Tôhoku Math. J.
43 (1991), 425–445

THE REPRESENTATION THEORY OF INNER ALMOST COMPACT FORMS OF KAC-MOODY ALGEBRAS

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(Received June 18, 1990)

1. Introduction. Let k be a field of characteristic 0, $\bar{k}$ its algebraic closure. In [A] we presented a construction by generators and relations of k-forms of (symmetrizable) "derived" Kac-Moody algebras over $\bar{k}$, under certain restrictions. This construction can be roughly described as *glueing together* suitably chosen three-dimensional simple Lie algebras (TDS for short) over k. (Let us recall that the TDS's over k are in one-to-one correspondence with the quaternion algebras over k; hence the notation $sq(a, b)$, see Section 1). It was shown in [AR] that, in the real case, these forms are inner "almost compact", using Rousseau's terminology.

Another approach is followed in [BP]. The classification of the real forms of the first kind of affine Lie algebras is contained in [L], see also [BR].

In this paper we extend the results of [A] and construct forms of (non-derived) Kac-Moody algebras. We drop also here the requirement $a_{ij} \geq -3$ of [A]. As in the quoted paper, we are also able to construct a symmetric bilinear invariant form.

Lie algebras become more interesting when (some of) its representation theory is understood. In the Kac-Moody case, the theory of highest weight modules, inspired by the finite case, has many deep connections with other areas of Mathematics: see for example [K]. Again, the theory relies on the $sl(2)$-case.

In this article, we propose a definition of "quadratic" highest weight modules for the introduced forms. As in the split case, we need first to understand the $sq(a, b)$-case, (cf. Section 4).

Let us emphasize that we have no longer the notion of Borel subalgebra, nor is Lie's theorem applicable, and the action of the Cartan subalgebra is not in general diagonalizable. We can however, manage the situation and define a "quadratic" highest weight module for each non-zero element of the dual of the Cartan subalgebra as a cyclic one, subject to some quadratic relations. In Sections 6 and 7 we extend this definition to the general case.

As a first application, we give a presentation of the "derived" forms of Kac-Moody algebras (in the spirit of Gabber-Kac's theorem) generalyzing formulas (13), $\cdots$, (16), (23), $\cdots$, (26) of [A], (cf. Section 5).

This work was supported by CONICET, CONICOR and FAMAF (Argentina).

We also obtain a classification of the finite dimensional irreducible modules, in the finite case. In particular, we get the classification of the irreducible real finite dimensional representations of a real inner form of a complex simple (finite dimensional) Lie algebra. Of course, there are other methods to obtain these results. The classification of the irreducible finite dimensional representations of a simple real finite dimensional Lie algebra was first obtained by E. Cartan [Ca]; over an arbitrary field this task was accomplished by Tits [T]. Their method can be succinctly described as "Galois descent". Satake also obtained some important results in this direction, see [Sa]. On the other hand, Seligman [Se] constructed explicitly finite dimensional modules, using for example non-associative algebra. But for the moment, his method does not apply to the Kac-Moody case.

We feel however that our approach could have some additional interest. For example, the square of any element of the Cartan subalgebra diagonalizes in a quadratic highest weight module. Thus every such module has a formal character, whose computation can be done exactly as in the split case.

Let us also remark that despite the fact that our statements are formulated without mentioning Galois actions or involutions, some of our proofs uses this tools in an elementary way. In other words, we take in mind all the time the split case results. We do not know if it is possible to avoid this, i.e. reproving the split case theorems as a particular case of our more general situation.

Part of this work was done during a visit of l'Ecole Politechnique (Paris); I would like to express here my gratitude to Alain Guichardet for his kind hospitality and his stimulating interest in this work. I extend also my thanks to Fernando Levstein, Raphael Freitas, Guy Rousseau and Victor Kac for interesting conversations.

2. Preliminaries and definitions. Let X, Y, Z be a basis of a 3-dimensional k-vector space V. For fixed $a, b \in k^* = k - \{0\}$ we can define a Lie algebra structure, which we shall call $sq(a, b)$ on V by the rule:

$$[X, Y] = 2Z, \qquad [Y, Z] = -2bX, \qquad [Z, X] = -2aY.$$

Let $(d_1, \cdots, d_n)$ denote the quadratic space (k^n, q), where q is the quadratic form such that $q(\sum_h \lambda_h e_h) = \sum_h d_h \lambda_h^2$. ($\{e_h\}$ is the canonical basis.) In addition let $\langle\!\langle -a, -b \rangle\!\rangle$ denote the quaternion algebra having a basis $\{1, I, J, K\}$ with the multiplication table

$$I^2 = a, \qquad J^2 = b, \qquad IJ = -JI = K.$$

Endowed with the usual norm, it is a quadratic space isomorphic to $(1, -a, -b, ab)$, which is in turn the Pfister 2-form $\langle\!\langle -a, -b \rangle\!\rangle$ (hence the notation). Then it is well known that $sq(a, b)$ is isomorphic to $sq(c, d)$ if and only if the quadratic spaces $(-a, -b, ab)$ and $(-c, -d, cd)$ are; moreover $sq(a, b)$ is simple and every 3-dimensional simple Lie algebra over k arises in this way. In fact, $sq(a, b)$ can be realized as the Lie algebra of the traceless elements of the quaternion algebra $\langle\!\langle -a, -b \rangle\!\rangle$; it is the Lie algebra of the group $SQ(a, b)$ of the elements of $\langle\langle -a, -b\rangle\rangle$ having norm equal to one. $sl(2, k)$ is isomorphic to $sq(1, -1)$ and if $k=\boldsymbol{R}$, $sq(-1, -1)$ is $su(2, \boldsymbol{R})$.

We will use the notation of [K]. Let $A=(a_{ij})\in \boldsymbol{Z}^{n\times n}$ be a generalized Cartan matrix, i.e.

$$a_{ii}=2$$
$$a_{ij}\leq 0\,, \qquad i\neq j$$
$$a_{ij}=0\Rightarrow a_{ji}=0\,.$$

As usual, we will say that A is finite if it corresponds to a finite dimensional complex Lie algebra. Let A be a generalized Cartan matrix. We will assume that the corresponding Dynkin diagram is connected. A realization of A ([K, 1.1]) is a triple $(\boldsymbol{h}, \Pi, \Pi^\vee)$ where $\boldsymbol{h}$ is a k-vector space of dimension $2n-\operatorname{rk} A$, $\Pi=\{\alpha_1, \cdots, \alpha_n\}\subset \boldsymbol{h}^*$, $\Pi^\vee=\{\alpha_1^\vee, \cdots, \alpha_n^\vee\}\subset \boldsymbol{h}$ are linearly independent indexed sets and

$$\langle a_i', \alpha_j\rangle = a_{ij} \qquad (i, j=1, \cdots, n)\,.$$

As in the proof of ([K, 1.1]) we will fix a realization $(\boldsymbol{h}, \Pi, \Pi^\vee)$ of A as follows: first, after reordering the indices if necessary, we will assume that

$$A=\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

where A_1 is a non-degenerate $\operatorname{rk} A\times \operatorname{rk} A$ matrix. Thus we can choose $\{\alpha_{n+1}^\vee, \cdots, \alpha_{2n-\operatorname{rk} A}^\vee\}\subset \boldsymbol{h}$ in such a way that $\{\alpha_1^\vee, \cdots, \alpha_{2n-\operatorname{rk} A}^\vee\}$ is a basis of $\boldsymbol{h}$ and

$$\langle \alpha_i, \alpha_{n+j}^\vee\rangle = \delta_{i, \operatorname{rk} A+j}$$

for $1\leq i\leq n$, $1\leq j\leq n-\operatorname{rk} A$.

Let us recall the definition of a Kac-Moody algebra:

DEFINITION 1. $\tilde{g}_k(A)$ is the Lie algebra over k with generators $\{E_i, F_i : 1\leq i\leq n\}$ and $\boldsymbol{h}$, and defining relations

$$[H, H']=0 \tag{1}$$
$$[E_i, F_j]=\delta_{ij}\alpha_i^\vee \tag{2}$$
$$[H, E_i]=\langle \alpha_i, H\rangle E_i \tag{3}$$
$$[H, F_i]=-\langle \alpha_i, H\rangle F_i \tag{4}$$

for all $H, H'\in \boldsymbol{h}$ and $i, j=1, \cdots, n$. There exists a unique maximal ideal $r_k(A)$ of $\tilde{g}_k(A)$ among the ideals intersecting $\boldsymbol{h}$ trivially (see [K, 1.2]). Then

$$g_k(A) := \tilde{g}_k(A)/r_k(A)\,.$$

Let us also recall the following notation: $g_k'(A)=[g_k(A), g_k(A)]$.

Let us fix $a, b \in k^*$. Our first task is to define a form of $g_{\bar{k}}(A)$ with $sq(a, b)$ playing the role of $sl(2)$.

