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DIRICHLET SERIES CORRESPONDING TO SIEGEL'S MODULAR FORMS OF DEGREE n WITH LEVEL N

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0. Introduction.

0.1. Koecher [KO] introduced Dirichlet series corresponding to Siegel's modular forms and presented explicit formulas for the Dirichlet series which express the location of the poles and the residues in a satisfactory manner, but did not succeed in proving them. Maass in his lecture notes [MA, § 15] studied those Dirichlet series in full generality and obtained the analytic continuation and the functional equations for them. His method is based upon the theory of invariant differential operators acting on real symmetric matrices, which gives a powerful tool in investigating those Dirichlet series and their functional equations. However one cannot have precise information on the residues of the poles by his method. In [AR] we have proved Koecher's explicit formulas by using Klingen's Eisenstein series and the structure theorem for the space of Siegel's modular forms due to Klingen [KL1]. Recently Weissauer [WE] studied Koecher-Maass Dirichlet series corresponding to Siegel's cusp forms with level N and solved a certain converse problem concerning the correspondence between those Dirichlet series with grössen characters and Siegel's cusp forms.

Our aim of the present paper is to prove Koecher's explicit formulas for the Dirichlet series corresponding to Siegel's modular forms (not necessarily cusp forms) with level N without using Klingen's Eisenstein series (Klingen in [KL2, p. 235] suggested the problem of obtaining Koecher's formulas without the help of Klingen's Eisenstein series). We also obtain an explicit formula for the Epstein-Koecher zeta function.

Another more arithmetic aspect of Koecher-Maass Dirichlet series is discussed in Böcherer [BÖ].

0.2. We summarize our results.

Let $\mathfrak{H}_n$ be the Siegel upper half plane of degree n, on which the Siegel modular group $\Gamma^{(n)}=Sp(n, \boldsymbol{Z})$ of degree n acts in a usual manner. Let $\Gamma_0^{(n)}(N)$ be the congruence subgroup of $\Gamma^{(n)}$ with level N given by

$$\Gamma_0^{(n)}(N)=\left\{M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\Gamma^{(n)} \,\middle|\, C\equiv 0 \bmod N\right\}. \tag{0.1}$$

For a Dirichlet character $\varepsilon \bmod N$ and a positive integer k, denote by $M_k(\Gamma_0^{(n)}(N), \varepsilon)$ the space of all holomorphic functions f on $\mathfrak{H}_n$ satisfying

(i) $f(M\langle Z\rangle)=\varepsilon(\det(D))\det(CZ+D)^k f(Z)$ for any $M=\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in\Gamma_0^{(n)}(N)$,

(ii) if $n=1$, f is holomorphic at all cusps of $\Gamma_0^{(1)}(N)$.

Each $f\in M_k(\Gamma_0^{(n)}(N),\varepsilon)$ has a Fourier expansion of the form

$$f(Z)=\sum_{T=T^{(n)}\geq 0} a_f(T)e[\mathrm{tr}(TZ)]\,, \tag{0.2}$$

where T runs over all half-integral semi-positive definite symmetric matrices of size n. Assume that $\varepsilon(-1)=(-1)^k$. Koecher [KO], Maass [MA], and Weissauer [WE] introduced the Dirichlet series $D_n(f,s)$ associated with $f\in M_k(\Gamma_0^{(n)}(N),\varepsilon)$:

$$D_n(f,s)=\sum_{\{T\}>0}\frac{a_f(T)}{\epsilon(T)\det(T)^s}\,, \tag{0.3}$$

where the summation indicates that T runs over the Γ_n-equivalence classes of half-integral positive definite symmetric matrices of size n and $\epsilon(T)$ denotes the order of the unit group $\{U\in GL_n(\mathbf{Z})\mid T[U]=T\}$ of T. Let Q be a positive definite symmetric matrix of size m. We define the Epstein-Koecher zeta function $D_n(Q;s)$ to be the sum

