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GRÖBNER BASES OF TORIC VARIETIES

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Abstract. In this article projective toric varieties are studied from the viewpoint of Gröbner basis theory and combinatorics. We characterize the radicals of all initial ideals of a toric variety $X_{\mathscr{A}}$ as the Stanley-Reisner ideals of regular triangulations of its set of weights $\mathscr{A}$. This implies that the secondary polytope $\Sigma(\mathscr{A})$ is a Minkowski summand of the state polytope of $X_{\mathscr{A}}$. Here the lexicographic (resp. reverse lexicographic) initial ideals of $X_{\mathscr{A}}$ arise from triangulations by placing (resp. pulling) vertices. We also prove that the state polytope of the Segre embedding of $\boldsymbol{P}^{r-1}\times\boldsymbol{P}^{s-1}$ equals the secondary polytope $\Sigma(\Delta_{r-1}\times\Delta_{s-1})$ of a product of simplices.

1. Introduction. A cornerstone for the interaction between combinatorics and algebraic geometry is the theory of toric varieties [8], [16] which relates algebraic torus actions to the combinatorial study of convex polytopes. In the present paper we investigate the class of projective toric varieties from the point of view of Gröbner basis theory [1], [7], [11], [15], [20], [21]. The methods used to study Gröbner bases here are combinatorial rather than algebraic. Recent results on regular triangulations and secondary polytopes [5], [10], [13] will be applied to describe, as explicitly as possible, the initial ideals with respect to all term orders of a given projective toric variety.

Let $\mathscr{A}=\{a_1, a_2, \cdots, a_n\}$ be a fixed subset of the lattice $\boldsymbol{Z}^{d-1}\times\{1\}$ with the property that $\mathscr{A}$ linearly spans $\boldsymbol{R}^d$. A diagonal action of the d-dimensional torus $(\boldsymbol{C}^*)^d$ on $\boldsymbol{C}^n$ is obtained by interpreting $\mathscr{A}$ as the set of weights. Since the a_i all lie in an affine hyperplane, we get an induced $(\boldsymbol{C}^*)^d$-action on projective space $\boldsymbol{P}^{n-1}$. We define the *projective toric variety* $X_{\mathscr{A}}$ to be the closure of the orbit $(\boldsymbol{C}^*)^d\cdot(1, 1, \cdots, 1)$ in $\boldsymbol{P}^{n-1}$. The vanishing ideal of $X_{\mathscr{A}}$ is the homogeneous prime ideal

$$\mathscr{I}_{\mathscr{A}}:=\mathrm{Kernel}(\boldsymbol{C}[y_1,y_2,\cdots,y_n]\to\boldsymbol{C}[x_1,\cdots,x_d,x_1^{-1},\cdots,x_d^{-1}],\quad y_i\mapsto x^{a_i}=x_1^{a_{i1}}x_2^{a_{i2}}\cdots x_d^{a_{id}}).$$

Note that here we allow $\mathscr{A}$ to be any set of lattice points, which means that the embedded toric variety $X_{\mathscr{A}}$ need not be projectively normal.

We shall be interested in Gröbner bases of the *toric ideal* $\mathscr{I}_{\mathscr{A}}$ with respect to an arbitrary term order on $\boldsymbol{C}[y]:=\boldsymbol{C}[y_1, y_2, \cdots, y_n]$. In Section 2 we prove a singly-exponential (in d) degree upper bound for these Gröbner bases and thus for the Castelnuovo regularity of $\mathscr{I}_{\mathscr{A}}$. In addition, we construct an explicit universal Gröbner basis $\mathscr{U}_{\mathscr{A}}$ which satisfies this bound.

Our main result is a natural correspondence, to be established in Section 3, between the distinct Gröbner bases of $\mathscr{I}_{\mathscr{A}}$ and the regular triangulations of $\mathscr{A}$. As a corollary we find that the secondary polytope of $\mathscr{A}$ is a Minkowski summand of the state polytope of $\mathscr{I}_{\mathscr{A}}$.

In Section 4 these results are illustrated for the example where $X_{\mathscr{A}}$ is the *twisted cubic* in $\boldsymbol{P}^3$. Hence $\mathscr{A}$ consists of four equidistant points on the affine line, and the state polytope of $\mathscr{I}_{\mathscr{A}}$ is seen to be a planar octagon.

In Section 5 we focus our attention on *lexicographic* and *reverse lexicographic* Gröbner bases of the toric ideal $\mathscr{I}_{\mathscr{A}}$, and we show that these correspond to triangulations of $\mathscr{A}$ which are obtained by *placing* and *pulling* of vertices [12], [13] respectively.

Section 6 deals with the case where $\mathscr{A}$ is the vertex set of a product of simplices $\Delta_{r-1}\times\Delta_{s-1}$. Its toric variety $X_{\mathscr{A}}\subset\boldsymbol{P}^{rs-1}$ is the Segre embedding of $\boldsymbol{P}^{r-1}\times\boldsymbol{P}^{s-1}$, and its toric ideal $\mathscr{I}_{\mathscr{A}}$ is the ideal of 2×2-minors of a generic $r\times s$-matrix. Using results from algebraic combinatorics [4], [20], we prove that all initial ideals of $\mathscr{I}_{\mathscr{A}}$ are square-free. This implies that the state polytope of $\mathscr{I}_{\mathscr{A}}$ equals the secondary polytope $\Sigma(\mathscr{A})$. It is an open problem, suggested by Gel'fand, Kapranov and Zelevinsky [10], to find an explicit description for the face lattice of this polytope.

2. A singly-exponential degree bound. In this section we estimate the total degree of the polynomials in the reduced Gröbner bases of a projective toric variety. To begin with, we recall some results from Gröbner basis theory. For details and further references see [1], [7], [11], [15], [20], [21]. Fix a homogeneous ideal $\mathscr{I}$ in $\boldsymbol{C}[y]=\boldsymbol{C}[y_1,y_2,\cdots,y_n]$, and let "$\prec$" be a term order on $\boldsymbol{C}[y]$. The monomial ideal which is generated by the set $\{\mathrm{init}_{\prec}(f)\mid f\in\mathscr{I}\}$ of leading monomials is denoted $\mathrm{init}_{\prec}(\mathscr{I})$ and is called the *initial ideal* of $\mathscr{I}$ with respect to "$\prec$". A finite subset $\mathscr{G}\subset\mathscr{I}$ is a *Gröbner basis* of $\mathscr{I}$ (with respect to "$\prec$") provided $\mathrm{init}_{\prec}(\mathscr{I})$ is generated by $\{\mathrm{init}_{\prec}(g)\mid g\in\mathscr{G}\}$. It is known (see, e.g., [1], [11], [15]) that for monomials of bounded degree every term order "$\prec$" can be represented by a *weight vector* $\omega=(\omega_1,\cdots,\omega_n)\in\boldsymbol{R}^n$, where the *weight* of a monomial $y_1^{\alpha_1}\cdots y_n^{\alpha_n}$ equals $\omega_1\alpha_1+\cdots+\omega_n\alpha_n$. Two term orders ω and υ are *equivalent* if they define the same initial ideal $\mathrm{init}_\omega(\mathscr{I})=\mathrm{init}_\upsilon(\mathscr{I})$. The resulting equivalence classes

$$\mathscr{C}(\mathscr{I},\omega)=\{\upsilon\in\boldsymbol{R}^n\mid \mathrm{init}_\omega(\mathscr{I})=\mathrm{init}_\upsilon(\mathscr{I})\} \tag{2.1}$$

are open, convex, polyhedral cones in $\boldsymbol{R}^n$. The collection of cones $\{\mathscr{C}(\mathscr{I},\omega)\}$ is finite and defines a polyhedral fan $\mathscr{F}(\mathscr{I})$, called the *Gröbner fan* of $\mathscr{I}$ [1], [15].

