rho,\beta,\gamma}(\vartheta+\pi/2,-\eta)=\rho_{\pi/2}\circ f_{\alpha,\rho,\beta,\gamma}(\vartheta,\eta)$, where ρ_ϑ still denotes the rotation of angle ϑ around $\overrightarrow{OZ}$.

(ii) $f_{\alpha,\rho,\lambda\beta,\lambda\gamma}(\vartheta,\eta)=a_\lambda\circ f_{\alpha,\rho,\beta,\gamma}(\vartheta,\eta)$, where a_λ is the linear mapping from $\boldsymbol{R}^3$ into $\boldsymbol{R}^3$ represented by the matrix

$$\begin{pmatrix}\lambda&0&0\\0&\lambda&0\\0&0&1\end{pmatrix}.$$

(iii) $s_X \circ f_{\alpha,\rho,\beta,\gamma} = f_{-\alpha,\rho,\beta,\gamma} \circ s_X$ where s_X denotes the reflection with respect to the plane $X=0$.

In addition, since the conic $\mathscr{E}_\vartheta$ has to be an ellipse, i.e., since the mapping $f_{\alpha,\rho,\beta,\gamma}$ must be defined everywhere, one has to assume that $\beta > \gamma\rho$. Let us set

$$\Delta = \{(\alpha, \rho, \beta, \gamma) \in \boldsymbol{S}^1 \times \boldsymbol{R}^3 : \sin\alpha < 0,\ \rho > 0,\ \gamma > 0,\ \beta > \gamma\rho,\ -2\sqrt{2}\,\rho\sin\alpha > |2 + \rho\cos\alpha|\}\ .$$

PROPOSITION 1. *If* $(\alpha, \rho, \beta, \gamma) \in \Delta$ *then* $f_{\alpha,\rho,\beta,\gamma}$ *induces a* C^1*-immersion of* $\boldsymbol{S}^2$ *into* $\boldsymbol{R}^3$ *whose image has a fourfold symmetry around OZ. Moreover, the mapping* $f: \Delta \to \mathscr{I}$ *defined by* $(\alpha, \rho, \beta, \gamma) \mapsto f_{\alpha,\rho,\beta,\gamma}$ *is continuous.*

PROOF. The symmetric property is a consequence of the relation (i) above (see Remark 1). In order to show that $f_{\alpha,\rho,\beta,\gamma}$ is a C^1-immersion, set $\rho = \beta\rho'/\gamma$, so that

$$f_{\alpha,\rho,\beta,\gamma}(\vartheta, \eta) = (1 - \rho' \sin(2\eta)\cos(2\vartheta + \alpha))^{-1} \cdot \cos\eta \begin{pmatrix} (2\gamma\cos\eta + \beta\sin\eta)\cos\vartheta \\ (2\gamma\cos\eta - \beta\sin\eta)\sin\vartheta \\ \cos\eta \end{pmatrix},$$

where $\rho' < 1$. In order to control the rank of $f_{\alpha,\rho,\beta,\gamma}$ outside the poles, write

$$f_{\alpha,\rho,\beta,\gamma}(\vartheta, \eta) = r(\vartheta, \tau) \cdot \begin{pmatrix} X \\ Y \\ 1 \end{pmatrix} \quad \text{and} \quad g(\vartheta, \tau) = \begin{pmatrix} X \\ Y \end{pmatrix},$$

where $r(\vartheta, \tau) = (1 + \tau^2 - 2\tau\rho'\cos(2\vartheta + \alpha))^{-1}$ and $\tau = \tan\eta$. Then equation of the critical locus of g is $\beta\tau = 2\gamma\cos 2\vartheta$. Along this critical locus, the jacobian matrix of

$$(g, r): \boldsymbol{S}^1 \times \boldsymbol{R} \to \boldsymbol{R}^3$$

has the same rank as the matrix

$$\begin{pmatrix} -\sin\vartheta(1 + \cos 2\vartheta) & \beta\cos\vartheta \\ \cos\vartheta(1 - \cos 2\vartheta) & -\beta\sin\vartheta \\ 2\rho'\cos 2\vartheta\sin(2\vartheta + \alpha) & 2\gamma\cos 2\vartheta - \beta\rho'\cos(2\vartheta + \alpha) \end{pmatrix}.$$

Since the upper 2×2 minor of this matrix vanishes, we replace the second row (row_2) by $(\text{row}_2)\sin\vartheta - (\text{row}_1)\cos\vartheta$, so that the lower 2×2 minor now reads

$$\begin{vmatrix} \sin 2\vartheta & -\beta \\ 2\rho'\cos 2\vartheta\sin(2\vartheta + \alpha) & 2\gamma\cos 2\vartheta - \beta\rho'\cos(2\vartheta + \alpha) \end{vmatrix}$$
$$= \beta\rho'\sin\alpha(1 + \cos^2 2\vartheta) + \sin 2\vartheta\cos 2\vartheta(\beta\rho'\cos\alpha + 2\gamma)\ .$$

If we multiply this expression by $1 + \tau'^2$, where $\tau' = \tan 2\vartheta$, we get

$$\beta\rho'\tau'^2\sin\alpha + \tau'(\beta\rho'\cos\alpha + 2\gamma) + 2\beta\rho'\sin\alpha\ ,$$

a polynomial of degree 2 with respect to τ'. Since the discriminant of this polynomial is equal to

$$(\beta\rho'\cos\alpha+2\gamma)^2-8\beta^2\rho'^2\sin^2\alpha=\gamma^2((\rho\cos\alpha+2)^2-8\rho^2\sin^2\alpha)<0\,,$$

one sees that, outside the poles, the rank of $f_{\alpha,\rho,\beta,\gamma}$ is equal to 2. In order to control the behaviour of $f_{\alpha,\rho,\beta,\gamma}$ near each pole, set

$$f_{\alpha,\rho,\beta,\gamma}(\vartheta,\eta)=\hat{f}(u,v,w) \qquad \text{with} \quad u=\cos\vartheta\cos\eta,\ v=\sin\vartheta\cos\eta,\ \text{and}\ w=\sin\eta\,,$$

so that in a neighbourhood of the pole where $w=\pm 1$, the three following relations hold:

$$\hat{f}=\pm\beta\begin{pmatrix} u \\ -v \\ 0 \end{pmatrix}+\nu\vec{\varepsilon}(\nu)\,, \quad \lim_{u,v\to 0}\partial_u\hat{f}=\pm\beta\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \lim_{u,v\to 0}\partial_v\hat{f}=\pm\beta\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix},$$

where $\nu=(u^2+v^2)^{1/2}$ and $\lim_{\nu\to 0}\vec{\varepsilon}(\nu)=\vec{0}$. These relations imply that $\hat{f}$ is of class C^1 and of rank 2 at each of the poles, so that we may conclude that $f_{\alpha,\rho,\beta,\gamma}\in\mathscr{I}$.

The above formulae yield that the derivative of $f_{\alpha,\rho,\beta,\gamma}\colon \boldsymbol{S}^2\to\boldsymbol{R}^3$, depends continuously on the coefficients α, ρ, β, γ, and hence, that $f\colon \Delta\to\mathscr{I}$ is continuous. ■

REMARK 2. Since when $\sin\alpha=0$, the mapping $f_{\alpha,\rho,\beta,\gamma}$ cannot be an immersion, the subset of all immersions $f_{\alpha,\rho,\beta,\gamma}$ has two connected components. Now Remark 1 (iii) shows that the model $f_{-\alpha,\rho,\beta,\gamma}(\boldsymbol{S}^2)$ is the mirror image of $f_{\alpha,\rho,\beta,\gamma}(\boldsymbol{S}^2)$. On the other hand, for the open halfway model f_O, one has $\sin\alpha=-1$, since $f_O=f_{\alpha,\rho,\beta,\gamma}$ with

$$(\alpha,\rho,\beta,\gamma)=(-\pi/2,\,1,\,1/2,\,1/2\sqrt{2})\in\Delta\,.$$

Hence, by discarding the α's such that $\sin\alpha>0$ we throw away the models having the wrong twist.

PROPOSITION 2. *The set Δ introduced in Proposition* 1 *is a contractible open subset of $\boldsymbol{S}^1\times\boldsymbol{R}^3$.*

PROOF. By construction, Δ is an open subset of $\boldsymbol{S}^1\times\boldsymbol{R}^3$. In order to show that Δ is contractible, set

$$\Delta_1=\{(\alpha,\rho,\beta,\gamma)\in\Delta:\beta=\sqrt{2}\,\gamma\rho\}\,.$$

We prove first that Δ_1 is a deformation retract of Δ, and next, that Δ_1 is homeomorphic to $\Delta'_1\times]0,+\infty[$, where

$$\Delta'_1=\{z\in\boldsymbol{C}:\Im z<0,\ -2\sqrt{2}\,\Im z>|2+\Re z|\}$$

will turn out to be contractible. For all $t\in[0,1]$, set $\beta_t=(1-t)\beta+\sqrt{2}\,t\gamma\rho$. For $(\alpha,\rho,\beta,\gamma)\in\Delta$, one has $(\alpha,\rho,\beta_t,\gamma)\in\Delta$ so that the mapping

$$H_1\colon \Delta\times[0,1]\to\Delta$$

defined by

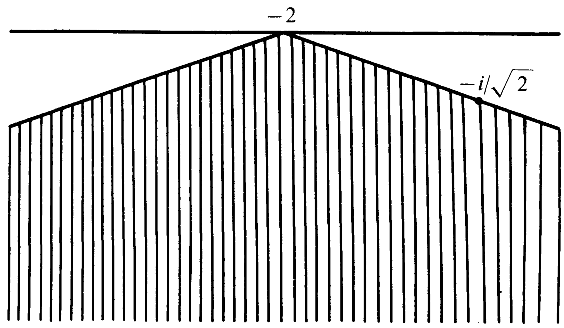

FIGURE 9. The contractible set Δ_1'.

$$((\alpha, \rho, \beta, \gamma), t) \mapsto (\alpha, \rho, \beta_t, \gamma)$$

is a deformation retract of Δ to Δ_1. Moreover, the projection $(\alpha, \rho, \beta, \gamma) \mapsto (\alpha, \rho, \gamma)$, yields a homeomorphism from Δ_1 to

$$\{(\alpha, \rho) \in \boldsymbol{S}^1 \times \boldsymbol{R} : \sin\alpha < 0, \rho > 0, -2\sqrt{2}\,\rho\sin\alpha > |2 + \rho\cos\alpha|\} \times]0, +\infty[\,.$$

Now, the mapping

$$(\alpha, \rho) \mapsto z = \rho e^{i\alpha}$$

is a homeomorphism from $\{(\alpha, \rho) \in \boldsymbol{S}^1 \times \boldsymbol{R} : \sin\alpha < 0, \rho > 0, -2\sqrt{2}\,\rho\sin\alpha > |2 + \rho\cos\alpha|\}$ to Δ_1'. Clearly the subset Δ_1' of $\boldsymbol{C}$ is convex and hence contractible (see Figure 9). ■

The homotopy starting from f_O and leading to the immersion f_F, announced in §3 and constructed in §5, is a path entirely contained in $f(\Delta) \subseteq \mathscr{I}$.

5. The closed halfway model. The existence of the closed halfway model is mentioned in [M-P]. Although topologically more complicated than the open halfway model, it gives rise to a more compact eversion of the sphere, i.e., to a psychologically shorter one, which one can grasp more easily, since, in the associated sequence of types, all the transitions of type $\boldsymbol{D}_1$ occur simultaneously at the halfway stage:

$$\boldsymbol{D}_0\boldsymbol{D}_0\boldsymbol{T}^+\boldsymbol{T}^+\begin{pmatrix} \boldsymbol{D}_1 & & \boldsymbol{D}_1 & & \\ & & \boldsymbol{Q} & & \boldsymbol{D}_1 \\ & \boldsymbol{D}_1 & & \boldsymbol{D}_1 & \end{pmatrix}\boldsymbol{T}^-\boldsymbol{T}^-\boldsymbol{D}_2\boldsymbol{D}_2\,.$$

In order to define f_F, we first impose a condition which splits the multiple locus of the general immersion $f_{\alpha,\rho,\beta,\gamma}$ into three algebraic curves. Since the singular locus of the model we are looking for, must decompose into two ovals tangent at O to the plane $Z=0$ plus a third component, it is natural to require that the ovals should belong to the family of ellipses $(\mathscr{E}_\vartheta)$. Since $\mathscr{E}_\vartheta$ and $\mathscr{E}_{\vartheta'}$ are coplanar if and only if

$$\vartheta \equiv \vartheta' \quad (2\pi) \quad \text{or} \quad \vartheta \equiv \vartheta' \equiv 0 \quad (\pi) \quad \text{or} \quad \vartheta \equiv \vartheta' \equiv \pi/2 \quad (\pi),$$

one sees that the condition $\mathscr{E}_\vartheta$ contained in the multiple locus implies $\vartheta \equiv 0$ $(\pi/2)$. By Remark 1 (i), if $\mathscr{E}_0$ is in the multiple locus, then $\mathscr{E}_0 = \mathscr{E}_\pi$ and $\mathscr{E}_{\pi/2} = \mathscr{E}_{-\pi/2}$.

PROPOSITION 3. *One has $\mathscr{E}_0 = \mathscr{E}_\pi$ if and only if $2 + \rho \cos \alpha = 0$.*

PROOF. For all $f_{\alpha,\rho,\beta,\gamma}$, the ellipse $\mathscr{E}_0$ intersects OZ at O and at a second point denoted Q, corresponding to the value $\tau = \tan \eta = -2\gamma/\beta$, since $\mathscr{E}_0$ and $\mathscr{E}_\pi$ are symmetric with respect to OZ, these two curves coincide if and only if their tangent at Q are orthogonal to OZ, i.e., when

$$\frac{dZ}{d\tau}_{|\tau = -2\gamma/\beta} = 0,$$

where the third coordinate Z of the parametrization of $\mathscr{E}_0$ is equal to

$$Z = \left(1 - 2\rho\tau \frac{\gamma}{\beta} \cos \alpha + \tau^2\right)^{-1}.$$

Hence, one gets $2 + \rho \cos \alpha = 0$. ∎

Since $\mathscr{E}_0$ and $\mathscr{E}_{\pi/2}$ are vertical, by applying the linear mapping a_λ of Remark 1 (ii), one may concentrate only on models where these two ellipses are circles.