DEFINITION 2. $\tilde{g}_k(A, a, b)$ is the Lie algebra over k with generators $\{X_i, Y_i : 1 \le i \le n\}$ and $\boldsymbol{h}$, and defining relations

(5) $$[H, H'] = 0$$

(6) $$[X_i, Y_i] = 2\alpha_i^\vee$$

(7) $$[H, X_j] = -\langle \alpha_j, H \rangle a Y_j$$

(8) $$[Y_j, H] = -\langle \alpha_j, H \rangle b X_j$$

and if $i \neq j$

(9) $$[X_i, Y_j] = [Y_i, X_j]$$

(10) $$[X_i, X_j] = -ab^{-1}[Y_i, Y_j]$$

for all $H, H' \in \boldsymbol{h}$ and $i, j = 1, \cdots, n$. There exists a unique maximal ideal $r_k(A, a, b)$ of $\tilde{g}_k(A, a, b)$ among the ideals intersecting $\boldsymbol{h}$ trivially (see Lemma 1 below). Then

$$g_k(A, a, b) := \tilde{g}_k(A, a, b)/r_k(A, a, b)\,.$$

LEMMA 1. (i) *There is a natural isomorphism*

$$\tilde{g}_k(A, a, b) \otimes_k k' \simeq \tilde{g}_{k'}(A, a, b)$$

if k' is an extension of k.

(ii) *Let $t, s \in k^*$. Then $X'_i \mapsto tX_i$, $Y'_i \mapsto sY_i$, $H \mapsto tsH$ ($H \in \boldsymbol{h}$) provides an isomorphism between $\tilde{g}(A, at^2, bs^2)$ (with generators X'_i, Y'_i, $\boldsymbol{h}$) and $\tilde{g}(A, a, b)$.*

In particular, putting $t = -1$, $s = 1$, we obtain an automorphism of $\tilde{g}(A, a, b)$, called the Cartan involution.

(iii) *$X_i^* \mapsto Y_i$, $Y_i^* \mapsto X_i$, $H \mapsto -H$ ($H \in \boldsymbol{h}$), provide an isomorphism between $\tilde{g}(A, a, b)$ (with generators X_i^*, Y_i^*, $\boldsymbol{h}$) and $\tilde{g}(A, b, a)$.*

(iv) *Let us fix $J \subset \{h : 1 \le h \le n\}$. There exists an involution φ_J of $\tilde{g}_k(A, a, b)$ given by*

$$\varphi_J(X_i) = X_i\,, \qquad \varphi_J(Y_i) = Y_i \qquad \textit{if} \quad i \notin J$$

$$\varphi_J(X_i) = -X_i\,, \qquad \varphi_J(Y_i) = -Y_i \qquad \textit{if} \quad i \in J$$

$$\varphi_J(H) = H \qquad \textit{for all} \quad H \in \boldsymbol{h}\,.$$

(v) *$\tilde{g}_k(A, 1, -1)$ is isomorphic to $\tilde{g}_k(A)$.*

(iv) *Among the ideals of $\tilde{g}_k(A, a, b)$ intersecting $\boldsymbol{h}$ trivially there exists a unique maximal ideal $\tilde{r}_k(A, a, b)$. Moreover, it is preserved by the isomorphism given in* (i).

PROOF. (i) to (iv) are easy. (v): The applications $\tilde{g}_k(A) \to \tilde{g}_k(A, 1, -1)$

$$H \mapsto H\,, \qquad E_i \mapsto \frac{1}{2}(X_i - Y_i)\,, \qquad F_i \mapsto \frac{1}{2}(X_i + Y_i)$$

and $\tilde{g}_k(A, 1, 1) \to \tilde{g}_k(A)$

$$H \mapsto H\,, \qquad X_i \mapsto E_i + F_i\,, \qquad Y_i \mapsto -E_i + F_i$$

for all $H \in \boldsymbol{h}$ and $1 \leq i, j \leq n$ are well defined and inverse of each other.

(vi): This is true for $\bar{k}$: use [K, 1.2], (ii), (v). For the general case we only need to check:

$$\tilde{r} \otimes_k \bar{k} = \tilde{r}_{\bar{k}}(A, a, b)$$

(up to the canonical identification given in (i)) if $\tilde{r}$ is any maximal ideal among the ones that intersect $\boldsymbol{h}$ trivially (such an ideal exists thanks to Zorn's lemma). One inclusion is clear and the other follows because $\tilde{r}_{\bar{k}}(A, a, b)$ is stable under $\mathrm{Gal}(\bar{k}, k)$. ■

We get at once:

PROPOSITION 1. *The statements* (i), $\cdots$, (v) *of Lemma* 1 *hold for* $g_k(A, a, b)$ *instead of* $\tilde{g}_k(A, a, b)$.

Now let us recall from [PK]:

THEOREM 1. *Two maximal* ad-*diagonalizable subalgebras of a Kac-Moody algebra are conjugate.*

This suggests to define a Cartan subalgebra of an arbitrary Lie algebra as a maximal subalgebra in the set of abelian subalgebras consisting of ad-locally finite semisimple elements. We get from Proposition 1:

COROLLARY 1. $\boldsymbol{h}$ *is a Cartan subalgebra of* $g_k(A, a, b)$. *Moreover, the center of* $g_k(A, a, b)$ *is* $\{Z \in \boldsymbol{h} : \alpha_j(Z) = 0, \forall j\}$.

The following step is to define forms of Kac-Moody algebras by glueing together suitably chosen TDS. For this, we need some transition scalars. So let use fix elements $(s_{ij})_{i,j=1,\cdots,n}$ of k^* such that

(11) $$s_{ii} = 1$$

(12) $$s_{ji} = s_{ij}^{-1}$$

(13) $$s_{ij} = s_{ir} s_{rj}$$

for all i, j, r. ((11), (12) are special cases of (13)). Now let us also fix $a_i, b \in k^*$ $(1 \leq i \leq n)$ and set

$$b_1 = b\,, \qquad b_j = b_i a_i^{-1} a_j s_{ij}^2\,.$$

Thanks to (13), there is no ambiguity in the definition of b_j.

Let us also introduce the following notation:

$$a_i = a_{\operatorname{rk} A - n + i}\,, \qquad b_i = b_{\operatorname{rk} A - n + i}\,, \qquad n+1 \le i \le 2n - \operatorname{rk} A$$

$$s_{ij} = s_{\operatorname{rk} A - n + i\, j} \quad \text{if} \quad 1 \le j \le n\,, \qquad n+1 \le i \le 2n - \operatorname{rk} A$$

$$s_{ij} = s_{i\, \operatorname{rk} A - n + j} \quad \text{if} \quad 1 \le i \le n\,, \qquad n+1 \le j \le 2n - \operatorname{rk} A$$

$$s_{ij} = s_{\operatorname{rk} A - n + i\, \operatorname{rk} A - n + j} \quad \text{if} \quad n+1 \le i, j \le 2n - \operatorname{rk} A\,.$$

DEFINITION 3. $\tilde{g}_k(A, a_i, s_{ij}, b)$ is the Lie algebra over k given by generators $\{X_i, Y_i : 1 \le i \le n\}$ and $\boldsymbol{h}$ with relations

$$[H, H'] = 0 \tag{14}$$

$$[X_i, Y_i] = 2\alpha_i^\vee \tag{15}$$

$$[\alpha_k^\vee, X_j] = -\alpha_k s_{kj}^{-1} \langle \alpha_k^\vee, \alpha_j \rangle Y_j \tag{16}$$

$$[Y_j, \alpha_k^\vee] = -b_k s_{kj} \langle \alpha_k^\vee, \alpha_j \rangle X_j\,, \tag{17}$$

and if $i \neq j$

$$[Y_i, X_j] = s_{ij}[X_i, Y_j] \tag{18}$$

$$[X_i, X_j] = -a_i b_i^{-1} s_{ij}^{-1} [Y_i, Y_j] \tag{19}$$

for all $H, H' \in \boldsymbol{h}$ and $i, j = 1, \cdots, n$, $k = 1, \cdots, 2n - \operatorname{rk} A$. There exists a unique maximal ideal $\tilde{r}_k(A, a_i, s_{ij}, b)$ of $\tilde{g}_k(A, a_i, s_{ij}, b)$ among the ideals intersecting $\boldsymbol{h}$ trivially (see Lemma 2 below). Then

$$g_k(A, a_i, s_{ij}, b) := \tilde{g}_k(A, a_i, s_{ij}, b) / \tilde{r}_k(A, a_i, s_{ij}, b)\,.$$

LEMMA 2. (i) *If k' is an extension of k, there is a natural isomorphism*

$$\tilde{g}_k(A, a_i, s_{ij}, b) \otimes_k k' \simeq \tilde{g}_{k'}(A, a_i, s_{ij}, b)\,.$$

(ii) *Let $a \in k^*$ and set $a_i = a$, $s_{ij} = 1$. Then $\tilde{g}(A, a_i, s_{ij}, b)$ is isomorphic to $\tilde{g}(A, a, b)$ (cf. Definition 2).*

(iii) *Let $\gamma, \lambda_i, v_{ij} \in k^*$ $(1 \le i \le n)$ such that the v_{ij} satisfy (13). Let us put*

$$a_i' = a_i \lambda_i^2\,, \qquad b' = b\gamma^2\,, \qquad s_{ij}' = s_{ij} v_{ij}\,.$$

Then $\tilde{g}(A, a_i', s_{ij}', b')$ is isomorphic to $\tilde{g}(A, a_i, s_{ij}, b)$.