PROPOSITION 2.1 (Bayer and Morrison [1]). *The Gröbner fan $\mathscr{F}(\mathscr{I})$ of a homogeneous ideal $\mathscr{I}\subset\boldsymbol{C}[y]$ is strongly polytopal. In other words, there exists a polytope $S_{\mathscr{I}}\subset\boldsymbol{R}^n$ whose normal fan equals $\mathscr{F}(\mathscr{I})$.*

Any lattice polytope $S_{\mathscr{I}}$ having the property of Proposition 2.1 will be called a *state polytope* for the ideal $\mathscr{I}$. It is noteworthy that the state polytope itself is derived from a projective toric variety, namely, from the closed $(\boldsymbol{C}^*)^n$-orbit of $\mathscr{I}$ in the Hilbert scheme. In fact, there is a canonical family of state polytopes of $\mathscr{I}$ coming from the state polytopes of the Hilbert points of $\mathscr{I}$ in each degree.

A polynomial of the form $y^\alpha-y^\beta$ (the difference of two monomials) is called a *binomial.* A *binomial ideal* is an ideal which is generated by binomials. It is known that the word problem for commutative semigroups can be solved by (and is in fact equivalent to) computing Gröbner bases for binomial ideals. Note that each step in Buchberger's Gröbner basis algorithm (forming an S-polynomial or normal form reduction) takes binomials to binomials.

LEMMA 2.2. *Let $\mathscr{F}$ be a set of binomials in $\boldsymbol{C}[y]$. Then the output and all intermediate polynomials in a Gröbner basis computation for $\mathscr{F}$ are also binomials.*

Although binomial ideals have a very special structure, they behave as bad as arbitrary ideals when it comes to computing Gröbner bases. The famous doubly-exponential degree lower bound of Mayr & Meyer [14] is attained for a family of binomial ideals. Since the toric ideal $\mathscr{I}_{\mathscr{A}}$ is binomial (cf. Lemma 2.5), it is natural to ask whether the doubly-exponential degree bound can be improved for the subclass of toric ideals. The affirmative answer is given by the following result.

THEOREM 2.3. *The total degree of a polynomial in any reduced Gröbner basis of $\mathscr{I}_{\mathscr{A}}$ is at most $n(n-d)A^d$, where A is the maximum of the Euclidean norms $|a_1|, \cdots, |a_n|$.*

It has been pointed out by M. Stillman (private communication) that Theorem 2.3 implies an upper bound of $n^2(n-d)A^d$ for the Castelnuovo regularity of the toric ideal $\mathscr{I}_{\mathscr{A}}$. An unpublished result of D. Mumford states that the Castelnuovo regularity of any smooth projective variety has order at most bilinear in the degree and dimension. For smooth projective toric varieties this implies the upper bound $O(dA^{d-1})$.

In order to prove Theorem 2.3, we first derive some basis facts about toric ideals. Consider the vector space of affine dependencies on $\mathscr{A}$,

$$D(\mathscr{A}) := \{(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \boldsymbol{R}^n \mid \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n = 0\}\,, \tag{2.2}$$

and let $D_{\boldsymbol{Z}}(\mathscr{A}) := D(\mathscr{A}) \cap \boldsymbol{Z}^n$ denote the $\boldsymbol{Z}$-module of integral affine dependencies.

OBSERVATION 2.4. *Let α, β be nonnegative vectors in $\boldsymbol{Z}^n$. Then the binomial $y^\alpha - y^\beta$ is contained in the toric ideal $\mathscr{I}_{\mathscr{A}}$ if and only if $\alpha - \beta \in D_{\boldsymbol{Z}}(\mathscr{A})$.*

Note that every vector $\alpha \in \boldsymbol{R}^n$ can be written uniquely as a difference $\alpha = \alpha_+ - \alpha_-$ of two nonnegaitive vectors α_+, α_- with disjoint support.

LEMMA 2.5. *The toric ideal $\mathscr{I}_{\mathscr{A}}$ is generated by $\{y^{\alpha_+} - y^{\alpha_-} \mid \alpha \in D_{\boldsymbol{Z}}(\mathscr{A})\}$.*

PROOF. It follows from Observation 2.4 that the ideal $\langle y^{\alpha_+} - y^{\alpha_-} \mid \alpha \in D_{\boldsymbol{Z}}(\mathscr{A})\rangle$ is contained in $\mathscr{I}_{\mathscr{A}}$. In order to show the reverse inclusion, we consider the binomial ideal

$$\mathscr{J}_{\mathscr{A}} := \langle y_1 - x^{a_1}, y_2 - x^{a_2}, \cdots, y_n - x^{a_n}\rangle \tag{2.3}$$

in the $(n+d)$-variate polynomial ring $\boldsymbol{C}[x, y]$. The toric ideal $\mathscr{I}_{\mathscr{A}} = \mathscr{J}_{\mathscr{A}} \cap \boldsymbol{C}[y]$ is obtained from $\mathscr{J}_{\mathscr{A}}$ by eliminating the x-variables. Thus the reduced Gröbner basis of $\mathscr{J}_{\mathscr{A}}$ with respect to the lexicographic term order $y_1 \prec y_2 \prec \cdots \prec y_n \prec x_1 \prec \cdots \prec x_d$ contains a reduced Gröbner basis for $\mathcal{I}_{\mathcal{A}}$. Let g be any element of that Gröbner basis. By Lemma 2.2, g is a binomial, say $g=y^{\alpha}-y^{\beta}$. By Observation 2.4, we have $\alpha-\beta\in D_{\mathbf{Z}}(\mathcal{A})$. In order to complete the proof, we need to show that the nonnegative integer vectors α and β have disjoint support. Suppose they did not. Then there exists a variable y_i which divides both y^{α} and y^{β}. Then g/y_i is also a polynomial contained in the toric ideal $\mathcal{I}_{\mathcal{A}}$. This is a contradiction to g being an element of a reduced Gröbner basis for $\mathcal{I}_{\mathcal{A}}$. ■

PROOF OF THEOREM 2.3. Write $e_1, \cdots, e_n$ for the standard basis in $\boldsymbol{R}^n$. An integral affine dependency $\lambda=\lambda_1 e_1+\cdots+\lambda_n e_n\in D_{\mathbf{Z}}(\mathcal{A})$ is called *elementary* if

(1) λ is non-zero,

(2) λ is primitive lattice point, i.e., g.c.d.$(\lambda_1, \lambda_2, \cdots, \lambda_n)=1$, and

(3) its support $\operatorname{supp}(\lambda):=\{i \mid \lambda_i\neq 0\}$ is minimal with respect to inclusion.