PROPOSITION 4. *The ellipse $\mathscr{E}_0$, (and therefore the ellipse $\mathscr{E}_{\pi/2}$), is a circle if and only if* (i) $2 + \rho \cos \alpha = 0$, *and* (ii) $\beta^2 = 1 + 4\gamma^2$. *If such is the case, the intersection of $f_{\alpha,\rho,\beta,\gamma}(\boldsymbol{S}^2)$ and of the sphere $X^2 + Y^2 + Z^2 = \beta^2 Z$, is equal to $\mathscr{C}_1 \cup \mathscr{C}_2 \cup \mathscr{C}_3$ where $\mathscr{C}_1 = \mathscr{E}_0$, $\mathscr{C}_2 = \mathscr{E}_{\pi/2}$, and where $\mathscr{C}_3$ is the circle generated by J_ϑ.*

PROOF. The ellipse $\mathscr{E}_0$ is a circle if and only if its tangent at Q is horizontal and its two axes are equal, and hence if and only if (i) $2 + \rho \cos \alpha = 0$ (see Proposition 3), and (ii) $\beta^2 = 1 + 4\gamma^2$. When these two conditions are satisfied, we put $2\gamma = \sinh \mu$ and $e^{i\alpha} = (-1 + i \sinh \nu)/\cosh \nu$, so that we get

$$\mu > 0, \quad \nu < 0, \quad \beta = \cosh \mu, \quad \text{and} \quad \rho = 2 \cosh \nu.$$

Let us then set $f_{\alpha,\rho,\beta,\gamma}(\vartheta, \eta) = f_{\mu,\nu}(\vartheta, \tau)$ where $\tau = \tan \eta$, so that

$$f_{\mu,\nu}(\vartheta, \tau) = (1 + \tau^2 + 2\tau \tanh \mu (\cos 2\vartheta + \sinh \nu \sin 2\vartheta))^{-1} \cdot \begin{pmatrix} (\sinh \mu + \tau \cosh \mu) \cos \vartheta \\ (\sinh \mu - \tau \cosh \mu) \sin \vartheta \\ 1 \end{pmatrix}.$$

If we replace X, Y and Z, by the three components of $f_{\mu,\nu}$, the equation $X^2 + Y^2 + Z^2 = \beta^2 Z$ yields $\tau = 0$ and $\sin 2\vartheta = 0$. Notice first that $\tau = 0$ is the equation of the equator of the source sphere $\boldsymbol{S}^2$ whose image is the circle $\mathscr{C}_3$ of radius 2γ generated by J_ϑ, and last, that $\sin 2\vartheta$ vanishes on the four meridians of longitude $\vartheta \equiv 0$ $(\pi/2)$ whose images are the double circles $\mathscr{C}_1$ and $\mathscr{C}_2$ (see Propositon 3). ■

For convenience, let us call *closed halfway models*, the C^1-immersions of $\boldsymbol{S}^2$ into $\boldsymbol{R}^3$, induced by the mappings $f_{\mu,v}$ introduced in the proof of Propositon 4, where μ and v are assumed to satisfy

$$(*) \quad \mu>0\,, \quad v<0\,, \quad \text{and} \quad \sinh\mu\cdot\cosh v<\cosh\mu\,.$$

If we impose on $(\alpha, \rho, \beta, \gamma)\in \boldsymbol{S}^1\times\boldsymbol{R}^3$, to verify

$$e^{i\alpha}=\frac{1}{\cosh v}(-1+i\sinh v)\,, \qquad \rho=2\cosh v\,, \qquad \beta=\cosh\mu\,, \quad \text{and} \quad \gamma=\frac{1}{2}\sinh\mu\,,$$

so that

$$f_{\alpha,\rho,\beta,\gamma}(\vartheta, \eta)=f_{\mu,v}(\vartheta, \tau)\,,$$

then, the conditions $(*)$ are equivalent to $(\alpha, \rho, \beta, \gamma)\in\Delta$. Therefore, we have defined a one-to-one mapping from the set of (μ, v) satisfying the conditions $(*)$, into Δ.

PROPOSITION 5. *The multiple locus of the closed halfway model $f_{\mu,v}$, splits into two circles, $\mathscr{C}_1$ and $\mathscr{C}_2$, and into the image $\mathscr{C}_0$ of the two connected curves of $\boldsymbol{S}^2$, parametrized by*

$$\begin{cases} \vartheta_1=\sigma+\arcsin(\sqrt{2}\,\sin\sigma\cos\sigma)-\pi/2 \\ \tau_1=\tanh\mu\cdot(\cos 2\sigma-(1+\cos^2 2\sigma)^{1/2}) \end{cases} \qquad \begin{cases} \vartheta_{-1}=\sigma-\arcsin(\sqrt{2}\,\sin\sigma\cos\sigma)+\pi/2 \\ \tau_{-1}=\tanh\mu\cdot(\cos 2\sigma+(1+\cos^2 2\sigma)^{1/2})\,, \end{cases}$$

where $\sigma\in\boldsymbol{R}/2\pi\boldsymbol{Z}$ (see, for instance, Figure 11 below). Moreover, one has $f_{\mu,v}(\vartheta_1, \tau_1)=f_{\mu,v}(\vartheta_{-1}, \tau_{-1})$ and

$$\begin{cases} \vartheta_1(\sigma+\pi/2)=\vartheta_{-1}(\sigma)-\pi/2 \\ \tau_1(\sigma+\pi/2)=-\tau_{-1}(\sigma)\,. \end{cases}$$

PROOF. For a given longitude ϑ_0, let us look for the points $(\vartheta, \tau)\in\mathscr{E}_\vartheta\cap\mathscr{E}_{\vartheta_0}$ where $\tau=\tan\eta$. The image of the sphere, $f_{\mu,v}(\boldsymbol{S}^2)$, meets the plane $\mathscr{P}_{\vartheta_0}$ at the ellipse $\mathscr{E}_{\vartheta_0}$ and at a second curve $\mathscr{C}_{\vartheta_0}$ given by

$$\tau_\vartheta=\tanh\mu\cdot\cos\left(\frac{3\vartheta_0+\vartheta}{2}\right)\Big/\cos\left(\frac{\vartheta-\vartheta_0}{2}\right).$$

If we set

$$\begin{cases} r_0=(1+\tau^2+2\tau\tanh\mu(\cos 2\vartheta_0+\sinh v\sin 2\vartheta_0))^{-1} \\ r=(1+\tau_\vartheta^2+2\tau_\vartheta\tanh\mu(\cos 2\vartheta+\sinh v\sin 2\vartheta))^{-1} \end{cases}$$

and

$$f_{\mu,\nu}(\vartheta_0, \tau) = r_0 \cdot \begin{pmatrix} X_0 \\ Y_0 \\ 1 \end{pmatrix} \quad \text{and} \quad f_{\mu,\nu}(\vartheta, \tau_\vartheta) = r \cdot \begin{pmatrix} X \\ Y \\ 1 \end{pmatrix},$$

then, since $f_{\mu,\nu}(\vartheta, \tau_\vartheta) \in \mathscr{P}_{\vartheta_0}$, the relation $f_{\mu,\nu}(\vartheta_0, \tau) = f_{\mu,\nu}(\vartheta, \tau_\vartheta)$ is equivalent to

$$X_0 \sin \vartheta_0 - Y_0 \cos \vartheta_0 = X \sin \vartheta_0 - Y \cos \vartheta_0 \quad \text{and} \quad r_0^{-1} = r^{-1}.$$

The first of these two last equations yields

$$\text{(i)} \quad \tau = \tanh \mu \cdot \cos\left(\frac{3\vartheta + \vartheta_0}{2}\right) \Big/ \cos\left(\frac{\vartheta - \vartheta_0}{2}\right),$$

while, after ignoring the factor $\sin(\vartheta - \vartheta_0)$, the second boils down to

$$\text{(ii)} \quad \cos^2(\vartheta + \vartheta_0) + \cos(\vartheta - \vartheta_0) = 0.$$

Let us next set

$$\begin{cases} \vartheta_0 = \sigma + \xi - \pi/2 \\ \vartheta = \sigma - \xi + \pi/2 . \end{cases}$$

The equation (ii) now reads

$$\sin^2 2\sigma = 2 \sin^2 \xi ,$$

or equivalently

$$\sin \xi = \pm \sqrt{2} \sin \sigma \cos \sigma .$$

Meanwhile (i) can be written

$$\tau = \tanh \mu \cdot \frac{\sin(\xi - 2\sigma)}{\sin \xi} = \tanh \mu \cdot (\cos 2\sigma \mp \sqrt{2} \cos \xi) ,$$

since

$$\cos^2 \xi = 1 - \sin^2 \xi = \frac{1}{2}(1 + \cos^2 2\sigma) ,$$

we obtain the desired parametrizations. ∎

Definition of f_F: In order to carry through a more detailed study of our object, it is convenient to specialize the tuple (μ, ν). We first impose to the maximum value, $Z_{\max}$, of the third coordinate Z of $f_{\mu,\nu}$, to be equal to the corresponding value for the open halfway model f_O. Since the third coordinate Z of $f_{\alpha,\rho,\beta,\gamma}$ is equal to

$$\left(1 - \frac{2\gamma}{\beta} \tau \rho \cos(2\vartheta + \alpha) + \tau^2\right)^{-1},$$

considered as a function of $\tau=\tan\eta$, it reaches its maximum $(1-(\gamma^2\rho^2/\beta^2)\cos^2(2\vartheta+\alpha))^{-1}$ at $\tau=(\gamma\rho/\beta)\cos(2\vartheta+\alpha)$. If we vary both ϑ and τ, the maximum of Z is

$$Z_{\max}=\frac{\beta^2}{\beta^2-\gamma^2\rho^2}.$$

For the open halfway model, one has $\beta^2/(\beta^2-\gamma^2\rho^2)=2$, therefore, since β, γ, and ρ are positive, for the closed halfway model, we are led to set $\beta=\sqrt{2}\,\gamma\rho$, so that in terms of μ and ν, we get $\cosh\mu=\sqrt{2}\sinh\mu\cosh\nu$. Since $(\alpha,\rho,\beta,\gamma)\in\Delta$, we have $\sin\alpha<0$, while Proposition 4 (i) yields $\cos\alpha<0$. We therefore choose $\alpha=-3\pi/4$, so that we eventually get

$$(\alpha,\rho,\beta,\gamma)=(-3\pi/4,\,2\sqrt{2},\,2/\sqrt{3},\,1/2\sqrt{3}),$$

or equivalently

$$(\mu,\nu)=\left(\operatorname{arg\,sinh}\frac{1}{\sqrt{3}},\,-\operatorname{arg\,cosh}\sqrt{2}\right).$$

By definition, the corresponding C^1-immersion $f_{\alpha,\rho,\beta,\gamma}(\vartheta,\eta)=f_{\mu,\nu}(\vartheta,\tau)$ is called *the closed halfway model* f_F. Hence

$$f_F(\vartheta,\eta)=(1-\sin\eta\cos\eta(\sin 2\vartheta-\cos 2\vartheta))^{-1}\cdot\cos\eta\begin{pmatrix}(1/\sqrt{3})(\cos\eta+2\sin\eta)\cos\vartheta\\(1/\sqrt{3})(\cos\eta-2\sin\eta)\sin\vartheta\\\cos\eta\end{pmatrix},$$

where $\vartheta\in\boldsymbol{R}/2\pi\boldsymbol{Z}$ and $\eta\in[-\pi/2,\pi/2]$ are as in §3 and §4.

The homotopy connecting f_O to f_F: With the previous notation Δ_1 (see proof of Proposition 2 §4), the tuples $(\alpha_O,\rho_O,\beta_O,\gamma_O)$ and $(\alpha_F,\rho_F,\beta_F,\gamma_F)$ respectively corresponding to f_O and f_F clearly belong to Δ_1. In the proof of Proposition 2 §4, we constructed a homeomorphism from Δ_1 to $\Delta_1'\times]0,+\infty[$. The images of $(\alpha_O,\rho_O,\beta_O,\gamma_O)$ and $(\alpha_F,\rho_F,\beta_F,\gamma_F)$ by this homeomorphism can be connected by a segment contained in $\Delta_1'\times]0,+\infty[$ (see Figure 10).

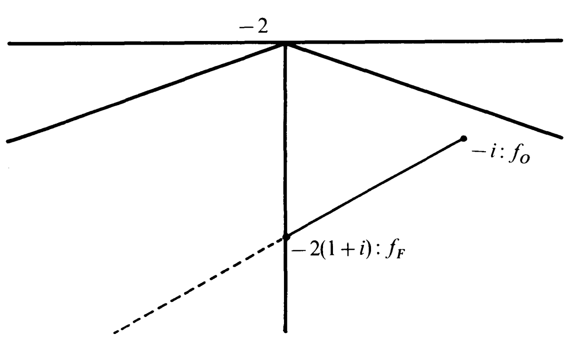

FIGURE 10. The homotopy connecting f_O to f_F.

This segment yields a regular homotopy from f_O to the immersion f_F, which, as we will see in Theorem 4 below, has four extra points of type $\boldsymbol{D}_1$.

REMARK 3. **Overclosing procedure:** The notion of general closed halfway models $f_{\mu,\nu}$ applies also to all immersions $f_{\alpha,\rho,\beta,\gamma}$ where

$$(\alpha, \rho, \beta, \gamma) \in \Delta^0 = \{(\alpha, \rho, \beta, \gamma) \in \Delta : 2 + \rho \cos\alpha = 0\},$$

since the multiple locus of such an immersion contains two ellipses (see Proposition 3). Notice that, by definition, the $f_{\mu,\nu}$'s are precisely the halfway models where these ellipses are requested to be circles. Let us denote

$$\Delta^+ = \{(\alpha, \rho, \beta, \gamma) \in \Delta : 2 + \rho \cos\alpha > 0\} \quad \text{and} \quad \Delta^- = \{(\alpha, \rho, \beta, \gamma) \in \Delta : 2 + \rho \cos\alpha < 0\}.$$

By an argument similar to the one used in the proof of Proposition 2 §4, one can prove that Δ^0 splits Δ into the two contractible open subsets Δ^+ and Δ^-. Since Δ^+ contains f_O, one is led to call *open halfway models*, all the immersions corresponding to $(\alpha, \rho, \beta, \gamma) \in \Delta^+$. The previous splitting induces a splitting of Δ'_1 into two contractible open subsets, $\Delta_1'^+$ and $\Delta_1'^-$ defined by

$$\Delta_1'^+ = \{z \in \Delta'_1 : \Re z > -2\} \quad \text{and} \quad \Delta_1'^- = \{(z \in \Delta'_1 : \Re z < -2\}.$$

Clearly, the segment of Δ'_1 previously used in order to obtain the homotopy from f_O to f_F, starts in $\Delta_1'^+$ and interesects transversally the line $\Re z = -2$ (corresponding to Δ^0), so that it can be extended in $\Delta_1'^-$ (see Figure 10). Therefore, one sees that the models of Δ^- are obtained by overclosing (so to speak) closed halfway models (see Remark 4 §A2).