(iv) *Let $c, d \in k^*$ and let us assume that there exist $\lambda_i, \gamma \in k^*$ satisfying:*

$$\lambda_i^2 = c a_i^{-1}\,, \qquad \gamma^2 = d b_1^{-1}\,.$$

Then $\tilde{g}(A, a_i, s_{ij}, b)$ is isomorphic to $\tilde{g}(A, c, d)$ (cf. Definition 2).

(v) *If k is algebraically closed, then $\tilde{g}(A, a_i, s_{ij}, b)$ is isomorphic to $\tilde{g}(A)$ (cf. Definition 1).*

(vi) *Among the ideals of $\tilde{g}_k(A, a_i, s_{ij}, b)$ intersecting $\boldsymbol{h}$ trivially there exists a unique* *maximal ideal* $\tilde{r}_k(A, a_i, s_{ij}, b)$. *Moreover, it is preserved by the isomorphism given in* (i).

(vii) *If A is a Cartan matrix, then $g_k(A, a_i, s_{ij}, b)$ is absolutely simple.*

PROOF. (i) and (ii) are clear.

(iii) We have:

$$b'_j = \gamma^2 \lambda_1^{-2} \lambda_j^2 \nu_{1j}^2 b_j \,.$$

That is, if $\gamma_j = \gamma \lambda_1^{-1} \lambda_j \nu_{1j}$ then $b'_j = \gamma_j^2 b_j$ and $b'_j = \gamma_i^2 \lambda_i^{-2} \lambda_j^2 \nu_{ij}^2 b_j$. Let us also put

$$\gamma_j = \gamma_{\operatorname{rk} A - n + j}, \qquad \lambda_j = \lambda_{\operatorname{rk} A - n + j}\,, \qquad n+1 \le j \le 2n - \operatorname{rk} A\,.$$

We will show that

$$X'_j \mapsto \lambda_j X_j\,, \qquad Y'_j \mapsto \gamma_j Y_j\,, \qquad \alpha_j^{\vee\prime} \mapsto \gamma_j \lambda_j \alpha_j^{\vee}$$

gives an isomorphism from $\tilde{g}(A, a_i \lambda_i^2, s_{ij} \nu_{ij}, b\gamma^2)$ (with generators X'_i, Y'_i, $\Pi^{\vee\prime} = \{\alpha_i^{\vee\prime}\}$, etc. and relations (14′), $\cdots$, (19′)) onto $\tilde{g}(A, a_i, s_{ij}, b)$.

We can reduce ourselves to showing that it is well defined, i.e. that the images of X'_j, Y'_j, Z'_j satisfy the relations (14′), $\cdots$, (19′). (14′), (15′) are obvious, and (16′), $\cdots$, (19′) are straightforward computations, taking into account that the ν_{ij} satisfy (13).

Now (iv) follows from (ii), (iii); (v) is a consequence of (iv) and Proposition 1; (vi) can be proved as in Lemma 1 and (vii) follows from (vi). ∎

PROPOSITION 2. *The statements* (i), $\cdots$, (vi) *of Lemma* 2 *hold for* $g_k(A, a_i, s_{ij}, b)$ *instead of* $\tilde{g}_k(A, a_i, s_{ij}, b)$.

COROLLARY 2. *h_k is a Cartan subalgebra of $g_k(A, a_i, s_{ij}, b)$.*

Let us denote $g'_k(A, a_i, s_{ij}, b) = [g_k(A, a_i, s_{ij}, b), g_k(A, a_i, s_{ij}, b)]$. For A symmetrizable the explicit presentation of $g'_k(A, a_i, s_{ij}, b)$ by generators and relations is given in Section 5.

3. The invariant bilinear form. Let us define $g_k^{\mathrm{loc}}(A, a_i, s_{ij}, b) = g^{\mathrm{loc}}$ as the linear subspace of $g_k(A, a_i, s_{ij}, b)$ spanned by $\{X_i, Y_i, \boldsymbol{h}\}$. The Lie algebra L freely generated by $\{X_i, Y_i, \boldsymbol{h}\}$ has an $\boldsymbol{N}_0$-graded structure given by $\deg(X_i) = \deg(Y_i) = 1$, $\deg(H) = 0$ for all $H \in \boldsymbol{h}$. Let us consider the ascending filtration on L given by

$$L_m = \{u \in L : \deg(u) \le m\}$$

and let $g_m = \pi(L_m)$, where $\pi : L \to g(A, a_i, s_{ij}, b) = g$ is the canonical projection. Thus $(g_m)_{m \in \boldsymbol{N}_0}$ is an ascending filtration of g. Moreover, $[g_m, g_n] = g_{m+n}$ and $g_1 = g^{\mathrm{loc}}$. Now, if $u \in g$, let us put

$$\nu(u) = \inf\{m : u \in g_m\}\,.$$

Let us also remark, though it is obvious, that the introduced filtration is compatible with the isomorphisms given by Propositions 1 and 2. On the other hand, let us consider the principal gradation of $g(A, 1, -1)=g(A)$ (cf. [K, 1.5]) denoted $(g_j(1))_{j\in\mathbf{Z}}$. In this case we have

$$g_m = \bigoplus_{-m\le j\le m} g_j(1)\,.$$

Let us recall that an $n\times n$ matrix A is called symmetrizable if there exists a non-degenerated diagonal $n\times n$ matrix $D=(d_1, \cdots, d_n)$ such that DA is symmetric. In the rest of the Section, A will denote a symmetrizable generalized Cartan matrix.

Let us define a symmetric bilinear form $(\ |\ )_0$ on $g_k^{\mathrm{loc}}(A, a_i, s_{ij}, b)=g^{\mathrm{loc}}$ as follows:

$$(\alpha_i^\vee|\alpha_j^\vee)_0 = -\frac{1}{2}s_{ij}b_j a_{ij} d_j^{-1} \qquad \text{if} \quad 1\le i\le n\,, \quad 1\le j\le n$$

$$(\alpha_i^\vee|\alpha_j^\vee)_0 = -\frac{1}{2}a_i b_i \delta_{i,\mathrm{rk}\,A-n+j} d_i^{-1} \qquad \text{if} \quad 1\le i\le n\,, \quad j>n$$

$$(\alpha_i^\vee|\alpha_j^\vee)_0 = 0 \qquad \text{if} \quad i>n\,, \quad j>n$$

$$(X_i|X_j)_0 = \delta_{ij} d_i^{-1} a_i$$

$$(Y_i|Y_j)_0 = \delta_{ij} d_i^{-1} b_i\,.$$

THEOREM 2. *There exists a unique symmetric bilinear form* $(\ |\ )$ *(adding a subscript* k *if necessary) on* $g_k(A, a_i, s_{ij}, b)$ *satisfying*

(i) $(\ |\ )$ *is invariant, i.e.* $([u, v]|w)=(u|[v, w])$.

(ii) $(\ |\ )|_{g^{\mathrm{loc}}\times g^{\mathrm{loc}}}=(\ |\ )_0$.

Moreover we have

(iii) $\nu(u)<\nu(v)\Rightarrow(u|v)=0$.

(iv) $(\ |\ )$ *is non-degenerate.*

PROOF. (As in [A]). First of all, let us observe that the isomorphisms given by Propositions 1 and 2 preserve $(\ |\ )_0$. (For Proposition 2, (iii) use (13)). Moreover, $(\ |\ )_0$ is invariant, i.e. satisfies (i) whenever $u, v, w, [u, v], [v, w]\in g^{\mathrm{loc}}$. Indeed, we only need to show that

$$(X_i|[\alpha_j^\vee, Y_i])_0 = ([X_i, \alpha_j^\vee]|Y_i)_0 = (\alpha_j^\vee|[Y_i, X_i])_0\,.$$

But if $j\le n$

$$(X_i|[\alpha_j^\vee, Y_i])_0 = a_j b_j s_{ji}^{-1} a_{ji} d_i^{-1} a_i$$

$$([X_i, \alpha_j^\vee]|Y_i)_0 = s_{ji} a_{ji} d_i^{-1} b_i$$

$$(\alpha_j^\vee|[Y_i, X_i])_0 = s_{ji} b_i a_{ji} d_i^{-1}\,,$$

and if $j>n$

$$(X_i|[\alpha_j^\vee, Y_i])_0 = a_i b_i \delta_{i,\mathrm{rk}\,A-n+j} d_i^{-1} = ([X_i, \alpha_j^\vee]|Y_i)_0 = (\alpha_j^\vee|[Y_i, X_i])_0\,.$$

Now for $g_k(A, 1, -1)$ the theorem is just [K, Th. 2.2]. Thus we only need to prove:

$$(\ |\ )_{\bar{k}}|_{g_k \times g_k} \subseteq k\ .$$

We can do this on g_m by induction on m; for $m=1$ it is clear and for the inductive step we use (i), (iii). ■

4. Representation theory of $sq(a, b)$. In this Section, we develop some representation theory of the TDS $sq(a, b)$. We begin by introducing *quadratic highest weight* modules associated to an element λ in $k-\{0\}$. As in the representation theory of $sl(2)$, every such module has a unique (up to isomorphism) irreducible quotient, denoted by $\mathscr{L}(\lambda)$; $\mathscr{L}(\lambda)$ is finite dimensional if and only if λ is a non-negative integer, and every finite dimensional irreducible module arises in this way. Let us denote by $U(a, b)$ the universal enveloping algebra of $sq(a, b)$. Let E, F, H be the canonical generators of $sl(2)$. They satisfy the following bracket relations:

$$[E, F]=H\ , \qquad [H, E]=2E\ , \qquad [H, F]=-2F\ .$$

Let us recall that a highest (resp., lowest) weight module for $sl(2, k)$ is cyclic, with a generator v satisfying $Hv=\lambda v$, $Ev=0$ (resp., $Hv=\lambda v$, $Fv=0$). We will denote by $M(\lambda)$ (resp., $m(\lambda)$) the highest (resp., lowest) weight module of highest (resp., lowest) weight λ which covers any other such module. Let us recall the well known description of $M(\lambda)$, $m(\lambda)$ (see [Hu, Ex. 7.7]). $M(\lambda)$ has a basis $(v_i)_{i\in \mathbf{N}_0}$ such that the module structure is defined by

$$Hv_i=(\lambda-2i)v_i\ , \qquad Fv_i=(i+1)v_{i+1}\ , \qquad Ev_i=(\lambda-i+1)v_{i-1}\ . \tag{20}$$

(By convention, $v_{-1}=0$). $M(\lambda)$ is irreducible if λ is not a non-negative integer; but if it is, $M(\lambda)$ has a unique submodule, which is irreducible, isomorphic to $M(-\lambda-2)$ and is spanned by $v_{\lambda+1}$.

