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ON SURFACES OF CLASS VII_0 WITH CURVES, II

Dedicated to Professor Friedrich Hirzebruch on his sixtieth birthday

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Introduction. This paper is a continuation of [14]. A compact complex surface is in class VII_0 if it is minimal and if its first Betti number b_1 is equal to one. We know many examples of surfaces of class VII_0 (VII_0 surfaces, for short) with the second Betti number b_2 positive [2]–[8], [15]–[18]. They are minimal surfaces with global spherical shells [7]. Any minimal surface with a global spherical shell is a VII_0 surface diffeomorphic to a blown-up primary Hopf surface and it is obtained as a smooth deformation of certain singular rational surfaces [7], [15], [16], [17]. Some of them have been characterized as VII_0 surfaces with certain kinds of curves on them [1], [13], [14]. For instance, a hyperbolic (or parabolic) Inoue surface is characterized as a VII_0 surface with a pair of cycles of rational curves (or a pair of a smooth elliptic curve and a cycle of rational curves). Any VII_0 surface with b_2 positive which we know so far has a global spherical shell and b_2 (possibly singular) rational curves, and a cycle of rational curves (possibly with branches). So it might not be too bold to pose the following conjecture:

CONJECTURE 1. *For an arbitrary* VII_0 *surface with* b_2 *positive the following three conditions are equivalent.*

(1) *It has a cycle of rational curves.*
(2) *It has at least* b_2 *rational curves.*
(3) *It contains a global spherical shell.*

The implications from (3) to the others and from (2) to (1) are known (see (3.4)). The implication from (2) to (3) was conjectured by Masahide Kato. When (2) is true, the surface is referred to as a *special* VII_0 surface. The main purpose of this article is to study special VII_0 surfaces and to give supporting evidences for the conjecture of Kato. This might be viewed as a step towards an affirmative solution of the conjecture of Kato. The consequences of this article were announced in [13, II]. See also [18].

The main consequences of this article are as follows: Let S be a VII_0 surface with a cycle C of rational curves. Then the deformation functor of S is unobstructed and the cycle C is deformed into a nonsingular elliptic curve in a suitable smooth family of deformations of the surface S. If a small deformation of S has a smooth elliptic curve which is an extension (a deformation) of the cycle C, it is isomorphic to either a blown-up parabolic Inoue surface or (generically) a blown-up primary Hopf surface. We see:

THEOREM. *Any* VII_0 *surface with a cycle of rational curves is a global analytic deformation of (hence is diffeomorphic to) a blown-up primary Hopf surface.*

If moreover S is special, that is, if S has at least b_2 (possibly singular) rational curves, then S has exactly b_2 rational curves and the weighted dual graph of the curves is completely determined. More precisely we show:

THEOREM. *Let S be a special* VII_0 *surface. Then the weighted dual graph of all the curves on S is the same as that of the dual graph of the maximal reduced curve on a minimal surface with a global spherical shell.*

The above theorems support Conjecture 1 and the conjecture of Kato. Except in some particular cases discussed in (2.1), S has a unique cycle of rational curves with nonempty branches, and the maximal reduced curve of S is connected. Thus the dual graph of curves is one of (3.8), (3.9), (4.2) and (4.11), which we call *global spherical graphs*. See also [4, pp. 144, 145], [15, (3.2)]. In view of these consequences, we are led to the following more precise conjectures.

CONJECTURE 2 (Existence). *Let Δ be an arbitrary global spherical graph with n vertices, U a strongly pseudoconvex open surface whose maximal curve has Δ as its weighted dual graph (or more precisely let U be a germ of a neighborhood of the maximal curve). Then there exists a minimal surface with b_2 equal to n containing a global spherical shell whose maximal curve ($=$the union of the n curves) has an open neighborhood isomorphic to U.*

CONJECTURE 3 (Uniqueness). *If two special* VII_0 *surfaces with equal positive b_2 are isomorphic to each other on sufficiently small neighborhoods of their maximal curves, then they are isomorphic globally. The local isomorphism near the maximal curves extends to a global one.*

Conjecture 2 will be discussed in a forthcoming article (part III in preparation).

This article is organized as follows: In Section 1, we recall some basic facts from [10] and [14] and verify two vanishing theorems for obstructions $H^2(S, \Theta_S)$ and $H^2(S, \Theta_S(-\log C))$, (1.2), (1.3). It follows from this that any cycle of rational curves on a VII_0 surface S can be deformed into a smooth elliptic curve by deforming S, (1.4), (1.5). This also proves the existence, unique up to permutation, of a sort of an "orthonormal" basis (referred to as a canonical basis) of $H^2(S, \mathbf{Z})$ which serves as a fundamental tool in subsequent study.

In Section 2, we study expressions of cohomology classes of rational curves on S in terms of the canonical basis of $H^2(S, \mathbf{Z})$.

In Sections 3–5, we study dual graphs of curves on a *special* VII_0 surface with b_2 positive. Then we see that S has exactly b_2 rational curves and at least a cycle of rational curves. We give a complete list of dual graphs of b_2 curves when S has a unique cycle of rational curves with at least a branch, see (3.8), (3.9), (4.2), (4.11). When a special

VII_0 surface S has a cycle C with no branches, then S is either a half Inoue surface (6.1), or $C^2=0$, or the surface S has another cycle of rational curves. See [1], [14, §6, (8.1), (10.3)] for the last two cases. See also (2.1) and (6.3) in this article. In Section 5, we construct minimal surfaces with global spherical shells so as to show that an arbitrary dual graph in the above list really appears on special VII_0 surfaces. See Figure 5.4 and (5.7)–(5.14).

In Section 6, we give a numerical characterization of Inoue surfaces with b_2 positive. More precisely, we see that a VII_0 surface S is isomorphic to an Inoue surface with b_2 positive if and only if the Dloussky number $\mathrm{Dl}(S)$ of S (roughly speaking, the sum of (-1) times the self-intersection numbers of all the curves on S) is equal to the possible maximum value $3b_2(S)$.

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NOTATION. We use the usual notation in analytic geometry or the same notation as in [14]. In addition to these, we use the following:

S, S^*, X, Y	compact complex surfaces.
$b_2(C)$	the number of irreducible components of a divisor C.
$c_1(D)$	the first Chern class in $H^2(S, \boldsymbol{Z})$ of $D \in H^1(S, \mathcal{O}_S^*)$.
$D \sim E$	$c_1(D)=c_1(E)$ for $D, E \in H^1(S, \mathcal{O}_S^*)$.
$[a, b]$	$:=\{k \in \boldsymbol{Z};\ a \leqq k \leqq b\}$.
A_i, B_i, I_i	see (2.9)–(2.10).
L_I, M_J	see (2.4), (4.1).
U_i	see (4.1), (4.6).
$E \# F$	see (5.7).

1. Smoothing a cycle of rational curves by deforming surfaces.

First we recall some basic facts from [10] and [14].

(1.1) LEMMA. *Let S be a VII_0 surface with $b_2>0$. Then*

(1.1.1) $h^q(S, \Omega_S^1)=0$ $(q=0, 2)$, $h^1(S, \Omega_S^1)=b_2$,

(1.1.2) $h^0(S, mK_S)=0$ *for* $m>0$,

(1.1.3) $K_S E \geqq 0$, $-E^2 \geqq 0$ *for an effective divisor E on S. Moreover $E^2=0$ if and only if $E=0$ in $H^2(S, \boldsymbol{R})$,*

(1.1.4) *S has no meromorphic functions except constants and $h^0(S, L) \leqq 1$ for any line bundle L on S.*

See [10, I, p. 755 & II, p. 683] or [14, (2.5), (12.1)] for the proofs.

(1.2) THEOREM. *Let S be a* VII_0 *surface with* $b_2>0$. *Then* $H^2(S, \Theta_S)=0$.

PROOF. Assume the contrary to derive a contradiction. By Serre duality, $H^0(S, \Omega_S^1(K_S))\neq 0$. Let D be the maximal effective divisor of S such that $H^0(S, \Omega_S^1(K_S-D))\neq 0$ and let ω be a nonzero element of $H^0(S, \Omega_S^1(K_S-D))$. By definition, zero($\omega$) is isolated.

(1.2.1) LEMMA. *The following is exact:*

$$0\to O_S(-K_S+D)\xrightarrow[f]{}\Omega_S^1\xrightarrow[g]{}O_S(2K_S-D)\,,$$

where $f(a)=a\omega$, $g(b)=b\wedge\omega$.

PROOF. Clearly f is injective. Take $b\in\operatorname{Ker} g$. Then $b\wedge\omega=0$. Hence $b=h\omega$ locally for a germ h of a meromorphic function. Then pole(h) is contained in isolated zero(ω), whence h is holomorphic. Hence b is contained in $\operatorname{Im} f$. q.e.d.

We continue the proof of (1.2). Let $H:=\operatorname{Coker} g$. Then supp($H$) consists of isolated points, so that $H^q(S, H)=0$ for any $q>0$. Therefore by taking the Euler-Poincaré characteristics, we see by (1.1)

$$\begin{aligned} b_2 &= -\chi(S, \Omega_S^1)=-\chi(S, -K_S+D)-\chi(S, 2K_S-D)+\chi(S, H) \\ &= -2\chi(S, -K_S+D)+h^0(S, H)=-2K_S^2+3K_SD-D^2+h^0(S, H)\,. \end{aligned}$$

Therefore by $-K_S^2=b_2$, we have, $b_2+3K_SD-D^2+h^0(S, H)=0$. By (1.1.3), we have $K_sD\geqq 0$, $-D^2\geqq 0$, so that $b_2=0$, $K_SD=D^2=h^0(S, H)=0$. This contradicts the assumption $b_2>0$. q.e.d.

(1.3) THEOREM. *Let S be a* VII_0 *surface with a cycle C of rational curves, and let E be a reduced effective divisor containing C. Then* $H^2(S, \Theta_S(-\log E))=0$.

PROOF. If S has another cycle or an elliptic curve, then the reduced effective maximal divisor D of S is anticanonical [14, (2.8)+(2.12)+(6.1)+(6.11)]. Hence $h^2(S, \Theta_S(-\log D))\leqq h^0(S, \Omega_S^1)=0$. From this, the assertion of (1.3) for general E follows immediately. So we may assume that S has a unique cycle C and no elliptic curves. We apply an argument similar to the proof of (1.2). We assume $H^2(S, \Theta_S(-\log E))\neq 0$ to derive a contradiction. By Serre duality, $H^0(S, \Omega_S^1(\log E)(K_S))\neq 0$. Let D be the maximal effective divisor of S such that $H^0(S, \Omega_S^1(\log E)(K_S-D))\neq 0$. Take $\omega\neq 0$ in $H^0(S, \Omega_S^1(\log E)(K_S-D))$. Then ω has isolated zeroes. As in (1.2) we have an exact sequence

$$0\to O_S(-K_S+D)\xrightarrow{f}\Omega_S^1(\log E)\xrightarrow{g}\Omega_S^2(E+K_S-D)\cong O_S(2K_S+E-D)\,,$$

where $f(a)=a\omega$, $g(b)=b\wedge\omega$. Let F (resp. H) be $\operatorname{Coker} f$ (resp. $\operatorname{Coker} g$). We see that supp(H) is finite so that $H^q(S, H)=0$ for $q>0$. We consider exact sequences

$$
\begin{aligned}
&0\to H^0(S, -K_S+D)\to H^0(S, \Omega_S^1(\log E))\to H^0(S, F)\\
(1.3.1)\qquad &\to H^1(S, -K_S+D)\to H^1(S, \Omega_S^1(\log E))\to H^1(S, F)\\
&\to H^2(S, -K_S+D)\to H^2(S, \Omega_S^1(\log E))\to H^2(S, F)\to 0
\end{aligned}
$$

$$
\begin{aligned}
&0\to H^0(S, F)\to H^0(S, 2K_S+E-D)\to H^0(S, H)\\
(1.3.2)\qquad &\to H^1(S, F)\to H^1(S, 2K_S+E-D)\to 0\\
&\to H^2(S, F)\to H^2(S, 2K_S+E-D)\to 0\,.
\end{aligned}
$$

(1.3.3) LEMMA. *$h^1(S, \Omega_S^1(\log E))=b_2-b_2(E)+\delta(C)$ where $\delta(C)$ equals 1 or 0 according as $C^2=0$ or $C^2<0$.*

PROOF OF (1.3.3). By [14, (3.3)] the following is exact:

$$0\to H^0(S, \Omega_S^1(\log E))\to H^0(\tilde{E}, O_{\tilde{E}})\to H^1(S, \Omega_S^1)\to H^1(S, \Omega_S^1(\log E))\to 0\,,$$

where $\tilde{E}$ is the normalization of E. Let $\delta(C)=h^0(S, \Omega_S^1(\log C))$. Then $h^0(S, \Omega_S^1(\log E))=\delta(C)$ where $\delta(C)=1$ or 0 according as $C^2=0$ or $C^2<0$ by [14, (3.3), (3.4)]. q.e.d.

Now we continue the proof of (1.3). We see $h^0(S, F)\leqq 1$, $h^2(S, -K_S+D)\leqq 1$ by (1.1) and (1.3.2). From (1.3.1)–(1.3.3) it follows that

$$
\begin{aligned}
b_2-b_2(E)+\delta(C)&\geqq h^1(S, -K_S+D)+h^1(S, F)-2\\
&\geqq h^1(S, -K_S+D)+h^1(S, 2K_S+E-D)+h^0(S, H)-3\\
&\geqq -\chi(S, -K_S+D)-\chi(S, 2K_S+E-D)-3\\
&\geqq 2b_2+3K_SD-3K_SE/2-D^2-E^2/2+DE-3\,,
\end{aligned}
$$

$$4\geqq b_2(E)+(K_SE+E^2)/2-(K_S+E-D/2)^2+2K_SD-3D^2/4$$

by $\delta(C)\leqq 1$. Therefore by (1.1), $4\geqq b_2(E)+(K_SE+E^2)/2$.

Let $E=C+H$, $H=\sum_\lambda H_\lambda$ with H_λ irreducible. Since $(K_S+C)\otimes O_S\cong O_C$ is the dualising sheaf of C, we have $(K_S+C)C=0$. Therefore

$$b_2(E)+(K_SE+E^2)/2=b_2(C)+CH+b_2(H)+(K_SH+H^2)/2=b_2(C)+CH+\sum_{\lambda<\nu} H_\lambda H_\nu\,.$$

Hence $4\geqq b_2(C)$. Take an unramified fivefold covering S^* of S. Then S^* is a VII_0 surface with a cycle C^* of rational curves, C^* being the pull-back of C. Moreover, $H^2(S^*, \Theta_{S^*}(-\log E^*))\neq 0$ for the pull-back E^* of E. Hence by the same argument as above we have $4\geqq b_2(C^*)$. However since $b_2(C^*)=5b_2(C)\geqq 5$, this is a contradiction.

q.e.d.

(1.4) THEOREM. *Let S be a VII_0 surface with a cycle C of rational curves, and let $E=C+H$ be a reduced divisor containing C. Then there is a smooth proper family $\pi:\mathscr{S}\to\Delta$ with π-flat divisors $\mathscr{C}$ and $\mathscr{H}$ of $\mathscr{S}$ such that*

(1.4.1) $(\mathscr{S}_0, \mathscr{C}_0, \mathscr{H}_0) \cong (S, C, H)$,

(1.4.2) $\mathscr{H}_t = H$ *for any* $t \in \Delta$,

(1.4.3) $\varpi(:=\pi_{|\mathscr{C}}): \mathscr{C} \to \Delta$ *is a versal deformation of* C.

PROOF. Let U be a strongly pseudoconverx open neighborhood of C in S. We prove that the canonical homomorphism

$$H^1(S, \Theta_S(-\log H)) \to H^1(U, \Theta_U) \tag{1.4.4}$$

is surjective. By [14, (4.3)], $H^1(U, \Theta_U) \cong H^1(C, \Theta_U \otimes O_C) \cong H^1(C, J_C)$, where $J_C = \Theta_S/\Theta_S(-\log C)$. Consider exact sequences

$$0 \to \Theta_S(-\log E) \to \Theta_S(-\log H) \to L \to 0\,, \tag{1.4.5}$$

$$0 \to L \to J_C \to J_C/L \to 0\,, \tag{1.4.6}$$

where $L := \Theta_S(-\log H)/\Theta_S(-\log E)$. It is clear that $\mathrm{supp}(J_C/L) = C \cap H$. Hence the homomorphism $H^1(C, L) \to H^1(C, J_C)$ is surjective. We have $H^2(S, \Theta_S(-\log E)) = 0$ by (1.3), whence $H^1(S, \Theta_S(-\log H)) \to H^1(C, L)$ is surjective. Hence (1.4.4) is surjective. This proves that the logarithmic deformation functor of (S, H) realizes any deformation of U near C (see [9], [12]). q.e.d.

From [14, (12.3) or (12.5)] and (1.4) we infer:

(1.5) THEOREM. *Let S be a* VII_0 *surface with a cycle C of rational curves. Then there is a smooth proper family $\pi: \mathscr{S} \to \Delta$ over a unit disc Δ with a π-flat Cartier divisor $\mathscr{C}$ such that*

(1.5.1) $(\mathscr{S}_0, \mathscr{C}_0) \cong (S, C)$,

(1.5.2) $\mathscr{S}_t$ *is a blown-up primary Hopf surface with a nonsingular elliptic curve* $\mathscr{C}_t$ $(t \neq 0)$.

PROOF. If $H_1(S, \boldsymbol{Z}) = i_* H_1(C, \boldsymbol{Z})$, then the same argument as in [14, (12.3)] applies because the assumption on the existence of E with $EC > 0$ is used only for showing $H_1(S, \boldsymbol{Z}) = i_* H_1(C, \boldsymbol{Z})$. So we are done in this case. If $H_1(S, \boldsymbol{Z}) \neq i_* H_1(C, \boldsymbol{Z})$, then S is isomorphic to a half Inoue surface by [14, (9.2)], whence the assertion is true as is well-known. See also (6.4). q.e.d.

(1.6) COROLLARY. *An arbitrary* VII_0 *surface with a cycle of rational curves is a global deformation of (hence diffeomorphic to) a blown-up primary Hopf surface.*

(1.7) COROLLARY. *Let S be a* VII_0 *surface with a cycle C of rational curves with $C^2 < 0$. Suppose that S is not a half Inoue surface. Then there exist complex line bundles L_j on S $(1 \leqq j \leqq n)$ such that*

(1.7.1) $E_j := c_1(L_j)$ $(1 \leqq j \leqq n)$ *is a* $\boldsymbol{Z}$*-basis of* $H^2(S, \boldsymbol{Z})$,

(1.7.2) $K_S L_j = -1$, $L_j L_k = -\delta_{jk}$, (1.7.3) $C=-(L_{r+1}+\cdots+L_n)$, $K_S=L_1+\cdots+L_n$ *in* $H^1(S, O_S^*)$ *for some* $1\leqq r\leqq n-1$, *where* $n=b_2(S)$.

PROOF. By (1.5), a general deformation $\mathscr{S}_t$ of S is a blown-up primary Hopf surface. Since S and $\mathscr{S}_t$ are diffeomorphic, by pulling back to S the total transforms of the j-th (-1)-curves on $\mathscr{S}_t$, we have a $\boldsymbol{Z}$-basis E_j $(1\leqq j\leqq n)$ of $H^2(S, \boldsymbol{Z})$ such that

(1.7.4) $$K_S E_j=-1\,,\quad E_j E_k=-\delta_{jk}\,,$$

(1.7.5) $$C=-(E_{r+1}+\cdots+E_n)\,,\quad K_S=E_1+\cdots+E_n\quad \text{in}\quad H^2(S, \boldsymbol{Z})\,.$$

Since S is not a half Inoue surface, we have $0<-C^2<n$ by [14, (9.3)]. Hence $0<r<n$. Therefore by choosing suitable line bundles L_j $(1\leqq j\leqq n)$ with $c_1(L_j)=E_j$, we have (1.7.2) and (1.7.3) q.e.d.

(1.8) REMARK. When S is a half Inoue surface, the assertion in (1.7) is true with (1.7.3) replaced by

(1.8.1) $$C=-(L_{r+1}+\cdots+L_n)+F_2\,,\quad K_S=L_1+\cdots+L_n\quad \text{in}\quad H^1(S, O_S^*)\,,$$

where F_2 is a line bundle of order two.

2. (Co)homology classes of curves.