$$\sum_G \det(Q[G])^{-s}\,,$$

where $G=G^{(m,n)}$ runs through a complete set of non-associated integral $m\times n$ matrices with $\mathrm{rank}(G)=n$, G and G_1 being said to be associated if $G_1=GU$ with some $U\in GL_n(\mathbf{Z})$. The Dirichlet series $D_n(f,s)$ (resp. $D_n(Q;s)$) is absolutely convergent for $\mathrm{Re}(s)>k+(n+1)/2$ (resp. $\mathrm{Re}(s)>m/2$). Set

$$\xi_n(f,s)=2N^{ns/2}(2\pi)^{-ns}\Gamma_n(s)D_n(f,s)\,, \tag{0.4}$$

$$\xi_n(Q;s)=2\pi^{-ns}\Gamma_n(s)D_n(Q;s)\,, \tag{0.5}$$

where

$$\Gamma_n(s)=\prod_{\mu=1}^{n}\pi^{(\mu-1)/2}\Gamma(s-(\mu-1)/2)\,.$$

Let Φ be the Φ-operator of Siegel giving a linear map from $M_k(\Gamma_0^{(n)}(N),\varepsilon)$ to $M_k(\Gamma_0^{(n-1)}(N),\varepsilon)$. For each f of $M_k(\Gamma_0^{(n)}(N),\varepsilon)$, we set

$$f\mid\omega_N^{(n)}(Z)=N^{nk/2}\det(NZ)^{-k}f(-(NZ)^{-1})\qquad(Z\in\mathfrak{H}_n)\,,$$

which is an element of $M_k(\Gamma_0^{(n)}(N),\bar{\varepsilon})$. Set, for any positive integer r,

$$v(r)=\begin{cases}\prod_{\nu=2}^{r}\pi^{-\nu/2}\zeta(\nu)\Gamma(\nu/2)\cdots r\geq 2\,,\\ 1\cdots r=1\end{cases} \tag{0.6}$$

We obtain the following theorem.

THEOREM. (i) *Suppose* $k>n-1$ *and* $\varepsilon(-1)=(-1)^k$. *Let* $f\in M_k(\Gamma_0^{(n)}(N),\varepsilon)$ *and set* $g=f|\omega_N^{(n)}$. *Then the function* $\xi_n(f,s)$ *can be continued analytically to a meromorphic function in the whole s-plane. Moreover the following formula holds*:

$$\xi_n(f,s)=I_n(f,s)+v(n)\left(\frac{i^{nk}a_g(0)}{s-k}-\frac{a_f(0)}{s}\right)$$
$$+\frac{1}{2}\sum_{\mu=1}^{n-1}v(n-\mu)\left(\frac{i^{nk}\xi_\mu(\Phi^{n-\mu}g,n/2)}{s-k+\mu/2}-\frac{\xi_\mu(\Phi^{n-\mu}f,n/2)}{s-\mu/2}\right),$$

where $I_n(f,s)$ *is a certain entire function of s.*

(ii) *Suppose* $m>2n-2$. *Then the function* $\xi_n(Q;s)$ *can be analytically continued to a meromorphic function in the whole s-plane and has the expression*:

$$\xi_n(Q;s)=I_n(Q;s)+v(n)\left(\frac{\det(Q)^{-n/2}}{s-m/2}-\frac{1}{s}\right)$$
$$+\frac{1}{2}\sum_{\mu=1}^{n-1}v(n-\mu)\left(\frac{\det(Q)^{-n/2}}{s-(m-\mu)/2}\cdot\xi_\mu(Q^{-1};n/2)-\frac{1}{s-\mu/2}\cdot\xi_\mu(Q;n/2)\right),$$

where $I_n(Q;s)$ *is a certain entire function of s.*

As an immediate corollary to the above theorem we obtain the functional equations for $\xi_n(f,s)$ and $\xi_n(Q;s)$ (see, for the explicit forms, Corollary 2.3 in §2).