In an analogous way, $m(\lambda)$ has a basis $(w_j)_{j\in \mathbf{N}_0}$ and the action is given by

$$Hw_j=(\lambda+2j)w_j\ , \qquad Ew_j=(j+1)w_{j+1}\ , \qquad Fw_j=(-\lambda-j+1)w_{j-1}\ . \tag{20'}$$

(By convention, $w_{-1}=0$). Here, $m(\lambda)$ is irreducible if λ is not a non-positive integer; otherwise, $m(\lambda)$ has a unique submodule, which is irreducible, isomorphic to $m(-\lambda+2)$ and is spanned by $w_{\lambda+1}$. (In fact, one has a pairing between $M(\lambda)$ and $m(-\lambda)$).

Clearly, a submodule of a highest (resp., lowest) weight module is again such one. A module which is both highest and lowest weight module (for different generators) is necessarily finite dimensional and irreducible and cannot be realized as a proper submodule of a highest or a lowest weight module.

There are various statements equivalent to the fact that $sq(a, b)$ is not isomorphic to $sl(2, k)$. We shall record for further use that if $-ab\in k^2$, then $sq(a, b)\simeq sl(2, k)$.

DEFINITION 4. Let $\lambda\in k-\{0\}$. $\mathscr{M}(\lambda)$ is the $U(a, b)$-module generated by v subject to the relations

$$XZv = \lambda a Yv\,, \qquad Z^2 v = -ab\lambda^2 v\,, \qquad YZv = -\lambda b Xv\,. \tag{21}$$

Any cyclic $U(a, b)$-module whose generator satisfies (21) is called a quadratic highest weight module (of highest weight λ). Thus a quadratic highest weight module is a quotient of $\mathcal{M}(\lambda)$.

Let us remark that the first two conditions of (21) imply the third, provided that $\lambda + 2 \neq 0$. Indeed, if the first two conditions are satisfied, then

$$\begin{aligned}\lambda a YZv &= (ZY - 2bX)\lambda av = ZXZv - 2\lambda abXv \\ &= XZ^2 v - 2aYZv - 2\lambda abXv = -2aYZv - a(\lambda+2)\lambda bXv\,.\end{aligned}$$

Now, if v satisfies (21), then Zv also does. On the other hand, if $sq(a, b) \not\simeq sl(2, k)$, v and Zv are linearly independent: $Zv = cv$ implies $-ab\lambda^2 v = c^2 v$.

Now let k' be an extension of k, V a k-vector space, $T \in \mathrm{End}_k(V)$ and $\sigma \in \mathrm{Gal}(k'|k)$. Let us also denote by T (resp. σ) the k'-endomorphism of $V \otimes k'$ (resp. the k-endomorphism) given by $T \otimes \mathrm{id}$ (resp. $\mathrm{id} \otimes \sigma$); clearly, such T and σ commute. We shall always identify V with $V \otimes 1$. In what follows, k' will be $k(\sqrt{-ab})$, where $\sqrt{-ab}$ is a fixed root of $T^2 + ab$ in $\bar{k}$. Thus, if $-ab \notin k^2$, then $\mathrm{Gal}(k'|k) = \{1, \sigma\}$ where $\sigma(\sqrt{-ab}) = -\sqrt{-ab}$.

Now let V be any $sq(a, b)$-module. Then $V \otimes k'$ is a $sq(a, b) \otimes k'$-module. We will exploit the fact that $sq(a, b) \otimes k'$ is isomorphic to $sl((2, k')$. Indeed

$$H = \frac{1}{\sqrt{-ab}} Z\,, \qquad E = \frac{1}{2}\left(X - \frac{a}{\sqrt{-ab}} Y\right), \qquad F = \frac{1}{2}\left(\frac{1}{a} X + \frac{1}{\sqrt{-ab}} Y\right)$$

provide one such isomorphism, which will be fixed from now on. (Note that for $a = 1$, $b = -1$, $\sqrt{-ab} = 1$ this is the identification claimed in Lemma 1).

EXAMPLES. 1) The adjoint representation is a quotient of $\mathcal{M}(2)$. X and Y are generators which satisfy (21).

2) As $[sq(a, b), sq(a, b)] = sq(a, b)$, if $sq(a, b) \to gl(2, k)$ is a non-trivial representation then $sq(a, b)$ is isomorphic to $sl(2, k)$.

3) Let us assume that $sq(a, b)$ and $sl(2, k)$ are not isomorphic. Then $\mathcal{M}(1)$ has a four-dimensional quotient. Indeed, the assignment

$$X \mapsto \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$Y \mapsto \begin{pmatrix} 0 & 0 & 0 & b \\ 0 & 0 & -a & 0 \\ 0 & -ba^{-1} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$Z \mapsto \begin{pmatrix} 0 & -b & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -a & 0 \end{pmatrix}$$

induces an irreducible representation of $sq(a, b)$ in k^4. Any non-zero vector is a generator satisfying (21), for $\lambda = 1$.

4) Let V, W be quadratic highest weight modules of highest weights λ, μ and generators v, w, respectively. Then the submodule of $V \otimes W$ spanned by $u = \mu Zv \otimes w + \lambda v \otimes Zw$ is a quadratic highest weight module of highest weight $\lambda + \mu$. Combining this with 1), 3) we get that $\mathscr{M}(n)$ has finite dimensional quotients for a non-negative integer n.

5) Let us consider a k-vector space $\mathscr{V}$, which has a basis $\{R_n\}_{n \in N_0} \cup \{S_m\}_{m \in N_0}$; i.e., $\mathscr{V}$ is the direct sum of two copies of a polynomial ring in one variable over k. For $\lambda \in k$, we can define a representation of $sq(a, b)$ on $\mathscr{V}$, by the rules

$$ZR_n = -ab(\lambda - 2n)S_n, \qquad ZS_n = (\lambda - 2n)R_n$$

$$XR_n = \begin{cases} a(R_{n+1} + n(\lambda - n + 1)R_{n-1}), & n \quad \text{even} \\ R_{n+1} + n(\lambda - n + 1)R_{n-1}, & n \quad \text{odd} \end{cases}$$

$$XS_n = \begin{cases} a(S_{n+1} + n(\lambda - n + 1)S_{n-1}), & n \quad \text{even} \\ S_{n+1} + n(\lambda - n + 1)S_{n-1}, & n \quad \text{odd} \end{cases}$$

$$YR_n = \begin{cases} -ab(S_{n+1} - n(\lambda - n + 1)S_{n-1}), & n \quad \text{even} \\ -b(S_{n+1} - n(\lambda - n + 1)S_{n-1}), & n \quad \text{odd} \end{cases}$$

$$YS_n = \begin{cases} R_{n+1} - n(\lambda - n + 1)R_{n-1}, & n \quad \text{even} \\ a^{-1}(R_{n+1} - n(\lambda - n + 1)R_{n-1}), & n \quad \text{odd}\,. \end{cases}$$

(By convention, $R_{-1} = S_{-1} = 0$). With this action, $\mathscr{V}$ becomes a quadratic highest weight module of highest weight λ, with a generator R_0 (or S_0).

PROPOSITION 3. (i) *For $sq(1, -1) \simeq sl(2, k)$ we have $\mathscr{M}(\lambda) \simeq M(\lambda) \oplus m(-\lambda)$.*

(ii) *$\mathscr{M}(\lambda)$ is isomorphic to the module $\mathscr{V}$ constructed in Example* 5.

(iii) *If $sq(a, b) \not\simeq sl(2, k)$, two irreducible quadratic highest weight modules V of highest weight $\lambda \in k - \{0\}$ are isomorphic.*

An irreducible module of highest weight λ will be denoted by $\mathscr{L}(\lambda)$.