Boy surfaces containing four circles: The method developped in order to construct halfway models containing three circles (in fact five if we think in terms of multiplicity), easily applies to Boy immersions which map the real projective plane $\boldsymbol{P}^2$ into $\boldsymbol{R}^3$, obtained by generalizing the construction presented in [A1]. In fact, the construction of Boy immersions is similar to the one given in §4, where the fourfold symmetry is now replaced by a threefold symmetry, so that we get a 4-parameter family $b_{\alpha,\rho,\beta,\gamma}$ of C^1-immersions of $\boldsymbol{S}^2$ into $\boldsymbol{R}^3$, invariant under the antipodal action on $\boldsymbol{S}^2$, and hence, factorizing through the canonical projection from $\boldsymbol{S}^2$ onto $\boldsymbol{P}^2$. Since the fourcuspidal hypocycloid $\mathscr{A}$ has now to be replaced by a deltoid, we are led to set

$$b_{\alpha,\rho,\beta,\gamma}(\vartheta, \eta) = \beta(\beta - \gamma\rho \sin(2\eta)\cos(3\vartheta + \alpha))^{-1} \cdot \cos\eta \begin{pmatrix} \gamma\cos\eta\cos 2\vartheta + \beta\sin\eta\cos\vartheta \\ \gamma\cos\eta\sin 2\vartheta - \beta\sin\eta\sin\vartheta \\ \cos\eta \end{pmatrix},$$

where, as ever, ϑ and η respectively denote the longitude and the latitude on $\boldsymbol{S}^2$, and

$$\Delta = \{(\alpha, \rho, \beta, \gamma) \in \boldsymbol{S}^1 \times \boldsymbol{R}^3 : \sin\alpha < 0, \rho > 0, \gamma > 0, \beta > \gamma\rho, -2\sqrt{3}\,\rho \sin\alpha > |3 + 2\rho\cos\alpha|\},$$

so that we have

PROPOSITION 6. *The set $\Delta \subseteq \boldsymbol{S}^1 \times \boldsymbol{R}^3$ is open and contractible. Moreover, for $(\alpha, \rho, \beta, \gamma) \in \Delta$, the mapping $b_{\alpha,\rho,\beta,\gamma}$ extends into a C^1-immersion of $\boldsymbol{S}^2$ into $\boldsymbol{R}^3$, which induces a C^1-immersion of $\boldsymbol{P}^2$ into $\boldsymbol{R}^3$ whose image has a threefold symmetry around OZ.*

The Boy immersion described in [A1], corresponds to

$$(\alpha, \rho, \beta, \gamma) = (-\pi/2, 3/2, \sqrt{2}/3, 2/3).$$

The statement corresponding to Proposition 4, claims that

PROPOSITION 7. *The image $\mathscr{E}_0 = b_{\alpha,\rho,\beta,\gamma}(0, \cdot)$ of the meridian of longitude 0, is a circle if and only if*

$$\text{(i)} \quad 1 + \rho \cos \alpha = 0 \quad \textit{and} \quad \text{(ii)} \quad \beta^2 = 1 + \gamma^2 .$$

If such is the case, $b_{\alpha,\rho,\beta,\gamma}(\boldsymbol{S}^2) \cap [X^2 + Y^2 + Z^2 = \beta^2 Z]$ is the union of the four circles $\mathscr{C}_1 = \mathscr{E}_0, \mathscr{C}_2 = \mathscr{E}_{2\pi/3}, \mathscr{C}_3 = \mathscr{E}_{4\pi/3}$, and $\mathscr{C}_4$, where $\mathscr{C}_4$ is given by the equations $Z = 1$ and $X^2 + Y^2 = \gamma^2$. Immersions satisfying (i) *only, are exactly those for which the ellipses $\mathscr{E}_0$, $\mathscr{E}_{2\pi/3}$ and $\mathscr{E}_{4\pi/3}$, have a vertical axis of symmetry.*

Since the proofs of Propositions 6 and 7 are analogous to those of Proposition 2 §4 and Propositions 3 and 4, they are left to the reader. By setting

$$b_{\mu,\nu} = b_{\alpha,\rho,\beta,\gamma} ,$$

where μ and ν satisfy

$$\mu > 0 , \quad \nu < 0 , \quad \sinh \mu \cdot \cosh \nu < \cosh \mu \quad \text{and} \quad -\sinh \nu > 1/2\sqrt{3} ,$$

and where

$$e^{i\alpha} = \frac{1}{\cosh \nu}(-1 + i \sinh \nu) , \quad \rho = \cosh \nu , \quad \beta = \cosh \mu \quad \text{and} \quad \gamma = \sinh \mu ,$$

Proposition 7 yields a 2-parameter family $b_{\mu,\nu}$ of Boy immersions containing four cospherical circles. Similar constructions can be given for immersions of $\boldsymbol{S}^2$ or $\boldsymbol{P}^2$ into $\boldsymbol{R}^3$, having higher order of symmetry (see [A1] and [A2]).

Now, with the closed halfway model f_F, computations become much easier and enable us to prove the following claims:

6. Statements and proofs of the properties of the closed halfway model f_F.

THEOREM 1. *The mapping $f_F : \boldsymbol{S}^2 \to \boldsymbol{R}^3$ is an immersion of class C^∞ except at the poles of $\boldsymbol{S}^2$, where it is C^1 but not C^2, such that the vertical axis OZ is a fourfold symmetry axis of $f_F(\boldsymbol{S}^2)$. Moreover $f_F(\boldsymbol{S}^2)$ is the union in $\boldsymbol{R}^3$ of a family of ellipses, all tangent to the horizontal plane $Z = 0$ at the origin.*

PROOF. Each ellipse $\mathscr{E}_\vartheta$ is the image of the meridian of longitude ϑ in $\boldsymbol{S}^2$

parametrized by the latitude η. Moreover $\mathscr{E}_\vartheta \subseteq \mathscr{P}_\vartheta$, where $\mathscr{P}_\vartheta$ is the plane of equation $X \sin\vartheta + Y\cos\vartheta = Z\sin 2\vartheta/\sqrt{3}$. Two ellipses $\mathscr{E}_\vartheta$ and $\mathscr{E}_{\vartheta'}$ are tangent at O if and only if $\vartheta - \vartheta' \in \pi\boldsymbol{Z}$; in addition, $\mathscr{P}_\vartheta = \mathscr{P}_{\vartheta'}$ if and only if $\vartheta \in (\pi/2)\boldsymbol{Z}$ (see Figure 7). Therefore the restriction of f_F to a great circle containing the two meridians of longitudes ϑ and $\vartheta + \pi$ is not of class C^2 at the poles of $\boldsymbol{S}^2$ whenever $\vartheta \notin (\pi/2)\boldsymbol{Z}$. The remaining statements of Theorem 1 are obtained by applying Proposition 1 (§4) to

$$(\alpha, \rho, \beta, \gamma) = (-3\pi/4, 2\sqrt{2}, 2/\sqrt{3}, 1/2\sqrt{3}).$$

∎

In order to prove Theorems 2 and 3, it is convenient to state the following Lemma.

LEMMA 3. *The set $f_F(\boldsymbol{S}^2)$ is contained in the real algebraic surface of equation $P_F = 0$.*

PROOF. Let us introduce an extra variable T in order to get the following parametrization of $f_F(\boldsymbol{S}^2)$ now expressed in homogeneous coordinates

$$\begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix} = \begin{pmatrix} (1/\sqrt{3})(1+2\tau)\cos\vartheta \\ (1/\sqrt{3})(1-2\tau)\sin\vartheta \\ 1 \\ 1 - \tau(\sin 2\vartheta - \cos 2\vartheta) + \tau^2 \end{pmatrix},$$

where, as before, $\tau = \tan\eta$. With the quantities A, B, C, D, E and G introduced in the statement of Theorem 3, where A and G are now homogenized by setting $A = Z(T-Z)$ and $G = 3(X^2 + Y^2 + Z^2) - 4ZT$, one obtains

$$G = 4\tau\sin 2\vartheta \quad \text{and} \quad 3E = 2(1-4\tau^2)\sin 2\vartheta .$$

From these relations one gets the following relation

$$G - 6E\tau - 4\tau^2 G = 0,$$

which is of degree two in τ. On the other hand, the following holds

$$3B - D + 12\tau C + 4\tau^2(3B + 2D) - 16\tau^4 D = 0 .$$

The polynomial P_F is obtained by eliminating τ between the two previous equations.

∎

This result shows only that $f_F(\boldsymbol{S}^2)$ is contained in $[P_F = 0]$ but not that f_F maps $\boldsymbol{S}^2$ onto the set of real zeros of P_F, a fact which will be proved in Theorem 3.

THEOREM 2. *The intersection of $f_F(\boldsymbol{S}^2)$ with the sphere $G = 0$ (where $G = 3(X^2 + Y^2 + Z^2) - 4Z$) is the union of the three circles $\mathscr{C}_1$, $\mathscr{C}_2$ and $\mathscr{C}_3$ of equations*

$$\mathscr{C}_1: \quad X = 0 \quad \textit{and} \quad Y^2 + Z^2 = 4Z/3,$$

$$\mathscr{C}_2: \quad Y = 0 \quad \textit{and} \quad X^2 + Z^2 = 4Z/3,$$

$$\mathscr{C}_3: \quad Z = 1 \quad \textit{and} \quad X^2 + Y^2 = 1/3 .$$

The multiple locus of f_F is the union of the two circles $\mathscr{C}_1$ and $\mathscr{C}_2$ and of the fourbladed propeller $\mathscr{C}_0$ parametrized by $\sigma \in \boldsymbol{R}/2\pi\boldsymbol{Z}$

$$\mathscr{C}_0 : 4(4+\sin 4\sigma+\cos 4\sigma)^{-1}\cdot\begin{pmatrix} -(\sqrt{2}/\sqrt{3})\cos 2\sigma\cos\sigma \\ (\sqrt{2}/\sqrt{3})\cos 2\sigma\sin\sigma \\ 1 \end{pmatrix}.$$

PROOF. We already know that, restricted to the sphere $[G=0]$, the sets $[P_F=0]$ and $f_F(\boldsymbol{S}^2)$ coincide, since, by Lemma 3

$$f_F(\boldsymbol{S}^2)\cap[G=0]\subseteq[P_F=0]\cap[G=0]=\mathscr{C}_1\cup\mathscr{C}_2\cup\mathscr{C}_3$$

and since $\mathscr{C}_1$, $\mathscr{C}_2$ and $\mathscr{C}_3$ are images by f_F of the great circles of $\boldsymbol{S}^2$ of equations

$$\vartheta\equiv\pi/2 \quad (\pi) \qquad \vartheta\equiv 0 \quad (\pi) \quad \text{and} \quad \eta=0\,.$$

Recall that Proposition 4 §5, already showed, in particular, that $f_F(\boldsymbol{S}^2)\cap[G=0]=\mathscr{C}_1\cup\mathscr{C}_2\cup\mathscr{C}_3$. In order to finish the proof of Theorem 2, we have now to study the multiple locus of f_F. Since, as a consequence of Lemma 3, this multiple locus is contained in the singular locus of $[P_F=0]$, one might consider to study this singular locus, i.e., to solve the system

$$\begin{cases} \partial_X P_F=0 \\ \partial_Y P_F=0 \\ \partial_Z P_F=0 \\ \partial_T P_F=0\,. \end{cases}$$

Unfortunately, the calculations needed in order to solve this system, turn out to be utterly complicated. It is much more convenient to determine the apparent contour of $f_F(\boldsymbol{S}^2)$ looked at from the origin. This is done by eliminating G between $P_F=0$ and $\partial_T P_F=0$, where T has been introduced in the proof of Lemma 3. From what precedes, we already selected the following components of this apparent contour:

(i) the planes $X=0$ and $Y=0$ yielding to the circles $\mathscr{C}_1$ and $\mathscr{C}_2$,

(ii) the two imaginary lines $X\pm iY=Z=0$ attached to the singularity of the algebraic surface $P_F=0$ at the origin,

(iii) the cone enveloped by the planes $\mathscr{P}_\vartheta$, i.e., the union of all lines connecting the origin with the fourcuspidal hypocycloid $\mathscr{A}\subseteq[Z=T]$ (§4) whose equation is

$$243E^2D+(3B-4D)^3=0\,.$$

As we now proceed to show, the remaining part of the apparent contour is the cone generated by the curve $\mathscr{C}_0$ defined in the statement of Theorem 2. While eliminating G between $P_F=0$ and $\partial_T P_F=0$, we factor out the terms corresponding to the components (i), (ii) and (iii) above, so that the remaining factor yields the equation

$$3B^3 - 2C^2D = 0 .$$

The intersection of the plane $Z = T$ and of the cone given by the previous equation, is a rational curve of degree six defined, in polar coordinates, by the relation

$$(X^2 + Y^2)^{1/2} = \frac{\sqrt{2}}{\sqrt{3}} \cos 2\sigma .$$

By lifting this curve on $f_F(\boldsymbol{S}^2)$, we get the following parametrization of $\mathscr{C}_0$:

$$m_\sigma = 4(4 + \sin 4\sigma + \cos 4\sigma)^{-1} \cdot \begin{pmatrix} -(\sqrt{2}/\sqrt{3}) \cos 2\sigma \cos \sigma \\ (\sqrt{2}/\sqrt{3}) \cos 2\sigma \sin \sigma \\ 1 \end{pmatrix} .$$

Notice that the inverse image of $\mathscr{C}_0$ in $\boldsymbol{S}^2$ has two connected components parametrized by (see Figure 11)

$$\begin{cases} \vartheta_1 = \sigma + \arcsin(\sqrt{2} \sin \sigma \cos \sigma) - \pi/2 \\ \tau_1 = (\cos 2\sigma - (1 + \cos^2 2\sigma)^{1/2})/2 \end{cases} \qquad \begin{cases} \vartheta_{-1} = \sigma - \arcsin(\sqrt{2} \sin \sigma \cos \sigma) + \pi/2 \\ \tau_{-1} = (\cos 2\sigma + (1 + \cos^2 2\sigma)^{1/2})/2 , \end{cases}$$

where $\sigma \in \boldsymbol{R}/2\pi\boldsymbol{Z}$. ■

The proof of Theorem 3 is based on the classical following fact:

LEMMA 4. *Let $P \in \boldsymbol{C}[x, y]$ be a non zero irreducible polynomial, for which there exist two rational functions $\varphi, \psi \in \boldsymbol{C}(t)$ satisfying the following conditions*:

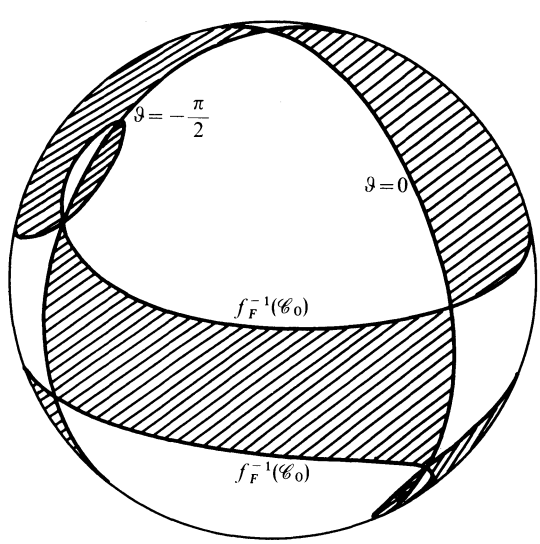

FIGURE 11. The inverse image of $\mathscr{C}_0$ in $\boldsymbol{S}^2$.

(i) *$P(\varphi(t), \psi(t))=0$ for all $t \in \boldsymbol{C}$ for which both φ and ψ are defined,*

(ii) *The set $D_P=\{t: \exists t' \neq t$ such that $\varphi(t)=\varphi(t')$ and $\psi(t)=\psi(t')\}$ is finite,*

(iii) *φ and ψ are not both constant.*

Then, for all $t \in D_P$, the point $(\varphi(t), \psi(t))$ is a singular point of the rational plane curve defined by $P=0$. Moreover, the set of zeros of P is equal to the image of $\boldsymbol{C} \cup \{\infty\}$ under (φ, ψ).

PROOF. According to the Euler homogeneity formula, each singular point of the curve $P=0$ is determined by a solution of the following system

$$\begin{cases} P=0 \\ \partial_X P=0 \\ \partial_Y P=0 \,. \end{cases}$$

Let $t_0 \in \boldsymbol{C}$, and, for convenience, assume that $\varphi(t_0)=\psi(t_0)=0$. Let us first prove that, if $P(0, 0)$ is not a singular point, namely, if one of the partial derivatives of P (say $\partial_X P$) does not vanish at $(0, 0)$, then $t_0 \notin D_P$. The implicit function Theorem asserts that, if

$$P(0,0)=0 \quad \text{and} \quad \partial_X P(0,0) \neq 0 \,,$$

then, in neighbourhoods of $(0, 0)$ and of 0, there exist analytic functions $g(x, y)$ and $\chi(y)$ such that

$$P(x, y)=(x-\chi(y)) \cdot g(x, y) \quad \text{and} \quad g(0,0) \neq 0 \,.$$

Hence, in a neighbourhood of t_0, one has $\varphi(t)=\chi \circ \psi(t)$. Now, if t_1 is such that $\varphi(t_1)=\psi(t_1)=0$, then the relation $\varphi(t)=\chi \circ \psi(t)$ also holds in a neighbourhood of t_1. In particular, we see that ψ cannot be constant, since, otherwise, φ would also be constant, a impossible fact by the hypothesis (iii). The open mapping Theorem then states that there exists $\varepsilon>0$ such that $\psi]t_0-\varepsilon, t_0+\varepsilon[$ and $\psi]t_1-\varepsilon, t_1+\varepsilon[$ are neighbourhoods of 0. Therefore, there exists a sequence of mutually disjoint complex numbers

$$y_n \in \psi]t_0-\varepsilon, t_0+\varepsilon[\, \cap\, \psi]t_1-\varepsilon, t_1+\varepsilon[\,,$$

and hence, two sequences $t_{0,n}$ and $t_{1,n}$ such that

$$\psi(t_{0,n})=\psi(t_{1,n})=y_n \quad \text{and} \quad |t_0-t_{0,n}|, |t_1-t_{1,n}|<\varepsilon \,.$$

Thus, for ε small enough, one has

$$\varphi(t_{0,n})=\chi \circ \psi(t_{0,n})=\chi \circ \psi(t_{1,n})=\varphi(t_{1,n}) \,.$$

Since the $t_{0,n}$'s are mutually distinct, while the set D_P is assumed to be finite, there exist m and n such that $t_{0,m}=t_{1,n}$, and therefore

$$|t_0-t_1| \leq |t_0-t_{0,m}|+|t_1-t_{1,n}|<2\varepsilon \,.$$

Hence, for ε sufficiently small, one gets $t_0=t_1$, which brings to an end the proof that $t_0 \notin D_P$.