Böcherer in his private letter suggested a possibility of obtaining some explicit formula of $\xi_n(Q;s)$ in the case of $m\leq 2n-2$. At the end of §2 we shall discuss that case a little further following his suggestion.

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NOTATION. Let, $\boldsymbol{N}$, $\boldsymbol{Z}$, $\boldsymbol{R}$, and $\boldsymbol{C}$ denote the set of positive integers, the ring of rational integers, the real number field, and the complex number field, respectively. For any commutative ring S, $M(m,n;S)$, $M_n(S)$, and $GL_n(S)$ denote the set of $m\times n$ matrices with entries in S, the ring of matrices of size n with entries in S, and the group of invertible elements in $M_n(S)$, respectively. For any element A of $M_n(S)$, let ${}^tA$, $\mathrm{tr}(A)$, $\det(A)$ denote the transposed matrix of A, the trace of A, and the determinant of A, respectively. We denote by E_n the identity matrix of $M_n(S)$. For A of $M(m,n;S)$ and B of $M_m(S)$, $B[A]$ denotes the matrix ${}^tABA$. For square matrices $A_1,\cdots,A_r$, we write $D(A_1,\cdots,A_r)$ for the matrix

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_r \end{pmatrix}.$$

For a real symmetric matrix T, $T>0$ (resp. $T\geq 0$) means that T is positive definite (resp. semi-positive definite). Let $\Gamma(s)$ and $\zeta(s)$ be the gamma function and the Riemann zeta function, respectively. The symbol $e[w]$ ($w\in \boldsymbol{C}$) is used as an abbreviation for $\exp(2\pi iw)$.

1. Siegel's modular forms and theta series. Let $GSp(n, \boldsymbol{R})$ be the real symplectic group of degree n with similitudes:

$$GSp(n, \boldsymbol{R})=\{M\in M_{2n}(\boldsymbol{R})\,|\,MJ_n{}^tM=\nu(M)J_n \quad \text{with} \quad \nu(M)>0\}\,,$$

where

$$J_n=\begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}.$$

The group $GSp(n, \boldsymbol{R})$ acts on the Siegel upper half plane $\mathfrak{H}_n=\{Z={}^tZ\in M_n(\boldsymbol{C})\,|\,\mathrm{Im}(Z)>0\}$ in a usual manner. Set

$$M\langle Z\rangle=(AZ+B)(CZ+D)^{-1} \qquad \text{and} \quad J(M, z)=\det(CZ+D)$$

for

$$M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in GSp(n, \boldsymbol{R}) \qquad \text{and} \qquad Z\in\mathfrak{H}_n\,.$$

Denote by $\Gamma^{(n)}$ the Siegel modular group $Sp(n, \boldsymbol{Z})$ of degree n and for a positive integer N let $\Gamma_0^{(n)}(N)$ be the congruence subgroup of $\Gamma^{(n)}$given by (0.1) in the introduction. Let ε be a Dirichlet character mod N. Then, ε gives rise to a character of the group $\Gamma_0^{(n)}(N)$ by

$$(M)= \ (\det(D)) \qquad \text{for} \quad M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\Gamma_0^{(n)}(N)\,.$$

Let k be a positive integer. We set, for any function $f(Z)$ on $\mathfrak{H}_n$ and $M\in GSp(n, \boldsymbol{R})$,

$$(f|_kM)(Z)=J(M, Z)^{-k}\det(M)^{k/2}f(M\langle Z\rangle)\,,$$

which is often written $f|M$ if there is no fear of confusion. Denote by $M_k(\Gamma_0^{(n)}(N), \varepsilon)$ the space of all holomorphic functions $f(Z)$ on $\mathfrak{H}_n$ which satisfy the following conditions:

(i) $f|_kM=\varepsilon(M)f$ for all $M\in\Gamma_0^{(n)}(N)$,

(ii) if $n=1$, f is holomorphic at all cusps of $\Gamma_0^{(1)}(N)$.