(iv) *$\mathscr{L}(\lambda)$ is finite dimensional if and only if λ is a non-negative integer.*

(v) *If V is an irreducible finite dimensional $sq(a, b)$-module, then V is isomorphic to $\mathscr{L}(\lambda)$, for some λ.*

(vi) *$\mathscr{L}(\lambda) \simeq \mathscr{L}(\lambda^{\#}) \Rightarrow \lambda = \lambda^{\#}$.*

PROOF. (i) Let v (resp. v_0, resp. w_0) be the generator of $\mathscr{M}(\lambda)$ (resp. $M(\lambda)$, resp. $m(-\lambda)$). The assignments

$$v_0 \mapsto \lambda v + Zv\,, \qquad w_0 \mapsto \lambda v - Zv$$

and

$$v \mapsto \frac{1}{2\lambda}(v_0 + w_0)$$

give rise to the claimed isomorphism and its inverse.

(ii) We need to prove that the surjection $\mathscr{M}(\lambda) \to \mathscr{V}$ with $v \mapsto R_0$ is actually an isomorphism. Tensoring with $\bar{k}$, we may assume that $a=1, b=-1$. Using (i) this is equivalent to proving that

$$M(\lambda) \oplus m(-\lambda) \to \mathscr{V}\,, \qquad v_0 \mapsto \lambda(R_0 + S_0)\,, \qquad w_0 \mapsto \lambda(R_0 - S_0)$$

is an isomorphism; and this can be deduced from the formulas

$$F(R_j + S_j) = 2(R_{j+1} + S_{j+1})\,, \qquad E(R_j - S_j) = 2(R_{j+1} - S_{j+1})\,.$$

(iii) Put

$$u_{\pm} = \lambda v \otimes 1 \pm Zv \otimes (\sqrt{-ab})^{-1} \in V \otimes k'$$

where $v \in V$ satisfies (21).

It is easy to see that $Eu_+ = Fu_- = 0$, $Hu_{\pm} = \pm\lambda u_{\pm}$. Thus $V \otimes k' = M_+ + M_-$, where $M_{\pm}$, the module spanned by $u_{\pm}$, is a highest (lowest) weight module over $sl(2, k')$.

It follows that $V = \bigoplus_{i \in N_0} V_{(\lambda - 2i)}$, where

$$V_{(\lambda-2i)} = \{t \in V : Z^2 t = -ab(\lambda - 2i)^2 t\}\,.$$

In other words, $V_{(\lambda-2i)}$ is a weight space for Z^2. Exactly as in [Hu, 20.2], each submodule of V is the direct sum of its weight spaces.

Let us assume first that $\lambda \notin N$. Then $\dim V_{(\lambda-2i)} \le 2$ for all i. We shall prove that the sum of all proper submodules of V is still proper. It suffices to show: if I_1, I_2 are proper submodules of V, so is $I_1 + I_2$. If $V = I_1 + I_2$, then $kv + kZv = (I_1)_\lambda + (I_2)_\lambda$ and hence there exist $\alpha, \beta \in k$ such that $\alpha v + \beta Zv \in I_i - \{0\}$ for, say, $i=1$. Then $-ab^2\lambda\beta v + \alpha Zv \in I_1$. The matrix

$$\begin{pmatrix} \alpha & \beta \\ -ab\lambda^2\beta & \alpha \end{pmatrix}$$

has determinant $\alpha^2 + ab\beta^2$, which is non-zero. Thus $v, Zv \in I_1$ and the claim follows.

There remains the case $\lambda \in N$. Using the description of $\mathscr{M}(\lambda)$ given by (ii), we see that $\mathscr{M}(n)$ contains a copy of $\mathscr{M}(-n-2)$, namely, the submodule spanned by R_{n+1}, S_{n+1}. Using the above introduced gradation, it is easy to see that any maximal submodule of $\mathscr{M}(n)$ contains $\mathscr{M}(-n-2)$; in other words, we have an epimorphism $\mathscr{M}(n)/\mathscr{M}(-n-2) \to \mathscr{V}$, for any irreducible quotient $\mathscr{V}$ of $\mathscr{M}(n)$. However, $(\mathscr{M}(n)/\mathscr{M}(-n-2)) \otimes k'$ is a direct sum of two isomorphic absolutely irreducible representations.

(iv) If $\mathscr{L}(\lambda)$ is finite dimensional, then so are $M_{\pm}$, and hence λ is a non-negative integer. The converse follows from the proof of (iii). (We have constructed finite dimensional modules of highest weight λ for each non-negative integer λ in the examples above).

(v) Let $u \in V \otimes k' - \{0\}$ such that $Eu=0$, $Hu=nu$, for some $n \in \boldsymbol{N}$. That is,

$$\sqrt{-ab}Xu = aYu\,, \qquad Zu = n\sqrt{-ab}u\,.$$

It follows easily that $u+\sigma u$ satisfies (21) for $\lambda=n$. And $u+\sigma u \neq 0$, because

$$u+\sigma u=0 \Rightarrow Zu+Z\sigma u=0 \Rightarrow n\sqrt{-ab}(u-\sigma u)=0 \Rightarrow u=0\,.$$

Therefore, V is spanned as a $U(a,b)$-module by $u+\sigma u$ and $V \simeq \mathscr{L}(n)$.

Finally, (vi) follows from the $sl(2)$ theory, in view of the proof of the preceding points. ■

The following fact will be useful later, when considering the presentation by generators and relations of a form of a *derived* Kac-Moody algebra, associated to a symmetrizable matrix. Let us first introduce the following notation. Let x be an element of a k-algebra, $n \in \boldsymbol{N}_0$, $t \in k$. Then we define

$$F_{n,t}(x) = \begin{cases} \prod\limits_{0 \le i \le j} (x^2-(2i+1)^2 t)\,, & \text{if} \quad n=2j+1 \quad \text{is odd} \\ x \prod\limits_{1 \le i \le j} (x^2-(2i)^2 t)\,, & \text{if} \quad n=2j \quad \text{is even}\,. \end{cases}$$

PROPOSITION 4. (i) *The following identities hold in $\mathscr{L}(n)$ for $n \in \boldsymbol{N}$:*

$$F_{n,-ab}(Z)=0\,, \qquad F_{n,a}(X)=0\,, \qquad F_{n,b}(Y)=0\,.$$

(ii) *Conversely, if $\mathscr{W}$ is a quadratic highest weight module of highest weight $n \in \boldsymbol{N}$ where any of the preceding identities holds, then it is finite dimensional.*

PROOF. (i) Arguing as in the proof of Proposition 3, we get that $F_{n,1}(H)=0$ in $\mathscr{L}(n) \otimes k'$. But $F_{n,\lambda^2}(\lambda x)=\lambda^{n+1}F_{n,1}(x)$ and hence $F_{n,-ab}(Z)=0$; the rest is similar. (Use the fact that over $\bar{k}$, X, Y are conjugated to $\sqrt{a}\,H$, $\sqrt{b}\,H$, respectively).

(ii) Let us recall that $\mathscr{M}(n)$ contains a copy of $\mathscr{M}(-n-2)$. We observed in the proof of Proposition 3 that $\mathscr{M}(n)=\bigoplus_{i \in \boldsymbol{N}_0}\mathscr{M}(n)_{(n-2i)}$. As in (i), we can deduce that

$$\mathscr{M}(n) = \bigoplus_{i \in \boldsymbol{N}_0} \{t \in \mathscr{M}(n) : X^2 t = a(n-2i)^2 t\}\,.$$

If, for example, $F_{n,a}(X)$ vanishes identically on $\mathscr{W}$ and $n=2j+1$ is odd, then

$$\mathscr{W} = \bigoplus_{0 \le i \le j} \{w \in \mathscr{W} : X^2 w = a(n-2i)^2 w\}\,.$$

Thus a copy of $\mathcal{M}(-n-2)$ is contained in the kernel of any epimorphism $\mathcal{M}(n)\to\mathcal{W}$ and hence $\mathcal{W}$ is finite dimensional. ∎

Let us remark that $F_{n,a}(X)$ vanishes identically on a quadratic highest weight module generated by v if

$$F_{n,a}(X)(v)=0\,, \qquad F_{n,a}(X)(Zv)=0\,,$$

as follows from the formulas in Example 5.

Finally, let us observe that we can *integrate* the finite dimensional representations of $sq(a,b)$ to the group $SQ(a,b)$ of elements of the quaternion algebra $\langle\!\langle -a, -b\rangle\!\rangle$ of norm one. For this, let us recall first that $SQ(1,-1)$ is $SL(2,k)$. Taking k' as above, we have $\langle\!\langle -a, -b\rangle\!\rangle_k \otimes k' = \langle\!\langle -a, -b\rangle\!\rangle_{k'}$, and hence $SQ_k(a,b)$ is a subgroup of $SQ_{k'}(a,b)\simeq SL(2,k')$. So, let V be a finite dimensional representation of $sq_k(a,b)$; then $V\otimes k'$ is a finite dimensional representation of $sq_{k'}(a,b)\simeq sl(2,k')$ and, a fortiori, of $SQ_{k'}(a,b)\simeq SL(2,k')$. Now Galois argument shows that $T(V)\subseteq V$ for any $T\in SQ_k(a,b)$.

5. The presentation of the derived algebra. For simplicity, let us denote in this Section $g'_k = g'_k(A, a_i, s_{ij}, b)$. We shall give here a presentation by generators and relations of g'_k, when A is assumed to be symmetrizable. Let us recall first what happens in the split case.