In order to prove now that (φ, ψ) maps $\boldsymbol{C} \cup \{\infty\}$ *onto* the set of zeros of P, let (x_0, y_0) be such a zero, and write $\varphi = \varphi_1/\varphi_2$ and $\psi = \psi_1/\psi_2$, where the polynomials φ_i's and ψ_i's are such that

$$\gcd(\varphi_1, \varphi_2) = 1 = \gcd(\psi_1, \psi_2).$$

Since P is irreducible and satisfies $P(\varphi, \psi) = 0$, Hilbert's Nullstellensatz asserts that there exists $k \geq 1$ such that P^k is the resultant of the two polynomials $x\varphi_2 - \varphi_1$, $y\psi_2 - \psi_1 \in \boldsymbol{C}[t]$. Therefore, we are led to the following dilemma, each horn of which yields our conclusion:

(i) either there exists $t_0 \in \boldsymbol{C}$ such that

$$x_0\varphi_2(t_0) - \varphi_1(t_0) = 0 \quad \text{and} \quad y_0\psi_2(t_0) - \psi_1(t_0) = 0,$$

(ii) or the terms of higher degree in t of $x\varphi_2 - \varphi_1$ and $y\varphi_2 - \varphi_1$ vanish at t_0, a fact which reads $x_0 = \varphi(\infty)$ and $y_0 = \psi(\infty)$.

■

THEOREM 3. *The set $f_F(\boldsymbol{S}^2)$ is equal to the set of real zeros of the polynomial P_F of degree eight*

$$P_F(X, Y, Z) = -72ADE^2 - 18DEG(C+E) + 3BG^2(3B-4D) + 4G^4$$

where $A = Z(1-Z)$, $B = (2X^2 + Y^2)$, $C = 2(Y^2 - X^2)$, $D = 2Z^2$, $E = 4XY$, and $G = 3(X^2 + Y^2 + Z^2) - 4Z$.

PROOF. In order to check that P_F has no other real zeros than those belonging to $f_F(\boldsymbol{S}^2)$, it sufficies to show that the (real) intersections of the surface $[P_F = 0]$ with the planes $\mathscr{P}_{\vartheta_0}$ are in $f_F(\boldsymbol{S}^2)$. It is indeed sufficient since the union of the $\mathscr{P}_{\vartheta_0}$'s is equal to $\boldsymbol{R}^3$. These planes intersect the surface $[P_F = 0]$ along the curve of degree eight which splits into the conic $\mathscr{E}_{\vartheta_0}$ and into the rational curve of degree six $\mathscr{C}_{\vartheta_0}$ which is the image by f_F of the curve contained in $\boldsymbol{S}^2$ and of equation

$$2\tan\eta = -\sin 2\vartheta_0 \cdot \tan\left(\frac{\vartheta - \vartheta_0}{2}\right) + \cos 2\vartheta_0,$$

where $\vartheta \in \boldsymbol{R}/2\pi\boldsymbol{Z}$ and $\eta \in [-\pi/2, \pi/2]$ are as above. Since each real point lying on $\mathscr{P}_{\vartheta_0}$ and satisfying the equation of the conic $\mathscr{E}_{\vartheta_0}$, belongs to $\mathscr{E}_{\vartheta_0}$, a real point $m \in [P_F = 0] \cap \mathscr{P}_{\vartheta_0}$ outside $\mathscr{C}_{\vartheta_0} \cup \mathscr{E}_{\vartheta_0}$ must satisfy the equation of $\mathscr{C}_{\vartheta_0}$. Moreover, such a point m is not in the image of the parametrization (φ, ψ) obtained by replacing $2\tan\eta$ by

$$-\sin 2\vartheta_0 \cdot \tan\left(\frac{\vartheta - \vartheta_0}{2}\right) + \cos 2\vartheta_0,$$

now considered as a function of $\tan(\vartheta/2)$. Since the parametrization (φ, ψ) satisfies the conditions of Lemma 4, the point m can be written $m = (\varphi(t), \psi(t))$, where $t \in \boldsymbol{C} \setminus \boldsymbol{R}$. Since the coefficients of the rational functions φ and ψ are real, one has

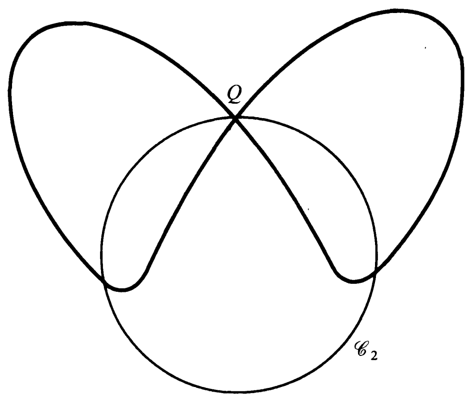

FIGURE 12. The intersection $[P_F=0]\cap\mathscr{P}_0$.

$$m=(\varphi(t),\psi(t))=(\varphi(\bar{t}),\psi(\bar{t}))\,,$$

a fact which, according to Lemma 4, proves that m is a singular point of $\mathscr{C}_{\vartheta_0}$.

Let us now prove that all points of this type belong to $f_F(S^2)$. Notice first that when $\vartheta_0\equiv 0\ (\pi/2)$, the intersection $[P_F=0]\cap\mathscr{P}_{\vartheta_0}$ splits into the union of one of the double circles $\mathscr{C}_1$ and $\mathscr{C}_2$, with a rational quartic having two singular points, namely, an ordinary singular point at $Q=(0,0,4/3)$ and a tacnode at the origin (see Figure 12).

Therefore, when $\vartheta_0\equiv 0\ (\pi/2)$, there exists no real singular point of $[P_F=0]\cap\mathscr{P}_{\vartheta_0}$ not contained in $f_F(S^2)$ so that, in order to complete the proof, one may assume that $\sin 2\vartheta_0\neq 0$. With this assumption, the equation of $\mathscr{C}_{\vartheta_0}$ reads

$$A_0+A_1G+A_2G^2+4G^4=0\,,$$

where

$$\begin{cases} A_0=18\sqrt{3}\,ZE^2(X\sin 3\vartheta_0+Y\cos 3\vartheta_0)/\sin^2 2\vartheta_0 \\ A_1=4D\sin 2\vartheta_0(D\sin 2\vartheta_0-3E)+3B(3B-4D) \\ A_2=8\sqrt{3}\,Z(X\sin\vartheta_0-Y\cos\vartheta_0)\,, \end{cases}$$

and where A, B, C, D, E, G, have the same meaning as in the statement of Theorem 3. This formula shows that the origin is the unique real zero of P_F belonging to the plane $Z=0$. It therefore suffices to look for singular points of $\mathscr{C}_{\vartheta_0}$ whose third cooordinate Z is different from 0. Such points are solutions of the following system (considered as linear with respect to the unknowns $1, G, G^2$):

$$\begin{cases} A_1+2A_2G+12G^2=0 \\ \partial_X A_0+\partial_X A_1\cdot G+\partial_X A_2\cdot G^2=0 \\ \partial_Y A_0+\partial_Y A_1\cdot G+\partial_Y A_2\cdot G^2=0\,. \end{cases}$$

After the multiplication by the quantity $2^{-14}\cdot 3^{-5}\cdot\sin^5 2\vartheta_0$, the determinant of this system becomes equal to

$$XY(X(\cos\vartheta_0-\sin\vartheta_0)+Y(\cos\vartheta_0+\sin\vartheta_0))(Y(\cos\vartheta_0-\sin\vartheta_0)-X(\cos\vartheta_0+\sin\vartheta_0))D_0\,.$$

Now, if we multiply by $-2^{-20}\cdot 3^{-8}\cdot\sin^8 2\vartheta_0$, the resultant of the two last equations of the system regarded as polynomial of degree two in G, we get

$$X^2Y^2(X(\cos\vartheta_0-\sin\vartheta_0)+Y(\cos\vartheta_0+\sin\vartheta_0))(Y(\cos\vartheta_0-\sin\vartheta_0)-X(\cos\vartheta_0+\sin\vartheta_0))R_0\,.$$

The two linear factors

$$X(\cos\vartheta_0-\sin\vartheta_0)+Y(\cos\vartheta_0+\sin\vartheta_0)\quad\text{and}\quad Y(\cos\vartheta_0-\sin\vartheta_0)-X(\cos\vartheta_0+\sin\vartheta_0)$$

correspond to the two points where the fourbladed propeller $\mathscr{C}_0$ intersects the plane $\mathscr{P}_{\vartheta_0}$. On the other hand, after a suitable combination of the equations $D_0=0$ and $R_0=0$, if we set

$$c=2\cos^2\vartheta_0\,,\qquad s=2\sin^2\vartheta_0\,,\quad x=X\sin\vartheta_0\quad\text{and}\quad y=Y\cos\vartheta_0\,,$$

we get

$$ax^2+2bxy+dy^2=0\,,$$

where

$$\begin{cases} a=12c^5-29c^4-16c^3+62c^2+28c+7\\ b=6c^5-4c^4-59c^3+67c^2+53c+17\\ d=25c^4-100c^3+86c^2+28c+25\,.\end{cases}$$

The rotation $\rho_{\pi/2}$ of angle $\pi/2$ around $\overrightarrow{OZ}$ which maps (c, s, x, y) onto (s, c, y, x), yields a new equation of degree two

$$a'x^2+2b'xy+d'y^2=0\,,$$

where

$$\begin{cases} a'=12s^5-29s^4-16s^3+62s^2+28s+7\\ b'=6s^5-4s^4-59s^3+67s^2+53s+17\\ d'=25s^4-100s^3+86s^2+28s+25\,.\end{cases}$$

The resultant of the previous polynomial and this new polynomial is equal to

$$2^8\cdot 3^2\cdot(1-t^2)(25t^7-11t^6-114t^5-70t^4+89t^3+237t^2+256t+100)\,,$$

where $t=cs=\sin^2 2\vartheta_0$. One readily sees that the factor of degree seven of this expression does not vanish on $]0, 1[$, by noticing that this can be written

$$(25t+39)t^4(1-t)^2+(61t^2+170t+81)t^2(1-t)+156t^2+256t+100\,.$$

Last, one checks that for $t^2=1$, i.e., for $\vartheta_0 \equiv \pi/4$ $(\pi/2)$, there are no real singular points in $[P_F=0] \cap \mathscr{P}_{\vartheta_0}$ not belonging to the image of f_F. This brings to an end the proof of Theorem 3. ■

THEOREM 4. *The two circles $\mathscr{C}_1$ and $\mathscr{C}_2$ of the multiple locus of f_F intersect at $O=(0,0,0)$ and at the point $Q=(0,0,4/3) \in \mathscr{C}_0$, which is the quadruple point of f_F, at which the tangent planes to f_F of equations*

$$3Z+2\sqrt{3}\,X=4\,,\quad 3Z-2\sqrt{3}\,X=4\,,\quad 3Z+2\sqrt{3}\,Y=4\,,\quad 3Z-2\sqrt{3}\,Y=4\,,$$

are in general position, so that Q is of type $\boldsymbol{Q}$. Each of the four blades of the propeller $\mathscr{C}_0$ intersects one of the four meridians contained in $\mathscr{C}_1 \cup \mathscr{C}_2 \setminus \{O, Q\}$ at a point D_1^i $(1 \le i \le 4)$ such that, if $D_1^1=(4\sqrt{2}/5\sqrt{3}, 0, 4/5)$, one obtains the other D_1^i's by applying to D_1^1 the fourfold symmetry of f_F. The five points O and D_1^i's are all of type $\boldsymbol{D}_1$. On the self-intersection curve of f_F, each point $P \neq O, Q, D_1^1, \ldots, D_1^4$ is such that the two sheets of f_F intersecting at P have tangent planes transverse to each other.

PROOF. On the closed halfway model f_F, with the notation of §5, one has

$$G=4\tau\sin 2\vartheta\cdot(1-\tau(\sin 2\vartheta-\cos 2\vartheta)+\tau^2)^{-2} \quad \text{where} \quad \tau=\tan\eta\,.$$

Therefore, when they cross each other along the circles $\mathscr{C}_1$ and $\mathscr{C}_2$, the two sheets of the model also cross the sphere $G=0$. Using an orientation-preserving local diffeomorphism around the origin in $\boldsymbol{R}^3$ of class C^1, but not C^2 at the origin, which maps a neighbourhood of the origin in the sphere $G=0$ into the horizontal plane $Z=0$, one therefore gets that the origin O is of type $\boldsymbol{D}_1$ in f_F. The expression of the 4-jet of P_F around the point $Q=(0,0,4/3)$

$$3^{-2}\cdot 2^{10}\cdot(\sqrt{3}(Z-4/3)+2X)(\sqrt{3}(Z-4/3)-2X)(\sqrt{3}(Z-4/3)+2Y)(\sqrt{3}(Z-4/3)-2Y),$$

shows that this point is of type $\boldsymbol{Q}$. The 3-jet of P_F at the point $D_1^1=(4\sqrt{2}/5\sqrt{3}, 0, 4/5)$ has the form

$$-5^{-6}\cdot 2^{15}\cdot l\cdot(2l+5(3x^2-18xy+4\sqrt{6}\,xz+3y^2+5z^2))\,,$$

where

$$l=2x\sqrt{6}-2y\sqrt{6}+z\,,\quad X=x+4\sqrt{2}/5\sqrt{3}\,,\quad Y=y\,,\quad \text{and} \quad Z=z+4/5\,.$$

The quadric showing up in the expression of this jet, is a saddle with tangent $l=0$ at D_1^1, so that this point is of type $\boldsymbol{D}_1$ (as well as the other D_1^i's for symmetry reasons).