(2.1) Let S be a VII_0 surface with a cycle C of rational curves. First we notice $b_2>0$. Indeed, if $b_2=0$, then there are only elliptic curves but no rational curves on S by [10, II, p. 699]. If S has an elliptic curve, then S is a parabolic Inoue surface by [14, (7.1)]. If S has a cycle D of rational curves distinct from C, then S is a hyperbolic Inoue surface by [14, (8.1)]. If $C^2=0$, then S is an exceptional compactification of an affine line bundle over an elliptic curve by [1]. If $b_2(C)=b_2$, and if $C^2<0$, then S is a half Inoue surface by [14, (9.2)]. If $C^2\leqq -b_2$, then $C^2=-b_2$ and S is a half Inoue surface by [14, (9.3)]. Under one of these additional assumptions, the structure of S is in any case completely known. In particular, they all contain a global spherical shell [7]. So we may exclude these cases in subsequent study. Summarizing these, we obtain (and make) the following:

(2.2) PROPOSITION–ASSUMPTION. *Let S be a* VII_0 *surface with a cycle C of rational curves. Assume that S is isomorphic to none of the above surfaces. Then,*

(2.2.1) *S has no curves of positive genus and no cycles of rational curves other than C,*

(2.2.2) $b_2>-C^2>0\,,\quad b_2>b_2(C)\,,$

(2.2.3) $H_1(S, \boldsymbol{Z})\cong H_1(C, \boldsymbol{Z})\cong \boldsymbol{Z}\,,\quad H^1(S, \boldsymbol{C}^*)\cong H^1(C, \boldsymbol{C}^*)\cong \boldsymbol{C}^*\,,$
where the isomorphisms are induced from the natural inclusion of C into S,

(2.2.4) *any unramified finite covering $\pi: S'\to S$ is cyclic and π^*C is a unique cycle of* *rational curves on S'.*

See [14, (2.13), (9.2)] for (2.2.3) and (2.2.4).

In what follows, we assume that (2.2.1)–(2.2.4) *are true.* See (2.13). Let $s=b_2(C)$, $n=b_2(S)$, and let r and L_i $(1\leqq i\leqq n)$ be the same as in (1.7). We note $0<r<n$ by (2.2.2).

(2.3) LEMMA. *Let L be a line bundle on S. Suppose that $L\sim a_1L_1+\cdots+a_nL_n$ and $a_1+\cdots+a_n>0$. Then $H^0(S, L)=0$.*

PROOF. Suppose $H^0(S, L)\neq 0$. Then there is an effective divisor D such that $[D]=L$. By (1.1) we have $K_SD\geqq 0$. By (1.7), we get $K_SD=K_SL=-(a_1+\cdots+a_n)<0$, which is absurd. q.e.d.

(2.4) LEMMA. *Let $L_I:=\sum_{i\in I}L_i$, $L=L_I+F$, $F\in H^1(S, \boldsymbol{C}^*)$ for a nonempty subset I of $[1, n]$. Then we have:*

(2.4.1) *If $I\neq[1, n]$, then $H^q(S, L)=0$ for any q.*

(2.4.2) *If $L\otimes O_C=O_C$, then $I=[1, r]$, and $F=O_S, K_S-L+C=O_S$.*

(2.4.3) *If $LC_i=0$ for any irreducible component C_i of C, then $I=[1, r]$.*

PROOF. $H^0(S, L)=0$ by (2.3). If $I\neq[1, n]$, then $h^2(S, L)=h^0(S, K_S-L)=0$ by (2.3). Hence by the Riemann-Roch theorem, $h^1(S, L)=-\chi(S, L)=-(-K_SL+L^2)/2=0$, whence (2.4.1). Suppose $L\otimes O_C=O_C$. Then $0=LC=L_IC=-\#(I\cap[r+1, n])$. Therefore I is a subset of $[1, r]$. Hence $H^q(S, L)=0$ for any q by (2.2.2) and (2.4.1). By the exact sequence

$$\begin{aligned}0&\to H^0(S, L-C)\to H^0(S, L)\to H^0(C, O_C)\\ &\to H^1(S, L-C)\to H^1(S, L)\to H^1(C, O_C)\\ &\to H^2(S, L-C)\to H^2(S, L)\to 0\,,\end{aligned}$$

we see $H^2(S, L-C)\cong H^1(C, O_C)\cong\boldsymbol{C}$. Since $K_S-L+C=L_{[1,r]}-L_I-F$, we have $K_S-L+C=O_S$, $I=[1, r]$, $F=O_S$ by (2.3). This proves (2.4.2). If $LC_i=0$ for any irreducible component C_i of C, then L_C is contained in $H^1(C, \boldsymbol{C}^*)$. By (2.2.3), there exists $G\in H^1(S, \boldsymbol{C}^*)$ such that $G_C=L_C$. Hence $(L-G)\otimes O_C=O_C$, whence $L-G=K_S+C$, $F=G$, $I=[1, r]$ by (2.4.2). This proves (2.4.3). q.e.d.

(2.5) LEMMA. *Let D be a nonsingular rational curve. Suppose $D\sim a_1L_1+\cdots+a_nL_n$. Then there exists a unique a_i such that $a_i=1$ or -2, $a_j=0$ or -1 for $j\neq i$.*

PROOF. By (1.7.2) we see $0=K_SD+D^2+2=2-\sum_{k=1}^n(a_k^2+a_k)$. Hence $a_i^2+a_i=2$ for a unique i, and $a_j^2+a_j=0$ for $j\neq i$. q.e.d.

(2.6) LEMMA. *Let D be a nonsingular rational curve which is not contained in the cycle C. Then $D\sim L_i-L_I$ for some $i\in[1, n]$ and $I\subset[1, n]$ with $i\notin I$.*

PROOF. Let $E=C+D$. Then by (1.4) there is a proper smooth family $\pi:\mathscr{S}\to\Delta$ over a unit disc Δ with Δ-flat divisors $\mathscr{C}$ and $\mathscr{D}$ such that $(\mathscr{S},\mathscr{C},\mathscr{D})_0=(S,C,D)$, $\mathscr{D}_t=D$ for any t, and $\mathscr{S}_t$ is a blown-up primary Hopf surface with a smooth elliptic curve $\mathscr{C}_t$ for $t\neq 0$. Since any primary Hopf surface has at most elliptic curves which are homologically trivial, $D=\mathscr{D}_t$ is a proper transform of a (-1)-curve by repeated blowing-ups. This implies that $D\sim L_i-L_I$ for some i and I with $i\notin I$. (Geometrically D is on $\mathscr{S}_t$ a proper transform of the i-th (-1)-curve on which the j-th blowing-ups ($j\in I$) are repeated.) q.e.d.

(2.7) LEMMA. *Let D_1 and D_2 be distinct nonsingular rational curves such that $D_j\sim L_{\alpha_j}-L_{I_j}$ ($\alpha_j\notin I_j$, $j=1,2$). Then $\alpha_1\neq\alpha_2$.*

PROOF. If $\alpha_1=\alpha_2$, then $D_1D_2=-1-\#(I_1\cap I_2)<0$, which is absurd. q.e.d.

(2.8) LEMMA. *Let S be a (not necessarily minimal) surface with $b_1=1$ having a cycle C of rational curves, and let C_i ($1\leqq i\leqq s$) be all the irreducible components of C. Suppose that $s\geqq 2$ and for some $2\leqq l\leqq r$,*

(2.8.1) $$C_i\sim L_i-L_{j(i)}-L_{A'_i}\ (1\leqq i\leqq l,\ j(i)\in[1,r],\ A'_i\subset[r+1,n]),$$

(2.8.2) $$C_i\sim\sum_{k=r+1}^{n}a_{ik}L_k\qquad(l+1\leqq i\leqq s),$$

where we do not require C_i to be in the cyclic order. Then we have $a_{ik}=\pm 1$.

PROOF. If $s=l$, then (2.8.2) is vacuous and any irreducible component C' of C is of the form (2.8.1) so that there is nothing to prove. We assume $s>l$. Suppose by (2.5) that there is an irreducible component C' of C of the form $-2L_i-L_I$ for some i and I, hence of the type (2.8.2). Since $s\geqq 3$, there is an irreducible component C'' of C with $C'C''=1$. Then C'' is not of the form $-2L_j-L_J$ because $(-2L_i-L_I)(-2L_j-L_J)\leqq 0$. Hence $C''\sim L_k-L_J$ for some k and J with $k\notin J$. We have,

$$1=C'C''=-2L_iL_k-L_kL_I+2L_iL_J+L_IL_J\,.$$

Therefore either $k\neq i$, $k\in I$, $i\notin J$, $I\cap J=\varnothing$ or $k=i$, $k\notin I$, $i\notin J$, $\#(I\cap J)=1$. In either case, C'' is of the form (2.8.2). In the first case, $C'+C''\sim -2L_i-L_{I\setminus\{k\}}-L_J$. In the second case, $C'+C''\sim -L_i-2L_j-L_{I\setminus\{j\}}-L_{J\setminus\{j\}}$, where $\{j\}=I\cap J$. In either case, $C^{(2)}:=C'+C''\sim -2L_a-L_A$ for some $a\in[r+1,n]$ and $A\subset[r+1,n]$ with $a\notin A$. If $s\geqq 4$, then there exists an irreducible component C''' of C different from C' and C'' with $C'''C^{(2)}=1$. Then by the same argument as above, C''' is of the form (2.8.2) and $C^{(3)}:=C^{(2)}+C'''\sim -2L_b-L_B$ for some $b\in[r+1,n]$ and $B\subset[r+1,n]$ with $b\notin B$. By repeating the same argument, we eventually obtain a straight chain $C^{(s-l)}$ of $s-l$ rational curves contained in C such that $C^{(s-l)}\sim -2L_e-L_E$ for some $e\in[r+1,n]$ and $E\subset[r+1,n]$ with $e\notin E$.

Since C is connected, $C^{(s-l)}$ meets one of C_i ($1\leqq i\leqq l$). However $C^{(s-l)}C_i\leqq 0$ by

(2.8.1). This is a contradiction. q.e.d.

(2.9) LEMMA. *Let S be a* VII_0 *surface with a cycle C of rational curves. Assume* $(K_S+C)^2 \leqq -2$ *and* $s:=b_2(C)\geqq 2$. *Let C_j be an irreducible component of C. Then there exist i and I such that* $C_j \sim L_i - L_I$.

PROOF. Let $L=K_S+C=L_{[1,r]}$. Then $L_C=(K_S+C)\otimes O_C$ is the dualizing sheaf of C so that L_C is trivial. Hence $LC_i=0$ for any irreducible component C_i of C. By (2.5), this shows that by suitable indexing for L_j,

$$C_i \sim L_i - L_{j(i)} - L_{A'_i} \quad (1\leqq i\leqq l, j(i)\in[1,r], A'_i\subset[r+1,n]), \tag{2.9.1}$$

$$C_i \sim \sum_{k=r+1}^{n} a_{ik}L_k \qquad (l+1\leqq i\leqq s), \tag{2.9.2}$$

where $l\leqq r$. Since $C=-L_{[r+1,n]}$, we see that $\sum_{i=1}^{l} C_i$ is a linear combination of L_j $(r+1\leqq j\leqq n)$. Hence $l=0$ or $l\geqq 2$. If $l\geqq 2$, then (2.9) follows from (2.8). So suppose $l=0$ to derive a contradiction. If $l=0$, then $L_1C_i=0$ for any i by $r>0$, hence $K_S-L_1+C\sim 0$ by (2.4.3). This implies that $(K_S+C)^2=-1$, which contradicts the assumption. q.e.d.

NOTATION. By (2.9), we write $C_i\sim L_i-L_{A_i}$ for a subset A_i of $[1,n]\setminus\{i\}$. Let $A'_i=A_i\cap[r+1,n]$, $B_i=A_i\cap[r+1,s]$, and $I_i=A_i\cap[s+1,n]$.

(2.10) LEMMA. *Suppose* $(K_S+C)^2\leqq -2$ *and* $s:=b_2(C)\geqq 2$. *Then* $s\geqq r\geqq 2$. *By suitable indexing for* L_j $(L_r=L_0)$, *we have*

$$C_i\sim L_i-L_{i-1}-L_{B_i}-L_{I_i} \qquad (1\leqq i\leqq r) \tag{2.10.1}$$

$$C_i\sim L_i-L_{B_i}-L_{I_i} \qquad (r+1\leqq i\leqq s) \tag{2.10.2}$$

$$I_i\cap I_k=\varnothing \quad \textit{for} \quad i\neq k \quad \textit{and} \quad I_1\cup\cdots\cup I_s=[s+1,n]\,. \tag{2.10.3}$$

PROOF. We use the same notation as in the proof of (2.9). We have $l\geqq 2$ by the proof of (2.9). Since $C_1+\cdots+C_l$ has no terms of L_i $(1\leqq i\leqq r)$, the set $[1,l]$ is decomposed into a disjoint union of $R_1,\cdots,R_N$ such that

$$j(R_k)=R_k \qquad (1\leqq k\leqq N) \tag{2.10.4}$$

(2.10.5) no R_k is decomposed into proper subsets with the property (2.10.4) where $j(R_k)=\{j(i);\ i\in R_k\}$ and $j(i)$ is defined in (2.9.1).

By suitable indexing for L_j, we may assume $R_1=[1,r']$ $(r'\leqq l\leqq r)$. Let $L'=L_{R_1}$. Then $L'C_i=0$ for any $i\in[1,s]$ by (2.9.1), (2.9.2) and (2.10.4). Hence by (2.4.3), we get $R_1=[1,r]$, $r=r'=l\geqq 2$, $N=1$. This shows that we may assume $j(i)=i-1$ $(1\leqq i\leqq r)$ by suitable indexing for L_j, where we may view $j(1)=0=r$. Hence in particular, $s\geqq r$. The assertions (2.10.1) and (2.10.2) are thus proved. (2.10.3) is clear from (1.7). q.e.d.

(2.11) LEMMA. *Let the assumptions and notation be the same as in* (2.10). *Assume*

$s>r$. Then there is $j\in[r+1, s]$ with $B_j=\varnothing$.

PROOF. By (2.10), $r\geqq 2$. First we show the following:

(2.11.1) SUBLEMMA. *There exist $j\in[r+1, s]$, and $i, k\in[1, s]$ such that $C_iC_j=C_jC_k=1$, $C_jC_l=0$ $(l\neq i, j, k)$, and that $B_i\cap B_k$ contains j.*

PROOF OF (2.11.1). Since $s>r$, there is a connected subcurve $C''=C_{j_1}+\cdots+C_{j_m}$ of C such that $j_k\in[r+1, s]$ and that $C''(C_1+\cdots+C_r)=2$. We may assume $C_{j_k}C_{j_{k+1}}=1$ $(1\leqq k\leqq m-1)$, and $C_{j_i}C_{j_k}=0$ (otherwise). Hence there exist i_1 and i_2 in $[1, r]$ such that $C_{j_1}C_{i_1}=C_{j_m}C_{i_2}=1$, $C_{j_k}C_i=0$ (otherwise). Let $C_j\sim L_j-L_{A_j}$ $(1\leqq j\leqq s)$. Since $A_{j_1}\cap[1, r]=\varnothing$ by (2.10.2), $C_{j_1}C_{i_1}=1$ implies that $A_{i_1}\ni j_1$, $A_{i_1}\cap A_{j_1}=\varnothing$.

We now prove that there exists j_α $(1\leqq\alpha\leqq m)$ such that both $A_{j_{\alpha-1}}$ and $A_{j_{\alpha+1}}$ contain j_α, where $j_0=i_1$ and $j_{m+1}=i_2$. If A_{j_2} contains j_1, then we can take $\alpha=1$. If $A_{j_2}\not\ni j_1$, then $A_{j_1}\ni j_2$ by $C_{j_1}C_{j_2}=1$. By repeating this argument, we either obtain α $(\leqq m-1)$ such that $A_{j_{k-1}}\ni j_k$ $(1\leqq k\leqq\alpha)$ and $A_{j_{\alpha+1}}\ni j_\alpha$, or we have $A_{j_k}\ni j_{k+1}$, $A_{j_{k+1}}\not\ni j_k$ for any k $(\leqq m-1)$. In the second case, since $C_{j_m}C_{i_2}=1$, we have either $A_{j_m}\ni i_2$ or $A_{i_2}\ni j_m$. Since $A_{j_m}\cap[1, r]=\varnothing$ by (2.10.2), we have $A_{i_2}\ni j_m$. Hence we may take $\alpha=m$. (2.11.1) is proved by taking $i=j_{\alpha-1}$, $j=j_\alpha$ and $k=j_{\alpha+1}$.

The following is easy to see:

(2.11.2) SUBLEMMA. (i) *$\lambda\in[r+1, s]$ if and only if $\lambda\in A_i$ for exactly two i $(1\leqq i\leqq s)$.*

(ii) *$\lambda\notin[r+1, s]$ if and only if $\lambda\in A_i$ for a unique i $(1\leqq i\leqq s)$.*

Next we prove:

(2.11.3) SUBLEMMA. *Let i, j, k be the same as in* (2.11.1). *Suppose $B_j\neq\varnothing$. Then there exist $l_1, \cdots, l_m$ in B_j such that*

$$A_j\cap A_{l_h}=\{l_{h+1}\}\,, \quad C_{l_h}\sim L_{l_h}-L_{l_{h+1}}-L_{A'_{l_h}} \qquad (1\leqq h\leqq m)\,,$$

where $l_{m+1}=l_1$, $A'_{l_h}=A_{l_h}\setminus\{l_{h+1}\}$, $l_h\neq i, j, k$, and $A_{l_h}\cap[1, r]=\varnothing$.

PROOF OF (2.11.3). Take l_1 from $A_j\cap[r+1, s]$. Then $l_1\neq i, k$. Indeed, if $l_1=i$, then B_i contains j by (2.11.1), and $\#(B_i\cap B_j)=1$ by $C_iC_j=1$, hence $C_i+C_j\sim -2L_v-L_J$ for some v and J. By deforming S, we obtain a cycle C^*, one of whose irreducible component is $C_i+C_j\sim -2L_v-L_J$ homologically. This is absurd by (2.8). If $l_1=k$, then by the same argument we derive a contradiction. Hence $l_1\neq i$, k, and $B_j\not\ni i, k$.

Hence $C_jC_{l_1}=0$, so that $-L_{A_j}L_{l_1}-L_{A_{l_1}}L_j+L_{A_j}L_{A_{l_1}}=0$. By (2.11.2) and by $j\in A_i\cap A_k$, we have $A_{l_1}\not\ni j$. Hence $\{l_2\}=A_j\cap A_{l_1}$ for some $l_2\in[r+1, s]$, since $A_{l_1}\cap[1, r]=\varnothing$ by (2.10.2). Clearly $l_2\neq l_1, i, k$. We note $j\notin A_{l_2}$ by (2.11.2). By $C_jC_{l_2}=0$, we have $\#(A_j\cap A_{l_2})=1$. Hence $A_j\cap A_{l_2}=\{l_1\}$ or $\{l_3\}$, where $l_3\neq l_1, l_2, i, k$. Repeating this argument, we eventually obtain $l_1, \cdots, l_m\in A_j\cap[r+1, s]$ such that (2.11.3) holds.

q.e.d.

(2.11.4) SUBLEMMA. *Suppose* $B_j \neq \varnothing$. *Let* C_λ *be an irreducible component of* C *with* $C_\lambda C_{l_h} = 1$. *Then* $\lambda = l_{h-1}$ *or* $\lambda \in B_{l_h}$.

PROOF. By the proof of (2.11.3), we have $B_j \not\ni i, k$ and $l_h \neq i, k$. Thus $C_j C_{l_h} = 0$, whence $\lambda \neq j$. Supose $\lambda \neq l_{h-1}$. Then since $l_h \in A_j \cap A_{l_{h-1}}$, we have $l_h \notin A_\lambda$ by (2.11.2) so that

$$1 = C_\lambda C_{l_h} = (L_\lambda - L_{A_\lambda})(L_{l_h} - L_{A_{l_h}}) = -L_\lambda L_{A_{l_h}} + L_{A_\lambda} L_{A_{l_h}}.$$

Thus $\lambda \in A_{l_h} \cap [r+1, s] (= B_{l_h})$ and $A_\lambda \cap A_{l_h} = \varnothing$ by $A_{l_h} \cap [1, r] = \varnothing$. q.e.d.

(2.11.5) SUBLEMMA. *Let* j *be the same as in* (2.11.1). *Then* $B_j = \varnothing$.