Then, $M_k(\Gamma_0^{(n)}(N), \varepsilon)$ is a $\boldsymbol{C}$-vector space of finite dimension. We take a particular element

$$\omega_N^{(n)}=\begin{pmatrix} 0 & -E_n \\ NE_n & 0 \end{pmatrix}$$

of $GSp(n, \boldsymbol{R})$ which normalizes the group $\Gamma_0^{(n)}(N)$: $\omega_N^{(n)}\Gamma_0^{(n)}(N)(\omega_N^{(n)})^{-1}=\Gamma_0^{(n)}(N)$. It is easily verified from the above property (i) that, if f is an element of $M_k(\Gamma_0^{(n)}(N), \varepsilon)$, then $f|\,\omega_N^{(n)}$ belongs to $M_k(\Gamma_0^{(n)}(N), \bar{\varepsilon})$, $\bar{\varepsilon}$ being the complex conjugate of ε. We denote by $M_k(\Gamma^{(n)})$

the space $M_k(\Gamma_0^{(n)}(1), 1)$, the usual space of Siegel's modular forms of degree n and weight k with respect to $\Gamma^{(n)}$. Each $f \in M_k(\Gamma_0^{(n)}(N), \varepsilon)$ satisfies the transformation formula

$$f \mid \omega_N^{(n)} \mid \omega_N^{(n)} = (-1)^{nk} f. \tag{1.1}$$

Moreover, $f \in M_k(\Gamma_0^{(n)}(N), \varepsilon)$ has a Fourier expansion of the form (0.2) in the introduction. By virtue of Theorem 2.3.4 of Andrianov [AN], the Fourier coefficients $a_f(T)$ have the estimates

$$|a_f(T)| < C_1 \det(T)^k \qquad \text{if} \quad T > 0 \tag{1.2}$$

with a certain positive constant C_1 independent of T. For each integer μ with $0 \leq \mu \leq n$, we define $f_\mu(Z)$ to be the subseries of (0.2) given by

$$f_\mu(Z) = \sum_T a_f(T) e[\mathrm{tr}(TZ)], \tag{1.3}$$

where T is taken over all $n \times n$ half-integral semi-positive definite symmetric matrices with $\mathrm{rank}(T) = \mu$. Obviously,

$$f(Z) = \sum_{\mu=0}^{n} f_\mu(Z).$$

We write especially $f^*(Z)$ instead of $f_n(Z)$:

$$f^*(Z) = \sum_{T>0} a_f(T) e[\mathrm{tr}(TZ)].$$

The Φ-operator of Siegel given by

$$\Phi f(Z_1) = \lim_{\lambda \to +\infty} f\begin{pmatrix} Z_1 & 0 \\ 0 & i\lambda \end{pmatrix} \qquad (Z_1 \in \mathfrak{H}_{n-1}, f \in M_k(\Gamma_0^{(n)}(N), \varepsilon))$$

defines a linear map from $M_k(\Gamma_0^{(n)}(N), \varepsilon)$ to $M_k(\Gamma_0^{(n-1)}(N), \varepsilon)$. For each interger μ $(1 \leq \mu \leq n)$, $\Phi^\mu f \in M_k(\Gamma_0^{(n-\mu)}(N), \varepsilon)$ has the Fourier expansion

$$\Phi^\mu f(Z_1) = \sum_{T = T^{(\mu)} \geq 0} a_f\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} e[\mathrm{tr}(TZ_1)] \qquad (Z_1 \in \mathfrak{H}_{n-\mu}).$$