A tangent vector $\vec{t}_\sigma$ to the fourbladed propeller $\mathscr{C}_0$, at the point m_σ of parameter σ, is given by the expression

$$\vec{t}_\sigma=\frac{\sqrt{3}}{4}q^2\cdot\frac{\overrightarrow{dm}}{d\sigma} \quad \text{where} \quad q=1+(\sin 4\sigma+\cos 4\sigma)/4\,.$$

Similarly, the vectors

$$\vec{t_i}=(\sin 2\sigma(\cos 2\sigma+\sin 2\sigma)-2t_i\cos 2\sigma)\cdot\vec{u}_i+2q\cdot\vec{v}_i\,,$$

where

$$4t_1t_2=-1\,,\quad t_1+t_2=-\cos 2\sigma\,,\quad \vec{u}_i=\begin{pmatrix}-\cos 2\sigma\cos\sigma\\ \cos 2\sigma\sin\sigma\\ \sqrt{3}/\sqrt{2}\end{pmatrix},\quad \vec{v}_i=\begin{pmatrix}\sin\sigma(t_i+\cos^2\sigma)\\ \cos\sigma(t_i-\sin^2\sigma)\\ 0\end{pmatrix},$$

and where $i=1, 2$, are respectively tangent to the ellipses $\mathscr{E}_{\vartheta_1}$ and $\mathscr{E}_{\vartheta_2}$ intersecting each other at m_σ. Since the determinant

$$\sqrt{3}\,q^2\cdot(1+\cos^2 2\sigma)^{1/2}\cdot\sin^3 2\sigma\,,$$

of the three vectors $\vec{t_1}$, $\vec{t_2}$, and $\vec{t_\sigma}$, is not equal to 0 when $m_\sigma\neq Q$, $D_1^1, \ldots, D_1^4$, one sees that f_F is transversal along $\mathscr{C}_0\setminus\{Q, D_1^1, \ldots, D_1^4\}$. The 2-jet of P_F at the point $(X_\sigma, 0, Z_\sigma)\in\mathscr{C}_2$, where

$$X_\sigma=(1+\sigma+\sigma^2)^{-1}\cdot(1+2\sigma)/\sqrt{3}\quad\text{and}\quad Z_\sigma=(1+\sigma+\sigma^2)^{-1}\,,$$

is given by

$$\begin{aligned}48X_\sigma^2Z_\sigma^4\{4\sigma\sqrt{3}\,yZ_\sigma+(2\sigma-1)(3xX_\sigma+z(3Z_\sigma-2))\}\\ \times\{-4(\sigma+1)\sqrt{3}\,yZ+(2\sigma+3)(3xX_\sigma+z(3Z_\sigma-2))\}\,,\end{aligned}$$

where

$$X=x+X_\sigma\,,\qquad Y=y\,,\quad\text{and}\quad Z=z+Z_\sigma\,.$$

This shows that the immersion f_F is transversal along $\mathscr{C}_2$ except at the points such that $4\sigma^2+4\sigma-1=0$, namely, at the points of type $\boldsymbol{D}_1$ belonging to $\mathscr{C}_2$, at the origin, and at the quadruple point. For symmetry reasons, f_F is also transverse along $\mathscr{C}_1$ except at four points. ■

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Appendix by Bernard MORIN

A1. Coding of a generic differentiable regular homotopy from a surface into a 3-dimensional manifold. This paragraph introduces the needed ingredients in order to be able to say in §A2 along which differentiable tracks the algebraic programme mentioned in the introduction should be carried out.

General background: generic homotopies and generic regular homotopies.

(a) The following is needed in order to state (e) ii. Let $r \geq 2$ be an integer or infinity. Recall that on a C^r manifold W (with $r \geq 2$) a *Morse function h* is a differentiable function whose Hessian (the determinant of the Hessian matrix of its second derivatives with respect to any chart) does not vanish at singular points $Dh = 0$ of h. In particular, this requirement implies that the critical points of h are isolated, and hence finite whenever W is compact. The *index* of a critical point of a Morse function is the index of its Hessian matrix. Hence, for a surface, the points of index 0 are the local minima, the points of index 1 the saddles and the points of index 2 the local maxima of the function, while for a curve, the points of indices 0 and 1 are the local minima and the local maxima of the function respectively.

(b) Let M be a closed C^r surface and V a 3-dimensional C^r manifold; let $C^r(M, V)$ denote the space of differentiable mappings M into V equipped with C^r topology (a topology which poses no problems since M is assumed to be compact). Let f_{-1} and f_1 be differentiable mappings from M into V and let f_t: $M \to V$, $t \in [-1, 1]$ be a family of mappings connecting f_{-1} to f_1. Let $\bar{f}$: $M \times [-1, 1] \to V \times [-1, 1]$ be defined by $\bar{f}(m, t) = (f_t(m), t)$ for all $(m, t) \in M \times [-1, 1]$. The map $\bar{f}$ will be referred to as the map *associated to* $\{f_t\}$. We say that the family $\{f_t\}$ is a differentiable deformation or a differentiable homotopy when the associated map $\bar{f}$ is differentiable. The differentiable homotopies are the differentiable paths in $C^r(M, V)$. While studying the homotopy groups of $C^r(M, V)$, a well-known fact asserts that it suffices to restrict one's attention to differential deformations of differentiable homotopies.

(c) In the case where f_{-1} and f_1 are immersions, we say that the differential homotopy $\{f_t\}$ is a (differentiable) *regular homotopy*, when all the f_t's are immersions, namely when $\bar{f}$ is an immersion. The regular homotopies are the differentiable paths in the subspace $\mathrm{Im}(M, V) \in C^r(M, V)$ of C^r immersions M into V. By differentiably deforming such paths, one can study the homotopy properties of $\mathrm{Im}(M, V)$.

(d) The following is needed in order to state (e) i. The differentiable immersion $\bar{f}$ of the 3-dimensional manifold $M\times[-1,1]$ into the 4-dimensional manifold $V\times[-1,1]$ is said to be a *transverse immersion*, if the following two conditions hold:

i. The map $\bar{f}$ has only double, triple and quadruple points at which the tangent hyperplanes are in general position.

ii. The mapping $\bar{f}$ has no quadruple points on $V\times\{-1,1\}=\partial(V\times[-1,1])$. At a simple, double or triple point of $\bar{f}$ lying on $\partial(V\times[-1,1])$ the tangent hyperplanes are also in general position with the tangent hyperplane to $\partial(V\times[-1,1])$.

If such is the case, all the f_t are immersions and f_{-1} and f_1 are transverse. Moreover, the multiple locus of $\bar{f}(M\times[-1,1])$ is an immersed surface D contained in $V\times[-1,1]$ transversely intersecting $\partial(V\times[-1,1])$ along its boundary ∂D. The multiple locus of D is an immersed curve T intersecting itself at a finite set $Q\subset V\times]-1,1[$ of quadruple points and intersecting $\partial(V\times[-1,1])$ transversely at its boundary ∂T, which is also a finite set. Notice that D is not transversely immersed since along $T\setminus Q$ three sheets of D intersect in such a way that six of them meet at every point of Q. Each point of Q is also a quadruple point for the immersed curve T which, by this reason, is not transverse in $V\times[-1,1]$.

(e) Here we define the central notion of this section. Denote by $\tau_{|D}$ and $\tau_{|T}$ the restriction of the projection $\tau\colon V\times[-1,1]\to[-1,1]$ to D and T respectively. In the case where the immersions f_{-1} and f_1 are transverse (see §1), the regular homotopy $\{f_t\}$ is said to be a generic (differentiable) regular homotopy when the associated mapping $\bar{f}$ satisfies the following three conditions:

i. The differentiable mapping $\bar{f}$ is a transverse immersion.

ii. The functions $\tau_{|D}$ and $\tau_{|T}$ are Morse functions with no critical points at ∂D and ∂T.

iii. For each $t\in]-1,1[$, the image of f_t in $V\times\{t\}$ contains at most one point belonging either to the singular set $\mathrm{Sing}(\tau_{|D})$ of $\tau_{|D}$, either to the singular set $\mathrm{Sing}(\tau_{|T})$ of $\tau_{|T}$ or to Q.

Notice that by construction $\tau\circ\bar{f}\colon M\times[-1,1]\to[-1,1]$ has no critical points. All the immersions of a generic regular homotopy $\{f_t\}$ are transverse except a finite number of them corresponding to the values of the parameter t for which either

i. $\tau_{|D}$ has a minimum and hence f_t a point of type $\boldsymbol{D}_0$
ii. $\tau_{|D}$ has a saddle and hence f_t a point of type $\boldsymbol{D}_1$
iii. $\tau_{|D}$ has a maximum and hence f_t a point of type $\boldsymbol{D}_2$
iv. $\tau_{|T}$ has a minimum and hence f_t a point of type $\boldsymbol{T}^+$
v. $\tau_{|T}$ has a maximum and hence f_t a point of type $\boldsymbol{T}^-$
vi. f_t has a quadruple point, namely a point of type $\boldsymbol{Q}$.

If we weaken the notion of generic regular homotopy by replacing iii of the definition by

iii′. $T\cap\mathrm{Sing}(\tau_{|D})=Q\cap\mathrm{Sing}(\tau_{|T})=\varnothing$, then we are led to a slightly more general class of regular homotopies of great interest, since one can then look for deformations for which $\tau_{|D}$ and $\tau_{|T}$ are nice functions in the sense of S. Smale (see his work on the Poincaré conjecture in high dimensions). The regular homotopies satisfying i, ii and iii′ will be called *quasigeneric*.

(f) For the sake of completeness let us mention the following notions. A *generic differentiable map* $f: M \to V$ is a transverse immersion except at a finite number of simple points called *cusps* or *pinch points* around which there exist local charts both in M and V, where f reads

$$(x, y) \to (x, xy, y^2).$$

A generic deformation (or *generic homotopy*) $\{f_t\}$ from a generic map f_{-1} to a generic map f_1 can be defined in such a way that the associated surface D has new boundary points corresponding to the various cusps and confluence of cusps of the f_t's. Associated to each cusp point of f_{-1} and f_1, the surface D has corners (see types and archetypes below). The restriction of τ to the closure of the curve $\partial D \cap V \times]-1, 1[$ has to be a Morse function whose extrema correspond to the hyperbolic and elliptic confluences of cusps studied in [A2]. Moreover, D is no longer immersed, but has pinch points all contained in $V \times]-1, 1[$ at which the curve T has boundary points. These points correspond to births and deaths of single triple points. Such births and deaths occur when a moving pinch point of $\{f_t\}$ hits another sheet of the deformed surface. Of course, the condition similar to (e) iii should also be written down. In order to control such generic deformations, one therefore has to introduce 6 additional types of modifications, corresponding respectively to births and deaths of pairs of cusps (both in the hyperbolic and elliptic case) as well as births and deaths of single triple points. Of course, conditions similar to (e) iii′ can be written in order to yield the notion of *quasigeneric homotopy*. An example of quasigeneric homotopy is Apéry's Romboy homotopy (see [A2]).

(g) With the help of the previous definitions, we are able to state the following well-known claims. The subspace $\mathrm{Gen}(M, V)$ of generic differentiable mappings of M into V is open and dense in $C^r(M, V)$. The subspace $\mathrm{Im}(M, V)$ of $C^r(M, V)$ is open but not dense in the space $C^r(M, V)$; moreover, it is not contained in $\mathrm{Gen}(M, V)$. The subspace $\mathrm{Im}_t(M, V)$ of transverse immersions is open and dense in $\mathrm{Im}(M, V)$; moreover, $\mathrm{Im}_t(M, V) = \mathrm{Im}(M, V) \cap \mathrm{Gen}(M, V)$ is an open and closed subset of $\mathrm{Gen}(M, V)$. The subspace of generic homotopies of M into V is open and dense in the space of differentiable homotopies. The subspace of generic regular homotopies of M into V is open and dense in the space of regular homotopies, while the set of quasigeneric regular homotopies is, of course, dense but not open. The set of quasigeneric deformations is dense, while the set of generic homotopies is open and dense in the space of differentiable paths in $C^r(M, V)$.

(h) The following is intended to provide the reader with a good intuition of the geometry of $C^r(M, V)$. Since, by compactness of M, the space $C^r(M, V)$ is a C^r manifold modeled on the Frechet space $C^r(M, \boldsymbol{R}^3)$, (which turns out to be Banach when $r < \infty$), the following considerations make sense. As noticed above, along a generic regular homotopy all immersions f_t are transverse except a finite number of them called immersions of codimension 1. According to the general theory of differentiable singularities due to John Mather, these exceptional immersions lie on the strata of codimension 1 of the stratified object $\mathrm{Im}(M, V) \setminus \mathrm{Im}_t(M, V)$. A generic regular homotopy f_t hits these strata transversely. The strata are of four types

i. the type $\mathfrak{D}_{0,2}$ corresponding to extrema of the Morse function $\tau_{|D}$
ii. the type $\mathfrak{D}_1$ corresponding to saddles of $\tau_{|D}$
iii. the type $\mathfrak{T}$ corresponding to extrema of the Morse function $\tau_{|T}$
iv. the type $\mathfrak{Q}$ corresponding to quadruple points of $\bar{f}$.

Since the strata of type $\mathfrak{D}_{0,2}$ and $\mathfrak{T}$ have a prescribed preferred side $\mathfrak{D}_0$ and $\mathfrak{T}^+$ (corresponding to the minima of $\tau_{|D}$ and $\tau_{|T}$), by calling the opposite side $\mathfrak{D}_2$ and $\mathfrak{T}^-$, we are led to the list of six modifications given in §2 with their local chart. The space $C^r(M, V) \setminus \mathrm{Gen}(M, V)$ is also a stratified object which contains the previous one. In addition to the four types of strata of codimension one listed above, it has strata of three new types with preferred sides corresponding respectively to the hyperbolic confluence of cusps, to the elliptic confluence of cusps and to the birth or death of a single triple point.

(i) One further wishes to deform generic homotopies and generic regular homotopies in a generic manner. In particular, it is most interesting to check whether or not a generic homotopy connecting transverse immersions f_{-1} and f_1 can be deformed to a generic regular homotopy. Such deformations yield also obstructions which can be defined in terms of singularities and which show up when it is impossible to deform a generic regular homotopy into another one by using only immersions. In order to carry out such a programm in the case of generic regular homotopies, one has to introduce immersions of codimension 2 and therefore to assume $r \geq 3$. In particular, one is led to modify by surgeries both the curve T and the surface D. One also has to deform Morse functions on regular parts of T and D and hence to use techniques similar to those introduced by J. Cerf in his work on the group Γ_n. Following the same tracks, one can also define quasigeneric deformations of either quasigeneric homotopies or quasigeneric regular homotopies. Statements similar to those given in (g) concerning generic and quasigeneric families of mappings and immersions hold for an arbitrary number of parameters in the family.

The notion of quasigeneric deformation of quasigeneric regular homotopies is a tool of great practical interest which will be needed at the end of [A2]. For example, in 1967 the French physisict Marcel Froissart imagined an eversion of the sphere which was a simplification of Antony Phillips' eversion [PH]. In order to be able to communicate Froissart's eversion it turned out to be indispensable to specify it into a generic regular homotopy. By deforming quasignerically this generic eversion, it became very easy to discover the open and closed halfway models and to device several variations for the eversion. One of these has been used by Michael Pugh when he constructed the eight models used by Nelson Max in order to produce his celebrated movie.

(j) While working on regular homotopies, the replacement of (e) iii by (e) iii′ also forces one to introduce immersions with finite codimension k, (where k is the number of points in V at which a given $f_t(M)$ is not transverse but only of codimension 1). But contrary to what happens while deforming Morse functions, such immersions of codimension k do not require $r \geq 3$. They lie on strata which are normal crossings of smooth hypersurfaces in the manifold $\mathrm{Im}(M, V)$ entirely contained in the stratified object $\mathrm{Im}(M, V) \setminus \mathrm{Im}^0(M, V)$.

Of course, as for the explicit immersions studied in the present paper (see §3 and §4), when it is impossible to assume $r \geq 2$, the notion of stratum of type $\mathfrak{D}_{0,2}$, $\mathfrak{D}_1$ and $\mathfrak{T}$ becomes fuzzy, a case which yields complications (see §2). However, one may always speak about topological extrema and saddles of functions of class C^1.