PROOF. We prove this by induction on $\delta(S) := s - r = s(S) - r(S) = b_2(C) + (K_S + C)^2$. If $s = r+1$, then $B_s = \varnothing$ by (2.10) and $s \notin B_s$. Next we assume that $s > r+1$ and that (2.11.5) is true for $\delta(S) \leqq s - r - 1$. Assume $B_j \neq \varnothing$ to derive a contradiction. By (2.11.3), we choose l_h. Suppose $C_\lambda C_{l_1} = 1$. By (2.11.4), we may assume $\lambda = l_m$ or $\lambda \in B_{l_1}$. If $\lambda = l_m$, then $C_{l_1} + C_\lambda \sim L_{l_m} - L_{l_2} - L_\Lambda$ for some Λ. If $\lambda \in B_{l_1}$, then $C_{l_1} + C_\lambda \sim L_{l_1} - L_{l_2} - L_{\Lambda'}$ for some Λ'. By deforming S suitably, we have a (not necessarily minimal) surface $\mathcal{S}_t$ with $b_1(\mathcal{S}_t) = 1$ and a cycle $\mathcal{C}_t = \mathcal{C}'_t + (C - C_{l_1} - C_\lambda)$, $\mathcal{C}'_t$ being a nonsingular rational curve for $t \neq 0$, $\mathcal{C}'_0 = C_{l_1} + C_\lambda$. In this situation, C_h $(1 \leqq h \leqq r)$ survives on $\mathcal{S}_t$. However if $\lambda = l_m$ (resp. $\lambda \in B_{l_1}$), then there is no irreducible component of $\mathcal{C}_t$ homologically equivalent to $L_{l_1} - L_{\Lambda'}$ (resp. $L_\lambda - L_{\Lambda'}$) for any Λ'. This implies that the index sets B_j and $[1, s]$ are changed into $B_j \setminus \{l_1\}$ and $[1, s] \setminus \{l_1\}$ (resp. $B_j \setminus \{\lambda\}$) and $[1, s] \setminus \{\lambda\}$) on $\mathcal{S}_t$. If $\lambda \neq i, k$ (resp. $\lambda = i$), then $C_j C_i = C_j C_k = 1$, $C_j \mathcal{C}'_t = C_j C_l = 0$ (resp. $C_j \mathcal{C}'_t = C_j C_k = 1$, $C_j C_l = 0$) on $\mathcal{S}_t$ for $l \neq i, k, \lambda, l_1$, and the condition in (2.11.1) for $\mathcal{S}_t$ is satisfied. The case $\lambda = k$ is similar. We note $\delta(\mathcal{S}_t) = s - r - 1 < \delta(S)$. By the induction hypothesis on $\mathcal{S}_t$, either $B_j \setminus \{l_1\}$ or $B_j \setminus \{\lambda\}$ is empty. This implies $\#(B_j) = 1$, which contradicts (2.11.3). q.e.d.

(2.12) LEMMA. *Let* D_j *be irreducible curves not contained in* C.

(2.12.1) *Suppose* $D_j \sim L_j - L_{k_j}$ $(1 \leqq j \leqq l, 1 \leqq k_j \leqq m)$. *Then* $l \leqq m-1$. *If* $l = m-1$, *then* $D_{j-1} D_j = 1$ $(2 \leqq j \leqq m-1)$, *and* $D_i D_j = -2\delta_{ij}$ $(i \neq j \pm 1)$ *by suitable indexing, and moreover either* $D_1 \sim L_1 - L_m$ *and* $D_j \sim L_j - L_{j-1}$ $(2 \leqq j \leqq m-1)$ *or* $D_j \sim L_j - L_{j+1}$ $(1 \leqq j \leqq m-1)$.

(2.12.2) *Supposse* $D_j \sim L_j - L_{k_j}$ $(2 \leqq j \leqq l, 1 \leqq k_j \leqq m)$. *Then* $l \leqq m+1$. *If* $l = m$ *or* $m+1$, *then* $D_{j-1} D_j = 1$ $(2 \leqq j \leqq l)$, $D_i D_j = -2\delta_{ij}$ $(i \neq j \pm 1)$ *by suitable indexing. If* $l = m+1$, *then* $D_j \sim L_j - L_{j-1}$ $(2 \leqq j \leqq l)$. *If* $l = m$, *then either* $D_j \sim L_j - L_{j-1}$ $(2 \leqq j \leqq l)$ *or* $D_j \sim L_j - L_{j+1}$ $(2 \leqq j \leqq l-1)$ *and* $D_l \sim L_l - L_1$.

Note that each of the two cases in (2.12.1) $l = m-1$ as well as those in (2.11.2) $l = m$ is reduced to the other by suitable indexing for L_j.

PROOF OF (2.12.1). If there is a pair of i and j such that $k_i = k_j$, then $D_i D_j < 0$, which is absurd. Hence $k_i \neq k_j$ for $i \neq j$ so that $l \leqq m$. If $l = m$ and $k_i \neq k_j$ for $i \neq j$, then

$D_1+\cdots+D_l\sim 0$ which contradicts (2.2) by [14, (2.10)].

Assume $l=m-1$. Let $\{k_1,\cdots,k_{m-1}\}=[1,m]\setminus\{a\}$. Then $D_1+\cdots+D_l\sim L_a-L_m$ whence $m\neq a$. We have

$$-2=(D_1+\cdots+D_l)^2=-2(m-1)+2\sum_{i<j}D_iD_j\,.$$

Hence $\sum_{i<j}D_iD_j=m-2$. This shows that $D_1+\cdots+D_l$ is connected by (2.2.1). Take mutually distinct $j,\lambda,\nu,\mu\in[1,l]$. Then $D_\lambda+D_\nu+D_\mu\sim L_I-L_J$, where $\#I=\#J\leqq 3$, $I\cap J=\varnothing$. Hence

$$D_j(D_\lambda+D_\nu+D_\mu)=(L_j-L_{k_j})(L_I-L_J)\leqq -L_jL_J-L_{k_j}L_I\leqq 2\,.$$

This shows that $D_1+\cdots+D_l$ is a straight chain, that is, $D_{j-1}D_j=1$ $(2\leqq j\leqq l)$, $D_iD_j=-2\delta_{ij}$ (otherwise) by suitable indexing. Then if $1<a<m-1$, then we have $k_a=a-1$ by $D_{a-1}D_a=1$, while $k_a=a+1$ by $D_aD_{a+1}=1$. This is absurd. Consequently $a=1$ or $m-1$. The rest is clear.

PROOF OF (2.12.2). By the same argument as above, $k_i\neq k_j$ for $i\neq j$, and $l\leqq m+1$. If $l=m+1$ or m, then $\sum_{i<j}D_iD_j=l-2$. It follows that $D_2+\cdots+D_l$ is a connected straight chain of (-2)-curves, that is, $D_{j-1}D_j=1$ $(2\leqq j\leqq l)$, $D_iD_j=-2\delta_{ij}$ (otherwise) by suitable indexing. One sees readily that if $l=m+1$, then $D_j\sim L_j-L_{j-1}$ $(2\leqq j\leqq l)$. If $l=m$, then either $D_j\sim L_j-L_{j-1}$ $(2\leqq j\leqq l)$ or $D_j\sim L_j-L_{j+1}$ $(2\leqq j\leqq l-1)$ and $D_l\sim L_l-L_1$. q.e.d.

(2.13) DEFINITION. A reduced connected divisor D is called a *branch* of the cycle C if $CD=1$ and if D has no components common with C.

In the rest of this section, we consider the case where C has at least a branch, for instance, and a nonsingular rational curve D with $CD=1$. *If a* VII_0 *surface has a cycle of rational curves with branches, then it satisfies the conditions* (2.2.1)–(2.2.4).

(2.14) LEMMA. *Let S be a* VII_0 *surface with a rational curve C with a node. Suppose that there is a nonsingular rational curve D with $CD=1$. Then by indexing L_j suitably, we have*

$$C=-(L_2+\cdots+L_n)\,,\quad D\sim L_2-L_1\,.$$

PROOF. By (1.7) we may assume

$$C=-(L_{r+1}+\cdots+L_n)\,,\quad K_S=L_1+\cdots+L_n\,,\quad 1\leqq r\leqq n-1\,.$$

Assume that $D\sim a_1L_1+\cdots+a_nL_n$ with $CD=1$. Then we have $1=CD=a_{r+1}+\cdots+a_n$. By (2.5), $(a_{r+1},\cdots,a_n)=(1,0,\cdots,0)$ up to permutation. Since $D^2\leqq -2$, there is a nonzero a_j $(1\leqq j\leqq r)$. We may assume $a_1\neq 0$. Since $L_1C=0$, we get $r=1$ by (2.4.3). Thus $D\sim L_i-L_1$ $(2\leqq i\leqq n)$. q.e.d.

(2.15) LEMMA. *Let S be a* VII_0 *surface with a cycle $C=C_1+C_2$ of two rational* *curves. If* $(K_S+C)^2 \leqq -2$ *and if there is a nonsingular rational curve* D *with* $C_1D=0$, $C_2D=1$, *then by indexing* L_j *suitably. we have*

$$C=-(L_3+\cdots+L_n)\,, \quad C_1\sim L_1-L_2-L_I\,, \quad C_2\sim L_2-L_1-L_J\,,$$

as well as either $D\sim L_i-L_2$ *and* $i\in I$, *or* $D\sim L_i-L_1-L_2$ *and* $i\in J$, *where* $I\cap J=\varnothing$, $I\cup J=[3, n]$, $n=b_2(S)$.

PROOF. By (2.9), we may set $C_1\sim L_1-L_{I'}$, $C_2\sim L_2-L_{J'}$ for some I' and J' such that $1\notin I'$, $2\notin J'$, Since $C_1C_2=2$, we have $2=-L_1L_{J'}-L_2L_{I'}+L_{I'}L_{J'}$. Hence $1\in J'$, $2\in I'$ and $I'\cap J'=\varnothing$. By setting $I=I'\setminus\{2\}$, $J=J'\setminus\{1\}$, we obtain the expressions for C_1 and C_2.

Let $L'=L_1+L_2$. Then $L'C_1=L'C_2=0$. Hence $r=2$ and $I\cup J=[3, n]$ by (2.4.3). Let D be a nonsingular rational curve with $C_2D=1$. By (2.6), we have $D\sim L_i-L_\Lambda$ for some i and Λ. By (2.7), $i\geqq 3$. By $C_1D=0$, $C_2D=1$, we have

$$-L_iL_I-L_1L_\Lambda+L_2L_\Lambda+L_IL_\Lambda=0\,, \tag{2.15.1}$$

$$-L_iL_J+L_1L_\Lambda-L_2L_\Lambda+L_JL_\Lambda=1\,. \tag{2.15.2}$$

From (2.15.1)+(2.15.2) and $i\in I\cup J$, it follows that $(L_I+L_J)L_\Lambda=1+(L_I+L_J)L_i=0$. Thus $L_IL_\Lambda=L_JL_\Lambda=0$, whence $I\cap\Lambda=J\cap\Lambda=\varnothing$. Consequently, Λ is a subset of $\{1, 2\}$. If $\Lambda=\{1\}$, then

$$C_2D=(L_2-L_1-L_J)(L_i-L_1)=-1-L_iL_J\leqq 0\,,$$

which is absurd. Hence $\Lambda=\{2\}$ or $\{1, 2\}$. The rest is clear by (2.15.1) and (2.15.2).

q.e.d.

(2.16) EXAMPLES. Let S be a VII_0 surface with $b_2=2$ or 3 containing a global spherical shell. Suppose that there is a cycle $C=C_1+\cdots+C_r$ with a branch $D_{r+1}+\cdots+D_{b_2}$ on S. Then by [4], [8], [15], [16] the dual graph of curves on S is given in Figure 2.16 below.

In Figure 2.16, a black vertex (resp. a white vertex) denotes a rational curve with a node (resp. a nonsingular rational curve). An edge stands for transversal intersection at a point, while a double edge stands for transversal intersection at two distinct points. Each integer below a vertex denotes the self-intersection number of the corresponding curve.

By (2.6), (2.14) and (2.15) we can express the curves C_i and D_j in terms of a canonical basis L_1, L_2 (and L_3) as follows:

$$C=-L_1\,,\ D_2\sim L_1-L_2\,, \tag{2.16.1}$$

$$C=-L_1-L_2\,,\ D_2\sim L_1-L_3\,,\ D_3\sim L_2-L_1\,, \tag{2.16.2}$$

$$C_1\sim L_1-L_2\,,\ C_2\sim L_2-L_1-L_3\,,\ D_3\sim L_3-L_1-L_2\,, \tag{2.16.3}$$

$$C_1\sim L_1-L_2-L_3\,,\ C_2\sim L_2-L_1\,,\ D_3\sim L_3-L_2\,. \tag{2.16.4}$$

The cycle C consists of two rational curves C_1 and C_2 in (2.16.3) and (2.16.4), while the cycle consists of a single rational curve with a node in (2.16.1) and (2.16.2). We note that the two cases in (2.15) are really possible as (2.16.3) and (2.16.4) show and that (2.16.1)–(2.16.4) exhaust all the possible dual graphs of b_2 $(\leqq 3)$ curves on special VII_0 surfaces.

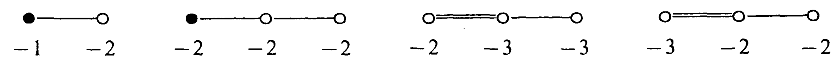


FIGURE 2.16

(2.17) LEMMA. *Let S be a VII_0 surface with a cycle $C=C_1+\cdots+C_s$ of s rational curves ($s\geqq 3$). Suppose that $(K_S+C)^2\leqq -2$ and that there is a nonsingular rational curve D with $C_hD=1$ and $C_jD=0$ $(j\neq h)$. We choose a canonical basis L_j subject to* (2.10.1)–(2.10.3). *Let $D\sim L_\alpha-L_A$ and $\alpha\in I_l$ for some α, A and l. Then*

(2.17.1) $1\leqq h\leqq r\,,\ 1\leqq l\leqq r$ *and* $A\subset[1,r]\,,$

(2.17.2) $$A=\begin{cases}[1,r] & \text{if } l=h\,,\\ [h,l-1] & \text{if } h<l\leqq r\,,\\ [h,r]\cup[1,l-1] & \text{if } 1\leqq l<h\,;\end{cases}$$

(2.17.3) *If $A=[1,r]$, then D is a unique irreducible branch of C.*

PROOF. By (2.6), let $D\sim L_\alpha-L_A$. Then $\alpha\geqq s+1$, and $\alpha\notin A$ by (2.7). Thus $1=CD=-L_{[r+1,n]}(L_\alpha-L_A)=1+L_{[r+1,n]}L_A$, whence $[r+1,n]\cap A=\varnothing$. Therefore A is a nonempty subset of $[1,r]$. There is a unique $h\in[1,s]$ such that $C_hD=1$. Then $C_jD=0$ for any $j\neq h$.

We now prove $1\leqq h\leqq r$. Suppose $r+1\leqq h\leqq s$. Since $C_hD=1$ and $h\notin A\subset[1,r]$, we have

$$1=(L_h-L_{B_h}-L_{I_h})(L_\alpha-L_A)=-L_\alpha L_{I_h}+L_{B_h}L_A\,.$$

Hence $\alpha\in I_h$ and $B_h\cap A=\varnothing$. By (2.10.3), $\alpha\notin I_j$ $(j\neq h)$. Since $C_jD=0$ $(j\neq h)$, we have $L_AC_j=L_\alpha C_j-DC_j=-L_\alpha L_{I_j}-DC_j=0$. On the other hand, $L_AC_h=L_A(L_h-L_{B_h})=0$. Therefore $A=[1,r]$ by (2.4.3). Let $\pi\colon S'\to S$ be an unramified double covering of S, and let $\pi^*L_j=L'_j+L''_j$, $\pi^*C_j=C'_j+C''_j$, $\pi^*D=D'+D''$. Then L'_j and L''_j $(1\leqq j\leqq n)$ form a canonical basis of $H^2(S',\boldsymbol{Z})$. Moreover,

$$\pi^*C=-L'_{[r+1,n]}-L''_{[r+1,n]}\,,\quad K_{S'}=\pi^*K_S=L'_{[1,n]}+L''_{[1,n]}\,.$$

Hence $K_{S'}+\pi^*C=L'_{[1,r]}+L''_{[1,r]}$. We may assume $D'C'_h=D''C''_h=1$. By the same argument as above, letting $D'\sim L'_\alpha-L'_I-L''_J$, we get $L'_I+L''_J=L'_{[1,r]}+L''_{[1,r]}$, whence $(D')^2=-2r-1$. This is absurd because $(D')^2=D^2=-r-1$, and $r>0$. Thus $1\leqq h\leqq r$.

Suppose $l>r$ next. Then by $C_lD=0$ and (2.10), we have

$$0=(L_l-L_{B_l}-L_{I_l})(L_\alpha-L_\Lambda)=-L_\alpha L_{I_l}+L_{B_l}L_\Lambda=1\ ,$$

which is absurd. Thus we complete the proof of (2.17.1).

Assume $l=h$. Then $\alpha\in I_h$. By $C_hD=1$, we have $1=-L_\alpha L_{I_h}-L_hL_\Lambda+L_{B_h}L_\Lambda$. Hence $L_\Lambda C_h=0$ and $L_\Lambda(L_h-L_{B_h}-L_{I_h})=0$. Since $\alpha\in I_h$, we see that I_j $(j\neq h)$ does not contain α. Therefore for $j\neq h$,

$$L_\Lambda C_j=(L_\Lambda+D)C_j=L_\alpha(L_j-L_{B_j}-L_{I_j})=-L_\alpha L_{I_j}=0\ .$$

By (2.4.3), we have $\Lambda=[1, r]$.

Assume next $h<l\leqq r$. Then from $C_hD=1$ it follows that $h\in\Lambda$, $h-1\notin\Lambda$ and $B_h\cap\Lambda=\varnothing$. (Here if $h=1$, then $h-1\notin\Lambda$ means $r\notin\Lambda$.) By $C_jD=0$ for $h+1\leqq j\leqq l-1$, we have $j-1\in\Lambda$ if and only if $j\in\Lambda$. This implies that Λ contains $[h, l-1]$. Similarly, $j-1\in\Lambda$ if and only if $j\in\Lambda$ for $l+1\leqq j\leqq r$ or $1\leqq j\leqq h-1$. Hence $\Lambda=[h, l-1]$. If $h>l$, then $\Lambda=[h, r]\cup[1, l-1]$ by the same argument. This completes the proof of (2.17.2).

Finally we prove (2.17.3). Assume that $D\sim L_\alpha-L_{[1,r]}$ and that there is another irreducible curve D' with $CD'=1$. Then by (2.17.1), we see that $D'\sim L_\beta-L_\Gamma$ for a nonempty subset Γ of $[1, r]$. Hence $DD'=L_{[1,r]}L_\Gamma=-\#(\Gamma)<0$, which is absurd.

q.e.d.

(2.18) COROLLARY. *Let S be a* VII_0 *surface with a cycle C of rational curves. Then for any irreducible component C_i of C $(1\leqq i\leqq r)$, there exists at most one irreducible branch D of C with $C_iD=1$.*

PROOF. We assume $(K_S+C)^2\leqq-3$ and $b_2(C)\geqq3$. Suppose that there exist two irreducible curves D, D' such that $DC_h=D'C_h=1$. Then by (2.17), we see that $D\sim L_\alpha-L_\Lambda$, $D'\sim L_\beta-L_\Gamma$ for some $\alpha, \beta\in[s+1, n]$, and $\Lambda, \Gamma\subset[1, r]$. By (2.17.2), $\Lambda\cap\Gamma$ contains h, whence $DD'=L_\Lambda L_\Gamma<0$. This is absurd. If $(K_S+C)^2\geqq-2$ or $b_2(C)\leqq2$, then take a triple covering S^* of S. By the above, any irreducible component C_i^* of the pull-back C^* of C has at most one irreducible branch, hence so does any irreducible component of C.

q.e.d

3. Dual graphs of curves (1).

(3.1) LEMMA. *Suppose that there exist a positive integer m, an effective divisor D and a flat line bundle F such that $mK_S+D=mF$. Then D_{red} is connected, and D_{red} contains a cycle of rational curves.*

PROOF. Suppose $m=1$, $F\neq O_S$. Then $p_a(D)=(K_SD+D^2)/2+1=1$, whence by [14, (2.7)], D_{red} contains a cycle C of rational curves. Let E be a connected component of D containing C. Consider the exact sequence

$$\begin{aligned}0 &\to H^0(S, F-D) \to H^0(S, F) \to H^0(D, O_D(F))\\ &\to H^1(S, F-D) \to H^1(S, F) \to H^1(D, O_D(F))\\ &\to H^2(S, F-D) \to H^2(S, F) \to 0\,.\end{aligned}$$

By (2.2) and [14, (2.10)], we have $H^0(S, F)=0$ and $H^0(S, -F)=0$. Hence $h^2(S, F)=h^0(S, K_S-F)=h^0(S, -D)=0$. By the Riemann-Roch theorem, we have $H^1(S, F)=0$. Hence $h^1(D, O_D(F))=h^2(S, F-D)=h^0(S, K_S+D-F)=1$. Let E' be a connected component of D_{red} with $E' \cap C=\varnothing$. Then E' is simply connected by (2.2). Therefore the line bundle F is trivial on a small neighborhood of E'. Hence $H^1(D', O_{D'}(F)) \cong H^1(D', O_{D'})=0$ for any divisor D' with $\mathrm{supp}(D')=E'$. Hence $H^1(D, O_D(F)) \cong H^1(E, O_E(F))$ for some $E \leqq D$ with E_{red} connected. Now consider the exact sequence

$$\begin{aligned}0 &\to H^0(S, F-E) \to H^0(S, F) \to H^0(E, O_E(F))\\ &\to H^1(S, F-E) \to H^1(S, F) \to H^1(E, O_E(F))\\ &\to H^2(S, F-E) \to H^2(S, F) \to 0\,.\end{aligned}$$

Hence $h^0(S, -D+E)=h^0(S, K_S+E-F)=h^2(S, F-E)=h^1(E, O_E(F))=1$. This shows that $D=-D''+E$ for an effective D''. Hence $E=D$, $D''=0$ and D_{red} is connected.