Denote by Γ_n the group $GL_n(\boldsymbol{Z})$ of unimodular matrices of size n. For $r+1$ positive integers $h_1, \cdots, h_{r+1}$ with $h_1 + \cdots + h_{r+1} = n$, we set

$$\Gamma_{h_1 \cdots, h_{r+1}}^{\infty} = \left\{ (R_{\mu\nu}) \in \Gamma_n \,\middle|\, \begin{array}{ll} R_{\mu\nu} \in M(h_\mu, h_\nu; \boldsymbol{Z}) & \text{for} \quad 1 \leq \mu, \nu \leq r+1 \\ R_{\mu\nu} = 0 & \text{for} \quad 1 \leq \nu < \mu \leq r+1 \end{array} \right\},$$

which is a subgroup of Γ_n. For instance, $\Gamma_{h_1, h_2}^{\infty}$ is a subgroup of Γ_n consisting of unimodular matrices whose lower left $h_2 \times h_1$ blocks are zero. Denote by $\Gamma_n / \Gamma_{h_1, \cdots, h_{r+1}}^{\infty}$ a complete set of representatives of the left cosets of Γ_n modulo $\Gamma_{h_1, \cdots, h_{r+1}}^{\infty}$. Let $\mathfrak{P}_n$ be the symmetric space of positive definite symmetric matrices of size n, on which the general linear group $GL(n, \boldsymbol{R})$ acts via $Y \rightarrow Y[g]$ $(g \in GL(n, \boldsymbol{R}))$. We define $\mathfrak{R}_n$ to be the set of all reduced matrices in $\mathfrak{P}_n$. Then, $\mathfrak{R}_n$ is a fundamental domain of $\mathfrak{P}_n$ by the above action of Γ_n (see [MA, §9] for reduced matrices).

For $f \in M_k(\Gamma_0^{(n)}(N), \varepsilon)$ and an integer μ $(1 \leq \mu \leq n-1)$, $f_\mu(iY)$ in (1.3) has the following expression

$$(1.4) \qquad f_\mu(iY) = \sum_{U \in \Gamma_n/\Gamma_{\mu,n-\mu}^\infty} (\Phi^{n-\mu} f)^* \left(i(Y[U]) \begin{bmatrix} E_\mu \\ 0 \end{bmatrix} \right) \qquad \text{(see [KO])}.$$

The Fourier coefficients of $f \in M_k(\Gamma_0^{(n)}(N), \varepsilon)$ have the property

$$(1.5) \qquad a_f(UT{}^tU) = \varepsilon(\det(U)) \det(U)^k a_f(T) \qquad \text{for any} \quad U \in \Gamma_n .$$

Since f satisfies $f = \varepsilon(-1)^n(-1)^{nk} f$, one necessarily has

$$\varepsilon(-1)^n = (-1)^{nk} .$$

In the sequel we assume that

$$\varepsilon(-1) = (-1)^k .$$

Let Q be a positive definite symmetric matrix of size m (i.e., $Q \in \mathfrak{P}_m$). We define the theta series $\Theta_n(Q; Y)$ $(Y \in \mathfrak{P}_n)$ by the equality:

$$\Theta_n(Q; Y) = \sum_{G \in M(m,n; \boldsymbol{Z})} \exp(-\pi \mathrm{tr}(Y \cdot Q[G])) .$$

We set, for each integer μ $(0 \leq \mu \leq n)$,

$$\Theta_{n,\mu}(Q; Y) = \sum_{G \in M(m,n; \boldsymbol{Z}), \mathrm{rank}(G) = \mu} \exp(-\pi \mathrm{tr}(Y \cdot Q[G])) .$$

We understand $\Theta_{n,0}(Q, Y) = 1$, if $\mu = 0$. We see easily that

$$\Theta_{n,\mu}(Q, Y) = \sum_{U \in \Gamma_n/\Gamma_{\mu,n-\mu}^\infty} \Theta_{\mu,\mu} \left(Q; Y \left[U \begin{pmatrix} E_\mu \\ 0 \end{pmatrix} \right] \right) .$$