THEOREM 3 (Gabber-Kac [GK]). *$[g(A), g(A)]$ is isomorphic to the Lie algebra given by generators $\{E_i, F_i, H_i : 1\le i\le n\}$ and relations*

$$[H_i, H_j]=0 \tag{22}$$

$$[E_i, F_j]=\delta_{ij}H_j \tag{23}$$

$$[H_i, E_j]=a_{ij}E_j \tag{24}$$

$$[H_i, F_j]=-a_{ij}F_j\,, \tag{25}$$

and for $i\neq j$

$$(\mathrm{ad}\,E_i)^{1-a_{ij}}E_j=0 \tag{26}$$

$$(\mathrm{ad}\,F_i)^{1-a_{ij}}F_j=0\,. \tag{27}$$

THEOREM 4. *Let g_0 be the Lie algebra over k given by generators $\{X_i\, Y_i, Z_i : 1\le i\le n\}$ and relations*

$$[Z_i, Z_j]=0 \tag{28}$$

$$[X_i, Y_i]=2Z_i \tag{29}$$

$$[Z_i, X_j]=-a_i s_{ij}^{-1} a_{ij} Y_j \tag{30}$$

$$[Y_j, Z_i]=-b_i s_{ij} a_{ij} X_j\,, \tag{31}$$

and for $i \neq j$

$$[X_i, Y_j] = s_{ij}[Y_i, X_j] \tag{32}$$

$$[X_i, X_j] = -a_i b_i^{-1} s_{ij}^{-1}[Y_i, Y_j] \tag{33}$$

$$F_{-a_{ij}, a_i}(\operatorname{ad} X_i)(X_j) = 0 \tag{34}$$

$$F_{-a_{ij}, a_i}(\operatorname{ad} X_i)(Y_j) = 0\,. \tag{35}$$

Then g_0 *is isomorphic to* g'_k.

PROOF. First of all, using $[g \otimes_k f, g \otimes_k f] \simeq [g, g] \otimes_k f$ for any field extension $f \supset k$ and any Lie algebra g, we can reduce ourselves to the cases $a=1, b=-1$. Taking Gabber-Kac's result in mind, we only need to prove that the assignments

$$H_i \mapsto Z_i\,, \qquad E_i \mapsto \frac{1}{2}(X_i - Y_i)\,, \qquad F_i \mapsto \frac{1}{2}(X_i + Y_i)$$

and

$$Z_i \mapsto H_i\,, \qquad X_i \mapsto E_i + F_i\,, \qquad Y_i \mapsto -E_i + F_i$$

provide an isomorphism between g_0 and $[g(A), g(A)]$. That is, we must check that relations (22), $\cdots$, (27) imply (28), $\cdots$, (35) and vice versa. That relations (22), $\cdots$, (25) are equivalent to (28), $\cdots$, (33) is easy. So, let g_1 (resp. g_2) be the Lie algebra generated by $\{E_i, F_i, H_i : 1 \leq i \leq n\}$ (resp. $\{X_i, Y_i, Z_i : 1 \leq i \leq n\}$) with relations (22), $\cdots$, (25) (resp. (28), $\cdots$, (33)) and let I_1 (resp. I_2) be the ideal generated by the relations (26), (27) (resp. (34), (35)). The above assignments give rise to an isomorphism ϕ from g_1 onto g_2; we want to show that ϕ maps I_1 onto I_2. Let s_i (resp. t_i) be the Lie subalgebra of g_1 (resp. g_2) spanned by $\{E_i, F_i, H_i\}$ (resp. $\{X_i, Y_i, Z_i\}$); clearly $\phi(s_i) = t_i$.

Let us first assume that $a_{ij} \neq 0$ and consider the s_i-module V_{ij} spanned by the images of E_j, F_j in g_1/I_1. V_{ij} is finite dimensional and a sum of copies of $L(-a_{ij})$. Thus, via ϕ, $F_{-a_{ij}, a_i}(\operatorname{ad} X_i)$ and $F_{-a_{ij}, b_i}(\operatorname{ad} Y_i)$ vanish identically on V_{ij}; in particular (34) and (35) hold, i.e. $\phi^{-1}(I_2) \subseteq I_1$.

Reciprocally, let U_{ij} be the t_i-module spanned by the images of X_j, Y_j. By (34), (35) and Proposition 4 it is finite dimensional and hence (26), (27) are true.

It only remains to treat the case $a_{ij} = 0$, which is very easy. ∎

REMARK 1. In the case $a_{ij} \geq -3$ the preceding theorem was obtained in [A] by elementary computations.

6. Representation theory of $g(A, a, b)$**.** In the next sections we will begin to investigate some modules over the introduced forms of Kac-Moody algebras, which are in certain sense generalizations of the highest weight modules (cf. [K, Ch. 9]). Conceptually, its structure relies heavily on the $sq(a, b)$-modules introduced in Section 4, as well as in the split case. Let us refer to [K, Ch. 9], for the definition of the *highest weight* Verma module $M(\lambda)$ and the *lowest* one, which we shall denote here $m(\lambda)$. The irreducible quotient of $M(\lambda)$ (resp. $m(\lambda)$) will be denoted $L(\lambda)$ (resp. $l(\lambda)$). For simplicity, we begin by $g(A, a, b)$ which will be denoted in this Section as g. Also, $U(a, b)$ will be its universal enveloping algebra and $k' = k(\sqrt{-ab})$ as in Section 4.

DEFINITION 5. Let $\lambda \in \boldsymbol{h}^* - \{0\}$. $\mathcal{M}(\lambda)$ is the $U(a, b)$-module generated by v subject to relations

$$X_i Zv = a\lambda(Z) Y_i v\,, \qquad Z^2 v = -ab\lambda^2(Z)v\,, \qquad Y_i Zv = -b\lambda(Z) X_i v \tag{36}$$

for all $Z \in \boldsymbol{h}$ and $1 \leq i \leq n$.

(The first two relations imply the third at least if $\lambda + \alpha_i \neq 0$ for all i). A quadratic highest weight module (of highest weight λ) is a quotient of $\mathcal{M}(\lambda)$; i.e., a cyclic $U(a, b)$-module whose generator satisfies (36).

Before collecting some fundamental facts about this quadratic highest weight modules, let us introduce the following notation: let V be an $\boldsymbol{h}$-module, $\lambda \in \boldsymbol{h}^* - \{0\}$. Then

$$V_{(\lambda)} = \{v \in V : H^2 = -ab\lambda(H)^2 v\,, \qquad \text{for all} \quad H \in \boldsymbol{h}\}\,.$$

But if $\lambda = 0$, put $V_{(0)} = \{v \in V : H = 0, \text{ for all } H \in \boldsymbol{h}\}$; i.e. $V_{(0)} = V_0$.

Let us also recall the following notions: $Q = \sum \boldsymbol{Z}\alpha_i \subset \boldsymbol{h}^*$ is the root lattice; $Q_+ = \sum \boldsymbol{Z}_+ \alpha_i$; $P = \{\lambda \in \boldsymbol{h}^* : \langle \lambda, \alpha_i^\vee \rangle \in \boldsymbol{Z}\}$; $P_+ = \{\lambda \in P : \langle \lambda, \alpha_i^\vee \rangle \geq 0\}$. There is a partial ordering in $\boldsymbol{h}^*$ given by $\lambda \geq \mu \Leftrightarrow \lambda - \mu \in Q_+$.

Let V be a quadratic highest weight module of highest weight λ. Let v be a generator of V satisfying (36); let $Z \in \boldsymbol{h}$ such that $\lambda(Z) \neq 0$. Let us set

$$u_\pm = v \otimes 1 \pm Zv \otimes (\lambda(Z)\sqrt{-ab})^{-1} \in V \otimes k'\,.$$

$u_\pm$ do not depend on Z; we get from (36) that for any Z'

$$(ZZ')^2 v = (ab\lambda(Z)\lambda(Z'))^2 v \Rightarrow Z'v = (-ab)^{-1}\lambda(Z)^{-2} Z'Z^2 v = \lambda(Z)^{-1}\lambda(Z')Zv\,.$$

As in the $sq(a, b)$-case, v and $\lambda(Z)^{-1}Zv$ are linearly independent if $g(A, a, b) \not\simeq g(A)$.

Clearly, $(\boldsymbol{h} \otimes k', \sqrt{-ab}\Pi, (\sqrt{-ab})^{-1}\Pi^\vee)$ is a realization of A over k' which we will use in this Section to construct $g_{k'}(A)$. With respect to this realization $\sqrt{-ab}Q$ (resp. $\sqrt{-ab}P$) is the root (weight) lattice. Moreover, we shall fix the isomorphism from $g_{k'}(A)$ to $g \otimes k'$ given by

$$E_i \mapsto \frac{1}{2}\left(X_i - \frac{a}{\sqrt{-ab}}\, Y_i\right), \qquad F_i \mapsto \frac{1}{2}\left(\frac{1}{a} X_i + \frac{1}{\sqrt{-ab}}\, Y_i\right)$$

and is multiplication by $(\sqrt{-ab})^{-1}$ (resp. by $\sqrt{-ab}$) in $\boldsymbol{h} \otimes k'$ (resp. $(\boldsymbol{h} \otimes k')^*$).