Assume next $m=1$ and $F=O_S$. It follows easily from [14, (2.6)] that $h^1(O_D)=2$. Let E be a connected component of D, E_{red} containing a cycle of rational curves. If $h^1(O_E)=1$, then $h^1(O_G)=1$ for $G:=D-E$. By [14, (2.3)], G contains an elliptic curve or a cycle of rational curves, a contradiction to (2.2). Hence $h^1(O_E)=2$. Therefore $h^0(S, K_S+E)=h^2(S, -E)=1$ by [14, (2.8)], whence $-D+E$ is effective (or zero). This shows that $E=D$, and D is connected.

Next we consider the case $m>1$. Consider an m-fold cyclic covering $X=\{(\zeta, x) \in -K_S+F;\ \zeta^m=d(x)\}$ of S where ζ (resp. $d(x)$) is the fiber coordinate of $-K_S+F$ (resp. a defining equation for D). Take a minimal resolution Y of singularities of X. Let Z be the minimal model of Y. Then by the same argument as in [14, (12.4)] we can show that Y is a surface with $b_1=1$ and $K_Y=-H+G$ for an effective H and a flat line bundle G on Y, and that Z is a VII_0 surface with $K_Z=-H'+G'$ for an effective H' and a flat line bundle G' on Z. By the above argument in the case $m=1$, it follows that H' is connected.

Let A be an exceptional curve on Y with $A^2=-1$. If A is contained in H, then the number of connected components of H is stable in blowing A down. If A is not contained in H, then by $K_Y A=-1$, we have $HA=1$. Hence the number of connected components of H is stable in blowing A down. Since H' is connected, so is H. Hence the image D of H is also connected. It follows from the proof of [14, (12.4)] that D_{red} contains a cycle of rational curves. q.e.d.

(3.2) LEMMA. *Suppose that $mK_S+D=G$ for an effective divisor D and a flat line* *bundle G. If an irreducible curve E intersects D_{red}, then E is contained in D_{red}. In particular, if $E^2 \leqq -3$, then E is contained in D_{red}.*

PROOF. If E is not contained in D_{red} and if E intersects D_{red}, then $DE>0$. But $-DE=mK_SE\geqq 0$, which is absurd. If $E^2\leqq -3$, then $K_SE\geqq 1$, whence $ED<0$ Hence E is contained in D. q.e.d.

(3.3) DEFINITION. A VII_0 surface S with $b_2>0$ is said to be *special* if S has at least b_2 rational curves.

By [14, (3.5)], any special VII_0 surface has exactly b_2 rational curves. Any VII_0 surface with a global spherical shell is special. See [4], [8], [16] as well as (5.2).

(3.4) LEMMA. *An arbitrary special* VII_0 *surface has a cycle of rational curves.*

PROOF. By (2.2), there exist no elliptic curves and no cycles C of rational curves with $C^2=0$. Hence the intersection matrix (C_iC_j) is negative definite by [14, (2.10)], where C_j $(j=1,\cdots,b_2)$ are all rational curves on S. Hence C_j is a $\boldsymbol{Q}$-basis of $H^2(S,\boldsymbol{Q})$. Thus there exist a positive integer m, an effective divisor D and a flat line bundle $F\in H^1(S,\boldsymbol{C}^*)$ $(\cong \boldsymbol{C}^*)$ such that $mK_S=-D+F$ in $H^1(S,O_S^*)$. Hence by (3.1), D_{red} contains a cycle of rational curves. q.e.d.

In the rest of § 3 and § 4 throughout, *we always assume that S is a special* VII_0 *surface satisfying* (2.2.1)–(2.2.4).

(3.5) LEMMA. *Let E be a connected effective divisor such that $E\sim L_k-L_A$ for some $k\geqq s+1$ and $A\subset[1,r]$. Let D be a reduced (possibly reducible) curve which contains none of E and the irreducible components of C. If $D\sim L_j-L_k-L_J$ for $j\geqq s+1$, then $ED=1$, $J\subset[1,r]$ and $A\cap J=\varnothing$.*

PROOF. Let $D=D'_1+D'_2+\cdots+D'_m$ with D'_i irreducible. Then we may assume $D'_i\sim L_i-L_{A_i}$ for some $i\geqq s+1$ with $i\neq k$. By assumption, there exists D'_i with $k\in A_i$. We may assume $k=s+1$ and $i=s+2$. Suppose $ED=0$. Then $ED'_i=0$. Hence $D_1:=D'_i\sim L_{s+2}-L_{s+1}-L_{J_1}$ for $J_1\subset J$, and $J_1\cap A\neq\varnothing$. Then by $CD_1\geqq 0$, we have $J_1\cap[r+1,n]=\varnothing$ so that $\varnothing\neq J_1\subset[1,r]$ and $CD_1=0$. We also see that there exist no irreducible curves $D'\sim L_i-L_{s+1}-L_{J'}$ for $i\geqq s+3$. Indeed, otherwise, we have $J'\subset[1,r]$ by the same argument as above, whence $D'D_1=L_{s+1}^2-\#(J'\cap J_1)<0$. If an irreducible curve D_2 not contained in C meets D_1, then $D_1D_2=1$, $CD_2=0$, $D_2\sim L_i-L_{s+2}-L_{J_2}$ for some $i\geqq s+3$ and $J_2\subset[1,r]$, $J_1\cap J_2=\varnothing$. Moreover, D_2 is a unique irreducible curve meeting D_1, because, if D' meets D_1, then $D'\sim L_p-L_{s+2}-L_{J'}$ for some p $(\geqq s+3)$, $J'\subset[1,r]$ and $D'D_2\leqq -1$, therefore $D'=D_2$. Now we may assume $i=s+3$ and $D_2\sim L_{s+3}-L_{s+2}-L_{J_2}$. If there exists an irreducible curve D_3 $(\neq D_1, C_i)$ meeting D_2, then $D_3\sim L_j-L_{s+3}-L_{J_3}$ for some $j\geqq s+4$ by (2.7), and D_3 is a unique irreducible curve other than D_1 which intersects D_2.

Repeating this argument, we obtain irreducible curves $D_2,\cdots,D_m$ such that (by indexing suitably), $D_j \sim L_{j+s+1} - L_{j+s} - L_{J_j}$ $(2 \leqq j \leqq m)$ and $CD_j = 0$, where J_p is a subset of $[1, r]$ with $J_p \cap J_q = \varnothing$ $(p \neq q)$, and there exist no irreducible curves meeting $D_1 + \cdots + D_m$. In particular, $C(D_1 + \cdots + D_m) = 0$. Hence there exist no connected divisors containing both C and D_1. However, since $D_1^2 \leqq -3$ by $J_1 \neq \varnothing$, D_1 is contained in a connected numerical antipluricanonical divisor which contains C by (3.2). This is absurd. Consequently $ED = 1$ and $\Lambda \cap J = \varnothing$. q.e.d.

(3.6) THEOREM. *Let S be a special* VII_0 *surface with a cycle $C = C_1 + \cdots + C_s$ of rational curves. Then $r = s$.*

PROOF. We assume first $(K_S + C)^2 \leqq -2$ and $s \geqq 2$. By (2.10) we take a canonical basis L_j subject to (2.10.1)–(2.10.3). Assume $s > r$ to derive a contradiction. By (2.11) there exists an irreducible component C_j $(r+1 \leqq j \leqq s)$ of C such that $B_j = \varnothing$ and $C_j \sim L_j - L_{I_j}$ with $I_j \subset [s+1, n]$. Since S is special, there exists an irreducible curve $D_k \sim L_k - L_{A_k}$ for any $k \in I_j$ by (2.7). Then

$$C_j D_k = (L_j - L_{I_j})(L_k - L_{A_k}) = 1 - L_j L_{A_k} + L_{I_j} L_{A_k}$$
$$CD_k = -L_{[r+1,n]}(L_k - L_{A_k}) = 1 + L_{[r+1,n]} L_{A_k}\,.$$

Suppose $CD_k = 1$. Then $[r+1, n] \cap A_k = \varnothing$, whence $A_k \subset [1, r]$. Hence $C_j D_k = 1$, which contradicts (2.17.1). Therefore, $CD_k = 0$ and $\#(A_k \cap [r+1, n]) = 1$. Hence $C_j D_k = 0$ so that $j \notin A_k$ and $\#(I_j \cap A_k) = 1$. Let $\{k'\} = I_j \cap A_k = [r+1, n] \cap A_k$. Then $D_k \sim L_k - L_{k'} - L_{A'_k}$ where $A'_k := A_k \setminus \{k'\}$ is a subset of $[1, r]$. By indexing suitably, we have a subset $\{k_1, \cdots, k_m\}$ of I_j such that $D_{k_i} \sim L_{k_i} - L_{k_{i+1}} - L_{A'_{k_i}}$ $(1 \leqq i \leqq m)$ with $CD_{k_i} = 0$, where $k_{m+1} = k_1$, $A'_{k_i} \subset [1, r]$.

Let D' be an irreducible curve different from D_{k_i}, C_j $(1 \leqq i \leqq m, 1 \leqq j \leqq s)$. Then by $CD' \leqq 1$, we have either $D' \sim L_k - L_{k'} - L_{A'_k}$ $(k \in [s+1, n], k' \in [r+1, n], A'_k \subset [1, r])$ or $D' \sim L_k - L_{A_k}$ $(k \in [s+1, n], A_k \subset [1, r])$, where $k \neq k_i$ by (2.7). In the first case, $k' \neq k_i$, because $D' D_{k_{i-1}} < 0$ if $k' = k_i$. In either case, $D' D_{k_i} = 0$ for any i. Since D_{k_i}'s do not form a cycle of rational curves by (2.2), there exists i such that $D_{k_i} D_{k_{i+1}} = 0$ or $D_{k_m} D_{k_1} = 0$. Hence there exists i such that $A'_{k_i} \neq \varnothing$ and $(D_{k_i})^2 \leqq -3$. By (3.1) and (3.4), D_{k_i} is contained in a connected divisor containing C. However as was shown above, $CD_{k_i} = 0$ and no curves except D_{k_j} meet D_{k_i}, which is absurd.

Hence $r = s$ if $(K_S + C)^2 \leqq -2$ and $s \geqq 2$. If $(K_S + C)^2 = -1$ or $s = 1$, then take an unramified double covering S^* of S, and the pull-back C^* of C. Then $(K_{S^*} + C^*)^2 = 2(K_S + C)^2$, $b_2(C^*) = 2b_2(C)$, whence $2r = 2s$. q.e.d.

(3.7) COROLLARY. *Let S be a special* VII_0 *surface with a cycle C of rational curves. Then*

(3.7.1) $(K_S + C)^2 = -b_2(C)$ *and* $b_2(C) - C^2 = b_2(S)$,

(3.7.2) $C_i \sim L_i - L_{i-1} - L_{I_i}$ $(1 \leqq i \leqq r)$ *in* (2.10), $C_{i-1} C_i = 1$, $C_i C_j = 0$ $(i \neq j, j \pm 1 \bmod r)$, *where $C_{k+r} = C_k$ for any k.*

PROOF. Clear from (3.6).

(3.8) THEOREM. *Let S be a special* VII_0 *surface with a rational curve C with a node. If C has an irreducible branch D_2, then by suitably indexing the remaining curves D_j $(3 \leqq j \leqq n)$ and a canonical basis L_j $(1 \leqq j \leqq n)$, we have*

$$C = -(L_2 + \cdots + L_n), \quad D_j \sim L_j - L_{j-1} \qquad (2 \leqq j \leqq n),$$

where $n = b_2(S)$. The dual graph of n curves is as in Figure 3.8, *where a black vertex (resp. a white vertex) stands for C (resp. D_j).*

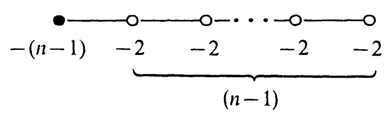


FIGURE 3.8

PROOF OF (3.8). By (2.14), we have $C = -(L_2 + \cdots + L_n)$ and $D_2 \sim L_2 - L_1$. Let D_j $(3 \leqq j \leqq n)$ be the remaining irreducible curves on S. Let D' be one of them. Then $D'C = 0$ by (2.18). Therefore $D' \sim L_j - L_k$ or $L_j - L_1 - L_2$ for $j \geqq 3$, $k \geqq 2$.

(3.8.1) SUBLEMMA. *$D' \sim L_j - L_k$ for some j, k.*

PROOF OF (3.8.1) Otherwise, we may assume $D_3 \sim L_3 - L_1 - L_2$. Then there exists no irreducible curve $D'' \sim L_j - L_1 - L_2$ with $D'' \neq D_3$. Hence we may assume $D_j \sim L_j - L_{k_j}$ $(4 \leqq j \leqq n, 3 \leqq k_j \leqq n)$. By indexing suitably, $D_j \sim L_j - L_{j-1}$ $(4 \leqq j \leqq n)$ by (2.12.2). Since $D_3 D_4 = 1$, the curve $D_3 + D_4 + \cdots + D_n$ is connected but $(C + D_2)(D_3 + \cdots + D_n) = 0$. This contradicts $D_3^2 = -3$ by (3.2). Consequently $D' \sim L_j - L_k$. q.e.d.

We continue the proof of (3.8). Now assume $D_j \sim L_j - L_{k_j}$ $(3 \leqq j \leqq n, 2 \leqq k_j \leqq n)$ by (3.8.1). Hence again by (2.12.2), we have, by indexing suitably, $D_j \sim L_j - L_{j-1}$, $D_{j-1} D_j = 1$ $(3 \leqq j \leqq n)$ and $D_j D_k = 0$ $(k \neq j, j \pm 1)$. q.e.d.

(3.9) THEOREM. *Let S be a special* VII_0 *surface with a cycle $C = C_1 + C_2$ of two rational curves. If S has an irreducible curve D_3 with $C_2 D_3 = 1$, then by indexing suitably, we have one of the following cases:*

(3.9.1) $$C_1 \sim L_1 - L_2 - L_{[3,l]}, \quad C_2 \sim L_2 - L_1 - L_{[l+1, l+m-2]},$$
$$D_j \sim L_j - L_{j-1} \ (3 \leqq j \leqq l+m-2, j \neq l+1), \quad D_{l+1} \sim L_{l+1} - L_1,$$

(3.9.2) $$C_1 \sim L_1 - L_2 - L_{[3,l]}, \quad C_2 \sim L_2 - L_1 - L_{[l+1, l+m-2]},$$
$$D_j \sim L_j - L_{j-1} \ (3 \leqq j \leqq l+m-2, j \neq l+1), \quad D_{l+1} \sim L_{l+1} - L_l - L_1,$$

(3.9.3) $$C_1 \sim L_1 - L_2, \quad C_2 \sim L_2 - L_1 - L_{[3,n]},$$
$$D_3 \sim L_3 - L_1 - L_2, \quad D_j \sim L_j - L_{j-1} \ (4 \leqq j \leqq n),$$

(3.9.4) $$C_1 \sim L_1 - L_2 - L_{[3,n]}, \quad C_2 \sim L_2 - L_1, \quad D_j \sim L_j - L_{j-1} \ (3 \leqq j \leqq n),$$

where $l, m, n \geqq 3$ and $b_2(S)$ equals $l+m-2$ or n, C_1D_{l+1} equals 1 *(resp.* 0*) in* (3.9.1) *(resp.* (3.9.2)*). The dual graph of b_2 curves are as in Figure* 3.9.

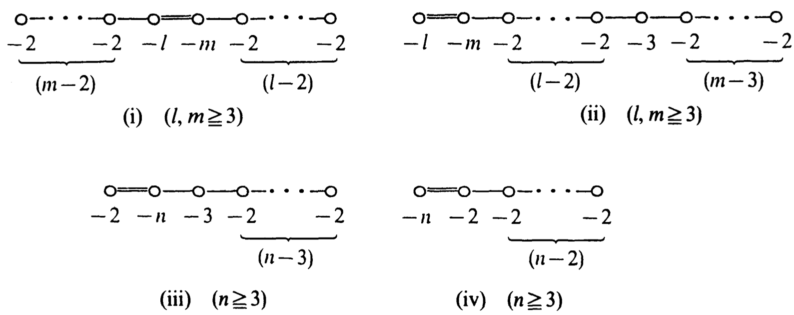


FIGURE 3.9

PROOF OF (3.9). The proof is divided into several cases.

Case 1. First we consider the case where C has two irreducible branches. By (2.18), we may assume that there exist two irreducible curves D and D' with $C_1D = C_2D' = 1$ and $DD' = 0$. We are able to apply (2.15) by (3.6). With the notation in (2.15), we may assume

$$C_1 \sim L_1 - L_2 - L_I\,, \quad C_2 \sim L_2 - L_1 - L_J\,, \quad D \sim L_k - L_1\,, \quad D' \sim L_3 - L_2\,,$$
$$I \cup J = [3, n]\,, \quad I \cap J = \varnothing\,, \quad 3 \in I\,, \quad k \in J$$

by indexing suitably. So by letting $C_1^2 = -l$ and $C_2^2 = -m$, we may assume $k = l+1$, $I = [3, l]$ and $J = [l+1, l+m-2]$. Let D'' be an irreducible curve different from D, D' and not contained in C. Then by (2.18) we have $C_1D'' = C_2D'' = 0$, $DD'' \geqq 0$, and $D'D'' \geqq 0$. Hence $D'' \sim L_j - L_{k_j}$ for some k_j. We note that $k_j \in [3, l]$ (resp. $[l+1, l+m-2]$) if and only if $j \in [4, l]$ (resp. $[l+2, l+m-2]$). Since S is special, we may suitably re-index and assume

$$D'' = D_j \sim L_j - L_{j-1} \qquad (4 \leqq j \leqq l+m-2, j \neq l+1)$$

in view of (2.12.2). Let $D_3 = D'$, $D_{l+1} = D$. It is easy to see that $D_1D_{l+1} = D_{j-1}D_j = 1$ $(4 \leqq j \leqq l)$, $C_2D_3 = D_{j-1}D_j = 1$ $(l+2 \leqq j \leqq l+m-2)$ and $D_iD_j = -2\delta_{ij}$ (otherwise). This is (3.9.1). The dual graph of n curves is as in Figure 3.9 (i).

Next we consider the case where C has a unique branch with $C_1D = 0$ and $C_2D = 1$. By (2.15), we may assume $C_1 \sim L_1 - L_2 - L_I$ and $C_2 \sim L_2 - L_1 - L_J$. We have either $D \sim L_i - L_2$ or $D \sim L_i - L_1 - L_2$.

Case 2. Consider the case $D \sim L_i - L_2$. Let $C_1^2 = -l$ and $C_2^2 = -m$. Then $\#(I) = l-2$, $\#(J) = m-2$. By (2.15), we have $i \in I$, $i \notin J$, $I \cup J = [3, n]$ and $I \cap J = \varnothing$. So we may assume $i = 3$, $I = [3, l]$ and $J = [l+1, l+m-2]$. Let D' be an irreducible curve different from D, C_1, C_2. Then by $C_iD' = 0$ and $DD' \geqq 0$, we have $D' \sim L_j - L_k - L_\Lambda$ for some j ($\geqq 4$), k ($\geqq 3$) and $\Lambda \subset \{1, 2\}$. Notice that if $\Lambda=\{2\}$, then $k=3$ by $DD' \geqq 0$, a contradiction to $C_2 D'=0$. Therefore the following three cases are possible:

Case 2-1. There exists an irreducible curve $D' \sim L_j - L_3 - L_1 - L_2$ for some $j \in I$.

Case 2-2. There exists an irreducible curve $D' \sim L_j - L_k - L_1$ for some $k \in I$ and $j \in J$.

Case 2-3. Any irreducible curve D' different D, C_1 and C_2 satisfies $D' \sim L_j - L_k$ for some j and k.