(k) Now, the space $\mathrm{Im}_t(M, V) = \mathrm{Im}^0(M, V)$, as well as the union $\mathrm{Im}^1(M, V)$ of strata of codimension 1, (and also the further $\mathrm{Im}^k(M, V)$ hinted at lately) split into connected components which are orbits under the action of the group $\mathrm{Diff}_0(M) \times \mathrm{Diff}_0^c(V)$, where Diff_0 denotes the connected component of the identity of the group of diffeomorphisms and Diff_0^c the same component in the group of diffeomorphism with compact support. Hence one may consider the graph whose vertices are the connected components of $\mathrm{Im}^0(M, V)$ and whose edges (later called modifications, see Lemma 5) are those of $\mathrm{Im}^1(M, V)$. The following material is introduced in order to investigate this graph since a generic regular homotopy is entirely coded, up to isotopy both of source and target, by a path in this graph.

Types and archetypes.

Let $\mathfrak{D}_0$ be the closed unit interval of the real line $\boldsymbol{R}$, let $\mathfrak{D}_1$ be the intersection in the real plane $\boldsymbol{R}^2$ of two closed disks of radius 1 each of them being centered at one of the two ends of $\mathfrak{D}_0 \subseteq \boldsymbol{R} \subseteq \boldsymbol{R}^2$, let $\mathfrak{D}_2$ be the intersection in $\boldsymbol{R}^3$ of two distinct closed balls of radius 1 centered at the ends of $\mathfrak{D}_0 \subseteq \boldsymbol{R}^2 \subseteq \boldsymbol{R}^3$, let $\mathfrak{T}^+$ be the convex hull of an equilateral triangle of $\boldsymbol{R}^2$ having sides of length 1 and let $\mathfrak{T}^-$ be the intersection in $\boldsymbol{R}^3$ of three distinct closed balls of radius 1 centered at the vertices of $\mathfrak{T}^+ \subseteq \boldsymbol{R}^2 \subseteq \boldsymbol{R}^3$. Let $\mathfrak{Q}$ be the convex hull of some regular tetrahedron in $\boldsymbol{R}^3$ with sides of length equal to 1. These six objects are examples of compact manifolds with boundary, edges and corners of dimensions 1, 2 and 3 as defined in [D]. In the sequel these six manifolds with corners will be called *archetypes* and will be denoted by using capital gothic letters.

Let $g: M \to V$ be a transverse immersion of the previously introduced closed surface M into the 3-dimensional manifold V defined up to isotopies both of M and of V. Let N_j $(1 \leq j \leq s)$ be the closures of the finitely many connected components of $V \setminus g(M)$ and notice that each N_j is a 3-dimensional manifold with corners.

Let D_g be the multiple locus and T_g the set of triple points of g so that the $D_g \setminus T_g$ is the set of double points of g. With this notation the open faces of each N_j are contained in $g(M) \setminus D_g$, the open edges in $D_g \setminus T_g$, and the corners in T_g. Recall that there exists a notion of embedding of a manifold with corners $\mathfrak{A}$ into a manifold with corners N mapping corners of codimension k into corners of codimension k with obvious transversality conditions along the image of $\partial\mathfrak{A}$ (where the corners of codimension 0 are by definition the connected components of the manifold itself). Now let $\tau: \mathfrak{A} \to N_j$ be an embedding of manifolds with corners defined on an archetype $\mathfrak{A}$ with values in one of the previously defined N_j's. With the help of a tubular neighbourhood U of $\tau(\mathfrak{A})$ adapted to the manifold with corners N_j, one associates to τ an *elementary modification* g_t of $g = g_{-1}$ in the following manner. Let

$s=2$ if $\mathfrak{A} = \mathfrak{D}_0, \mathfrak{D}_1, \mathfrak{D}_2$,

$s=3$ if $\mathfrak{A} = \mathfrak{T}^+, \mathfrak{T}^{-1}$

and

$s=4$ if $\mathfrak{A} = \mathfrak{Q}$.

Let U_1 and U_2 be open balls such that $U \subset U_1$ and $\bar{U}_1 \subset U_2 \subset V$ and such that for $g^{-1}(U_i)$ is the union of s non intersecting open disks in M. For $i=1,2$ let $M_i = g^{-1}(V \setminus \bar{U}_i)$ and assume that $g_t | M_2 = g | M_2$ for all $t \in [-1, 1]$. In U_1 choose local coordinates adapted to U and prescribed that for these coordinates $g_t | U_1$ is of the form given in §2 clearly corresponding to the shape assumed by the archetype $\mathfrak{A}$. Then g_t can be extended to an immersion of M into V differentiably depending on t. The family $\{g_t\}$ is a generic regular homotopy such that g_t is transverse for all $t \neq 0$ and that g_0 is of the type corresponding to $\mathfrak{A}$. Notice that g_t is completely characterized up to isotopy.

LEMMA 5. (i) *Up to isotopy the possible elementary modifications of the transverse immersion g (namely the arrows of the graph mentioned in* (k) *above starting at the vertex defined by g) are in one-to-one correspondence with the isotopy classes of the various τ's modulo any reparametrization of the source $\mathfrak{A}$ of τ*; (ii) *up to isotopy, any generic regular homotopy from a transverse immersion g_1 to a transverse immersion g_2 of M into V, decomposes into a sequence of such elementary modifications.*

While coding a regular homotopy, we allow ourselves to perform simultaneously a finite number of elementary modifications associated to maps $\tau_i: \mathfrak{A}_i \to N_{j_i}$, provided that the images $\tau_i(\mathfrak{A}_i)$ are disjoint (see §2). In such a case the homotopy is no longer generic but only quasigeneric (see (e)).

A2. Coding of an eversion of the sphere. Let $M = \boldsymbol{S}^2$ and $V = \boldsymbol{R}^3$, and let f_{-1} and f_1 be the standard and the antipodal embeddings of the sphere. The present Paragraph uses the previous material of §A1 in order to code a quasigeneric eversion of the sphere (f_t). Although defined up to isotopy of the source $\boldsymbol{S}^2$ and of the target $\boldsymbol{R}^3$, the eversion (f_t) will be assumed to satisfy the equivariant condition given in §3, which reads

$$f_{-t} = \rho_{\pi/2} \circ f_t \circ \rho' \qquad \forall t \in [-1, 1].$$

This requirement will force us to perform simultaneously more than one elementary modification.

Since we also want the halfway model f_0 to be the closed halfway model, in a neighbourhood of 0 in $[-1, 1]$ we will have to perform simulatneously six of these elementary modifications. The sequence of types associated to the eversion f_t that we intend to describe is

$$(\boldsymbol{D}_0 \quad \boldsymbol{D}_0\boldsymbol{D}_0)(\boldsymbol{T}^+\boldsymbol{T}^+)(\boldsymbol{D}_1\boldsymbol{D}_1 \quad \boldsymbol{D}_1\boldsymbol{Q} \quad \boldsymbol{D}_1\boldsymbol{D}_1)(\boldsymbol{T}^-\boldsymbol{T}^-)(\boldsymbol{D}_2\boldsymbol{D}_2 \quad \boldsymbol{D}_2),$$

where the spaces between groups of types inside a given parenthesis indicate that the simultaneity is possible although not imposed by the equivariance constraint. Our journey (which starts in the connected component $\mathscr{I}_0 \subseteq \mathrm{Im}_t(\boldsymbol{S}^2, \boldsymbol{R}^3)$ of the standard embedding, and ends in the component $\mathscr{I}_A$ of the antipodal embedding) will therefore visit four other connected components of $\mathrm{Im}_t(\boldsymbol{S}^2, \boldsymbol{R}^3)$ denoted by $\mathscr{I}_D$, $\mathscr{I}_T$, $\mathscr{I}'_T$, and $\mathscr{I}'_D$. One enters in $\mathscr{I}_D$ through a stratum of codimension 3, in $\mathscr{I}_T$ through a stratum of codimension 2, in $\mathscr{I}'_T$ through a stratum of codimension 6, in $\mathscr{I}'_D$ through a stratum of codimension 2, and in $\mathscr{I}_A$ through a stratum of codimension 3. Although not described, this quasigeneric eversion is mentioned in [M-P]. This eversion uses 16 elementary moves, while [M-P] describes an alternative version with only 14 elementary modifications (see §3 and §5). After describing the three steps leading to the central model of the present eversion, we will give the reasons why in fact, among equivariant quasigneric eversions, f is simpler than the eversion thoroughly described in [M-P] although it looks slightly more complicated. By carefully controlling the behaviour of the saddle points of the third coordinate function restricted to each f_t, one is able to apply Nelson Max's criterion in order to prove the main result of [B-M] without referring to pictures. A polyhedral regular eversion of the suitably triangulated cuboctahedron has been deviced. Handmade models of this eversion have been constructed by Richard Denner. Each model, including the halfway model which is closed, has therefore twelve vertices and twenty triangles. The polyhedral eversion mimics the present one in an almost generic manner where the phrase "almost generic" has a precise meaning (to appear later).

First step: The coding of the D_0's of the eversion.

Let $n=(0, 0, 1)$ and $s=(0, 0, -1)$ be the poles of the standard oriented sphere $\boldsymbol{S}^2$ in $\boldsymbol{R}^3$. Let

$$e=(0,1,0), \quad e_+=\frac{\sqrt{3}}{3}(-1,1,1), \quad e_-=\frac{\sqrt{3}}{3}(1,1,-1),$$

$$w=(0,-1,0), \quad w_+=\frac{\sqrt{3}}{3}(1,-1,1), \quad \text{and} \quad w_-=\frac{\sqrt{3}}{3}(-1,-1,-1),$$

which all belong to $\boldsymbol{S}^2$. Let C_e and C_w be the two segments in $\boldsymbol{R}^3$ connecting respectively e_- to e_+ and w_- to w_+. Let C_0 be the complement in $\boldsymbol{R}^3$ of the intersection with the open unit ball of the circle centered at n and containing e and w. Notice that C_0, C_e

and C_w hit $\boldsymbol{S}^2$ transversally. Assume that these three curves are the images of $\mathfrak{D}_0$ by three embeddings of manifolds with corners τ_0, τ_e and τ_w, and perform equivariantly the three modifications of type $\boldsymbol{D}_0$ associated with these embeddings in such a way that the three births of these modifications occur at time $t=-2/3$. Assume, moreover, that no new modification occurs before the time $t=-1/3$. For each $t\in]-2/3, -1/3[$ the transverse immersion f_t belongs to $\mathscr{J}_D$ and is such that $\boldsymbol{R}^3\setminus f_t(\boldsymbol{S}^2)$ has five connected components. The closure of three of these components N_0, N_e, and N_w are diffeomorphic to the archetype $\mathfrak{D}_2$. Denote by $D(e)$ and $D(w)$ the faces of N_0 containing respectively the points e and w, by $D(e_+)$ and $D(e_-)$ the faces of N_e containing respectively the points e_+ and e_-, and by $D(w_+)$ and $D(w_-)$ those containing respectively the points w_+ and w_-. These six faces are disks in $f_t(\boldsymbol{S}^2)$. The complement of these six disks will be called the *noman's land* of $f_t(\boldsymbol{S}^2)$. The fourth connected component N' of $\boldsymbol{R}^3\setminus f_t(\boldsymbol{S}^2)$ is bounded by the noman's land and by $D(e)$ and $D(w)$. Moreover, N' is homeomorphic to the interior of a 3-dimensional ball to which two handles have been attached. The last of these five connected components N'' is the unbounded one which has the homeomorphic type of an open solid torus minus one point, the point at infinity. Call Γ_0, Γ_e and Γ_w the respective creases of N_0, N_e and N_w. Of course the equivariance requirement imposes that $\rho_\pi(\Gamma_0)=\Gamma_0$ and $\rho_\pi(\Gamma_e)=\Gamma_w$, where ρ_π denotes the rotation of $\boldsymbol{R}^3$ of angle π around the vertical axis.

Call $\bar{C}_e$ and $\bar{C}_w$ the shortest oriented geodesic arcs connecting respectively e_- to e_+ and w_- to w_+. It is convenient to assume that $f_t(\bar{C}_e)$ (resp. $f_t(\bar{C}_w)$) has a double point $\bar{a}_e\in\Gamma_e$ (resp. $\bar{a}_w\in\Gamma_w$). Let L_e and L_w be the oriented embedded subloops of $f_t(\bar{C}_e)$ and $f_t(\bar{C}_w)$ starting and ending respectively at $\bar{a}_e$ and at $\bar{a}_w$. Assume further that, in $N'\cup N_0$, the loops L_e and L_w bound teardrops, i.e., smooth disks with respective corners at $\bar{a}_e$ and $\bar{a}_w$. Let $\{\bar{b}_w, \bar{c}_w\}$ equal $L_e\cap\Gamma_0$ and $\{\bar{b}_e, \bar{c}_e\}$ equal $L_w\cap\Gamma_0$; in the closed curve Γ_0 the pair $\{\bar{b}_e, \bar{c}_e\}$ links the pair $\{\bar{b}_w, \bar{c}_w\}$. Suppose last that L_e and L_w have linking number equal to 1. With linking number -1, one would be constructing mirror images of the models we have in mind.

Second step: the coding of the T^+'s of the eversion.

For all $f'\in\mathscr{J}_D$, let $N'\subset\boldsymbol{R}^3\setminus f'(\boldsymbol{S}^2)$ be as in the previous step and let $\bar{N}'$ be its closure. Let $\tau'_e\colon\mathfrak{T}^+\to N'$ (resp. $\tau'_w\colon\mathfrak{T}^+\to\bar{N}'$) denote an embedding of manifold with corners mapping one of the corners of $\mathfrak{T}^+$ into a point $a_e\in\Gamma_e$ (resp. $a_w\in\Gamma_w$) and the two other corners into points b_e and c_e (resp. b_w and c_w) all belonging to the curve Γ_0.

Let $X\to\mathscr{J}_D$ be the fiber space of ordered pairs (τ'_e, τ'_w) of embeddings with non intersecting images and such that the pair $\{b_e, c_e\}$ links the pair $\{b_w, c_w\}$ in Γ_0.

LEMMA 6. (i) *With its canonical C^∞-topology, the quotient fiber space $\bar{X}\to\mathscr{J}_D$ of X under the action of the group of reparametrizations of the disjoint union of the two copies of $\mathfrak{T}^+$ is arcwise connected.* (ii) *The subspace $\bar{X}_0$ of $\bar{X}$ of ordered pairs (τ'_e, τ'_w) such that $\rho_\pi\circ\tau'_w=\tau'_e$ is nonempty and has two distinct connected components.*

Notice that if τ'_e and τ'_w are as in Lemma 6, the triangular submanifolds $\tau'_e(\mathfrak{T}^+)$

and $\tau'_w(\mathfrak{T}^+)$ of the manifold with corners $\bar{N}'$ necessarily have two of their sides contained in the noman's land of $f'(\boldsymbol{S}^2)$, while the last sides of $\tau'_e(\mathfrak{T}^+)$ and $\tau'_w(\mathfrak{T}^+)$ must be respectively contained into $D(w)$ and $D(e)$.

Lemma 6 (ii) implies that, while forcing the ordered pair (τ'_e, τ'_w) to satisfy $\tau'_w = \rho_\pi \circ \tau'_e$, one has to make a choice between the two components of $\bar{X}_0$. The choice we want to make here is obtained by setting $a_e = \bar{a}_e$, $a_w = \bar{a}_w, \ldots$ and $c_w = \bar{c}_w$ where $\bar{a}, \bar{b}, \bar{c}$ have been defined in the previous step.

Perform equivariantly and therefore simultaneously the two modifications of type $\boldsymbol{T}^+$ associated to τ_e and τ_w with τ_e and τ_w as defined in the previous step. This should be done in such a way that the contact giving birth to the two pairs of triple points occur at time $t = -1/3$ and that no other modifications occur before the time $t = 0$.