This the elementary affine dependencies are the scaled *elementary vectors* [17] of the $(n-d)$-dimensional linear subspace $D(\mathcal{A})$ of $\boldsymbol{R}^n$. Using Cramer's rule of linear algebra, we find that, for any given elementary affine dependency λ, there exist indices $i_1, i_2, \cdots, i_{d+1}$ and a multiplier $c\in\boldsymbol{Z}$ such that

$$c\cdot\lambda=\sum_{j=1}^{d+1}(-1)^j\cdot\det(a_{i_1}, \cdots, a_{i_{j-1}}, a_{i_{j+1}}, \cdots, a_{i_{d+1}})\cdot e_j\,. \tag{2.4}$$

By Hadamard's inequality, this implies

$$|\lambda_i|\leq|\det(a_{i_1}, \cdots, a_{i_{j-1}}, a_{i_{j+1}}, \cdots, a_{i_{d+1}})|\leq|a_{i_1}|\cdots|a_{i_{j-1}}||a_{i_{j+1}}|\cdots|a_{i_{d+1}}|\leq A^d\,. \tag{2.5}$$

Now suppose that $f(y)$ is an element of the reduced Gröbner basis of $\mathcal{I}_{\mathcal{A}}$ with respect to some term order on $\boldsymbol{C}[y]$. We need to show that the degree of $f(y)$ is less than $n(n-d)A^d$. Lemma 2.2 implies that $f(y)$ is a binomial, and, with the same argument as in the proof of Lemma 2.5, we find that $f(y)=y^{\alpha_+}-y^{\alpha_-}$ for some affine integral dependency $\alpha\in D_{\mathbf{Z}}(\mathcal{A})$.

Consider the set of elementary vectors in the $(n-d)$-dimensional vector space $D(\mathcal{A})$. It is known [17] that $\alpha\in D_{\mathbf{Z}}(\mathcal{A})\subset D(\mathcal{A})$ can be expressed as a conformal linear combination

$$\alpha=q_1\lambda^1+q_2\lambda^2+\cdots+q_{n-d}\lambda^{n-d} \tag{2.6}$$

of $n-d$ elementary integral affine dependencies $\lambda^1, \lambda^2, \cdots, \lambda^{n-d}$. The sum in (2.6) being *conformal* means that $q_1, \cdots, q_{n-d}$ are positive rational numbers and that no cancellations occur in any coordinate slot. Equivalently,

$$\alpha_+=q_1\lambda^1_+ +\cdots+q_{n-d}\lambda^{n-d}_+ \quad\text{and}\quad \alpha_-=q_1\lambda^1_- +\cdots+q_{n-d}\lambda^{n-d}_-\,. \tag{2.7}$$

We will next show that all coefficients $q_1, q_2, \cdots, q_{n-d}$ in the representation (2.6) must be less or equal to 1. Suppose on the contrary that $q_i>1$. Then the nonnegative integer vector α_+ is componentwise larger than the nonnegative integer vector λ^i_+, and similarly α_- is larger than λ^i_-. In other words, the monomial y^{α_+} is a proper multiple of the monomial $y^{\lambda^i_+}$, and y^{α_-} is a proper multiple of $y^{\lambda^i_-}$. Hence the binomial $f(y)=y^{\alpha_+}-y^{\alpha_-}$ can be reduced by $y^{\lambda^i_+}-y^{\lambda^i_-}\in\mathscr{I}_{\mathscr{A}}$. This is a contradiction to the assumption that $f(y)$ belongs to a reduced Gröbner basis of $\mathscr{I}_{\mathscr{A}}$.

We have shown that $0\le q_1, q_2, \cdots, q_{n-d}\le 1$. For the j-th coordinate of the vector equation (2.6) we find

$$(2.8)\quad |\alpha_j|\le q_1|\lambda^1_j|+q_2|\lambda^2_j|\cdots+q_{n-d}|\lambda^{n-d}_j|\le|\lambda^1_j|+|\lambda^2_j|\cdots+|\lambda^{n-d}_j|\le(n-d)A^d.$$

Hence the degree of $f(y)$ is bounded above by $n(n-d)A^d$. ■

This result yields an explicit finite *universal Gröbner basis* $\mathscr{U}_{\mathscr{A}}$ for the toric ideal $\mathscr{I}_{\mathscr{A}}$. Here "universal" means that $\mathscr{U}_{\mathscr{A}}$ is a Gröbner basis of $\mathscr{I}_{\mathscr{A}}$ for *all* term order on $\boldsymbol{C}[y]$ simultaneously [21]. Consider the $(n-d)$-dimensional zonotope $\mathscr{E}_{\mathscr{A}}:=[0,\lambda^1]+[0,\lambda^2]+\cdots+[0,\lambda^m]$ which is generated by the set $\{\lambda^1,\lambda^2,\cdots,\lambda^m\}$ of all elementary integral affine dependencies in $D_{\boldsymbol{Z}}(\mathscr{A})\subset\boldsymbol{R}^n$. We call $\mathscr{E}_{\mathscr{A}}$ the *elementary zonotope* of $\mathscr{A}$. It follows from our proof of Theorem 2.3 that the lattice points in the elementary zonotope define a universal Gröbner basis.

COROLLARY 2.6. *The set* $\mathscr{U}_{\mathscr{A}}:=\{y^{\alpha_+}-y^{\alpha_-}\mid\alpha\in\mathscr{E}_{\mathscr{A}}\cap\boldsymbol{Z}^n\}$ *is a universal Gröbner basis for* $\mathscr{I}_{\mathscr{A}}$.

It must be remarked that the universal Gröbner basis exhibited here is not minimal. An algorithm for computing a minimal universal Gröbner basis for *any* projective variety has been given by Bayer and Morrison [2].

Given two polytopes P_0 and P_1, we say that P_1 is a *Minkowski summand* of P_0 provided $P_0=\lambda P_1+P_2$ for some polytope P_2 and $\lambda\in\boldsymbol{R}_+$. Recall from [11] that the Newton polytope of a finite set of polynomials equals the Minkowski sum of the individual Newton polytopes. By [11, Corollary 3.23], the state polytope $S_{\mathscr{I}_{\mathscr{A}}}$ of the toric ideal is a Minkowski summand of the Newton polytope of the universal Gröbner basis $\mathscr{U}_{\mathscr{A}}$. These observations imply the following structure theorem for state polytopes of toric varieties.

PROPOSITION 2.7. *The state polytope $S_{\mathscr{I}_{\mathscr{A}}}$ of the toric ideal $\mathscr{I}_{\mathscr{A}}$ is a Minkowski summand of the derived zonotope* $\sum\{[\alpha_+,\alpha_-]\mid\alpha\in\mathscr{E}_{\mathscr{A}}\cap\boldsymbol{Z}^n\}$.

Proposition 2.7 follows from Corollary 2.6 because the Netwon polytope of the binomial $y^{\alpha_+}-y^{\alpha_-}$ is the line segment $[\alpha_+,\alpha_-]$.