As before we check easily that $E_i u_+ = F_i u_- = 0$, $Hu_\pm = \pm\Lambda(H)u_\pm$. Thus $V \otimes k' = M_+ + M_-$, where $M_\pm$, the module spanned by $u_\pm$, is a highest (lowest) module over $g_{k'}(A)$ of highest (lowest) weight $\pm\Lambda = \pm\sqrt{-ab}\lambda$. It follows (cf. [K, 9.2]) that

$$V = \bigoplus_{\mu \leq \lambda} V_{(\mu)} .$$

The following fact is inspired by [K, Ch. 9]. Let V^* be the g-module contragredient to V. Then $V^* = \prod_\mu V^*_{(\mu)}$. Let us define

$$V^{\mathscr{D}} = \bigoplus_{\mu} V^*_{(\mu)} \subset \prod_{\mu} V^*_{(\mu)} .$$

Clearly, $V^{\mathscr{D}}$ is a quadratic highest weight module of highest weight λ generated by $\phi \in V^*_{(\lambda)}$, $\phi(v) = 1$, $\phi(Zv) = 0$. In particular, $\mathscr{M}(\lambda) \simeq \mathscr{M}(\lambda)^{\mathscr{D}}$.

PROPOSITION 5. *Let $g(A, a, b) \not\simeq g(A)$ and $\lambda \in \boldsymbol{h}^* - \{0\}$.*

(i) *If $\lambda \notin Q$, then $\mathscr{M}(\lambda)$ has a unique irreducible quotient. If $\lambda \in Q$, two irreducible quadratic highest weight modules of highest weight λ are isomorphic.*

Let us denote by $\mathscr{L}(\lambda)$ an irreducible quadratic highest weight module of highest weight λ.

(ii) *If $a = 1$, $b = -1$, we have $\mathscr{M}(\lambda) \simeq M(\lambda) \oplus m(-\lambda)$.*

Now let us assume that A is of finite type.

(iii) *$\mathscr{L}(\lambda)$ is finite dimensional if and only if $\lambda \in P_+$. In such a case, $\mathscr{L}(\lambda)$ is isomorphic to $\mathscr{L}(\lambda')$ if and only if $L(\sqrt{-ab}\lambda)$ and $L(\sqrt{-ab}\lambda')$ are isomorphic or contragredient.*

(iv) *Any irreducible finite dimensional non-zero g-module is isomorphic to $\mathscr{L}(\lambda)$, for some $\lambda \in P_+$.*

PROOF. The proofs of (i) (for $\lambda \notin Q$) and (ii) follow as in the $sq(a, b)$-case. So let $\lambda \in Q - \{0\}$ and let $\mathscr{L}$, $\mathscr{L}^\#$ be two irreducible quadratic highest weight modules of highest weight λ.

Let us assume that A is not finite. Then $\mathscr{L} \otimes k' \simeq M_+ + M_-$ as above. Now the weights of $M_\pm$ are contained in the cone $\{\pm\sqrt{-ab}\lambda \mp \alpha : \alpha \in \sqrt{-ab}Q_+\}$. Thus the weights of $M_+ \cap M_-$ are contained in $\{\sqrt{-ab}\mu : -\lambda \leq \mu \leq \lambda\}$. It follows that $M_+ \cap M_- = 0$ and hence $\mathscr{L} \otimes k' \simeq L(\sqrt{ab}\lambda) \oplus l(\sqrt{-ab}\lambda)$ and we are done.

Let us assume now that A is finite. There are two possibilities:

$$\mathscr{L} \otimes k' \simeq L(\sqrt{-ab}\lambda) \otimes l(\sqrt{-ab}\lambda) \quad \text{or} \quad \mathscr{L} \otimes k' \simeq L(\sqrt{-ab}\lambda) \simeq l(\sqrt{-ab}\lambda) .$$

Looking at $L(\sqrt{-ab}\lambda)$, $l(\sqrt{-ab}\lambda)$ as g-modules, we see that $\mathscr{L}^\#$ cannot achieve a different possibility that $\mathscr{L}$ and the statement follows.

For (iii), it is clear that $\mathscr{L}(\lambda)$ finite dimensional implies $\lambda \in P_+$. Conversely, if $\lambda \in P_+$, the irreducible highest and lowest weight $g_{k'}(A)$-modules of highest weight and lowest weight Λ, $-\Lambda$ and generators v_0, w_0, respectively, are finite dimensional over k' and a fortiori over k. Thus in their direct sum $v_0 + w_0$ satisfies (36) and therefore the (finite dimensional) g-submodule spanned by $v_0 + w_0$ is a quotient of $\mathscr{M}(\lambda)$; hence $\mathscr{L}(\lambda)$ is finite dimensional.

It remains to prove (iv); this is very analogous to the proof of the $sq(a, b)$-case, taking in mind the proof of (iii). ∎

7. Representation theory of $g_k(A, a_i, s_{ij}, b)$. Let us remark first that $g_k(A, a_i, s_{ij}, b)=g_k$ splits after tensoring with a quadratic extension. Indeed, let us assume that $g_k \not\simeq g_k(A)$; then for some l we have $-a_l b_l \notin k^2$; but $-a_j b_j = -a_l b_l (a_j a_l^{-1} s_{lj})^2$. Let us denote by $\sqrt{-a_l b_l}$ a fixed root of the polynomial $T^2 + a_l b_l$ in $\bar{k}$, $k' = k(\sqrt{-a_l b_l})$, $\sqrt{-a_j b_j} = \sqrt{-a_l b_l} a_j a_l^{-1} s_{lj} \in k'$. Let us observe that

$$\sqrt{-a_j b_j} = \sqrt{-a_i b_i} a_j a_i^{-1} s_{ij}\,, \qquad \text{for all} \quad i, j\,.$$

Now we shall use a realization $(\boldsymbol{h}', \Pi, \Pi^\vee)$ of A over k' where $\boldsymbol{h}'$ is a k'-vector space of dimension $2n - \operatorname{rk} A$, $\Pi = \{\beta_1, \cdots, \beta_n\} \subset \boldsymbol{h}'^*$, $\Pi^\vee = \{\beta_1^\vee, \cdots, \beta_n^\vee\} \subset \boldsymbol{h}'$.

We will reorder the index set as in §2 and hence we will choose $\{\beta_{n+1}^\vee, \cdots, \beta_{2n-\operatorname{rk} A}^\vee\} \subset \boldsymbol{h}'$ in such a way that $\{\beta_1^\vee, \cdots, \beta_{2n-\operatorname{rk} A}^\vee\}$ is a basis of $\boldsymbol{h}'$ and

$$\langle \beta_i, \beta_{n+j}^\vee \rangle = \delta_{i, \operatorname{rk} A + j}$$

for $1 \le i \le n$, $1 \le j \le n - \operatorname{rk} A$. In the following, we shall consider $g_{k'}(A)$ as constructed using this realization.

Lemma 3 (See also [AR]). *$g_k \otimes k'$ is isomorphic to $g_{k'}(A)$ via*

$$\beta_i^\vee \mapsto \frac{1}{\sqrt{-a_i b_i}} \alpha_i^\vee\,, \qquad E_i \mapsto \frac{1}{2}\left(X_i - \frac{a_i}{\sqrt{-a_i b_i}} Y_i\right), \qquad F_i \mapsto \frac{1}{2}\left(\frac{1}{a_i} X_i + \frac{1}{\sqrt{-a_i b_i}} Y_i\right).$$

Let Δ be the root system of $g_{k'}(A)$ with respect to the Cartan subalgebra $\boldsymbol{h}'$; and as in the preceding Section, let $Q = \sum \boldsymbol{Z}\beta_i \subset \boldsymbol{h}'^*$ be the root lattice; $Q_+ = \sum \boldsymbol{Z}_+ \beta_i$; $P = \{\lambda \in \boldsymbol{h}'^* : \langle \lambda, \beta_i^\vee \rangle \in \boldsymbol{Z}\}$; $P_+ = \{\lambda \in P : \langle \lambda, \beta_i^\vee \rangle \ge 0\}$. We will consider the partial ordering in $\boldsymbol{h}'^*$ given by $\lambda \ge \mu \Leftrightarrow \lambda - \mu \in Q_+$.

Now we shall consider an application $\xi : \boldsymbol{h}'^* \to \boldsymbol{h}^* \otimes k'$ defined by

$$\langle \xi(\beta), \alpha_i^\vee \rangle = \langle \beta, \beta_i^\vee \rangle\,.$$

Clearly, $\xi(\Delta)$, $\xi(Q)$, $\xi(P)$ are contained in $\boldsymbol{h}^*$, because $\xi(\beta_j) = \alpha_j$.

Let V be a g_k-module, $\mu \in \boldsymbol{h}^* - \{0\}$, $\mu_i = \mu(\alpha_i^\vee)$. Let us define

$$V_{(\mu)} = \{v \in V : \alpha_i^\vee \alpha_j^\vee v = -a_i b_i s_{ij} \mu_i \mu_j v\}\,.$$

(If $\mu = 0$, put $V_{(0)} = \{v \in V : H = 0, \text{ for all } H \in \boldsymbol{h}\}$; i.e. $V_{(0)} = V_0$). Of course, $V_{(\mu)} = V_{(-\mu)}$. Lemma 3 enables us to consider a *generalized* root decomposition in g.