We are now in the component $\mathscr{J}_T$. For all $t \in]-1/3, 0[$ the complement in $\boldsymbol{R}^3$ of the image of S^2 by f_t has seven connected components. The closures of four of these components are diffeomorphic to the archetype $\mathfrak{T}^-$. The closure N_q of another one is diffeomorphic to the archetype $\mathfrak{Q}$. The two remaining components stemming from the previous N' and N'' are respectively diffeomorphic to the open ball and to the open solid torus minus one point (the point at infinity) and still denoted by N' and N''.

For two vertices a and b of the differentiable tetrahedron N_q, denote by $E[a, b]$ the (possibly curved) edge of N_q, connecting these two vertices. The equivariance constraint forces the tetrahedron N_q to satisfy the relation $\rho_\pi(N_q) = N_q$. Hence, the Z-axis hits two opposite edges of the tetrahedron at their midpoints m_1 and m_2. Choose the indexing so that s, n, m_1, and m_2 are above each other. By construction there exist two vertices ε_1 and ε_2 (resp. ω_1 and ω_2) of N_q such that $E[\varepsilon_1, \varepsilon_2] \subseteq \Gamma_e$ (resp. $E[\omega_1, \omega_2] \subseteq \Gamma_w$). Choose the indexing in such a way that m_i is the midpoint of $E[\varepsilon_i, \omega_i]$ for $i = 1, 2$. With the choice made on the linking number of the loops L_e and L_w at the end of the previous step, the orientation defined on N_q by the sequence $(\varepsilon_1, \varepsilon_2, \omega_1, \omega_2)$ is the standard one.

For $t \in]-1/3, 0[$, the four disks $D(e_+)$, $D(e_-)$, $D(w_+)$ and $D(w_-)$ are decomposed by arcs $A(e_+) = D(e_+) \cap \Gamma_0$, $A(e_-) = D(e_-) \cap \Gamma_0$, $A(w_+) = D(w_+) \cap \Gamma_0$ and $A(w_-) = D(w_-) \cap \Gamma_0$, respectively, into two submanifolds with corners $D'(e_+)$ and $D''(e_+)$, $D'(e_-)$ and $D''(e_-)$, $D'(w_+)$ and $D''(w_+)$, $D'(w_-)$ and $D''(w_-)$. These eight objects are diffeomorphic to the archetype $\mathfrak{D}_1$. The four half discs D'' share an edge with the tetrahedron N_q. Let us denote by Γ'_e and Γ'_w be the open arcs $\Gamma_e \setminus E[\varepsilon_1, \varepsilon_2]$ and $\Gamma_w \setminus E[\omega_1, \omega_2]$ so that the components $D'(e_+)$, $D'(e_-)$, $D'(w_+)$ and $D'(w_-)$ are respectively bounded by $\Gamma'_e \cup A(e_+)$, $\Gamma'_e \cup A(e_-)$, $\Gamma'_w \cup A(w_+)$ and $\Gamma'_w \cup A(w_-)$. The immersed surface $f_t(\boldsymbol{S}^2)$ is decomposed by its multiple locus D_t into fifteen submanifolds with corners. Four of these are diffeomorphic to the archetype $\mathfrak{T}^+$ (the four faces of the tetrahedron N_q). Ten others are diffeomorphic to the archetype $\mathfrak{D}_1$, namely, the eight half disks already described plus two others (bounded respectively by the curves $A(e_+) \cup A(e_-)$ and $A(w_+) \cup A(w_-)$).

The last component called M_0, is what remains of the noman's land. The manifold with corners M_0 is diffeomorphic to the cylinder and its boundary splits into two immersed piecewise differentiable closed curves. Along each of these closed curves M_0 has eight kinks and two opposite sides identified with each other. Equipped with the orientation inherited through f_t from the canonical orientation of $\boldsymbol{S}^2$, the two octagonal boundary components $\partial_i(M_0)$, decompose respectively into the eight following arcs:

first component $\partial_1(M_0)$:

$$\Gamma'_e,\ E[\varepsilon_1,\omega_2],\ A(w_-),\ E[\omega_1,\varepsilon_1],\ \Gamma'_e,\ E[\varepsilon_2,\omega_1],\ A(w_+),\ E[\omega_2,\varepsilon_2]$$

second component $\partial_2(M_0)$:

$$\Gamma'_w,\ E[\omega_1,\varepsilon_2],\ A(e_-),\ E[\varepsilon_1,\omega_1],\ \Gamma'_w,\ E[\omega_2,\varepsilon_1],\ A(e_+),\ E[\varepsilon_2,\omega_2]\,.$$

Notice last that the curve Γ_0 is decomposed by the four vertices of the tetrahedron N_q into the eight following curves: $A(e_-)$, $E[\varepsilon_1,\omega_2]$, $A(w_+)$, $E[\omega_1,\varepsilon_1]$, $A(e_+)$, $E[\varepsilon_2,\omega_1]$, $A(w_-)$ and $E[\omega_2,\varepsilon_2]$.

Third step: the coding of the D's of the eversion.

Assume $t\in]-1/3,0[$. Introduce four points $a(e_+)\in A(e_+)$, $a(e_-)\in A(e_-)$, $a(w_+)\in A(w_+)$ and $a(w_-)\in A(w_-)$. On the closure of the open arc Γ'_e oriented from ε_1 to ε_2, choose points $b(e_+), d(e)$ and $b(e_-)$ in such a way that the sequence $\varepsilon_1, b(e_+), d(e), b(e_-), \varepsilon_2$ is increasing. Similarly, on the closure of the open arc Γ'_w oriented from ω_1 to ω_2, choose points $b(w_+)$, $d(w)$ and $b(w_-)$ in such a way that the sequence $\omega_1, b(w_+), d(w), b(w_-), \omega_2$ is increasing. Choose disjoint curves $C'(e_+)$, $C'(e_-)$, $C'(w_+)$ and $C'(w_-)$ contained respectively in $D'(e_+)$, $D'(e_-)$, $D'(w_+)$ and $D'(w_-)$, and connecting respectively $a(e_+)$ to $b(e_+)$, $a(e_-)$ to $b(e_-)$, $a(w_+)$ to $b(w_+)$ and $a(w_-)$ to $b(w_-)$. In M_0 choose also non intersecting curves $C(e_+)$, $C(e_-)$, $C(w_+)$ and $C(w_-)$ connecting respectively $a(e_+)$ to $b(e_+)$, $a(e_-)$ to $b(e_-)$, $a(w_+)$ to $b(w_+)$, and $a(w_-)$ to $b(w_-)$, in such a way that the four closed curves $C'(e_+)\cup C(e_+)$, $C'(e_-)\cup C(e_-)$, $C'(w_+)\cup C(w_+)$ and $C'(w_-)\cup C(w_-)$ bound disks in the closure of the open punctured torus N''. Let $\tau_{e_+}, \tau_{e_-}, \tau_{w_+}$ and τ_{w_-} be four embeddings of manifolds with corners having disjoint images and mapping $\mathfrak{D}_1$ into $\bar{N}''$. These four embeddings decompose the open torus N'' into four connected components all homeomorphic to the open ball except one which is homemorphic to the punctured open ball. Choose these mappings equivariantly, and hence so that the punctured component of the complement of their images in N'', namely, the unbounded component, lies between $\tau_{e_+}(\mathfrak{D}_1)$ and $\tau_{w_+}(\mathfrak{D}_1)$. (Notice that, without the equivariance constraint, one can also make ones way to the central model by choosing the mappings so that the unbounded component lie between $\tau_{e_-}(\mathfrak{D}_1)$ and $\tau_{w_-}(\mathfrak{D}_1)$.) Let $\bar{N}'$ be the closure of the component N' of $\boldsymbol{R}^3\setminus f_t(\boldsymbol{S}^2)$ homeomorphic to the open 3-ball which has been described in step 2. Let $\tau_1: \mathfrak{D}_1\to\bar{N}'$ be an embedding of manifolds with corners mapping the corners of $\mathfrak{D}_1$ to d_e and d_w, such that $\tau_1(\mathfrak{D}_1)$ does not intersect the images of the previously defined maps $\tau_{e_+}, \tau_{e_-}, \tau_{w_+}$, and τ_{w_-}. Notice that one edge of $\tau_{e_+}, \tau_{e_-}, \tau_{w_+}$ and τ_{w_-} as well as the two edges of τ_1 are curves in M_0 connecting the two components $\partial_i(M_0)$ for $i=1,2$ of the boundary of M_0. Notice also that the present construction is unique up to isotopy and reparametrization and can be done equivariantly. We are therefore in a position to perform simultaneously and equivariantly the modifications of type D_1 associated to τ_{e_+}, τ_{e_-}, τ_{w_+}, τ_{w_-} and τ_1 as well as the modification of type Q obtained by shrinking to a point the tetrahedron N_q. The equivariance constraint forces these six modifications to occur at time $t=0$. This brings our work to an end since what happens on the closed interval $[0, 1]$ is already determined by applying the equivariant formula of §3 (see also the beginning of the present Paragraph) to the construction we made on the interval $[-1, 0]$.

REMARK 4. (i) The halfway stage f_0 is well defined and belongs to the class of halfway models namely to an orbit of codimension 6 of $\mathrm{Diff}_0(S^2)\times\mathrm{Diff}_0^c(\boldsymbol{R}^3)$ (see §A1 (i)). In $C^1(S^2, \boldsymbol{R}^3)$, this orbit embeds into the corresponding C^1-orbit which contains the family $\{f_{\mu,\nu}\}$ of §5 above.

(ii) Assume now that we perform the two modifications associated to τ_{e-} and τ_{w_-} at time say $t=-1/6$ and those associated to τ_1 and to $\tau_q\colon \mathfrak{Q}\to N_q$ at time $t=0$. The equivariance constraint forces the modifications corresponding to τ_{e_+} and τ_{w_+} to occur at time $t=1/6$. In such a case, the halfway stage is C^1-equivalent to the open halfway model f_O of §3 above. Similarly, by peforming the modifications associated to τ_{e_+} and τ_{w_+} at time $t=-1/6$ we are led to a halfway stage corresponding to an overclosed version of the halfway model (see Remark 3 §5). The mirror image of the open halfway model is also the image of the overclosed halfway model by an inversion centered anywhere on the Z-axis, but inside the model.

(iii) Suppose next we refuse to perform the modification associated to τ_1, which, in the equivariant situation, has to occur at time $t=0$. In such a case, we are led to a quasigeneric eversion with sequence of types

$$(\boldsymbol{D}_0\quad \boldsymbol{D}_0\boldsymbol{D}_0)(\boldsymbol{T}^+\boldsymbol{T}^+)(\boldsymbol{D}_1\boldsymbol{D}_1\quad \boldsymbol{Q}\quad \boldsymbol{D}_1\boldsymbol{D}_1)(\boldsymbol{T}^-\boldsymbol{T}^-)(\boldsymbol{D}_2\quad \boldsymbol{D}_2)\,.$$

In spite of the fact that this eversion is not equivariant, one has $f_t=\rho_\pi\circ f_t\circ\rho_\pi$ for each of its models f_t. If we apply to this eversion the symmetry with respect to the origin in $\boldsymbol{R}^3\times[-1, 1]\subseteq\boldsymbol{R}^4$, we get mirror image of the nonequivariant eversion mentioned in [M-P] whose sequence of types is

$$(\boldsymbol{D}_0\quad \boldsymbol{D}_0)(\boldsymbol{T}^+\boldsymbol{T}^+)(\boldsymbol{D}_1\boldsymbol{D}_1)\boldsymbol{Q}(\boldsymbol{D}_1\boldsymbol{D}_1)(\boldsymbol{T}^-\boldsymbol{T}^-)(\boldsymbol{D}_2\boldsymbol{D}_2\quad \boldsymbol{D}_2)\,.$$

If we introduce the opposite of the modification of type $\boldsymbol{D}_1$ associated to τ_1 in this last eversion, we end up with the equivariant quasigeneric eversion thoroughly described in [M-P] and whose sequence of types has been given in the very beginning of the present Paragraph. (By the mirror image of a deformation $\{f_t\}$ we mean the deformation $\{\mu\circ f_t\circ\mu\}$ where μ is the symmetry with respect to the plane $Y=0$ in $\boldsymbol{R}^3$.)

(iii) These last remarks as well as later manipulations are based on the fact that the modification associated to τ_1 can occur before the modifications of type $\boldsymbol{T}^+$ and also after those of type $\boldsymbol{T}^-$.

How to look at the complexity of an eversion.

In the [M-P] case, as well as in the present case, the triple and quadruple locus of the quasigeneric equivariant eversion, is an immersed circle in $\boldsymbol{R}^3 \times]-1, 1[$. More precisely, it is a fourbladed propeller for which the restriction of the function τ has two minima lying on opposite blades and two maxima on the other blades. For any quasigeneric eversion, since the multiple locus is an immersed closed surface in $\boldsymbol{R}^3 \times]-1, 1[$, the Morse formula which computes the Euler characteristic χ of the surface apply. One gets $\chi = n_0 - n_1 + n_2$ where n_i is the number of points of type $\boldsymbol{D}_i$ of the eversion, for $0 \le i \le 2$. Hence, in the present case, $\chi = 1$, while $\chi = -1$ in the [M-P] case. Moreover, by looking at the multiple locus of the closed or open halfway models (where the branches intersecting at the quadruple point have been separated from each other), one sees that our two multiple loci are immersed connected surfaces. Furthermore (as can be checked with the help of the pictures of §A3), in the present case, the inverse image in $\boldsymbol{S}^2 \times]-1, 1[$ of the immersed projective plane, is the image, by a transverse immersion, of a sphere Σ. The inverse image in Σ of the multiple locus of the closed halfway model, is the union of two orthogonal great circles intersecting, say at the north and south poles, and of the tropics. This union of four circles split Σ into four rectangles decomposing the equatorial zone, and into two systems of four triangles, each of these systems decomposing one of the polar region. At the center of each of these twelve polygons, sits the inverse image of a point of type $\boldsymbol{D}_0$ or $\boldsymbol{D}_2$ in such a way that, on the two sides of a given edge the two types are distinct. In Σ, the inverse image of the fourbladed propeller splits into two immersed circles having each four double points and intersecting each other at four extra double points. The complement in Σ of this inverse image has fourteen connected components. The closure of four of these components look like four rhombi centered at the inverse images of points of type $\boldsymbol{D}_{0,2}$. Each of these rhombi is contained in one of the rectangular region, with vertices at the midpoints of the sides of the rectangles. Teardrops are attached to the vertices of the rhombi lying on the tropics. Each teardrop lies in one of the eight polar triangles and does not contain the point of type $\boldsymbol{D}_{0,2}$ of that triangle. The two remaining components are what remains now of the nothern and southern hemispheres. The closure of each of these, is a dodecagon with pairs of consecutive vertices identified to each other in order to surround the teardrops. In the [M-P] case, the four pairs of triangles touching each other at the poles, have to be replaced by two hexagons each containing one point of type $\boldsymbol{D}_{0,2}$. Hence, Σ is now replaced by a surface of genus two.