Case 2-1. We show that this is impossible. We may assume $D'=D_4 \sim L_4 - L_3 - L_1 - L_2$ and $4 \leqq l$. First we assume $D'' \sim L_j - L_k - L_1$ for $j \in J$ and $k \in I$. Hence by $D'D'' \geqq 0$ we may assume $j=l+1$ and $k=4$. Then an irreducible curve G ($\neq D, D', D'', C_i$) is homologically equivalent to $L_p - L_q$ for some p, q ($\geqq 4$), where $p \in I$ (resp. $p \in J$) if and only if $q \in I$ (resp. $q \in J$). Since $q \neq 4$ by $GD'' \geqq 0$, we may assume $D_p \sim L_p - L_{k_p}$ ($5 \leqq p \leqq l$, $5 \leqq k_p \leqq l$). This is impossible by (2.12). Thus we see that D'' ($\neq D, D', C_i$) is equivalent to $L_p - L_q$ ($q \geqq 4$), where $p \in I$ (resp. $p \in J$) if and only if $q \in I$ (resp. $q \in J$). Consequently, $D'D = D''D = 0$, whence $C+D+D'$ is contained in no connected divisor. This contradicts (3.1), (3.2) and $(D')^2=-4$.

Case 2-2. We may assume $D' \sim L_{l+1} - L_l - L_1$. By Case 2-1, for any irreducible curve D'' different from D, D' and C_i, we have $D'' \sim L_p - L_q$ for some p and q. Here $q \neq l$ because if $q=l$, then $D''D'=-1$. By $C_i D''=0$, we have $p \in I$ (resp. $p \in J$) if and only if $q \in I$ (resp. $q \in J$). Hence by (2.12.2), we may assume that the remaining curves are $D_j \sim L_j - L_{j-1}$ ($4 \leqq j \leqq l$ or $l+2 \leqq j \leqq l+m-2$). We set $D_3 = D$ and $D_{l+1}=D'$, which is (3.9.2). The dual graph of n curves is as in Figure 3.9 (ii).

Case 2-3. By $CD'=0$, we have $D' \sim L_j - L_k$ with $j \geqq 4$, $k \geqq 3$. Let $n=b_2(S)$. By applying (2.12.2) to $D_j \sim L_j - L_{k_j}$ ($4 \leqq j \leqq n$, $3 \leqq k_j \leqq n$), we may assume $D_j \sim L_j - L_{j-1}$ ($4 \leqq j \leqq n$). By $C_i D_j = 0$, we have $j \in I$ (resp. $j \in J$) if and only if $j-1 \in I$ (resp. $j-1 \in J$). Hence $I=[3, n]$ and $J=\varnothing$. This is (3.9.4). The dual graph of n curves is as in Figure 3.9 (iv).

Case 3. Finally we consider the case where $C_1 \sim L_1 - L_2 - L_I$, $C_2 \sim L_2 - L_1 - L_J$ and $D \sim L_3 - L_1 - L_2$ with $3 \in J$. It follows from $C_i D'=0$ that $D' \sim L_j - L_{k_j}$ for any irreducible curve D' ($\neq D, C_i$), where $4 \leqq j \leqq n$, $3 \leqq k_j \leqq n$ and $n=b_2(S)$. By (2.12.2), we may assume $D_j \sim L_j - L_{j-1}$ ($4 \leqq j \leqq n$). Since $j \in J$ if and only if $j-1 \in J$, we see that $I=\varnothing$ and $J=[3, n]$. This is (3.9.3). The dual graph of n curves is as in Figure 3.9 (iii).

q.e.d.

4. Dual graphs of curves (2).

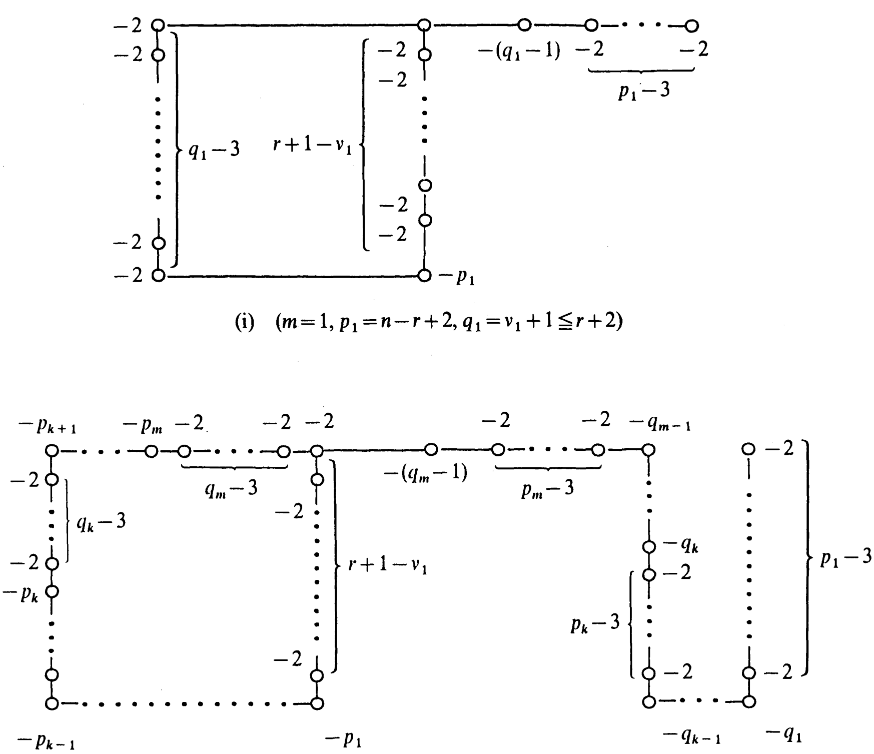
(4.1) Notation. Assume $r \geqq 3$. Let $M_j = L_j$ ($r+1 \leqq j \leqq n$) and $n=b_2(S)$. Define L_j ($j \in \mathbf{Z}$) by $L_{j+mr} = L_j$ ($m \in \mathbf{Z}$, $1 \leqq j \leqq r$). Notice that $L_i M_j = 0$, $L_{r+1} = L_1$, $L_{r+1} \neq M_1$ from now on. We write $L_I = \sum_{i \in I} L_i$, $M_J = \sum_{j \in J} M_j$, $C_i \sim L_i - L_{i-1} - M_{I_i}$ ($1 \leqq i \leqq r$) for a subset I (resp. subsets J and I_i) of $[1, r]$ (resp. $[r+1, n]$). For an irreducible curve D_j not contained in C, we write $D_j \sim M_j - L_{U_j} - M_{I_j}$ ($r+1 \leqq j \leqq n$). For a subset J of $[1, r]$

and $\mu<\nu<\mu+r$, we write $J=[\mu, \nu-1]$ if the images of J and $[\mu, \nu-1]$ in $\boldsymbol{Z}/r\boldsymbol{Z}$ coincide. See (4.4), (4.5).

(4.2) THEOREM. *Let S be a special* VII_0 *surface with a cycle $C=C_1+\cdots+C_r$ of r rational curves ($r\geqq 3$). Suppose that there is a unique irreducible curve D_{r+1} such that $D_{r+1}C=1$. Assume $D_{r+1}C_1=1$. Then by suitable indexing, we have*

$$C=-M_{[r+1,n]}, \quad K_S=L_{[1,r]}+M_{[r+1,n]},$$
$$C_{\nu_k}\sim L_{\nu_k}-L_{\nu_k-1}-M_{[i_k,j_k-1]} \qquad (1\leqq k\leqq m),$$
$$D_{r+1}\sim M_{r+1}-L_{[\nu_{m+1},\nu_m-1]}$$
$$D_{i_k}\sim M_{i_k}-M_{j_k}-L_{[\nu_{k+1},\nu_k-1]} \qquad (1\leqq k\leqq m-1)$$
$$C_i\sim L_i-L_{i-1}, \quad D_j\sim M_j-M_{j-1} \qquad (otherwise),$$

where $\nu_{m+1}=1<\nu_m<\cdots<\nu_1\leqq 1+r$, $j_k=i_k-1$ $(1\leqq k\leqq m)$ and $i_m=r+1<i_{m-1}<\cdots<i_1\leqq j_0=n$.



(i) $(m=1, p_1=n-r+2, q_1=\nu_1+1\leqq r+2)$

(ii) $(m\geqq 2, \nu_1\leqq r+1, p_k\geqq 3, q_k\geqq 3)$

FIGURE 4.2

A proof of (4.2) is given in (4.3)–(4.10). We notice that $C_{i-1}C_i=D_{j-1}D_j=1$ ($i\in\boldsymbol{Z}, r+1\leqq j\leqq n$). The dual graph of n curves in (4.2) is as in Figure 4.2, where $p_k=-C_{v_k}^2=j_{k-1}-i_k+3\geqq 3$ $(1\leqq k\leqq m)$, $q_k=-D_{i_k}^2=v_k-v_{k+1}+2\geqq 3$ $(1\leqq k\leqq m-1)$, $q_m=-D_{r+1}^2+1=v_m-v_{m+1}+2\geqq 3$.

(4.3) By (2.17), let $D_{r+1}\sim M_{r+1}-L_A$ for a subset A of $[1,r]$. By (2.17) and $D_{r+1}C_1=1$, we have $A=[1,a-1]$ with $a\leqq r+1$. In this paragraph we consider the case $A=[1,r]$. By (2.17.3), we have $D'C=0$ for any irreducible curve D' ($\neq D_{r+1}, C_1$). Hence $D'\sim M_j-M_k-L_I$ for some subset I of $[1,r]$. Suppose $I\neq\varnothing$ to derive a contradiction. By $D_{r+1}D'\geqq 0$, we have $k=r+1$. Hence in view of (3.5), we have $D_{r+1}D'=1$ and $I\cap[1,r]=\varnothing$. This is absurd. Therefore $I=\varnothing$. Let D_j be the irreducible curves ($r+2\leqq j\leqq n$). Then $D_j\sim M_j-M_{k_j}$ ($r+2\leqq j\leqq n$, $r+1\leqq k_j\leqq n$). By (2.12.2), we have $D_{j-1}D_j=1$ and $D_j\sim M_j-M_{j-1}$ ($r+2\leqq j\leqq n$) by indexing suitably. Let $C_i\sim L_i-L_{i-1}-M_{I_i}$ ($1\leqq i\leqq r$). By $D_{r+1}C_1=1$, we get $r+1\in I_1$. By $C_iD_j=0$ ($r+2\leqq j\leqq n$), we see that $j-1\in I_i$ if and only if $j\in I_i$. Hence $I_1=[r+1,n]$ and $I_i=\varnothing$ ($2\leqq i\leqq r$). This completes the proof of (4.2) in this case. The dual graph of n curves is as in Figure 4.2 (i), where $m=1$, $q_1=r+2$, $v_1=r+1$.

In (4.4)–(4.10) we consider the case $A=[1,a-1]$, $r+1\in I_a$ and $1<a\leqq r$. See (2.17).

(4.4) LEMMA. *Let D' be a reduced (possibly reducible) curve which contains none of D_{r+1} and the irreducible components C_i of C. Suppose that $D'\sim M_j-M_k-L_J$ for $J\subset[1,r]$, and that $j\in I_v$, $k\in I_\mu$ for some $\mu\leqq v\leqq\mu+r$. Then $J=[\mu,v-1]$, $J\cap[1,a-1]=\varnothing$. In particular, if $\mu=v$, then $J=\varnothing$.*

PROOF. First consider the case $\mu=v$. We have

$$0=C_vD'=(L_v-L_{v-1}-M_{I_v})(M_j-M_k-L_J)=(L_v-L_{v-1})(-L_J)\,,$$
$$0=C_\lambda D'=(L_\lambda-L_{\lambda-1}-M_{I_\lambda})(M_j-M_k-L_J)=(L_\lambda-L_{\lambda-1})(-L_J)\,.$$

Hence $\lambda=J$ if and only if $\lambda-1\in J$ for any $\lambda\in[1,r]$. This shows $J=\varnothing$ or $J=[1,r]$. If $J=[1,r]$, then $D_{r+1}D'=(M_{r+1}-L_{[1,a-1]})(M_j-M_k-L_{[1,r]})=-M_{r+1}M_k-a+1$. Hence $k=r+1$ and $a=2$. By (3.5), if $k=r+1$, then $D'_{r+1}D'=1$ and $J\cap[1,a-1]=\varnothing$, which is a contradiction. Thus we have $J=\varnothing$.

Next we consider the case $\mu<v<\mu+r$. We have

$$0=C_vD'=1-(L_v-L_{v-1})L_J\,,\quad 0=C_\mu D'=-1-(L_\mu-L_{\mu-1})L_J\,.$$

Hence $v\notin J$, $v-1\in J$, $\mu\in J$ and $\mu-1\notin J$. By $C_\lambda D'=0$, we have $\lambda\in J$ if and only if $\lambda-1\in J$ for $\mu<\lambda<v$ or $v<\lambda<\mu+r$. This implies that $J=[\mu,v-1]$. If $k=r+1$, then $J\cap[1,a-1]=\varnothing$ by (3.5). If $k\neq r+1$, then $J\cap[1,a-1]=\varnothing$ by $D_{r+1}D'\geqq 0$. q.e.d.

(4.5) LEMMA. *Let D' and D'' be irreducible curves different from D_{r+1}, C_j. Suppose that $D'\sim M_i-M_j-L_I$, $D''\sim M_k-M_l-L_J$ and $D'\neq D''$, where $\mu<v<\mu+r$, $\beta<\alpha<\beta+r$, $i\in I_v$, $j\in I_\mu$, $k\in I_\alpha$, $l\in I_\beta$, $I=[\mu,v-1]$ and $J=[\beta,\alpha-1]$. Then $I\cap J=\varnothing$.*

PROOF. Suppose $I\cap J\neq\varnothing$ to derive a contradiction. By

$$0\leqq D'D''=(M_i-M_j)(M_k-M_l)-\#(I\cap J)\,,$$

we have $(M_i-M_j)(M_k-M_l)>0$. We have three possibilities:

Case 1. $i=l$, $j=k$, $\#(I\cap J)=1$ or 2. We have $\nu=\beta$, $\mu=\alpha$ (mod r), $I=[\mu,\nu-1]$, and $J=[\nu,\mu+r-1]$. Hence $I\cap J=\varnothing$ and $D'D''=2$, which is absurd.

Case 2. $i=l, j\neq k$, $\#(I\cap J)=1$, $D'D''=0$. We may assume $\beta=\nu$, hence $I=[\mu,\nu-1]$ and $J=[\nu,\alpha-1]$. From $\#(I\cap J)=1$ it follows that $\alpha-1=\mu+r$ and $I\cup J=[1,r]$. However $(I\cup J)\cap[1,a-1]=\varnothing$ by applying (4.4) to D_{r+1} and $D'+D''$. But $[1,a-1]\neq\varnothing$, a contradiction.

Case 3. $i\neq l, j=k$, $\#(I\cap J)=1$, $D'D''=0$. This case is clearly reduced to Case 2.

q.e.d.

(4.6) Let $D_j\sim M_j-L_{U_j}-M_{T_j}$ $(r+2\leqq j\leqq n)$. Then by $CD_j=0$, we have $\#(T_j)=1$, so we let $T_j=\{k_j\}$ for some $k_j\geqq r+1$. Let $N=\{i\in[r+2,n]\,;\,D_i^2\leqq -3\}=\{i_1,i_2,\cdots,i_{m-2},i_{m-1}\}$. We let $G_k=D_{i_k}\sim M_{i_k}-M_{j_k}-L_{J_k}$ for some nonempty $J_k\subset[1,r]$. In view of (4.4), we may assume $J_k=[\mu_k,\nu_k-1]$, $i_k\in I_{\nu_k}$ and $j_k\in I_{\mu_k}$ for some $\mu_k\leqq\nu_k<\mu_k+r$. Let $G_m=D_{r+1}$, $i_m=r+1$, $\mu_m=1$, $\nu_m=a$ and $J_m=[1,a-1]$. In view of (4.4) and (4.5), $J_1,\cdots,J_m$ are mutually disjoint. Hence we may assume $\nu_{m+1}=\mu_m=1<\nu_m\leqq\mu_{m-1}<\nu_{m-1}\leqq\cdots\leqq\mu_2<\nu_2\leqq\mu_1<\nu_1\leqq r+1$.

(4.7) LEMMA. *$I_\lambda=\varnothing$ for $\lambda\neq\nu_k$ $(1\leqq k\leqq m)$.*

PROOF. Suppose $I_\lambda\neq\varnothing$ for some $\lambda\neq\nu_k$ $(1\leqq k\leqq m)$. Let $D_j\sim M_j-M_{k_j}-L_{U_j}$ for $j\in I_\lambda$. If $k_j\in I_\mu$ for $\mu\leqq\lambda<\mu+r$, we have $U_j=[\mu,\lambda-1]$ in view of (4.4). If $U_j\neq\varnothing$, then $U_j=[\mu_k,\nu_k-1]$ for some k, whence $\lambda=\nu_k$, a contradiction. Hence $U_j=\varnothing$, $\mu=\lambda$, $k_j\in I_\lambda$ and $D_j\sim M_j-M_{k_j}$. However by applying (2.12) to D_j for $j\in I_\lambda$, we infer a contradiction. Hence $I_\lambda=\varnothing$.

q.e.d.

(4.8) LEMMA. *$\mu_k=\nu_{k+1}$ and $J_k=[\nu_{k+1},\nu_k-1]$ $(1\leqq k\leqq m)$.*

PROOF. Assume $\nu_b<\mu_{b-1}$ for some $2\leqq b\leqq m$ to derive a contradiction. Let $I=\bigcup_{1\leqq\lambda\leqq\nu_b}I_\lambda$, $J=\bigcup_{\nu_b+1\leqq\lambda\leqq r}I_\lambda$, $D_I=\sum_{i\in I}D_i$ and $D_J=\sum_{i\in J}D_i$. We note $I\cup J=[r+1,n]$.

(4.8.1) SUBLEMMA. *Assume $i\neq r+1$. Then $i\in I$ (resp. $j\in J$) if and only if $k_i\in I$ (resp. $k_j\in J$).*

PROOF OF (4.8.1). It suffices to prove that *if $i\in I$ (resp. $j\in J$) then $k_i\in I$ (resp. $k_j\in J$)*. If $j\neq i_k$ for any $k\leqq m-1$ and if $j\in I_\lambda$, then $k_j\in I_\lambda$. Hence if $j\neq i_k$, then $j\in I$ (resp. $j\in J$) if and only if $k_j\in I$ (resp. $k_j\in J$). Suppose $j\in I_\lambda$ for $\lambda\leqq\nu_b$. Then if $j=i_k$ for some k $(1\leqq k\leqq m-1)$, we have $U_j=J_k=[\mu_k,\nu_k-1]$ for $j\in I_{\nu_k}$ and $k_j\in I_{\mu_k}$. Hence $\lambda=\nu_k\leqq\nu_b$ and $k_j\in I_{\mu_k}\subset I$. If $j\in I_\lambda\neq\varnothing$ for $\lambda\geqq\nu_b+1$, then $\lambda\geqq\nu_{b-1}$ in view of (4.7). If $j=i_k$ for some k, then $U_j=[\mu_k,\nu_k-1]$, $j\in I_{\nu_k}$ and $k_j\in I_{\mu_k}$. Since $\nu_k=\lambda\geqq\nu_{b-1}$, we have $\mu_k\geqq\mu_{b-1}\geqq\nu_{b-1}+1$ by the assumption. Hence $k_j\in I_{\mu_k}\subset J$.

q.e.d.

We continue the proof of (4.8). Notice $I, J \neq \emptyset$. By (4.8.1), for any pair of $i \in I$ and $j \in J$, we have

$$D_i D_j \leqq (M_i - M_{k_i})(M_j - M_{k_j}) = 0\ ,$$

which shows $D_I D_J = 0$. The curve D_I (resp. D_J) contains D_{r+1} (resp. G_{b-1}) and $G_{b-1}^2 \leqq -3$. Since $D_{r+1}C = 1$ and both G_k $(1 \leqq k \leqq m-1)$ and C are contained in a numerical antipluricanonical divisor by (3.1) and (3.2), both D_{r+1} and G_{b-1} are in one and the same connected divisor on S. However this contradicts $D_I D_J = 0$. Consequently, $\mu_k = \nu_{k+1}$ and $J_k = [\nu_{k+1}, \nu_k - 1]$ $(1 \leqq k \leqq m)$. q.e.d.

(4.9) LEMMA. *By suitable indexing, we have* $D_{j-1}D_j = 1$, $D_j \sim M_j - M_{j-1} - L_{U_j}$ $(r+2 \leqq j \leqq n)$.