The theta function $\Theta_n(Q; Y)$ satisfies the well-known theta transformation formula

$$(1.6) \qquad \Theta_n(Q^{-1}; Y^{-1}) = \det(Q)^{n/2} \det(Y)^{m/2} \Theta_n(Q; Y) .$$

To prove Koecher's formulas for $\xi_n(f, s)$ $(f \in M_k(\Gamma_0^{(n)}(N), \varepsilon))$ and $\xi_n(Q; s)$ in the introduction we would like to define the functions $P_n(f, Y)$ and $P_n(Q; Y)$ on the space $\mathfrak{P}_n$ *by induction on n* as follows:

$$(1.7) \qquad P_n(f, Y) = f^*(iY) - c^{-n} \det(Y)^{-k}$$
$$\times \left\{ a_g(0) + \sum_{\mu=1}^{n-1} \sum_{U \in \Gamma_n/\Gamma_{n-\mu,\mu}^\infty} P_\mu \left(\Phi^{n-\mu} g, ((NY)[U])^{-1} \begin{bmatrix} 0 \\ E_\mu \end{bmatrix} \right) \right\},$$

where $c = N^{k/2} i^{-k}$, $g = f | \omega_N^{(n)} (\in M_k(\Gamma_0^{(n)}(N), \bar{\varepsilon}))$, and $a_g(0)$ is the first Fourier coefficient of

g in (0.2) corresponding to 0. Moreover, we define

$$(1.8)\qquad P_n(Q;\,Y)=\Theta_{n,n}(Q;\,Y)-\det(Q)^{-n/2}\det(Y)^{-m/2}$$
$$\times\left\{1+\sum_{\mu=1}^{n-1}\sum_{U\in\Gamma_n/\Gamma_{n-\mu,\mu}^{\infty}}P_\mu\left(Q^{-1};\,(Y([U])^{-1}\begin{bmatrix}0\\E_\mu\end{bmatrix}\right)\right\}.$$

The proof for the well-definedness of $P_n(f,\,Y)$, $P_n(\mathrm{Q},\,Y)$ (especially the convergence of them) is complicated and will be postponed until the last section. However we state here the explicit assertions concerning the well-definedness.

THOREM 1.1. (i) *Suppose* $\varepsilon(-1)=(-1)^k$ *and* $k>n-1$. *Let* $f\in M_k(\Gamma_0^{(n)}(N),\,\varepsilon)$ *and set* $g=f|\omega_N^{(n)}$. *Then the infinite series on the right hand side of* (1.7) *is absolutely convergent and hence the function* $P_n(f,\,Y)$ *can be defined inductively by the equality* (1.7). *Moreover,* $P_n(f,\,Y)$ *satisfies*

$$(1.9)\qquad P_n(g,\,(NY)^{-1})=c^n\det(Y)^kP_n(f,\,Y)\qquad(Y\in\mathfrak{P}_n)$$

and

$$(1.10)\qquad P_n(f,\,Y[U])=P_n(f,\,Y)\qquad\text{for any}\quad U\in\Gamma_n,\ Y\in\mathfrak{P}_n\,.$$

(ii) *Suppose* $m>2n-2$. *Then the function* $P_n(Q;\,Y)$ *can be defined inductively with respect to n by the equality* (1.8) *and the infinite series on the right hand side of* (1.8) *is absolutely convergent for any* $Y\in\mathfrak{P}_n$. *Moreover,* $P_n(Q;\,Y)$ *satisfies*

$$(1.11)\qquad P_n(Q^{-1};\,Y^{-1})=\det(Q)^{n/2}\det(Y)^{m/2}P_n(Q;\,Y)\qquad(Y\in\mathfrak{P}_n)$$

and

$$(1.12)\qquad P_n(Q;\,Y[U])=P_n(Q;\,Y)\qquad\text{for any}\quad U\in\Gamma_n,\ Y\in\mathfrak{P}_n\,.$$

We shall prove Theorem 1.1 in §3.