3. Regular triangulations and initial ideals. In order to state our main result, we need to recall the definition of a regular triangulation. (For details see [5], [10], [13].) A *polyhedral subdivision* Δ of $\mathscr{A}$ is a polyhedral subdivision of the $(d-1)$-polytope $\mathrm{conv}(\mathscr{A})$ with vertices in $\mathscr{A}$. If all cells in Δ are simplices, then Δ is a *triangulation* of $\mathscr{A}$. Every vector $\omega=(\omega_1,\cdots,\omega_n)\in\boldsymbol{R}^n$ induces a subdivision Δ_ω of $\mathscr{A}$ as follows. Consider the convex d-polytope $P_\omega:=\mathrm{conv}\{(a_1,\omega_1),(a_2,\omega_2),\cdots,(a_n,\omega_n)\}$ which is obtained by lifting $\mathscr{A}$ according to the height vector ω. The lower envelope of P_ω is a polyhedral $(d-1)$-ball which maps bijectively onto $\operatorname{conv}(\mathscr{A})$. Let Δ_ω be the image under this projection.

If the height vector ω is chosen sufficiently generic, then Δ_ω is a triangulation of $\mathscr{A}$. A triangulation of $\mathscr{A}$ is called *regular* if it equals Δ_ω for some ω. For any regular triangulation Δ of $\mathscr{A}$, the set $\mathscr{C}(\mathscr{A}, \Delta) := \{\omega \in \boldsymbol{R}^n \mid \Delta_\omega = \Delta\}$ is a convex polyhedral cone. The cones $\mathscr{C}(\mathscr{A}, \Delta)$, where Δ ranges over all regular triangulations, are the maximal cells of a polyhedral fan $\mathscr{F}(\mathscr{A})$, called the *secondary fan* of $\mathscr{A}$. It was shown in [10] that there exists an $(n-d)$-polytope $\Sigma(\mathscr{A})$ in $\boldsymbol{R}^n$, called a *secondary polytope*, whose normal fan equals the secondary fan $\mathscr{F}(\mathscr{A})$. See also [5] for an alternative geometric construction.

The *Stanley-Reisner ideal* I_Δ of any simplicial complex Δ on $\{1, 2, \cdots, n\}$ is the ideal in $\boldsymbol{C}[y_1, y_2, \cdots, y_n]$ generated by all monomials of the form $y_{\tau_1} y_{\tau_2} \cdots y_{\tau_l}$ where $\{\tau_1, \tau_2, \cdots, \tau_l\}$ is not a face of Δ [18]. We establish the following correspondence between the regular triangulations of $\mathscr{A}$ and the distinct Gröbner bases of the toric ideal $\mathscr{I}_{\mathscr{A}}$.

THEOREM 3.1. *Let $\omega \in \boldsymbol{R}^n$ be a weight vector which defines a term order for the toric ideal $\mathscr{I}_{\mathscr{A}}$. Then the polyhedral subdivision $\Delta = \Delta_\omega$ is a regular triangulation of $\mathscr{A}$ whose Stanley-Reisner ideal I_Δ equals the radical of the initial ideal* $\operatorname{init}_\omega(\mathscr{I}_{\mathscr{A}})$.

We obtain the following two corollaries from Theorem 3.1.

COROLLARY 3.2. *The Gröbner fan $\mathscr{F}(\mathscr{I}_{\mathscr{A}})$ is a refinement of the secondary fan $\mathscr{F}(\mathscr{A})$.*

COROLLARY 3.3. *The secondary polytope $\Sigma(\mathscr{A})$ is a Minkowski summand of the state polytope $S_{\mathscr{I}_{\mathscr{A}}}$ of the toric ideal $\mathscr{I}_{\mathscr{A}}$.*

Corollary 3.2 follows directly from the definitions of the secondary fan and the Gröbner fan. Corollary 3.3 follows from Corollary 3.2 because the normal fan of a polytope P_1 refines the normal fan of another polytope P_2 if and only if P_2 is a Minkowski summand of P_1 [11, Lemma 2.1.5].

PROOF OF THEOREM 3.1. Let us first suppose that $\Delta = \Delta_\omega$ is a triangulation, and thus ω is in the interior of $\mathscr{C}(\mathscr{A}, \Delta)$. We will show that the ideal I_Δ is contained in $\operatorname{rad}(\operatorname{init}_\omega(\mathscr{I}_{\mathscr{A}}))$. Let $\tau = \{\tau_1, \tau_2, \cdots, \tau_l\}$ be a non-face of Δ, i.e., $y_{\tau_1} y_{\tau_2} \cdots y_{\tau_l} \in I_\Delta$. Then there exists a face $\sigma = \{\sigma_1, \sigma_2, \cdots, \sigma_k\}$ of Δ such that

$$\operatorname{relint} \operatorname{conv}\{a_{\sigma_1}, a_{\sigma_2}, \cdots, a_{\sigma_k}\} \cap \operatorname{relint} \operatorname{conv}\{a_{\tau_1}, a_{\tau_2}, \cdots, a_{\tau_l}\} \neq \emptyset.$$

Pick positive integers $\lambda_1, \lambda_2, \cdots, \lambda_k$ and $\mu_1, \mu_2, \cdots, \mu_l$ such that

$$\lambda_1 a_{\sigma_1} + \lambda_2 a_{\sigma_2} + \cdots + \lambda_k a_{\sigma_k} = \mu_1 a_{\tau_1} + \mu_2 a_{\tau_2} + \cdots + \mu_l a_{\tau_l}. \tag{3.1}$$

By Observation 2.4, the binomial $f(y) := y_{\tau_1}^{\mu_1} y_{\tau_2}^{\mu_2} \cdots y_{\tau_l}^{\mu_l} - y_{\sigma_1}^{\lambda_1} y_{\sigma_2}^{\lambda_2} \cdots y_{\sigma_k}^{\lambda_k}$ is contained in the toric ideal $\mathscr{I}_{\mathscr{A}}$. Since σ is a face while τ is not a face of Δ, we have the inequality

$\sum_{i=1}^{k} w_{\sigma_i}\lambda_i < \sum_{j=1}^{l} \omega_{\tau_j}\mu_j$. Therefore $\mathrm{init}_\omega(f) = y_{\tau_1}^{\mu_1} \cdots y_{\tau_l}^{\mu_l}$, and consequently $y_{\tau_1} y_{\tau_2} \cdots y_{\tau_l} \in \mathrm{rad}(\mathrm{init}_\omega(\mathscr{I}_{\mathscr{A}}))$.

Conversely suppose that $\mathrm{rad}(\mathrm{init}_\omega(\mathscr{I}_{\mathscr{A}}))$ is not contained in I_Δ. Then there exists a binomial $f(y) = y_{\tau_1}^{\mu_1} \cdots y_{\tau_l}^{\mu_l} - y_{\sigma_1}^{\lambda_1} \cdots y_{\sigma_k}^{\lambda_k}$ in the toric ideal $\mathscr{I}_{\mathscr{A}}$ such that τ is a face of Δ and $y_{\tau_1}^{\mu_1} \cdots y_{\tau_l}^{\mu_l}$ is the leading monomial of $f(y)$. This means that $\sum_{i=1}^{k} \omega_{\sigma_i}\lambda_i < \sum_{j=1}^{l} \omega_{\tau_j}\mu_j$, which is a contradiction to $\omega \in \mathscr{C}(\mathscr{A}, \Delta)$. Hence $\mathrm{rad}(\mathrm{init}_\omega(\mathscr{I}_{\mathscr{A}})) = I_\Delta$.