Lemma 4. *With the module structure given by the adjoint representation, we have*

$$g_k = \boldsymbol{h} \oplus \left(\bigoplus_{\alpha \in \xi(\Delta)} (g_k)_{(\alpha)} \right).$$

Proof. By Lemma 3, we have

$$g_k \otimes k' = \boldsymbol{h}' \oplus \left(\bigoplus_{\beta \in \Delta} (g_k \otimes k')_\beta \right).$$

Let $v \in (g_k \otimes k')_\beta$. Then

$$\alpha_i^\vee \alpha_j^\vee v = \sqrt{-a_i b_i}\sqrt{-a_j b_j}\beta_i^\vee \beta_j^\vee v = -a_i b_i s_{ij} \beta(\beta_i^\vee)\beta(\beta_j^\vee)v = -a_i b_i s_{ij} \xi(\beta)(\alpha_i^\vee)\xi(\beta)(\alpha_j^\vee)v .$$

■

DEFINITION 6. *Let $\lambda \in \boldsymbol{h}^* - \{0\}$, $\lambda_i = \lambda(\alpha_i^\vee)$. $\mathscr{M}(\lambda)$ is the g_k-module generated by v subject to relations*

$$(37) \qquad X_i \alpha_j^\vee v = a_j s_{ij} \lambda_j Y_i v , \qquad \alpha_i^\vee \alpha_j^\vee v = -a_i b_i s_{ij} \lambda_i \lambda_j v , \qquad Y_i \alpha_j^\vee v = -b_j s_{ji} \lambda_j X_i v .$$

(There are some redundancy, at least generically). A quadratic highest weight module (of highest weight λ) is a quotient of $\mathscr{M}(\lambda)$; i.e., a cyclic module whose generator satisfies (37).

PROPOSITION 6. *Let $g_k \not\simeq g_k(A)$ and $\lambda \in \boldsymbol{h}^* - \{0\}$.*

(i) *If $\xi^{-1}(\lambda) \notin Q$, then $\mathscr{M}(\lambda)$ has a unique irreducible quotient. If $\lambda \in \xi(Q)$, two irreducible quadratic highest weight modules of highest weight λ are isomorphic.*

Let us denote by $\mathscr{L}(\lambda)$ an irreducible quadratic highest weight module of highest weight λ.

(ii) *If A is not of finite type, then*

$$\mathscr{L}(\lambda) \otimes k' \simeq L(\xi^{-1}(\lambda)) \oplus l(\xi^{-1}(\lambda)) .$$

Now let us assume that A is of finite type.

(iii) *$\mathscr{L}(\lambda)$ is finite dimensional if and only if $\xi^{-1}(\lambda) \in P_+$. In such a case, $\mathscr{L}(\lambda)$ is isomorphic to $\mathscr{L}(\lambda')$ if and only if $L(\xi^{-1}(\lambda))$* and *$L(\xi^{-1}(\lambda'))$ are isomorphic or contragredient.*

(iv) *Any irreducible finite dimensional non-zero g-module is isomorphic to $\mathscr{L}(\lambda)$ for some $\lambda \in \xi(P_+)$.*

PROOF. Taking into account Lemma 3, the proof is quite similar to the $sq(a, b)$ and $g(A, a, b)$-cases.

REMARKS. 1. Compare Proposition 6 (iii), (iv) with [T, Th. 7.2].

2. Let us recall the following well-known theorem of Duflo:

Let g be a complex finite dimensional simple Lie algebra, $I \subset U(g)$ a primitive ideal (i.e., $I = \mathrm{Ann}(V)$ for some simple $U(g)$-module V), $h \subset g$ a Cartan subalgebra. Then $I = \mathrm{Ann}(L(\lambda))$ for some $\lambda \in h^*$. We can deduce a real analog of this fact from Proposition 6.

As in [K, Ch. 9], let us consider a subcategory (denoted here $\mathcal{O}_+$) of the category of all $g_{k'}(A)$-modules. V is an object of $\mathcal{O}_+$ if the following is satisfied:

(i) It is $\boldsymbol{h}'$-diagonalizable, i.e., it admits a weight space decomposition $V = \bigoplus_\mu V_\mu$ $(\mu \in \boldsymbol{h}'^*)$. Let $P(V) = \{\mu : V_\mu \neq 0\}$.

(ii) $\dim V_\mu$ is finite for all μ.

(iii) $P(V) \subset \bigcup_{1 \le j \le s} D(\Lambda^j)$ for some $\Lambda^1, \cdots, \Lambda^s \in \boldsymbol{h}'$. (Here $D(\Lambda) = \{\mu \in \boldsymbol{h}'^* : \mu \le \Lambda\}$.)

There is another subcategory $\mathcal{O}_-$, which is defined by replacing the above $D(\Lambda)$ by

$d(\Lambda)=\{\mu\in \boldsymbol{h}'^* : \mu\geq\Lambda\}$. $\mathcal{O}_+$ and $\mathcal{O}_-$ are (anti-) equivalent, by means of the functor $V\to V^{\mathscr{D}}$, defined exactly as in Section 6, before Proposition 5 (cf. [K, 9.4]). We have already pointed out that in the non-finite case, a module belonging to both categories must be trivial. This is related to the fact that the opposite Borel subalgebras are not conjugate.

DEFINITION 7. $\mathcal{O}$ is the subcategory of the category of all g_k-modules whose objects V satisfy the following:

(i) $V=\bigoplus_\mu V_{(\mu)}$ ($\mu\in\boldsymbol{h}^*$). Let $P(V)=\{\mu : V_{(\mu)}\neq 0\}$.
(ii) $\dim V_{(\mu)}$ is finite for all μ.
(iii) $P(V)\subset\bigcup_{1\leq j\leq s}\mathscr{D}(\lambda^j)$ for some $\lambda^1, \cdots, \lambda^s\in\boldsymbol{h}$.
(Here $\mathscr{D}(\lambda)=\{\mu\in\boldsymbol{h}^* : \pm\mu\in\xi(D(\xi^{-1}(\lambda)))\}$).

In this case, $V\to V^{\mathscr{D}}$ maps $\mathcal{O}$ into itself.

The condition (i) of Definition 7 implies that V is diagonalizable under the action of the abelian Lie algebra of degree 2 homogeneous elements in the symmetric algebra of $\boldsymbol{h}$.

Let $\{e(v) : v\in\boldsymbol{h}'^*\}$ be the canonical basis of the group ring $\boldsymbol{Z}[\boldsymbol{h}'^*]$ and let $\mathscr{E}$ be the group of all series of the form

$$\sum_{v\in\boldsymbol{h}'^*} c_v e(v)$$

with $c_v\in\boldsymbol{Z}$ and the support of the family $\{c_v\}$ contained in a finite union of sets of the form $D(\lambda)$. The multiplication of $\boldsymbol{Z}[\boldsymbol{h}'^*]$ can be extended to $\mathscr{E}$. Let us recall (see [K, 9.7]) that the formal character ch W of $W\in\mathcal{O}_+$ (resp. $W\in\mathcal{O}_-$) is $\operatorname{ch} W=\sum_v \dim V_v e(v)$ (resp. $\operatorname{ch} W=\operatorname{ch} W^{\mathscr{D}}$).

Let V be any g_k-module satisfying the conditions (i), (ii) of Definition 7. Let $\mu\in\boldsymbol{h}'^*$; let us fix some j such that $\mu\in\mathscr{D}(\lambda^j)$ and put $\xi_0(\mu)=\pm\xi^{-1}(\mu)$ if $\pm\mu\in\xi(D(\xi^{-1}(\lambda^j)))$. Let us define the formal character of V as

$$\chi(V)=\dim V_{(0)}e(0)+\frac{1}{2}\sum_{\mu\in\boldsymbol{h}^*-\{0\}}\dim V_{(\mu)}e(\xi_0(\mu)) .$$

On the other hand, let $s_i=kX_i+kY_i+k\alpha_i^\vee$ $(1\leq i\leq n)$; s_i is isomorphic to $sq(a_i, b_i)$. Let us say as in [A] that a g-module V is integrable if it is s_i-locally finite for all i: $1\leq i\leq n$.

We postpone to a forthcoming paper the study of the category $\mathcal{O}$, which can be derived from the split case. We will also give the definition of the quadratic Verma module of highest weight 0. Nevertheless, let us include here some results which can be proved without an extra effort (we are assuming that A is not of finite type):

PROPOSITION 7. *Let* $g_k\neq g_k(A)$.

(i) $\mathscr{L}(\lambda)$ *is integrable if and only if* $\xi^{-1}(\lambda)\in P_+$.
(ii) $\chi(\mathscr{L}(\lambda))=2\operatorname{ch}(L(\xi^{-1}(\lambda)))$. *In particular, if* $\xi^{-1}(\lambda)\in P_+$ *and* A *is symmetrizable*,

then $\chi(\mathscr{L}(\lambda))$ *can be expressed with the help of the Kac-Weyl character formula* (*cf.* [*K*, 10.4]).

(iii) $\mathscr{L}(\lambda)$ *and* $(\mathscr{L}(\lambda))^{\mathscr{D}}$ *are isomorphic; in particular* $\mathscr{L}(\lambda)$ *admits an invariant bilinear form* $B(,)$, *i.e.*

$$B(t(x), y) = -B(x, t(y)) \qquad \textit{for all} \quad x, y \in \mathscr{L}(\lambda), \ t \in g_k$$

(*see* [*K*, 9.4]).

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