PROOF. We define formally $D'_{r+1} := M_{r+1} - L_1$ and $D'_j := D_j + L_{U_j} = M_j - M_{k_j}$ $(r+2 \leqq j \leqq n)$. Then $D'_i D'_j = D_i D_j$ for any i, j. By the same argument as in (2.12), we have $k_i \neq k_j$ for $i \neq j$. Then $D'_{r+2} + \cdots + D'_n \not\sim 0$. Because, otherwise, $D_{r+2}, \cdots, D_n$ form a cycle of rational curves, contradicting (2.2). Hence there exists b $(> r+1)$ such that $\{k_{r+2}, \cdots, k_n\} = [r+1, n] \setminus \{b\}$ and therefore

$$D'_{r+1} + D'_{r+2} + \cdots + D'_n \sim M_b - L_1\ ,$$

$$-2 = (D'_{r+1} + \cdots + D'_n)^2 = -2(n-r) + 2 \sum_{i<j} D'_i D'_j\ .$$

Consequently, $\sum_{i<j} D'_i D'_j = \sum_{i<j} D_i D_j = n - r - 1$. We also notice $D_{r+1}(D_{r+2} + \cdots + D_n) = (M_{r+1} - L_1)(M_b - M_{r+1}) = 1$. It is shown by the same argument as in (2.12) that $D_{r+1} + \cdots + D_n$ is a connected straight chain, that is, by suitable indexing, we get $D_{j-1}D_j = 1$ $(r+2 \leqq j \leqq n)$. It follows that

$$D_{r+1} \sim M_{r+1} - L_{U_{r+1}}\ , \quad D_j \sim M_j - M_{j-1} - L_{U_j} \qquad (r+2 \leqq j \leqq n)\ . \qquad \text{q.e.d.}$$

(4.10) LEMMA. *In the same notation as in* (4.9), *we have* $i_m < i_{m-1} < \cdots < i_1$, $I_{\nu_k} = [i_k, i_{k-1} - 1]$ $(1 \leqq k \leqq m)$ *and* $I_\lambda = \emptyset$ *(otherwise), where* $i_0 = n+1$, $i_m = r+1$.

PROOF. By the definition in (4.6) and by (2.17), $i_m = r+1$ is contained in $I_{\nu_m} = I_a$. We also see $i_k \in I_{\nu_k}$ $(1 \leqq k \leqq m-1)$. Since $C_\lambda D_j = 0$ for $j \neq i_k$, we have $j \in I_\lambda$ if and only if $j-1 \in I_\lambda$. For $2 \leqq k \leqq m$, we define l_k as follows: If $i_k \geqq i_p$ for any p $(1 \leqq p \leqq m-1)$, then $l_k := n+1$. Otherwise, $l_k := \min\{i_p;\, i_k < i_p,\, 1 \leqq p \leqq m-1\}$. Then we see that I_{ν_k} contains $[i_k, l_k - 1]$ but not l_k. By $C_{\nu_k} D_{l_k} = 0$, we have $\nu_k \in U_{l_k}$ and $\nu_k - 1 \notin U_{l_k}$. If $l_k = i_p$, then $U_{l_k} = J_p = [\nu_{p+1}, \nu_p - 1]$ by (4.8), whence $\nu_k = \nu_{p+1}$, $k = p+1$, $l_k = i_{k-1}$ and $i_m < i_{m-1} < \cdots < i_1$. Thus I_{ν_k} contains $[i_k, i_{k-1} - 1]$ $(2 \leqq k \leqq m-1)$. On the other hand, I_{ν_1} contains $[i_1, n]$ by $C_{\nu_1} D_j = 0$ for any $i_1 \leqq j \leqq n$. Thus the union of I_{ν_k} $(1 \leqq k \leqq m)$ contains $[i_m, i_{m-1} - 1] \cup [i_{m-1}, i_{m-2} - 1] \cup \cdots \cup [i_1, n] = [r+1, n]$. This shows by (2.10.3) and (3.6) that $I_{\nu_k} = [i_k, i_{k-1} - 1]$ and $I_\lambda = \emptyset$ for $\lambda \neq \nu_k$. q.e.d.

Let $p_k=-C_{v_k}^2$, $q_k=-D_{i_k}^2$ and $q_m=-D_{i_m}^2+1$. Then $p_k, q_k \geqq 3$. By (4.6)–(4.10) we have the expressions for curves in (4.2). The dual graph turns out to be as in Figure 4.2 (i) ($m=1$) or Figure 4.2 (ii) ($m \geqq 2$). This completes the proof of (4.2).

(4.11) THEOREM. *Let S be a special* VII_0 *surface with a cycle C of rational curves. Suppose that there are d ($\geqq 2$) irreducible curves meeting C. Then the union $\mathscr{D}$ of all the curves on S is connected, and the complement of the cycle C in $\mathscr{D}$ consists of d connected (straight) chains of rational curves. More precisely, there exist a set of integers $s(l)$, $m(l)$, $i(l)$, $v(k,l)$, $i(k,l)$, $j(k,l)$ $(1 \leqq l \leqq d,\ 1 \leqq k \leqq m(l))$ and a canonical basis L_i $(1 \leqq i \leqq r)$, M_j $(r+1 \leqq j \leqq n)$ of $H^2(S, \boldsymbol{Z})$ such that*

$$1=s(1)<s(2)<\cdots<s(d)<r+1=s(d+1)\,,$$

$$r+1=i(1)<i(2)<\cdots<i(d)<j(0,d)=n<n+1=i(d+1)\,,$$

$$s(k)<v(m(k),k)<v(m(k)-1,k)<\cdots<v(1,k)\leqq s(k+1)\,,$$

$$i(k)<i(m(k)-1,k)<\cdots<i(1,k)\leqq j(0,k) \qquad (1 \leqq k \leqq d)\,,$$

$$i(l)=i(m(l),l)=j(0,l-1)+1,\ i(k,l)=j(k,l)+1$$

and such that (by suitable indexing)

(4.11.1) $$C_i \sim L_i - L_{i-1} - M_{I_i} \qquad (L_0=L_r,\ 1 \leqq i \leqq r)\,,$$

$$D_{i(l)} \sim M_{i(l)} - L_{J_l} \qquad (1 \leqq l \leqq d)\,,$$

$$D_j \sim M_j - M_{j-1} - L_{U_j} \qquad (r+1 \leqq j \leqq n,\ j \neq i(l))\,,$$

where

$$I_i=[i(k,l), j(k-1,l)] \qquad (i=v(k,l))\,, \quad =\varnothing \quad (\textit{otherwise})\,,$$

$$J_l=[s(l), v(m(l),l)-1]\,,$$

$$U_j=[v(k+1,l), v(k,l)-1] \qquad (j=i(k,l))\,, \quad =\varnothing \quad (\textit{otherwise})\,.$$

(4.11.2) *The intersection numbers among curves are given by,*

$$C_{s(l)}D_{i(l)}=1\,, \quad \textit{while} \quad C_iD_j=0 \qquad ((i,j)\neq(s(l),i(l)))\,,$$

$$C_iC_{i+1}=1\,, \quad \textit{while} \quad C_iC_j=0 \qquad (i\neq j, j\pm 1)\,,$$

$$D_jD_{j+1}=1 \ (i(l)\leqq j \leqq i(l+1)-2)\,, \quad D_iD_j=0 \quad (\textit{otherwise}, i\neq j)\,.$$

(4.11.3) *Let $\mathscr{D}_l=\sum_{j=i(l)}^{i(l+1)-1} D_j$ and $\mathscr{C}_l=\sum_{i=s(l)+1}^{s(l+1)} C_i$. Then we have $C=\mathscr{C}_1+\cdots+\mathscr{C}_d$, $\mathscr{D}=C+\mathscr{D}_1+\cdots+\mathscr{D}_d$. Moreover $\mathscr{C}_{l-1}$ (resp. $\mathscr{D}_l$) is a connected curve containing $C_{s(l)}$ (resp. $D_{i(l)}$) subject to $\mathscr{C}_d\mathscr{D}_1=1$ and $\mathscr{C}_i\mathscr{D}_j=\delta_{i,j-1}$ $((i,j)\neq(d,1))$.*

(4.11.4) $(-C^2_{s(l+1)}, -C^2_{s(l+1)-1}, \cdots, -C^2_{s(l)+1})$

$$=(\underbrace{2, \cdots, 2}_{\lambda(l)}, p_1, 2, \cdots, p_k, \underbrace{2, \cdots, 2}_{(q_k-3)}, p_{k+1}, \cdots, p_m, \underbrace{2, \cdots, 2}_{(q_m-3)}),$$

(4.11.5) $(-D^2_{i(l)}, -D^2_{i(l)+1}, \cdots, -D^2_{i(l+1)-1})$

$$=(q_m-1, \underbrace{2, \cdots, 2}_{(p_m-3)}, q_{m-1}, 2, \cdots, q_k, \underbrace{2, \cdots, 2}_{(p_k-3)}, q_{k-1}, \cdots, q_1, \underbrace{2, \cdots, 2}_{(p_1-3)}),$$

where $p_k := p(k, l) = -C^2_{v(k,l)} = j(k-1, l) - i(k, l) + 3 \geqq 3$ $(1 \leqq k \leqq m(l))$, $q_k := q(k, l) = -D^2_{i(k,l)} = v(k, l) - v(k+1, l) + 2 \geqq 3$ $(1 \leqq k \leqq m(l)-1)$, $q_m := q(m(l), l) = 1 - D^2_{i(l)} = v(m(l), l) - s(l) + 2 \geqq 3$, $m := m(l)$ *and* $\lambda(l) := s(l+1) - v(1, l)$. *The integers* p_k, q_k, m *and* $\lambda(l)$ *depend on* l. *The dual graph of n curves is as in Figure* 4.11.

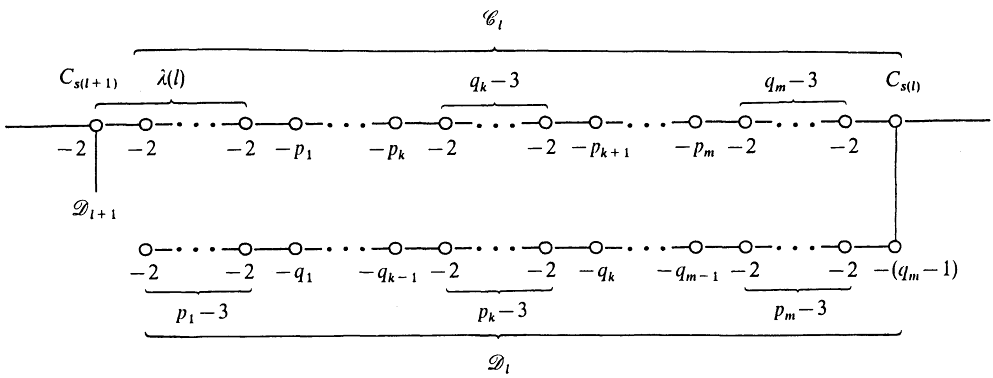


FIGURE 4.11

The dual graphs thus obtained are among dual graphs of curves on surfaces with global spherical shells. See [15, (3.2)]. In (4.12)–(4.18) below, we prove (4.11).

(4.12) For simplicity we consider the case where there are exactly two irreducible curves D and E such that $CD = CE = 1$. A similar argument proves (4.11) in the case where there are three or more irreducible curves meeting C. By (2.18), we may assume $C_1 D = 1$ and $C_{u+1} E = 1$ for $1 < u \leqq r$. Then by (2.17) we have

$$C_i \sim L_i - L_{i-1} - M_{I_i}, \quad D \sim M_{r+1} - L_{[1,a-1]}, \quad E \sim M_{R+1} - L_{[u+1,u+b-1]}$$

for some R $(\geqq r+1)$ and $a, b \geqq 2$, where $L_0 = L_r$.

(4.13) LEMMA. *Let* D' *be a reduced (possibly reducible) curve which contains none of* D, E *and* C_i $(1 \leqq i \leqq r)$. *Suppose that* $D' \sim M_j - M_k - L_J$ *for* $j, k \geqq r+1$ *and* $J \subset [1, r]$ *and that* $j \in I_v$, $k \in I_\mu$ *and* $\mu \leqq v < \mu + r$. *Then* $J = [\mu, v-1]$, $J \cap [1, a-1] = \varnothing$ *and* $J \cap [u+1, u+b-1] = \varnothing$.

PROOF. One sees $J=[\mu, \nu-1]$ in the same manner as in (4.4). By (3.5), if $k=r+1$ (or resp. $R+1$), then $J\cap[1, a-1]=\varnothing$ (resp. $J\cap[u+1, u+b-1]=\varnothing$). The other assertions are clear. q.e.d.

(4.14) LEMMA. *Let D', D'' be irreducible curves different from D, E, C_i. Suppose that $D'\neq D''$, $D'\sim M_i-M_j-L_I$ and $D''\sim M_k-M_l-L_J$. Then $I\cap J=\varnothing$.*

PROOF. The same as in (4.5). q.e.d.

For convenience, we employ the following notation:

$$D_j\sim M_j-M_{k_j}-L_{U_j} \quad \text{if} \quad j\in I_\nu \ (2\leqq\nu\leqq u+1)$$

$$E_j\sim M_j-M_{k_j}-L_{U_j} \quad \text{if} \quad j\in I_\nu \ (u+2\leqq\nu\leqq r+1)\,.$$

Let G_k $(1\leqq k\leqq m-1)$ and H_k $(1\leqq k\leqq l-1)$ be all the irreducible curves on S such that $G_k=D_{i_k}$, $H_k=E_{i'_k}$, $U_{i_k}\neq\varnothing$ and $U_{i'_k}\neq\varnothing$. We note $U_{i_k}\subset[a, u]$. Indeed, $U_{i_k}=[\mu, \nu-1]$, where $G_k\sim M_{i_k}-M_{j_k}-L_{U_{i_k}}$, $i_k\in I_\nu$ and $j_k\in I_\mu$. Since $1\leqq\nu-1\leqq u$, U_{i_k} is contained in $[a, u]$ by (4.13). Similarly $U_{i'_k}\subset[u+b, r]$. Hence we may assume

$$U_{i_k}=[\mu_k, \nu_k-1]\,, \quad U_{i'_k}=[\mu'_k, \nu'_k-1]\,,$$

$$a\leqq\mu_{m-1}<\nu_{m-2}\leqq\mu_{m-2}<\cdots\leqq\mu_2<\nu_2\leqq\mu_1<\nu_1\leqq u+1\,,$$

$$u+b\leqq\mu'_{l-1}<\nu'_{l-2}\leqq\mu'_{l-2}<\cdots\leqq\mu'_2<\nu'_2\leqq\mu'_1<\nu'_1\leqq r+1\,.$$

We let $\nu_{m+1}=\mu_m=1$, $\nu_m=a$, $\nu'_{l+1}=\mu'_l=u+1$, $\nu'_l=u+b$, $G_m=D_{r+1}=D$ and $H_l=E_{R+1}=E$. We note $U_j=\varnothing$ for $j\neq i_k$, i'_k.

(4.15) LEMMA. $\nu_k=\mu_{k-1}$ $(1\leqq k\leqq m)$, $\nu'_k=\mu'_{k-1}$ $(1\leqq k\leqq l)$, $U_{i_k}=[\nu_{k+1}, \nu_k-1]$ $(2\leqq k\leqq m)$, $U_{i'_k}=[\nu'_{k+1}, \nu'_k-1]$ $(2\leqq k\leqq l)$.

PROOF. The same as in (4.8). q.e.d.

(4.16) LEMMA. $D_pE_q=0$.

PROOF. We note $I_\lambda=\varnothing$ for $\lambda\neq\nu_k$ $(1\leqq k\leqq m)$ and $\lambda\neq\nu'_k$ $(1\leqq k\leqq l)$ in view of (4.7). Let $I=\bigcup_{2\leqq\lambda\leqq u+1}I_\lambda$ and $J=\bigcup_{u+2\leqq\lambda\leqq r+1}I_\lambda$. First we show that $p\in I$ (resp. $p\in J$) if and only if $k_p\in I$ (resp. $k_p\in J$). If $p\neq i_k$, i'_k, then by $C_\lambda D_p=0$, we see $p\in I_\lambda$ if and only if $k_p\in I_\lambda$, whence $p\in I$ if and only if $k_p\in I$. If $p=i_k$, then $k<m$, and $U_p=[\mu_k, \nu_k-1]$, $p\in I_{\nu_k}$ and $k_p\in I_{\mu_k}=I_{\nu_{k+1}}$. Hence $k_p\in I$. If $p=i'_k\in J$, then one sees $k_p\in J$ similarly. Now assume that $D_p\neq D$ and $E_q\neq E$. Since $U_{i_p}\cap U_{i'_q}=\varnothing$ by (4.14), we have $D_pE_q=(M_p-M_{k_p})\times(M_q-M_{k_q})$, whence $D_pE_q=0$. Since $r+1\in I_a$, $R+1\in I_{u+b}$, $k_p\in I$ and $k_q\in J$, we have by (4.13) $DE_q=M_{r+1}(M_q-M_{k_q})=0$, $D_pE=(M_p-M_{k_p})M_{R+1}=0$. q.e.d.

(4.17) LEMMA. *By indexing suitably, we have*

$$D_j\sim M_j-M_{j-1} \ (j\neq i_k, r+1\leqq j\leqq R)\,, \quad E_j\sim M_j-M_{j-1} \ (j\neq i'_k, R+1\leqq j\leqq n)\,,$$

$$G_k\sim M_{i_k}-M_{j_k}-L_{[\nu_{k+1}, \nu_k-1]} \ (1\leqq k\leqq m-1)\,, \quad G_m\sim M_{i_m}-L_{[\nu_{m+1}, \nu_m-1]}\,,$$

$$H_k \sim M_{i'_k} - M_{j'_k} - L_{[\nu'_{k+1},\, \nu'_k - 1]}\ (1 \leqq k \leqq l-1)\,, \quad H_l \sim M_{i'_l} - L_{[\nu'_{l+1},\, \nu'_l - 1]}\,,$$
$$C_i \sim L_i - L_{i-1} - M_{I_i}\ (1 \leqq i \leqq s)\,, \quad I_{\nu_k} = [i_k, j_{k-1}]\ (1 \leqq k \leqq m)\,,$$
$$I_{\nu'_k} = [i'_k, j'_{k-1}]\ (1 \leqq k \leqq l)\,, \quad \textit{while} \quad I_\lambda = \varnothing\ (\textit{otherwise})$$

and

$$i_k = j_k + 1\,, \quad i_m = r + 1 < i_{m-1} < \cdots < i_1 \leqq j_0 = R\,,$$
$$i'_k = j'_k + 1\,, \quad i'_l = R + 1 < i'_{l-1} < \cdots < i'_1 \leqq j'_0 = n\,.$$

PROOF. First we let $j_0 := R$ and $j'_0 := n$. Since $D_p E_q = 0$, we can apply the same argument as in (4.8) and (4.9) to $\sum_{p \in I} D_p$ and $\sum_{q \in J} E_q$. Hence we infer the above expressions for D_j, E_j, G_k and H_k. In the same manner as in (4.10), we can show that I_{ν_k} $(1 \leqq k \leqq m)$ (resp. $I_{\nu'_k}$ $(1 \leqq k \leqq l)$) contains $[i_k, i_{k-1} - 1]$ (resp. $[i'_k, i'_{k-1}]$), whence the union of I_{ν_j} and $I_{\nu'_k}$ $(1 \leqq j \leqq m,\ 1 \leqq k \leqq l)$ contains $[i_p, i_{p-1} - 1]$ and $[i'_q, i'_{q-1} - 1]$ for any p and q, hence it contains $[r+1, n]$. This proves $I_{\nu_k} = [i_k, i_{k-1} - 1]$ and $I_{\nu'_k} = [i'_k, i'_{k-1} - 1]$.

q.e.d.

(4.18) Compare (4.17) with (4.11) by setting

$$m(1) = m\,, \quad m(2) = l\,, \quad s(1) = 1\,, \quad s(2) = u + 1\,, \quad s(3) = r + 1\,,$$
$$\nu(m(1), 1) = a\,, \quad \nu(m(2), 2) = u + b\,, \quad \nu(k, 1) = \nu_k\,, \quad \nu(k, 2) = \nu'_k\,,$$
$$i(k, 1) = i_k\,, \quad i(k, 2) = i'_k\,, \quad j(k, 1) = j_k\,, \quad j(k, 2) = j'_k\,.$$

Thus we complete the proof of the first half of (4.11). The rest is easy to check. Since the argument in the general case is similar, we omit the details.

(4.19) PROBLEM. Is a VII_0 surface special if it has a cycle of rational curves? Does the equality $r = s$ in (3.6) hold?

5. Surfaces with global spherical shells.

(5.1) DEFINITION (cf. [7]). A nonempty subset Σ of a compact complex surface S is called *a global spherical shell* if

(5.1.1) Σ is isomorphic to a shell $S_\varepsilon = \{x \in \boldsymbol{C}^2\,;\ 1 - \varepsilon < \| x \| < 1 + \varepsilon\}$ for some ε $(0 < \varepsilon < 1)$,

(5.1.2) the complement of Σ in S is connected.