Suppose now that ω is contained in the common boundary of two cones $\mathscr{C}(\mathscr{A}, \Delta)$ and $\mathscr{C}(\mathscr{A}, \Delta')$. Then we can find disjoint faces σ of Δ and τ of Δ' which intersect in $\mathscr{A}$. Considering the binomial $f(y) \in \mathscr{I}_{\mathscr{A}}$ as above, this implies that the weight vector ω gives a tie between the two monomials of $f(y)$.

We finally suppose that $\omega \in \boldsymbol{R}^n$ is any vector which defines a term order for the toric ideal $\mathscr{I}_{\mathscr{A}}$. We have shown that ω lies in the interior of a unique cone $\mathscr{C}(\mathscr{A}, \Delta)$. This containment remains valid if ω is replaced by an equivalent term order, and therefore we have $\mathscr{C}(\mathscr{I}_{\mathscr{A}}, \omega) \subseteq \mathscr{C}(\mathscr{A}, \Delta)$. Clearly, $\Delta = \Delta_\omega$ is the regular triangulation induced by the height vector ω. We have seen that under this assumption, $\mathrm{rad}(\mathrm{init}_\omega(\mathscr{I}_{\mathscr{A}})) = I_\Delta$. This completes the proof of Theorem 3.1. ■

4. An Example: the twisted cubic in $\boldsymbol{P}^3$. In this section we illustrate our results by computing the Gröbner fan and state polytope of a toric curve in 3-space. Calculations of the state polytope of the twisted cubic in this and other coordinate systems were also presented by Bayer and Morrison at the 1988 Sundance Conference (see [2]).

The *twisted cubic* $X_{\mathscr{A}} \subset \boldsymbol{P}^3$ is the 1-dimensional projective toric variety defined by four equidistant points on the affine line, say, $\mathscr{A} = \{(0, 1), (1, 1), (2, 1), (3,1)\}$. Its vanishing ideal, the toric ideal $\mathscr{I}_{\mathscr{A}}$, is the kernel of the ring map

$$\boldsymbol{C}[a, b, c, d] \to \boldsymbol{C}[x_1, x_2], \qquad a \mapsto x_2, \quad b \mapsto x_1 x_2, \quad c \mapsto x_1^2 x_2, \quad d \mapsto x_1^3 x_2 .$$

(Here we use the variable names a, b, c, d rather than y_1, y_2, y_3, y_4.) In terms of generators, the ideal of the twisted cubic can be written as

$$\mathscr{I}_{\mathscr{A}} = \langle ac - b^2, ad - bc, bd - c^2 \rangle .$$

In the sequel we identify the variable a with the point (0, 1), b with (1, 1), c with (2, 1), and d with (3, 1).

The set $\mathscr{A}$ has four triangulations which are all regular. These triangulations are, in terms of their maximal simplices,

$$\Delta_1 = \{ab, bc, cd\}, \quad \Delta_2 = \{ac, cd\}, \quad \Delta_3 = \{ad\}, \quad \Delta_4 = \{ab, bd\} .$$

Using the methods of [5], [10], we see that the secondary polytope $\Sigma(\mathscr{A})$ is a quadrangle in $\boldsymbol{R}^4$ with vertices

$$\phi_{\Delta_1} = (1, 2, 2, 1), \quad \phi_{\Delta_2} = (2, 0, 3, 1), \quad \phi_{\Delta_3} = (3, 0, 0, 3), \quad \phi_{\Delta_4} = (1, 3, 0, 2) .$$

So, the maximal cells of the secondary fan $\mathscr{F}(\mathscr{A})$ are the cones of inner normals of $\Sigma(\mathscr{A})$:

$$\mathscr{C}(\mathscr{A}, \Delta_1)=\{(\omega_1, \omega_2, \omega_3, \omega_4)\in \boldsymbol{R}^4 : \omega_1-2\omega_2+\omega_3>0,\ \omega_2-2\omega_3+\omega_4>0\}$$

$$\mathscr{C}(\mathscr{A}, \Delta_2)=\{(\omega_1, \omega_2, \omega_3, \omega_4)\in \boldsymbol{R}^4 : \omega_1-3\omega_3+2\omega_4>0,\ -\omega_1+2\omega_2-\omega_3>0\}$$

$$\mathscr{C}(\mathscr{A}, \Delta_3)=\{(\omega_1, \omega_2, \omega_3, \omega_4)\in \boldsymbol{R}^4 : -2\omega_1+3\omega_2-\omega_4>0,\ -\omega_1+3\omega_3+2\omega_4>0\}$$

$$\mathscr{C}(\mathscr{A}, \Delta_4)=\{(\omega_1, \omega_2, \omega_3, \omega_4)\in \boldsymbol{R}^4 : -\omega_2+2\omega_3-\omega_4>0,\ 2\omega_1-3\omega_2+\omega_4>0\}\ .$$

It can be shown that the following polynomials form a universal Gröbner basis for $\mathscr{I}_{\mathscr{A}}$:

$$\mathscr{U}=\{ac-b^2,\ ad-bc,\ bd-c^2,\ ad^2-c^3,\ a^2d-b^3\}\ .$$

Up to equivalence, there are eight term orders for the ideal $\mathscr{I}_{\mathscr{A}}$. For each term order we list a representative weight vector $\omega\in\boldsymbol{R}^4$, its initial ideals $\mathrm{init}_\omega(\mathscr{I}_{\mathscr{A}})$, and its radical $\mathrm{rad}(\mathrm{init}_\omega(\mathscr{I}_{\mathscr{A}}))$.