Assume that $\{f_t\}$ is the quasigeneric equivariant eversion used by Max in his movie and recall that the corresponding sequence of types is

$$(\boldsymbol{D}_0)(\boldsymbol{D}_0)(\boldsymbol{T}^+)(\boldsymbol{D}_1\boldsymbol{D}_1 \quad \boldsymbol{T}^+)(\boldsymbol{D}_1\boldsymbol{Q})(\boldsymbol{T}^- \quad \boldsymbol{D}_1\boldsymbol{D}_1)(\boldsymbol{T}^-)(\boldsymbol{D}_2)(\boldsymbol{D}_2)\,.$$

This variation of the eversion (whose central model is the open one) has been deviced in order to minimize the amount of hidden happening during the process. For all $t \in [1, 3]$ let $f_t = \mu \circ f_{2-t} \circ \mu$, where μ is defined in Remark 4 (iii). The loop $\{g_t\}$ obtained by identifying f_{-1} and f_3 is freely homotopic in $\mathrm{Im}(\boldsymbol{S}^2, \boldsymbol{R}^3)$ to a loop $\{g'_t\}$ parametrised by $\boldsymbol{S}^1$, and having the sequence of types

$$(\boldsymbol{D}_1\boldsymbol{D}_1) \quad (\boldsymbol{D}_1\boldsymbol{D}_1)\,.$$

(In particular see Lemma 6.) The multiple locus of $\{g'_t\}$ is a nonorientable surface with $\chi=-4$. The triple locus of $\{g'_t\}$, which has two connected components, is embedded in $\boldsymbol{R}^3\times\boldsymbol{S}^1$, and covers $\boldsymbol{S}^1$ four times. The loop $\{g'_t\}$ is another good candidate one which the Max criterion can be applied in order to formalize the proof given in [B-M] of the following basic result:

LEMMA 7. *The loop* $\{g_t\}$ *generates the fundamental group of* $\mathrm{Im}(\boldsymbol{S}^2, \boldsymbol{R}^3)$.

Therefore, any quasigeneric eversion can be quasigenerically deformed into the path $\{g_t\}$ composed n times with itself and followd by $\{f_t\}$ where $\{f_t\}$ is as in the movie and $n\in\boldsymbol{Z}$. The eversion is said to be minimal when $n=-1$ or 0. Clearly, this Lemma implies that the space of eversions has two connected components (compare with Lemma 6). (However, it should be noted that the space of equivariant eversions has much more than two connected components; for instance, the [M-P]-eversion as well as the one used by Max, which are equivariantly equivalent to each other, can be deformed into the present one but not equivariantly.) During any deformation of the eversion (see §A1 (i)), the number of quadruple points as well as the characteristic χ remains constant modulo 2.

Hence, any quasigeneric eversion has at least one quadruple point, and its multiple locus is, at least, an immersed porjective plane. When the eversion has only one critical point, its triple locus is of course nonempty, and contained at least two points of type $\boldsymbol{T}^+$ and two others of type $\boldsymbol{T}^-$. When the minimal situation is attained for the point of type $\boldsymbol{T}$, by §A1 (e) iii′, the triple and quadruple locus of the eversion, has to be a fourbladed propeller in the equivariant case.

While quasigenerically deforming a quasigeneric eversion the multiple locus (as well as its inverse image in $\boldsymbol{S}^2\times\,]-1,1[$) gets modified by surgery (see §A1 (i)). A clear understanding and a complete list of the types of surgeries that may occur lead one to assert the following. The surface whose immersed image is the multiple locus of a minimal quasigeneric eversion has exactly one connected component with odd Euler characteristic. Moreover, the universal cover of this nonoriented component is transversely immersed onto the inverse image in $\boldsymbol{S}^1\times\,]-1,1[$ of that component.

Since any eversion necessarily has a point of type $\boldsymbol{T}^+$ of a prescribed type (details have to be omitted here), it must also have at least two points of type $\boldsymbol{D}_0$ occuring at values of the parameter less than the one at which the first point of type $\boldsymbol{T}^+$ occurs. These two points must have the same type of coding either as those shown in the Max movie or as those drawn in [M-P]. Notice that in the [M-P]-case, the coding is the same as the one prescribed by τ_0 and say τ_e in the first step above. By reversing the time orientation, one gets that the eversion must also have at least two points of type

$\boldsymbol{D}_2$ occuring after the last point of type $\boldsymbol{T}^-$ and related to each other either as the Max movie or as in [M-P]. Useful points of type $\boldsymbol{D}_1$ can occur neither before the first point of type $\boldsymbol{T}^+$ nor after the last point of type $\boldsymbol{T}^-$.

The number of points of type $\boldsymbol{D}_1$ of an equivariant quasigeneric eversion is equal to 1 modulo 4 and at least equal to 5. Hence, among all equivariant quasigeneric eversions, the one thoroughly described in [M-P], as well as the one presented in the movie, has the minimum number of modifications, while, with the condition $\chi=1$, the eversion described in this Paragraph is minimal in that respect.

Among all quasigeneric eversions, the nonequivariant one mentioned in [M-P] and characterized in Remark 4, seems to be minimal in all respects. In order to verify this statement, one should show that there exists no quasigeneric eversion with sequence of types containing exactly, two $\boldsymbol{D}_0$'s, two $\boldsymbol{T}^+$'s, two $\boldsymbol{T}^-$'s, two $\boldsymbol{D}_2$'s, one $\boldsymbol{Q}$, and three $\boldsymbol{D}_1$'s, an exercise which, for the moment, I am forced to leave to the reader.

A3. Stereographic representations of the inverse images in $\boldsymbol{S}^2$ of the multiple loci of the various f_t's of our eversion. Using the facilities offered by the stereographic projection we can give 2-dimensional representations of curves in $\boldsymbol{S}^2$. These representations are meant to help the reader in checking the correctness of claims made in Step 1, 2 and 3. On the other hand, the present set of pictures summarizes the many notations which had to be introduced in order to present the constructions. Moreover by piling on top of each other obvious interpolations of pictures 1–3, one is easily able to visualize the inverse image in $\boldsymbol{S}^2\times]-1,0]$ of the multiple locus D intersected with $\boldsymbol{R}^3\times]-1,0]$. Figure 0 is intended to prepare the reader to interpret stereographic views. Figure 5 helps the reader to visualize what is explained in the beginning of the section entitled "How to look at the complexity of an eversion" in §A2. Of course §A2 is supposed to be self-contained. Therefore in principle it does not require the help of any pictures. Nevertheless since a well-known Chinese proverb asserts that a picture is worth one hundred messages, we hope the five present ones will save five hundred tedious explanations.

Anyhow I thank the team of the Topology Seminar of the Kyushu University (K. and T. Motegi, H. Nishi, T. Takata, M. Wakui and Y. Yokota) who was patient enough to listen to my thousand explanations in order to produce under the direction of T. Kohno the following beautiful examples of what should be a full series of drawings illustrating differential topological expositions of the present kind.

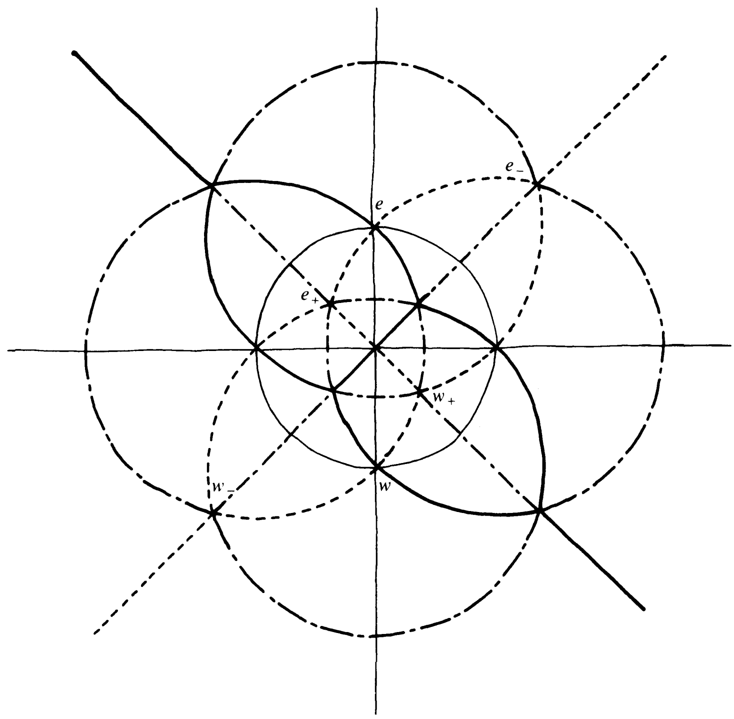

FIGURE 0. A stereographic projection of S^2 on the plane. The origin corresponds to the north pole of the sphere. Two tetrahedra and a cube inscribed on the sphere are projected on the plane.

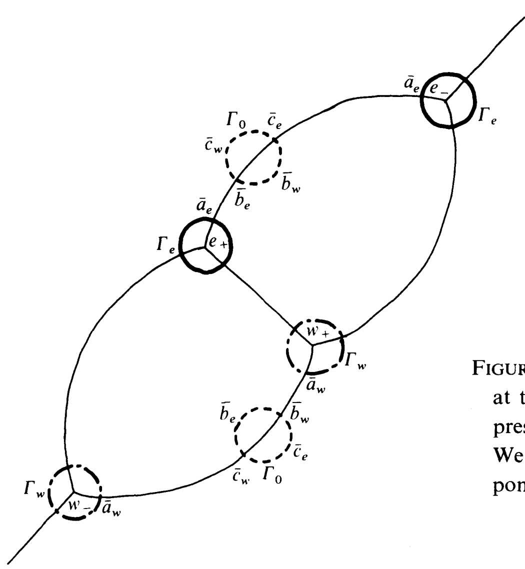

FIGURE 1. After three modifications of type D_0 at time $t = -2/3$. Six circles in the picture represent the inverse image of the double locus. We see that $R^3 - f_t(S^2)$ has five connected components.

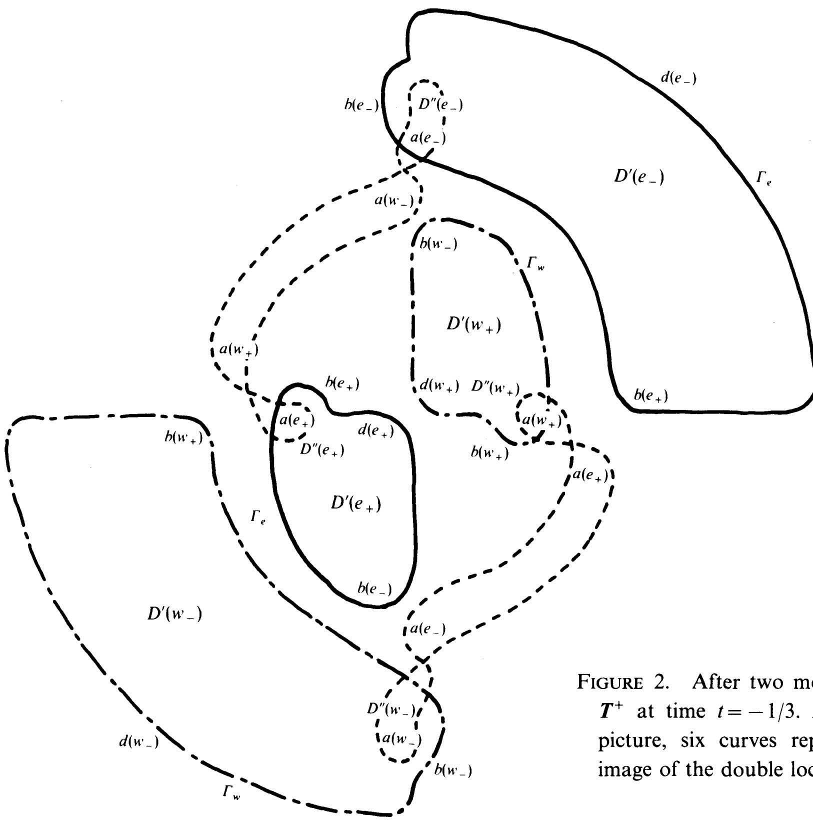

FIGURE 2. After two modifications of type $\boldsymbol{T}^+$ at time $t=-1/3$. As in the previous picture, six curves represent the inverse image of the double locus.

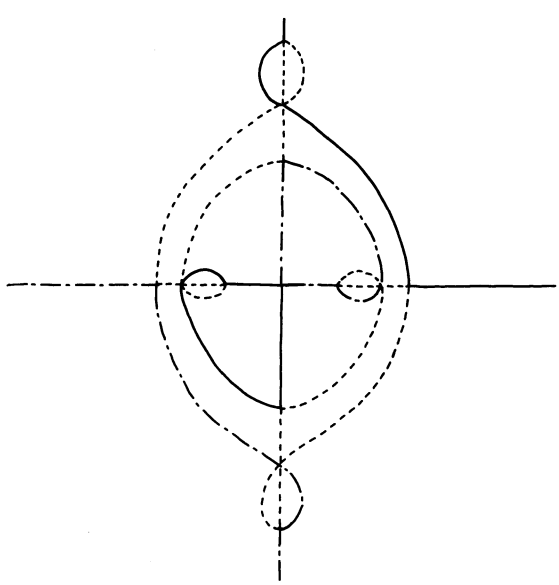

FIGURE 3. See also Figure 11. The halfway model at $t=0$. Five modifications of type $\boldsymbol{D}_1$ and a modification of type $\boldsymbol{Q}$ are performed.

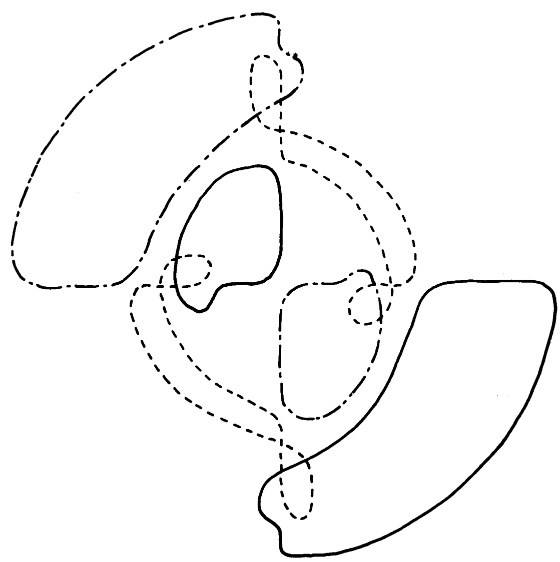

FIGURE 4. After the halfway model. We see that the picture is symmetric to Figure 2.

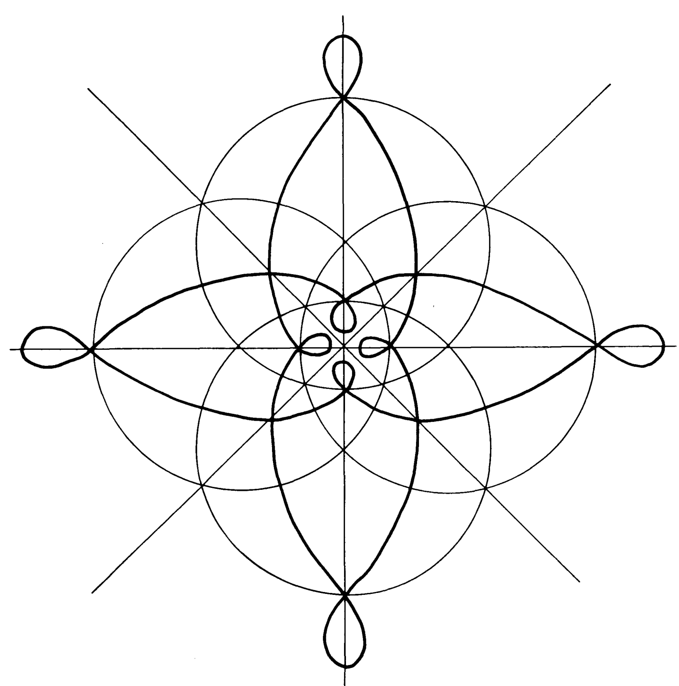

FIGURE 5. In the sphere Σ, the inverse image of the fourbladed propeller splits into two immersed circles. Each of them has four double points.

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