(5.2) THEOREM (Ma. Kato, see also [4], [8]). *Any surface with a global spherical shell is special.*

PROOF. We freely use the notation in [7, pp. 47–49, 54, 55]. Let X be a minimal surface with a global spherical shell. Then X is constructed as follows (cf. [7, p. 55]): Let $\sigma : Z^*_\varepsilon \to B_\varepsilon$ be a finite succession of blowing-ups, and $N' = \sigma^{-1}(S_\varepsilon)$. Let $\zeta :\ B_\varepsilon \to Z^*_\varepsilon \setminus N'$

be an embedding of B_ε, D_ε the image of B_ε, $N'':=\zeta(S_\varepsilon)$, and $K:=\zeta(B_{-\varepsilon})$, where $B_c=\{x\in \boldsymbol{C}^2;\ \|x\|<1+c\}$, $S_c=\{x\in \boldsymbol{C}^2;\ 1-c<\|x\|<1+c\}$. Let g_0 be the mapping $g_0=\zeta\circ\sigma: Z_\varepsilon^*\to Z_\varepsilon^*$ and let $g'=g_{0|N'}$. Then X is isomorphic to a quotient space E/g' with $E:=Z_\varepsilon^*\setminus K$. We may identify g_0, E, N' and N'' with g, E_0, N_{-1} and N_0 in [7, p. 47]. We see $b_1(X)=1$, $b_2(X)=b_2(X\setminus\Sigma)=b_2(Z_\varepsilon^*)$ (see [7, p. 47]). Let $n=b_2(X)$ (>0). By [7, Lemma 1 (ii)], there is a unique fixed point O^* of g_0 in Z_ε^*. Since $b_2(Z_\varepsilon^*)=n$, the maximal compact analytic subset A of dimension one consists of n rational curves, say, $A=A_1+\cdots+A_n$. The curve A is with normal crossing. Hence there are at most two A_j's passing through O^*, say, A_1 and A_2. Hence by [7, Lemma 1], there is a large integer l such that $g_0^l(Z_\varepsilon^*)\cap A_j=\emptyset$ for $3\leqq j\leqq n$. Let $\hat{E}=E_0\cup E_1\cup\cdots\cup E_{l-1}$, $\hat{Z}=\hat{E}\cup_\zeta B_\varepsilon$ (by identifying N_{l-1} with B_ε as in [7, p. 48]). Then we have natural mappings $f\colon \hat{E}\to X$ and $h=g_0^l\colon \hat{Z}\to Z_\varepsilon^*$. Since $\hat{Z}$ and Z_ε^* are strongly pseudoconvex manifolds with their boundaries $\partial\hat{Z}$ and ∂Z_ε^* standard spheres, the Remmert reduction $\mathrm{Rem}\colon \hat{Z}\to B$ (resp. $\sigma=\mathrm{Rem}_\varepsilon\colon Z_\varepsilon^*\to B_\varepsilon$) is a finite succession of blowing-ups of an open ball B (resp. B_ε). The open balls B and B_ε are naturally isomorphic near their boundaries (by the mapping induced from h), hence isomorphic globally. Hence h is a finite succession of blowing-ups of Z_ε^*. Hence we have proper transforms $[A_j]$ of A_j on $\hat{Z}$. By the choice of l, $[A_j]$'s $(3\leqq j\leqq n)$ are contained in $\hat{E}$. Thus we have $n-2$ curves $D_j:=f([A_j])$ $(3\leqq j\leqq n)$ on X. Hence the cardinality $\rho_r(X)$ of the set of rational curves on X is not less than $b_2(X)-2$. Consider an unramified triple covering X^* of X. Then X^* contains a global spherical shell. Hence $b_1(X^*)=1$ and $\rho_r(X^*)\geqq b_2(X^*)-2$. Since $\rho_r(X^*)=3\rho_r(X)$ and $b_2(X^*)=3b_2(X)$, we have $\rho_r(X)\geqq b_2(X)$. By a theorem of Kato [14, (3.5)], we have $\rho_r(X)=b_2(X)$.
q.e.d.

(5.3) THEOREM (cf. [7], [16]). *Any minimal surface with a global spherical shell is a (global) deformation of a blown-up primary Hopf surface.*

PROOF. By (5.2), the surface is special. Hence it has a cycle of rational curves. Hence by (1.6), it is a (global) deformation of a blown-up primary Hopf surface.
q.e.d.

In view of (5.2), either the dual graph of b_2 rational curves on a surface with a global spherical shell is one of (3.8), (3.9), (4.2) and (4.11), or the surface is one of the well-understood surfaces (2.1). We now prove the converse:

(5.4) THEOREM. *Let Γ be one of the weighted dual graphs with n vertices in* (3.8), (3.9), (4.2) *and* (4.11). *Then there exists a special* VII_0 *surface with $b_2=n$, having Γ as the weighted dual graph for n rational curves on it.*

We prove this in (5.7)–(5.12) below by constructing a minimal surface with a global spherical shell which has the desired property. See Figure 5.4 (ii), (iv), (vi), (viii) and (xii).

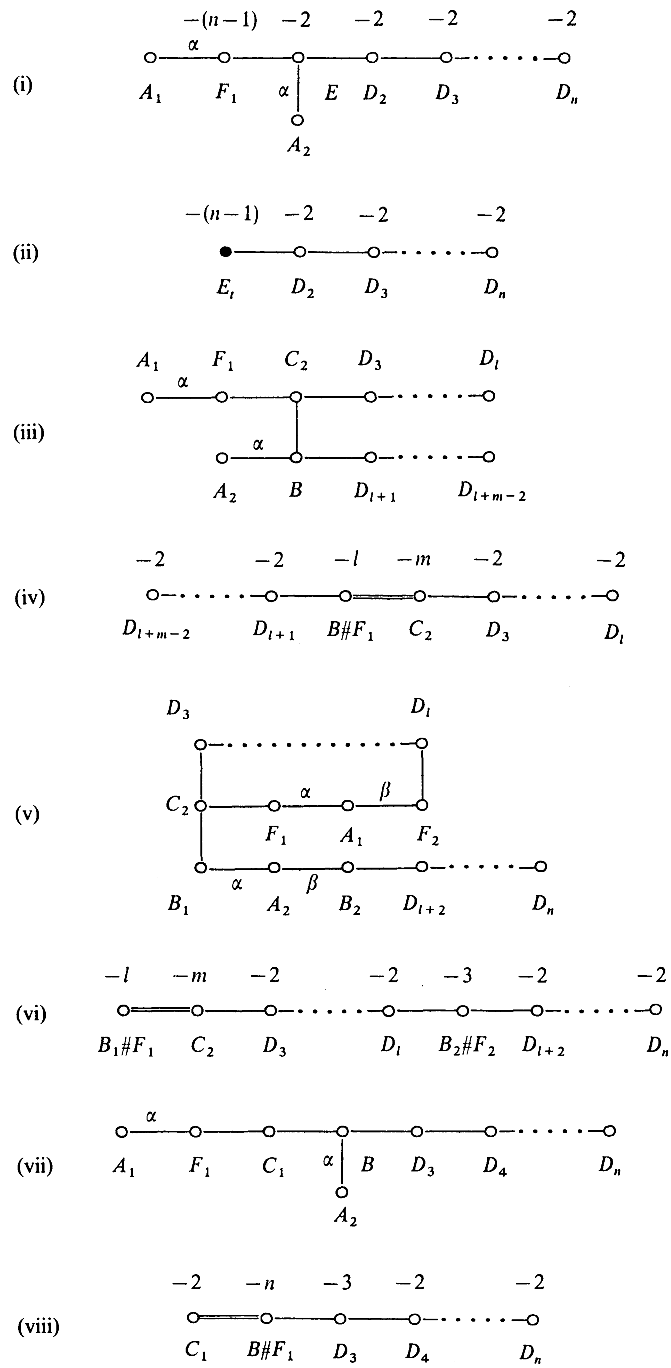
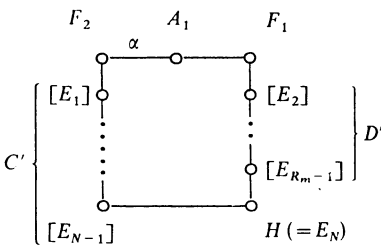
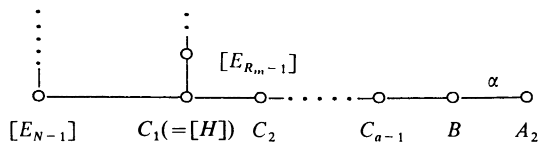

(i)
−(n−1) −2 −2 −2 −2
α
A1 F1 α E D2 D3 Dn
A2
(ii)
−(n−1) −2 −2 −2
Et D2 D3 Dn
(iii)
A1 F1 C2 D3 Dl
α
α
A2 B Dl+1 Dl+m−2
(iv)
−2 −2 −l −m −2 −2
Dl+m−2 Dl+1 B#F1 C2 D3 Dl
(v)
D3 Dl
α β
C2
F1 A1 F2
α β
B1 A2 B2 Dl+2 Dn
(vi)
−l −m −2 −2 −3 −2 −2
B1#F1 C2 D3 Dl B2#F2 Dl+2 Dn
(vii)
α
A1 F1 C1 α B D3 D4 Dn
A2
(viii)
−2 −n −3 −2 −2
C1 B#F1 D3 D4 Dn

FIGURE 5.4

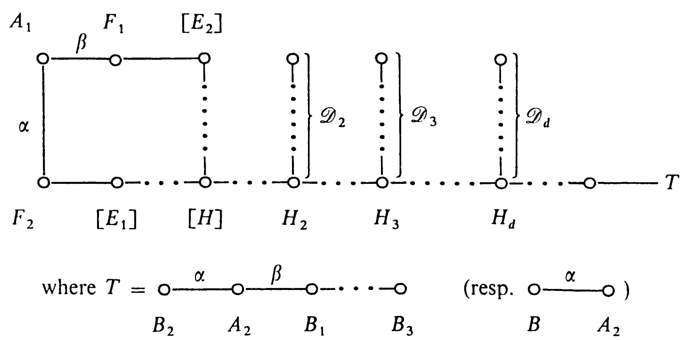
(ix)



(x)



(xi)



(xii)

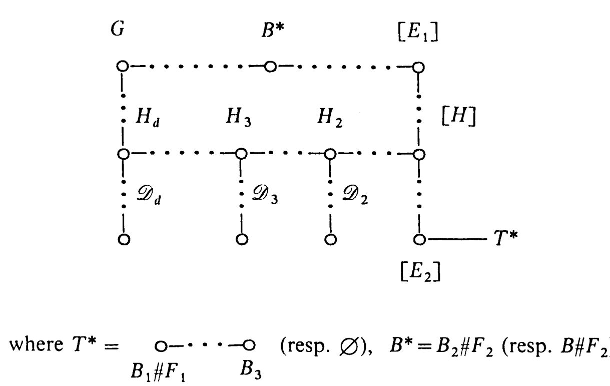


FIGURE 5.4 (Continued)

(5.5) DEFINITION (cf. [15], [16]). A quadruple (X, A_1, A_2, ψ) is said to be *admissible* if X is a nonsingular rational surface, A_k is a nonsingular rational curve with $A_1^2=1$, $A_2^2=-1$, $A_1A_2=0$ and ψ is an isomorphism of A_1 onto A_2.

The quadruple is said to be *minimal* if any (-1)-rational curve meets either A_1 or A_2.

(5.6) THEOREM. *Let (X, A_1, A_2, ψ) be a minimal admissible quadruple. Then there exists a proper flat family $\pi: \mathscr{S} \to \Delta$ over the disc such that*

(5.6.1) $\mathscr{S}_0 \cong X$ modulo ψ *with a double curve* $A \cong A_2 = \psi(A_1)$,

(5.6.2) $\mathscr{S}_t$ $(t \neq 0)$ *is a* VII_0 *surface with a global spherical shell,*

(5.6.3) *there exists an open neighborhood $\mathscr{U}$ of C in $\mathscr{S}$ such that $\mathscr{S} \setminus \mathscr{U}$ is Δ-isomorphic to $(\mathscr{S}_0 \setminus \mathscr{U}_0) \times \Delta$.*

See [16, (4.2)] and [17] for the proof.

(5.7) In the remainder of §5, we apply (5.6) to a suitable minimal admissible quadruple so as to construct a special VII_0 surface with a desired graph. Let $[X_0, X_1, X_2]$ be the homogeneous coordinate of $\boldsymbol{P}^2$ with $p_0=[1,0,0]$ and $l_k=\{X_k=0\}$.

There is an $(n+1)$-fold blowing-up $\sigma: X \to \boldsymbol{P}^2$ such that

$$\sigma^{-1}(p_0)=A_2+E+D_2+D_3+\cdots+D_n\,, \quad A_1=[l_0]\,,$$

$$A_2=E_{n+1}\,, \quad D_k=[E_{n-k+1}]\ (2 \leqq k \leqq n)\,, \quad E=[E_n]\,, \quad F_1=[l_1]\,,$$

$$A_1^2=1\,, \quad A_2^2=-1\,, \quad D_j^2=E^2=-2\,, \quad F_1^2=-(n-1)\,,$$

where $[H]$ stands for a proper transform of H, E_k the k-th (-1)-curve $(1 \leqq k \leqq n+1)$ (that is, the (-1)-curve arising from the k-th blowing-up) and the dual graph of these curves is as in Figure 5.4 (i), where we denote the points $A_1 \cap F_1$ and $A_2 \cap E$ by two α's. Let ψ be an isomorphism of A_1 onto A_2 with $\psi(A_1 \cap F_1)=A_2 \cap E$.

Consider a proper flat family $\pi: \mathscr{S} \to \Delta$ in (5.6) for the quadruple (X, A_1, A_2, ψ). By a suitable choice of π (cf. [16, (4.2)]), we have π-flat divisors $\mathscr{D}$ and $\mathscr{E}$ of $\mathscr{S}$ such that $\mathscr{D}_t=D_2+D_3+\cdots+D_n$ for any t, $\mathscr{E}_0=E+F_1$ and $\mathscr{E}_t$ is a rational curve with a node $(t \neq 0)$. This can be checked as follows: $\mathscr{S}$ is a complex manifold of dimension three and is covered with open charts V^λ near the double curve C (cf. (5.6), [16, (4.2)]). Let V be one of V^λ and let $W:=V \cap \mathscr{S}_0$. Then by the construction of the family $\mathscr{S}$, the normalization $\tilde{W}$ of W consists of two connected components W_1 and W_2. Then W_k is an open chart in the normailzation X of $\mathscr{S}_0$ such that

$$V=\{(x,y,z,t); xy=t\}\,, \quad W=\{(x,y,z); xy=0\}\,, \quad W_1=\{(x,z_1)\}\,, \quad W_2=\{(y,z_2)\}\,,$$

where the projection π (resp. the isomorphism ψ) is given by $\pi(x,y,z,t)=t$ (resp. $\psi(z_1)=z_2$). The chart W_1 (resp. W_2) is embedded into V by $(x,y,z,t)=(x,0,z_1,0)$ (resp. $=(0,y,z_2,0)$). Moreover the curvees A_1 and A_2 (resp. F_1 and E) of X are defined on W_1 (resp. on W_2) by

$$A_1 \cap W_1 = \{x=0\}\,,\quad F_1 \cap W_1 = \{z_1 = 0\}\,,$$
$$A_2 \cap W_2 = \{y=0\}\,,\quad E \cap W_2 = \{z_2 = 0\}\,.$$

Then the divisor $\mathscr{E}$ of $\mathscr{S}$ is locally just $\{xy=t, z=0\}$ in V. One sees readily that $\mathscr{E}_0$ has a unique singular point $(0,0,0,0)$ in V and that $\mathscr{E}_t$ $(t\neq 0)$ is smooth in V. Since E and F_1 meet transversally at a point $(=: q)$ different from p (cf. (5.6.3)), $\mathscr{E}_t$ $(t \neq 0)$ is a rational curve with a node q.

(5.7.1) LEMMA. *$\mathscr{E}_t^2 = (E+F_1)^2$ for $t\neq 0$.*

PROOF. Let $f\colon X \to \mathscr{S}_0$ be the normalization. Then from the exact sequence $0 \to O_{\mathscr{S}_0} \to f_* O_X \to O_A \to 0$, we infer

$$\begin{aligned}(E+F_1)^2 &= (f^*\mathscr{E}_0)^2 = \chi(X, f^*\mathscr{E}_0) + \chi(X, -f^*\mathscr{E}_0) - 2\chi(X, O_X)\\ &= \chi(\mathscr{S}_0, \mathscr{E}_0) + \chi(\mathscr{S}_0, -\mathscr{E}_0) - 2\chi(\mathscr{S}_0, O_{\mathscr{S}_0}) + \chi(A, \mathscr{E}_0 \otimes O_A)\\ &\quad + \chi(A, -\mathscr{E}_0 \otimes O_A) - 2\chi(A, O_A)\\ &= \chi(\mathscr{S}_t, \mathscr{E}_t) + \chi(S_t, -\mathscr{E}_t) - 2\chi(\mathscr{S}_t, O_{\mathscr{S}_t}) = (\mathscr{E}_t^2)\,.\end{aligned}$$

q.e.d.

Since $b_1(\mathscr{S}_t)=1$ and $b_2(\mathscr{S}_t)=n$ by [16, (3.4)], the curves D_k $(1\leqq k \leqq n-1)$ and $\mathscr{E}_t$ are all the irreducible rational curves on $\mathscr{S}_t$ $(t\neq 0)$. As was shown above, we have $D_k^2 = -2$ and $\mathscr{E}_t^2 = -(n-1)$, whence $\mathscr{S}_t$ is minimal. Thus $\mathscr{S}_t$ is a special VII_0 surface with the dual graph of b_2 $(=n)$ curves as in Figure 5.4 (ii), which is the dual graph in Figure 3.8.

It is now clear how to get in general a graph on $\mathscr{S}_t$ from a dual graph of curves for a (minimal) admissible quadruple (X, A_1, A_2, ψ).

In what follow, we use the following notation: If we are given a π-flat divisor $\mathscr{B}$ of $\mathscr{S}$ such that $\mathscr{B}_0 = B_1 + \cdots + B_r$ for B_i irreducible, while $\mathscr{B}_t$ $(t\neq 0)$ is irreducible, then we write $\mathscr{B}_t = B_1 \# B_2 \# \cdots \# B_r$. By a straightforward generalization of (5.7.1), we get $(B_1\#\cdots\#B_r)^2 = (B_1 + \cdots + B_r)^2$.

(5.8) There is an $(n+1)$-fold blowing-up $\sigma\colon X \to \boldsymbol{P}^2$ such that

$$\sigma^{-1}(p_0) = A + B + C_2 + D_3 + \cdots + D_n\,,\quad A_1 = [l_0]\,,\quad A_2 = E_{n+1}\,,$$
$$B = [E_n]\,,\quad C_2 = [E_{l-1}]\,,\quad D_j = [E_{l-j+1}]\ (3\leqq j \leqq l)\,,$$
$$D_j = [E_{2l+m-j-2}]\ (l+1 \leqq j \leqq l+m-2)\,,\quad F_1 = [l_1]\,,$$
$$B^2 = D_j^2 = -2\ (3 \leqq j \leqq n)\,,\quad C_2^2 = -m\,,\quad F_1^2 = -(l-2)\,,$$

where $n = l+m-2$, l, $m \geqq 3$, and the dual graph of these curves is as in Figure 5.4 (iii).

We take an isomorphism ψ of A_1 onto A_2 such that $\psi(A_1 \cap F_1) = A_2 \cap B$, which is indicated by two α's. This means that $\psi(\alpha) = \alpha$, when α is viewed as an intersection of curves denoted by two vertices connected by the edge α.

Therefore by the rule for deducing Figure 5.4 (ii) from Figure 5.4 (i), we obtain a dual graph of n curves on $\mathscr{S}_t$ $(t\neq 0)$ as in Figure 5.4 (iv), where $C_1:=B\# F_1$, $C_1^2=-l$, $C_2^2=-m$, $D_j^2=-2$. The dual graph for $m\geqq 3$ is as in Figure 3.9 (i), while we obtain the graph in Figure 3.9 (iv) by taking $m=2$.

(5.9) There is an $(n+1)$-fold blowing-up $\sigma: X\to \boldsymbol{P}^2$ such that

$$\sigma^{-1}(p_0)=A_2+B_1+B_2+C_2+D_3+D_4+\cdots+D_n\,,$$
$$A_1=[l_0]\,,\quad A_2=E_{n+1}\,,\quad B_1=[E_n]\,,\quad B_2=[E_{n-1}]\,,$$
$$C_2=[E_{l-1}]\,,\quad F_k=[l_k]\,,\quad D_j=[E_{l-j+1}]\ (3\leqq j\leqq l)\,,$$
$$D_j=[E_{2l+m-j-2}]\ (l+2\leqq j\leqq l+m-2)\,,$$
$$B_1^2=-2\,,\quad B_2^2=-3\,,\quad C_2^2=-m\,,\quad F_1^2=-(l-2)\,,\quad F_2^2=0\,,$$
$$D_j^2=-2\ (j\neq l+1,\, 3\leqq j\leqq l+m-2)\,,$$

where $n=l+m-2$, $l, m\geqq 3$, and the dual graph of these curves is as in Figure 5.4 (v). Hence by the rule in (5.7), we have a dual graph of n curves on $\mathscr{S}_t$ $(t\neq 0)$ as in Figure 5.4 (vi), where $C_1:=B_1\# F_1$, $D_{l+1}:=B_2\# F_2$, $C_1^2=-l$, $C_2^2=-m$, $D_{l+1}^2=-3$, $D_j^2=-2$ $(j\neq l+1)$. This is the graph in Figure 3.9 (ii).