(1) $\omega=(1, 0, 0, 1)$, $\mathrm{init}_\omega(\mathscr{I}_{\mathscr{A}})=\langle ac, ad, bd\rangle$, $\mathrm{rad}(\mathrm{init}_\omega(\mathscr{I}_{\mathscr{A}}))=\langle ac, ad, bd\rangle=I_{\Delta_1}$,
(2) $\omega=(1, 1, 0, 2)$, $\mathrm{init}_\omega(\mathscr{I}_{\mathscr{A}})=\langle ab, b^2, bd\rangle$, $\mathrm{rad}(\mathrm{init}_\omega(\mathscr{I}_{\mathscr{A}}))=\langle b, ad\rangle=I_{\Delta_2}$,
(3) $\omega=(0, 2, 0, 1)$, $\mathrm{init}_\omega(\mathscr{I}_{\mathscr{A}})=\langle ad^2, b^2, bc, bd\rangle$, $\mathrm{rad}(\mathrm{init}_\omega(\mathscr{I}_{\mathscr{A}}))=\langle b, ad\rangle=I_{\Delta_2}$,
(4) $\omega=(0, 2, 1, 1)$, $\mathrm{init}_\omega(\mathscr{I}_{\mathscr{A}})=\langle b^2, bc, bd, c^3\rangle$, $\mathrm{rad}(\mathrm{init}_\omega(\mathscr{I}_{\mathscr{A}}))=\langle b, c\rangle=I_{\Delta_3}$,
(5) $\omega=(0, 1, 1, 0)$, $\mathrm{init}_\omega(\mathscr{I}_{\mathscr{A}})=\langle b^2, bc, c^2\rangle$, $\mathrm{rad}(\mathrm{init}_\omega(\mathscr{I}_{\mathscr{A}}))=\langle b, c\rangle=I_{\Delta_3}$,
(6) $\omega=(1, 1, 2, 0)$, $\mathrm{init}_\omega(\mathscr{I}_{\mathscr{A}})=\langle ac, b^3, bc, c^2\rangle$, $\mathrm{rad}(\mathrm{init}_\omega(\mathscr{I}_{\mathscr{A}}))=\langle b, c\rangle=I_{\Delta_3}$,
(7) $\omega=(1, 0, 2, 0)$, $\mathrm{init}_\omega(\mathscr{I}_{\mathscr{A}})=\langle ac, a^2d, bc, c^2\rangle$, $\mathrm{rad}(\mathrm{init}_\omega(\mathscr{I}_{\mathscr{A}}))=\langle ad, c\rangle=I_{\Delta_4}$,
(8) $\omega=(2, 0, 1, 1)$, $\mathrm{init}_\omega(\mathscr{I}_{\mathscr{A}})=\langle ac, ad, c^2\rangle$, $\mathrm{rad}(\mathrm{init}_\omega(\mathscr{I}_{\mathscr{A}}))=\langle ad, c\rangle=I_{\Delta_4}$.

The Gröbner fan $\mathscr{F}(\mathscr{I}_{\mathscr{A}})$ has eight maximal cones and is a refinement of the secondary fan $\mathscr{C}(\mathscr{A})$. For instance, the cone $\mathscr{C}(\mathscr{A}, \Delta_2)$ consists of the two Gröbner regions given in (2) and (3). The quadrangle $\Sigma(\mathscr{A})$ is a Minkowski summand of the state polytope of $\mathscr{I}_{\mathscr{A}}$, an octagon with normal fan $\mathscr{F}(\mathscr{I}_{\mathscr{A}})$.

5. Lexicographic and reverse lexicographic term orders. In this section we show that the most prominent Gröbner basis term orders correspond to the most prominent triangulations. Fix a sign vector $\sigma=(\sigma_1, \cdots, \sigma_n)\in\{-, +\}^n$ and fix a permutation $\pi=(\pi_1, \pi_2, \cdots, \pi_n)$ of $\{1, 2, \cdots, n\}$. We define the *(σ, π)-lexicographic term order* on the polynomial ring $\boldsymbol{C}[y_1, \cdots, y_n]$ by saying that $y_{\pi_1}^{i_1}y_{\pi_2}^{i_2}\cdots y_{\pi_n}^{i_n}$ is larger than $y_{\pi_1}^{j_1}y_{\pi_2}^{j_2}\cdots y_{\pi_n}^{j_n}$ if, either $i_1+\cdots+i_n>j_1+\cdots+j_n$, or if $i_1+\cdots+i_n=j_1+\cdots+j_n$ and the first non-zero entry in the vector $(\sigma_1(i_1-j_1), \sigma_2(i_2-j_2), \cdots, \sigma_n(i_n-j_n))$ is positive. In the special case where $\sigma=(+, +, +, \cdots, +)$ we obtain the *degree lexicographic order* induced by $y_{\pi_1}\succ y_{\pi_2}\succ\cdots\succ y_{\pi_n}$, and in the case where $\sigma=(-, -, -, \cdots, -)$ we obtain the *reverse lexicographic order* induced by $y_{\pi_1}\prec y_{\pi_2}\prec\cdots\prec y_{\pi_n}$.

In the following we fix an affine set $\mathscr{A}\subset\boldsymbol{Z}^d$ and its toric ideal $\mathscr{I}_{\mathscr{A}}\subset\boldsymbol{C}[y_1, \cdots, y_n]$. Then the (σ, π)-lexicographic term order can be represented by the weight vector $\omega=\sum_{i=1}^n \sigma_i R^i\cdot e_{\pi_i}$ where $R\gg 0$ is a sufficiently large real number.

Lifting the affine set $\mathscr{A}$ according to the height vector ω corresponds to the *lexicographic extension* defined by the string $[\pi_1^{\sigma_1}, \pi_2^{\sigma_2}, \cdots, \pi_n^{\sigma_n}]$ in the *dual oriented* *matroid* to $\mathscr{A}$ (see [6]). (Note that dual oriented matroids correspond to *Gale transforms* of polytope theory [13] and that lexicographic extensions are called *principal* in [3].) The resulting triangulation $\Delta(\sigma, \pi) = \Delta_\omega$ of $\mathscr{A}$ depends only on the oriented matroid. We call $\Delta(\sigma, \pi)$ the *(σ, π)-lexicographic triangulation* of $\mathscr{A}$. The following is a direct consequence of Theorem 3.1.

PROPOSITION 5.1. *The radical of the initial ideal of $\mathscr{I}_{\mathscr{A}}$ with respect to the (σ, π)-lexicographic term order is the Stanley-Reisner ideal of the (σ, π)-lexicographic triangulation $\Delta(\sigma, \pi)$ of $\mathscr{A}$.*

A geometric description of the lexicographic triangulation $\Delta(\sigma, \pi)$ is given in [13, Section 2]. If we assume for simplicity that $\mathscr{A} = \{a_1, a_2, \cdots, a_n\}$ is in general position and that $\pi = (1, 2, \cdots, n)$, then this description specializes as follows:

(1) If $n = d$, then $\Delta(\sigma, \pi)$ consists just of the simplex $(a_1, a_2, \cdots, a_d)$.
(2) If $n > d$ and $\sigma_1 = -$, then $\Delta(\sigma, \pi)$ is the triangulation of $\mathscr{A}$ obtained by joining the point a_1 with every face of the simplicial polytope P.
(3) If $n > d$ and $\sigma_1 = +$, then $\Delta(\sigma, \pi)$ consists of the (inductively constructed) lexicographic triangulation $\Delta(\sigma \backslash \sigma_1, \pi \backslash \pi_1)$ of $\mathscr{A}' = \{a_2, \cdots, a_n\}$ plus all simplices $(a_1, a_{i_1}, \cdots, a_{i_k})$ such that $\{a_{i_1}, \cdots, a_{i_k}\}$ is a face of $\operatorname{conv}(\mathscr{A}')$ visible from a_1.

The operations (2) and (3) are known in polytope theory as *placing* and *pulling* of vertices. Of particular importance are the special cases where σ is all positive or all negative. If $\sigma = (+, +, +, \cdots, +)$ then $\Delta(\sigma, \pi)$ is the triangulation of $\mathscr{A}$ obtained by placing vertices in the order $a_n, a_{n-1}, \cdots, a_1$. If $\sigma = (-, -, -, \cdots, -)$, then $\Delta(\sigma, \pi)$ is the triangulation of $\mathscr{A}$ obtained by pulling vertices in the order $a_1, a_2, \cdots, a_n$. These canonical triangulations have been considered by many authors, including [3], [12], [19].

COROLLARY 5.2. *Under the correspondence of Theorem* 3.1, *the lexicographic Gröbner bases of the toric ideal $\mathscr{I}_{\mathscr{A}}$ correspond to triangulations by placing vertices, and the reverse lexicographic Gröbner bases correspond to triangulations by pulling vertices.*

We close this section with an example of a 2-dimensional toric variety in $\boldsymbol{P}^5$. All initial ideals have been computed using the computer algebra system MACAULAY.

EXAMPLE 5.3. Let $\mathscr{A} = \{a, b, c, d, e, f\} \subset \boldsymbol{Z}^5$ where $a = (0, 4, 1)$, $b = (7, 3, 1)$, $c = (4, 5, 1)$, $d = (0, 0, 1)$, $e = (3, 2, 1)$, $f = (4, 0, 1)$. Thus $\mathscr{A}$ is a pentagon in the affine plane with edges ad, df, fb, bc, ca and the point e in its center. As in Section 4, we identify each point with a variable via the ring map $\boldsymbol{C}[a, b, c, d, e, f] \to \boldsymbol{C}[x_1, x_2, x_3]$,

$$a \mapsto x_2^4 x_3, \quad b \mapsto x_1^7 x_2^3 x_3, \quad c \mapsto x_1^4 x_2^5 x_3, \quad d \mapsto x_3, \quad e \mapsto x_1^3 x_2^2 x_3, \quad f \mapsto x_1^4 x_3$$

whose kernel is the toric ideal $\mathscr{I}_{\mathscr{A}}$. We fix the permutation $\pi = (a, b, c, d, e, f)$.

First consider the sign vector $\sigma_1 = (+, +, +, +, +, +)$. The (σ_1, π)-lexicographic term order on $\boldsymbol{C}[a, b, c, d, e, f]$ is the degree lexicographic order induced by the variable ordering $a \succ b \succ c \succ d \succ e \succ f$. This term order is represented by the weight vector $\omega_1 = (R^5, R^4, R^3, R^2, R, 1)$ for $R \gg 0$ sufficiently large. The initial ideal $\mathrm{init}_{\omega_1}(\mathscr{I}_{\mathscr{A}})$ equals

$$\langle c^8 d^5 f^7, be^{11}, bc^3 d^3 f^2, b^2 de^2, b^3 cd^4, b^5 d^5, ae^4 f^4, ae^6 f, \\ ae^8, acf^3, ab, a^2 f^3, a^2 e^5 f, a^3 e^4 f \rangle .$$

Its radical, $\mathrm{rad}(\mathrm{init}_{\omega_1}(\mathscr{I}_{\mathscr{A}})) = \langle be, bd, ae, ab, af \rangle = I_{\Delta(\sigma_1, \pi)}$ is the Stanley-Reisner ideal of the triangulation $\Delta(\sigma_1, \pi) = \{acd, bcf, cef, cde, def\}$ by placing vertices in the order f, e, d, c, b, a.

The choice $R = 4$ is large enough for the (σ_1, π)-lexicographic triangulation, but it is *not* large enough for the (σ_1, π)-lexicographic term order. For $\omega_2 = (1024, 256, 64, 16, 4, 1)$ the initial ideal $\mathrm{init}_{\omega_2}(\mathscr{I}_{\mathscr{A}})$ becomes

$$\langle c^5 d^2 f^5, bc^3 d^3 f^2, b^2 de^2, b^3 e^{13}, b^3 cd^4, b^4 d^5, ae^6 f, ae^8, acf^3, ab, a^2 f^3, a^2 e^5 f, a^3 e^4 f \rangle$$

whose radical is still $I_{\Delta(\sigma_1, \pi)} = \langle be, bd, ae, ab, af \rangle$.

Note that a different choice of sign vector may still result in the same triangulation. For $\sigma_3 = (+, +, +, -, -, -)$ and $\omega_3 = (1024, 256, 64, -16, -4, -1)$ we get

$$\mathrm{init}_{\omega_3}(\mathscr{I}_{\mathscr{A}}) = \\ \langle c^5 d^2 f^5, bc^3 d^3 f^2, b^2 de^2, b^3 e^{13}, b^3 cd^4, b^5 d^5, ae^4 f^4, ae^6 f, ae^8, acf^3, ab, a^2 f^3, a^2 e^5 f, a^3 e^4 f \rangle$$

whose radical is still $I_{\Delta(\sigma_1, \pi)} = \langle be, bd, ae, ab, af \rangle$.

Let us now look at the reverse lexicographic term order induced by the variable ordering $a \prec b \prec \cdots \prec f$. This is the (σ_4, π)-lexicographic term order for $\sigma_4 = (-, -, -, -, -, -)$, and it is represented by the negated weight vector $\omega_4 := -\omega_1$. We compute

$$\mathrm{init}_{\omega_4}(\mathscr{I}_{\mathscr{A}}) = \langle b^2 c^2 d^4, bc^3 d^4, c^4 d^4, b^4 d^5, b^3 cd^5, ce, bde^3, de^4, e^9, e^7 f, c^4 d^3 f^2, c^2 f^3 \rangle .$$

Its radical, $\mathrm{rad}(\mathrm{init}_{\omega_4}(\mathscr{I}_{\mathscr{A}})) = \langle cd, bd, e, cf \rangle$, is the Stanley-Reisner ideal of the triangulation $\Delta(\sigma_4, \pi) = \{abc, abf, adf\}$ by pulling vertices in the order a, b, c, d, e, f.

We finally consider the (σ_5, π)-lexicographic term order given by the sign vector $\sigma_5 = (+, -, +, +, +, +)$. Here the radical of the initial ideal equals $\langle e, cf, af, ab \rangle$. This is the Stanley-Reisner ideal of the triangulation $\Delta(\sigma_5, \pi) = \{acd, bcd, bdf\}$, which is obtained by first placing a in $\mathscr{A}$ and afterwards pulling b in $\{b, c, d, e, f\}$.

6. On products of simplices. An important toric variety is the Segre embedding of the product of projective spaces $\boldsymbol{P}^{r-1} \times \boldsymbol{P}^{s-1}$ into $\boldsymbol{P}^{rs-1}$. Its set of weights is the point configuration $\mathscr{A} = \{e_i \oplus e'_j \in \boldsymbol{Z}^{r+s} \mid i = 1, \cdots, r; j = 1, \cdots, s\}$, where $e_1, \cdots, e_r$ denotes the standard basis of $\boldsymbol{R}^r$ and $e'_1, \cdots, e'_s$ denotes the standard basis of $\boldsymbol{R}^s$. Thus $\mathscr{A}$ consists of the vertices of the product $\Delta_{r-1} \times \Delta_{s-1}$ of a regular $(r-1)$-simplex with a regular $(s-1)$-simplex.