(5.10) There is an $(n+1)$-fold blowing-up $\sigma: X\to \boldsymbol{P}^2$ such that

$$\sigma^{-1}(p_0)=A_2+C_1+B+D_3+\cdots+D_n\,,\quad A_1=[l_0]\,,\quad A_2=E_{n+1}\,,$$
$$B=[E_n]\,,\quad C_1=[E_{n-1}]\,,\quad D_k=[E_{n-k+1}]\ (3\leqq k\leqq n)\,,\quad F_1=[l_1]\,,$$
$$B^2=C_1^2=D_k^2=-2\ (4\leqq k\leqq n)\,,\quad D_3^2=-3\,,\quad F_1^2=-(n-2)\,,$$

and the dual graph of these curves is as in Figure 5.4 (vii). Therefore by the rule in (5.7), we obtain a dual graph of n curves on $\mathscr{S}_t$ $(t\neq 0)$ as in Figure 5.4 (viii), where $C_2:=B\# F_1$, $C_1^2=-2$, $C_2^2=-n$, $D_3^2=-3$, $D_k^2=-2$ $(k\neq 3)$. This is the graph in Figure 3.9 (iii).

(5.11) Next we construct the graph in Figure 4.2. There is a two dimensional torus embedding Y with the following one dimensional orbits

$$Y\setminus(\boldsymbol{C}^*)^2=A_1+F_2+C'+H+D'+F_1\,,$$
$$A_1^2=1\,,\quad F_1^2=-1\,,\quad F_2^2=0\,,\quad H^2=-1\,,$$

and

$$C'=[E_1]+\sum_{k=1}^{m-1}\sum_{j=0}^{q_k-3}[E_{R_k+j}]+\sum_{j=0}^{q_m-4}[E_{R_m+j}]\,,$$

$$D'=\sum_{k=0}^{m}\sum_{j=0}^{p_k-3}[E_{S_k+j}]\,,\quad A_1=[l_0]\,,\quad F_k=[l_k]\,,\quad H=E_N\,,$$

The self-intersection numbers of $[E_j]$ are

$$[E_{S_k-1}]^2=-p_{k+1}\ (0\leqq k\leqq m-1)\,,\quad [E_{R_k-1}]^2=-q_k\ (1\leqq k\leqq m-1)\,,$$

$$[E_{R_m-1}]^2=-(q_m-1)\,,\quad [E_j]^2=-2\ (\text{otherwise})$$

$$(-C_i'^2)=(p_1,\underbrace{2,\cdots,2}_{(q_1-3)},p_2,\cdots,p_m,\underbrace{2,\cdots,2}_{(q_m-3)})$$

$$(-D_j'^2)=(\underbrace{2,\cdots,2}_{(p_1-3)},q_1,\underbrace{2,\cdots,2}_{(p_2-3)},q_2,\cdots,q_{m-1},\underbrace{2,\cdots,2}_{(p_m-3)},q_m)\,,$$

where $p_i,\ q_i\geqq 3,\ R_0=0,\ S_0=2,\ R_k=\sum_{i=1}^{k-1}(p_i+q_i-4)+p_k$, and $S_k=\sum_{i=1}^{k}(p_i+q_i-4)+2$. We note that $N=S_m-1$, $S_k-1=R_k+q_k-3$, and $R_k-1=S_{k-1}+p_k-3$. The dual graph of the above curves is as in Figure 5.4 (ix).

Moreover, we blow up at H successively to get the dual graph of curves as in Figure 5.4 (x), where $B^2=-2$, $C_j^2=-2$ $(1\leqq j\leqq a-1)$, $A_2^2=-1$. By using an isomorphism ψ of A_1 onto A_2 such that $\psi(A_1\cap F_2)=A_2\cap B$, we consider a proper flat family $\pi:\mathscr{S}\to\Delta$ in (5.6). Let $C_a:=B\# F_2$, $C_{a+1}:=[E_1]$, $C_{a+2}:=[E_{p_1}],\cdots,C_r:=[E_{N-1}]$, $D_{r+1}:=[E_{R_m-1}],\cdots,D_n:=[E_2]$, $C':=C_{a+1}+\cdots+C_r$ and $D':=D_{r+1}+\cdots+D_n$. Then $C=C_1+\cdots+C_a+C'$ is a cycle of rational curves and D' is the longest branch of C. Thus we obtain Figure 4.2 (ii) for $a=1+r-v_1>0$. The construction of Figure 4.2 (i) is similar. We omit the details.

(5.12) Finally we construct Figure 4.11. Since no new argument is necessary, we only give a sketch of the construction. We start with Y in (5.11). Continue to blow Y up over the previous centers. Eventually we obtain (by choosing a suitable process) a rational surface X with a graph of curves given as in Figure 5.4 (xi), where $A_1^2=1$, $A_2^2=-1$, $B_1^2=B_3^2=B^2=-2$.

Consider a minimal admissible quadruple (X, A_1, A_2, ψ) such that $\psi(A_1\cap F_2)=B_2\cap A_2$ (resp. $B\cap A_2$) and $\psi(A_1\cap F_1)=A_2\cap B_1$ (resp. $\neq B\cap A_2$). Then on $\mathscr{S}_t$ $(t\neq 0)$, we have a graph as in Figure 5.4 (xii). The graph has a unique cycle with d branches. Since $(B_1\# F_1)^2=B_1^2-1\leqq -3$ and $(B^*)^2\leqq -2$, the surface $\mathscr{S}_t$ $(t\neq 0)$ is minimal. By computing the self-intersection numbers of these curves, we see that Figure 5.4 (xii) is one of the graphs in Figure 4.11. We note that an arbitrary graph in Figure 4.11 is constructed in this way.

This completes the proof of (5.6).

(5.13) Here we take again the torus embedding Y in (5.11). With the notation in (5.11) we define $A_2=H=E_N$ and consider a minimal admissible quadruple (Y, A_1, A_2, ψ) such that $\psi(A_1\cap F_1)=A_2\cap[E_{R_m-1}]$ and $\psi(A_1\cap F_2)=A_2\cap[E_{N-1}]$.

Then by (5.6), we have two cycles A and B of rational curves of $\mathscr{S}_t$ $(t\neq 0)$ such that

$$\begin{aligned}(-A_i^2)&=(-[E_1]^2,\ -[E_{p_1}]^2,\cdots,\ -([E_{N-1}]\#F_2)^2)\\&=(p_1,\underbrace{2,\cdots,2}_{(q_1-3)},p_2,\cdots,p_m,\underbrace{2,\cdots,2}_{(q_m-3)}),\\(-B_j^2)&=(-[E_2]^2,\ -[E_3]^2,\cdots,\ -([E_{R_m-1}]\#F_1)^2)\\&=(\underbrace{2,\cdots,2}_{(p_1-3)},q_1,\underbrace{2,\cdots,2}_{(p_2-3)},q_2,\cdots,q_{m-1},\underbrace{2,\cdots,2}_{(p_m-3)},q_m).\end{aligned}$$

The surface $\mathscr{S}_t$ $(t\neq 0)$ is isomorphic to a hyperbolic Inoue surface by [14, (8.1)].

Parabolic Inoue surfaces and exceptional compactifications $S_{n,\beta,\lambda}$ appear as $\mathscr{S}_t$ $(t\neq 0)$ by taking the following Y and various ψ (see [16, p. 349]):

$$Y\setminus(C^*)^2=A_1+F_2+C'+A_2+F_1\,,\quad A_2=E_{n+1}\,,\quad C'=\sum_{k=1}^{n}[E_k]\,,$$

$$F_k=[l_k]\,,\quad A_1^2=1\,,\quad A_2^2=-1\,,\quad [E_k]^2=-2\,,\quad F_1^2=-n\,,\quad F_2^2=0\,.$$

(5.14) We take again the torus embedding Y in (5.11) and set $A_2=H=E_N$. Then we choose a minimal admissible quadruple (Y,A_1,A_2,ψ) such that $\psi(A_1\cap F_1)=A_2\cap[E_{N-1}]$ and $\psi(A_1\cap F_2)=A_2\cap[E_{R_m-1}]$. Then we have a unique cycle C on $\mathscr{S}_t$ $(t\neq 0)$. The cycle C has no branches and is given as

$$C=(C'-[E_{n-1}])+[E_{N-1}]\#F_1+(D'-[E_{R_m-1}])+[E_{R_m-1}]\#F_2\,,$$

where $([E_{N-1}]\#F_1)^2=-3$ (resp. $-(p_m+1)$) for $q_m>3$ (resp. $q_m=3$) and $([E_{R_m-1}]\#F_2)^2=-(q_m-1)$. Hence $(-C_i^2)$ is equal to

$$(p_1,\underbrace{2,\cdots,2}_{(q_1-3)},p_2,\cdots,p_m,\underbrace{2,\cdots,2}_{(q_m-4)},3,\underbrace{2,\cdots,2}_{(p_1-3)},q_1,\cdots,q_{m-1},\underbrace{2,\cdots 2}_{(p_m-3)},q_m-1)$$

or

$$(p_1,\underbrace{2,\cdots,2}_{(q_1-3)},p_2,\cdots p_{m-1},\underbrace{2,\cdots,2}_{(q_{m-1}-3)},p_m+1,\underbrace{2,\cdots,2}_{(p_1-3)},q_1,\cdots,q_{m-1},\underbrace{2,\cdots,2}_{(p_m-2)}).$$

This is the self-dual cycle (see (6.3)). If C' is irreducible and if $D'=\varnothing$, then we have a rational curve $C'\#F_1\#F_2$ with a node with $(C'\#F_1\#F_2)^2=(C'+F_1+F_2)^2=-1$. The surface $\mathscr{S}_t$ $(t\neq 0)$ is in any case a half Inoue surface by (6.2).

6. Inoue surfaces with positive b_2.

(6.1) THEOREM. *Let S be a special* VII_0 *surface with a unique cycle C of rational curves. Assume $C^2<0$ and that C has no branches. Then S is isomorphic to a half Inoue surface.*

PROOF. Let D be a divisor such that $mK_S+D=0$ in $H^2(S, \mathbf{Z})$ for some $m\in\mathbf{Z}$ with $m>0$. By (3.1) and the assumption, we have $D_{\mathrm{red}}=C$. Hence $D=\sum_i n_iC_i$. Then $(D-mC)C_i=-m(K_S+C)C_i=0$, whence $D=mC$ because (C_iC_j) is negative definite. Hence $b_2(S)=-K_S^2=-C^2$. It follows from [14, (9.3)] the S is isomorphic to a half Inoue surface. q.e.d.

(6.2) COROLLARY (cf. [15]). *Let S be a minimal surface with a global spherical shell. If S has a unique cycle C without branches and with $C^2<0$, then S is isomorphic to a half Inoue surface.*

(6.3) PROPOSITION. *Let $C=C_1+\cdots+C_n$ be the unique cycle of rational curves on the surface in* (6.1) *or* (6.2). *If $n=1$, then $C^2=-1$. If $n\geqq 2$, then there exist integers p_j $(\geqq 3)$ $(1\leqq j\leqq l+1)$, q_j $(\geqq 3)$ $(1\leqq j\leqq l)$ such that*

$$(-C_k^2)=(p_1, \underbrace{2,\cdots,2}_{(q_1-3)}, p_2,\cdots, p_l, \underbrace{2,\cdots,2}_{(q_l-3)}, p_{l+1}, \underbrace{2,\cdots,2}_{(p_1-3)}, q_1,\cdots, q_l, \underbrace{2,\cdots,2}_{(p_{l+1}-3)}).$$

PROOF. Although this follows also from (6.1) or (6.2), we give a direct proof by using (1.5), (1.7), and (1.8). If $n=1$, then $C^2=-b_2(S)=-n=-1$. Assume $n\geqq 2$. By applying (1.8) we have a canonical basis L_j $(1\leqq j\leqq n)$ of $H^2(S, \mathbf{Z})$ such that

$$K_S=L_{[1,n]}, \quad C=-L_{[1,n]}+F_2\sim -L_{[1,n]},$$

where F_2 is a flat line bundle of order two. Assume $C_j\sim L_j-L_{B_j}$ for some $B_j\subset[1,n]\setminus\{j\}$. Then by modifying [14, (6.8)] slightly, we have a canonical basis $\{N_j, M_j\ (1\leqq j\leqq n)\}$ of $H^2(S^*, \mathbf{Z})$ for an unramified double covering S^* of S such that

$$\pi^*C_j=C_j'+C_j'', \quad C_j'\sim N_j-N_{j-1}-M_{I_j'}, \quad C_j''\sim M_j-M_{j-1}-N_{I_j''}.$$

Since the involution ι of S^* transforms C_j' into C_j'' for any j, we have $\iota^*M_j=N_j$, $\iota^*N_j=M_j$, $I_j'=I_j''$ $(=:I_j)$, and $C_j\sim L_j-L_{j-1}-L_{I_j}$ $(1\leqq j\leqq n)$ on S and $\pi^*L_j=M_j+N_j$. We define $1=\nu_1<\nu_2<\cdots<\nu_m\leqq n$ by $I_{\nu_k}\neq\emptyset$, I and $I_{\nu_1}\cup\cdots\cup I_{\nu_m}=[1,n]$. Hence $C_{\nu_k}\sim L_{\nu_k}-L_{\nu_k-1}-L_{I_{\nu_k}}$, $C_\lambda\sim L_\lambda-L_{\lambda-1}$ (otherwise), where $I_{\nu_k}=[\beta_k, \beta_{k+1}-1]$ $(\subset\mathbf{Z}/n\mathbf{Z})$ and $\beta_1<\beta_2<\cdots<\beta_m$. We define ν_k and β_k $(k\in\mathbf{Z})$ by $\nu_{k+m}=\nu_k$, $\beta_{k+m}=\beta_k$. For simplicity, we further assume $C_{\nu_i}C_{\nu_j}=0$ for $i\neq j$. Then by $C_{\nu_k}C_\lambda=1$ $(\lambda=\nu_k\pm 1)$ and $C_{\nu_k}C_\lambda=0$ $(\lambda\neq\nu_k, \nu_k\pm 1)$, we see $[\beta_k, \beta_{k+1}-1]=[\nu_{j_k}, \nu_{j_k+1}-1]$ for some j_k $(1\leqq k\leqq m)$. Hence there is an l $(0\leqq l\leqq m-1)$ such that $\beta_k=\nu_{k+l}$ for any k. If $l>0$, then by $C_{\nu_k}C_{\nu_{l+k}}=0$, we have $\nu_k=\nu_{2l+1+k}$ mod n, whence $m=2l+1\geqq 3$. Letting $q_j:=\nu_{j+1}-\nu_j+2$, $p_k:=\nu_{k+l+1}-\nu_{k+l}+2$ $(1\leqq j\leqq l, 1\leqq k\leqq l+1)$, we have (6.3) with $l\geqq 1$. If $C_{\nu_i}C_{\nu_j}=1$ for some i and j, then we can prove (6.3) with some p_k or q_j equal to 3 similarly. If $l=0$, then $C_1\sim -2L_i-L_I$ and we can prove (by indexing suitably)

$$C_1\sim -2L_n-L_{[2,n-1]}, \quad C_j\sim L_j-L_{j-1} \qquad (2\leqq j\leqq n),$$

whence

$$(-C_k^2)=(n+2, \underbrace{2, \cdots, 2}_{(n-3)})=(p_1, \underbrace{2, \cdots, 2}_{(p_1-3)}) \qquad (p_1=n+2\geqq 4)\,.$$

q.e.d.

(6.4) THEOREM (cf. [7], [19]). *Any Inoue surface with $b_2>0$ contains a global spherical shell.*

PROOF. Let S be a hyperbolic Inoue surface, and let A and B be cycles of rational curves on S. Then Zykel(A) and Zykel(B) are given by [14, (6.8)]. For any pair of sequences

$$\mathrm{seq} \ =(p_1, \underbrace{2, \cdots, 2}_{(q_1-3)}, p_2, \cdots, p_n, \underbrace{2, \cdots, 2}_{(q_n-3)})$$

$$\mathrm{seq}^*=(\underbrace{2, \cdots, 2}_{(p_1-3)}, q_1, \underbrace{2, \cdots, 2}_{(p_2-3)}, q_2, \cdots, q_{n-1}, \underbrace{2, \cdots, 2}_{(p_n-3)}, q_n)$$

we have as in (5.13) a proper flat family $\pi: \mathscr{S}\to\Delta$ over the unit disc Δ such that $\mathscr{S}_t$ has two cycles A and B of rational curves with Zykel(A) = seq, Zykel(B) = seq*. By [14, (8.1)], $\mathscr{S}_t$ $(t\neq 0)$ is a hyperbolic Inoue surface isomorphic to S above. By (5.6), $\mathscr{S}_t$ contains a global spherical shell, hence so does S. The same argument applies to a half Inoue surface (resp. a parabolic Inoue surface) by using (6.2), (6.3) and (5.14) (resp. (5.1) and [14, (7.1)]). See also (5.13), [7] or [19] for parabolic Inoue surfaces. q.e.d.

(6.5) DEFINITION. Let S be a VII_0 surface with $b_2>0$. The *Dloussky number* of S is defined as

$$\mathrm{Dl}(S):=-\sum_D D^2+2\,\#(\text{rational curves with nodes})$$

with D running through all irreducible curves on S (see [4, p. 43]).

(6.6) LEMMA (cf. [4], [16]). *Let S be a special* VII_0 *surface with a cycle C with branches. Then* $\mathrm{Dl}(S)=3b_2(S)-d-\sum_{l=1}^d \lambda(l)$.

PROOF. Clear from (3.8), (3.9), (4.2) and (4.11). q.e.d.

(6.7) THEOREM. *Let S be a* VII_0 *surface with $b_2>0$. Then* $\mathrm{Dl}(S)\leqq 3b_2(S)$, *with the equality holding if and only if S is an Inoue surface with $b_2>0$.*

PROOF. It suffices to prove the assertion when S has no rational curves with nodes, by taking an unramified double covering of S if necessary. Let M be the reduced effective maximal divisor on S. Then $b_2(M)\leqq b_2(S)$ and M is with normal crossing. By [14, §4] we have an exact sequence

$$0 \to \Theta_S(-\log M) \to \Theta_S \to J_M \to 0 .$$

We see $J_M \cong \bigoplus_{D \subset M} O_D(D)$ with D ranging over all the irreducible components D of M. Hence $h^0(M, J_M)=0$ and $h^1(M, J_M)=\mathrm{Dl}(S)-b_2(M)$. We also have $h^2(S, \Theta_S(-\log M)) \leqq 2$ by [11], and $h^2(S, \Theta_S)=0$ by (1.2). On the other hand, $h^0(S, \Theta_S) \leqq 2$ by [11] and $\chi(S, \Theta_S)=-2b_2$ so that $h^1(S, \Theta_S) \leqq 2b_2+2$. This shows

$$\mathrm{Dl}(S) \leqq b_2(M)+2b_2(S)+4 \leqq 3b_2(S)+4 .$$

This inequality holds for any VII_0 surface. So we take an unramified fivefold covering S^* of S. Let M^* be the pullback of M. Then M^* is clearly the reduced effective maximal divisor of S^*. Hence

$$\mathrm{Dl}(S^*) \leqq b_2(M^*)+2b_2(S^*)+4 \leqq 3b_2(S^*)+4$$

so that $5\mathrm{Dl}(S) \leqq 15b_2(S)+4$. Therefore $\mathrm{Dl}(S) \leqq 3b_2(S)$. If moreover $\mathrm{Dl}(S)=3b_2(S)$, then $\mathrm{Dl}(S^*)=3b_2(S^*)$. Hence $15b_2(S) \leqq 5b_2(M)+10b_2(S)+4$. This implies $b_2(S)=b_2(M)$, that is, S is special. By (6.6), $\mathrm{Dl}(S)<3b_2(S)$ if S satisfies (2.2). Consequently, S is either an Inoue surface with $b_2>0$ or an exceptional compactification of an affine bundle (cf. [1] and (2.1)). However in the second case, $\mathrm{Dl}(S)=2b_2(S)<3b_2(S)$. It is easy to check that any Inoue surface with $b_2>0$ satisfies $\mathrm{Dl}(S)=3b_2(S)$ (see [5], [6]). q.e.d.

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