Let $\boldsymbol{C}[Y]$ be the polynomial ring on a generic $(r \times s)$-matrix $Y = (y_{ij})$. The toric ideal $\mathcal{I}_{\mathcal{A}}$ is the kernel of the ring map

$$\boldsymbol{C}[Y] \to \boldsymbol{C}[x_1, \cdots, x_r, x'_1, \cdots, x'_s], \qquad y_{ij} \mapsto x_i \cdot x'_j,$$

and it is generated by the 2×2-minors $y_{ik}y_{jl} - y_{il}y_{jk}$ of Y. For a combinatorial study of the determinantal ring

$$\boldsymbol{C}[Y]/\mathcal{I}_{\mathcal{A}} = \boldsymbol{C}[x_1x'_1, x_1x'_2, x_1x'_3, \cdots, x_rx'_s] = \boldsymbol{C}[\boldsymbol{P}^{r-1} \times \boldsymbol{P}^{s-1}]$$

we refer to [4] and [20]. In this section we prove the following result.

THEOREM 6.1. *For any term order ω on $\boldsymbol{C}[Y]$, $\operatorname{init}_\omega(\mathcal{I}_{\mathcal{A}})$ is a square-free monomial ideal.*

Using the results of Section 3, this implies a combinatorial characterization of the Gröbner fan and state polytope of the ideal $\mathcal{I}_{\mathcal{A}}$.

COROLLARY 6.2. *The state polytope $S_{\mathcal{I}_{\mathcal{A}}}$ of the 2×2-determinantal ideal $\mathcal{I}_{\mathcal{A}}$ equals the secondary polytope $\Sigma(\mathcal{A})$. Hence the distinct Gröbner bases of $\mathcal{I}_{\mathcal{A}}$ are in one-to-one correspondence with the regular triangulations of the product of simplices $\Delta_{r-1} \times \Delta_{s-1}$.*

Recall that the *f-vector* of a d-dimensional simplicial complex Δ is the vector $f(\Delta) = (f_0(\Delta), f_1(\Delta), \cdots, f_d(\Delta))$ where $f_i(\Delta)$ denotes the number of i-dimensional faces of Δ [18]. The key ingredient in our proof of Theorem 6.1 is the following result due to Billera, Cushman and Sanders [4, p. 387] (see also [19, Corollary 2.7]).

LEMMA 6.3. *All triangulations of $\Delta_{r-1} \times \Delta_{s-1}$ have the same f-vector.*

For the purposes of this section, we need the following reformulation of Lemma 6.3.

LEMMA 6.4. *Let Δ and $\tilde{\Delta}$ be any two triangulations of $\Delta_{r-1} \times \Delta_{s-1}$. Then their Stanley-Reisner ideals I_Δ and $I_{\tilde{\Delta}}$ have the same Hilbert function.*

Lemma 6.4 follows direcly from Lemma 6.3 because the Hilbert function of the Stanley-Reisner ideal of any simplicial complex can be read off from its f-vector (see [18]). We will now describe a specific regular triangulation of a product of simplices (see [4] and [9, page 67]). Let $\omega = (\omega_{11}, \omega_{12}, \cdots, \omega_{rs})$ where $\omega_{ij} = R^{i(s-j)}$ for some large $R \gg 0$. This height vector induces the *staircase triangulation* Δ_ω whose maximal cells are

$$\{ y_{i_1j_1}y_{i_2j_2}y_{i_3j_3}\cdots y_{i_{r+s-2}j_{r+s-2}}y_{i_{r+s-1}j_{r+s-1}} \mid (i_{k+1} = i_k + 1 \text{ and } j_{k+1} = j_k) \\ \text{or } (i_{k+1} = i_k \text{ and } j_{k+1} = j_k + 1)\} .$$

These simplices are indexed by the staircase paths through the matrix Y starting at $y_{11} = y_{i_1j_1}$ and ending at $y_{i_{r+s-1}j_{r+s-1}} = y_{rs}$. The following lemma is a special case of the results in [20].

LEMMA 6.5. *The weight vector ω defines a term order for $\mathcal{I}_{\mathcal{A}}$, and the set of 2×2-minors of Y is the reduced Gröbner basis with respect to ω. This implies*

$$\mathrm{init}_\omega(\mathscr{I}_\mathscr{A}) = \mathrm{rad}(\mathrm{init}_\omega(\mathscr{I}_\mathscr{A})) = I_{\Delta_\omega} = \langle \{ y_{ik} y_{jl} \mid i<j \text{ and } k>l \} \rangle .$$

PROOF OF THEOREM 6.1. Let $\tilde{\omega} \in \boldsymbol{R}^{rs}$ be any term order for $\mathscr{I}_\mathscr{A}$, and let $\tilde{\Delta}$ be the corresponding regular triangulation of $\Delta_{r-1} \times \Delta_{s-1}$. By Lemma 6.4, its Stanley-Reisner ideal $I_{\tilde{\Delta}}$ has the same Hilbert function as the ideal I_{Δ_ω} in Lemma 6.5. By a well-known result of Gröber basis theory, the homogeneous ideal $\mathscr{I}_\mathscr{A}$ and all its initial ideals have the same Hilbert function. This implies that $I_{\tilde{\Delta}}$ and $\mathrm{init}_{\tilde{\omega}}(\mathscr{I}_\mathscr{A})$ have the same Hilbert function. On the other hand, $I_{\tilde{\Delta}}$ equals the radical of $\mathrm{init}_{\tilde{\omega}}(\mathscr{I}_\mathscr{A})$ by Theorem 3.1. But an ideal can have the same Hilbert function as its radical only if it is equal to its radical. Hence

$$\mathrm{init}_{\tilde{\omega}}(\mathscr{I}_\mathscr{A}) = \mathrm{rad}(\mathrm{init}_{\tilde{\omega}}(\mathscr{I}_\mathscr{A}))$$

is square-free. This completes our proof of Theorem 6.1. ∎

It is an open problem due to Gel'fand, Kapranov & Zelevinsky [10] to find an explicit description of all triangulations of $\Delta_{r-1} \times \Delta_{s-1}$. As an illustration for this question, we give a lexicographic triangulation of $\Delta_2 \times \Delta_2$ which is not a staircase triangulation.

Let v denote the lexicographic term order on $\boldsymbol{C}[Y]$ induced by the variable ordering $y_{11} \succ y_{22} \succ y_{33} \succ y_{23} \succ y_{12} \succ y_{21} \succ y_{13} \succ y_{32} \succ y_{31}$. Then

$$\mathrm{init}_v(\mathscr{I}_\mathscr{A}) =$$
$$\langle y_{11}y_{22}, y_{11}y_{23}, y_{11}y_{32}, y_{11}y_{33}, y_{12}y_{23}y_{31}, y_{12}y_{33}, y_{13}y_{22}, y_{21}y_{33}, y_{22}y_{31}, y_{22}y_{33} \rangle .$$

The corresponding lexicographic triangulation equals

$$\Delta_v = \{ y_{11}y_{12}y_{13}y_{21}y_{31}, y_{12}y_{13}y_{21}y_{31}y_{32}, y_{12}y_{13}y_{21}y_{23}y_{32},$$
$$y_{12}y_{21}y_{22}y_{23}y_{32}, y_{13}y_{21}y_{23}y_{31}y_{32}, y_{13}y_{23}y_{31}y_{32}y_{33} \} .$$

This triangulation is not isomorphic to any staircase triangulation of $\Delta_2 \times \Delta_2$ because no vertex y_{ij} lies in all six 4-simplices. This example also shows that the 2×2-minors of a 3×3-matrix are not a universal Gröbner basis.

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