REMARK 1.1. If $n=1$, one immediately has

$$P_1(f,\,Y)=f^*(iY)-c^{-1}Y^{-k}a_g(0)\qquad(g=f|\,\omega_N^{(1)})\,,$$
$$P_1(Q;\,Y)=\Theta_{1,1}(Q;\,Y)-\det(Q)^{-1/2}Y^{-m/2}\,,$$

and the properties (1.9), (1.11) are verified from the transformation formulas (1.1), (1.6), respectively. Suppose $n=2$. Since the real analytic Eisenstein series

$$\sum_{U\in\Gamma_2/\Gamma_{1,1}^{\infty}}\left((Y[U])^{-1}\begin{bmatrix}0\\1\end{bmatrix}\right)^s=\sum_{U\in\Gamma_2/\Gamma_{1,1}^{\infty}}\left((Y^{-1}[U])\begin{bmatrix}1\\0\end{bmatrix}\right)^s$$

on $\mathfrak{P}_2$ is absolutely convergent for $\mathrm{Re}\,(s)>1$, the functions $P_2(f,\,Y)$ ($f\in M_k(\Gamma_0^{(2)}(N),\,\varepsilon)$) and $P_2(Q;\,Y)$ are well-defined by (1.7), (1.8), respectively. Further, it is not difficult to prove the identities (1.9), (1.11) from the transformation formulas (1.1), (1.6).

REMARK 1.2. Suppose $N=1$. Then $f|\omega_N^{(n)}=f$ for each $f\in M_k(\Gamma^{(n)})$. Denote by $S_k(\Gamma^{(n)})$ the space of cusp forms of degree n and weight k with respect to $\Gamma^{(n)}$, i.e., $S_k(\Gamma^{(n)})=\{f\in M_k(\Gamma^{(n)})\,|\,\Phi f=0\}$. For any integer r with $0\leq r\leq n-1$, let $E^k_{n,r}(Z,\varphi)$ be the Klingen Eisenstein series associated with $\varphi\in S_k(\Gamma^{(r)})$ (see [KL1]). It is not difficult to see from [AR, Main Lemma] that if $f(z)=E^k_{n,r}(Z,\varphi)$,

$$P_n(f,Y)=P^k_{n,r}(iY,\varphi)\qquad (Y\in\mathfrak{P}_n)$$

(see [AR, p. 159] for the function $P^k_{n,r}(Z,\varphi)$ $(Z\in\mathfrak{H}_n)$).

2. Explicit formulas for the Koecher-Maass Dirichlet series. We assume that, for a Dirichlet character ε defined mod N and a positive integer k,

$$\varepsilon(-1)=(-1)^k.$$

Let $f\in M_k(\Gamma_0^{(n)}(N),\varepsilon)$. In view of (1.5) the Dirichlet series $D_n(f,s)$ associated with f is defined by the equality (0.3) in the introduction. By virtue of the estimates (1.2) for $a_f(T)$, $D_n(f,s)$ is absolutely convergent for $\mathrm{Re}(s)>k+(n+1)/2$. Suppose that $m\geq n$. Two matrices $G_1$and G_2 of $M(m,n;\boldsymbol{Z})$ are said to be associated if $G_2=G_1U$ with some $U\in\Gamma_n$. Let Q be a positive definite symmetric matrix of size m. Koecher [KO] studied the zeta function

$$D_n(Q;s)=\sum_G\det(Q[G])^{-s},$$

where G is taken over a complete set of non-associated matrices of $M(m,n;\boldsymbol{Z})$ with $\mathrm{rank}(G)=n$. The zeta function $D_n(Q;s)$ is absolutely convergent for $\mathrm{Re}(s)>m/2$. We define the functions $\xi_n(f,s)$ and $\xi_n(Q;s)$ by the equalities (0.4) and (0.5) in the introduction. Then the functions $\xi_n(f,s)$, $\xi_n(Q;s)$ have the integral expressions:

$$\xi_n(f,s)=\int_{\mathfrak{R}_n}\det(Y)^s f^*\left(\frac{iY}{\sqrt{N}}\right)dv_n(Y)\,, \tag{2.1}$$

$$\xi_n(Q;s)=\int_{\mathfrak{R}_n}\det(Y)^s\Theta_{n,n}(Q;Y)dv_n(Y)\,, \tag{2.2}$$

where $dv_n(Y)=\det(Y)^{-(n+1)/2}\prod_{i\leq j}dy_{ij}$ for $Y=(y_{ij})$. The integral on the right hand side of (2.1) (resp. (2.2)) is absolutely convergent for $\mathrm{Re}(s)>k+(n+1)/2$ (resp. $\mathrm{Re}(s)>\mathrm{m}/2$).

Assuming Theorem 1.1 in § 1, we shall prove the following theorems including the explicit formulas for $\xi_n(f,s)$ and $\xi_n(Q;s)$ in the introduction. The constant $v(r)$ for a positive integer r is given by (0.6) in the introduction.

THEOREM 2.1. *Assume $k>n-1$ and $\varepsilon(-1)=(-1)^k$. Let $f\in M_k(\Gamma_0^{(n)}(N),\varepsilon)$ and set $g=f|\omega_N^{(n)}$. Then the function $\xi_n(f,s)$ can be analytically continued to a meromorphic function in the whole s-plane and has the integral expression in the vertical strip $(n-1)/2<\mathrm{Re}(s)<k-(n-1)/2$:*

$$\xi_n(f, s) = \int_{\mathfrak{R}_n} \det(Y)^s P_n\left(f, \frac{Y}{\sqrt{N}}\right) dv_n(Y), \tag{2.3}$$

where the integral is absolutely convergent in the same strip of s. *Moreover, the residue formula for* $\xi_n(f, s)$ *holds*:

$$\xi_n(f, s) = I_n(f, s) + v(n)\left(\frac{i^{nk} a_g(0)}{s-k} - \frac{a_f(0)}{s}\right) \tag{2.4}$$

$$+ \frac{1}{2} \sum_{\mu=1}^{n-1} v(n-\mu) \left(\frac{i^{nk} \xi_\mu(\Phi^{n-\mu} g, n/2)}{s-k+\mu/2} - \frac{\xi_\mu(\Phi^{n-\mu} f, n/2)}{s-\mu/2}\right),$$

where we put

$$I_n(f, s) = \int_{\substack{\mathfrak{R}_n \\ \det(Y) \geq 1}} \left\{\det(Y)^s f^*\left(\frac{iY}{\sqrt{N}}\right) + i^{nk} \det(Y)^{k-s} g^*\left(\frac{iY}{\sqrt{N}}\right)\right\} dv_n(Y),$$

$I_n(f, s)$ *being an entire function of* s.

Similarly, the following theorem for the function $\xi_n(Q; s)$ holds.

THEOREM 2.2 *Suppose* $m > 2n-2$ *and let* $Q \in \mathfrak{P}_m$. *Then the function* $\xi_n(Q; s)$ *can be continued analytically to a meromorphic function in the whole* s*-plane and has the integral expression in the vertical strip* $(n-1)/2 < \mathrm{Re}(s) < m/2 - (n-1)/2$:

$$\xi_n(Q; s) = \int_{\mathfrak{R}_n} \det(Y)^s P_n(Q; Y) dv_n(Y),$$

where the integral is absolutely convergent in the same strip. Moreover,

$$\xi_n(Q; s) = I_n(Q; s) + v(n)\left(\frac{\det(Q)^{-n/2}}{s-m/2} - \frac{1}{s}\right) \tag{2.5}$$

$$+ \frac{1}{2} \sum_{\mu=1}^{n-1} v(n-\mu) \left(\frac{\det(Q)^{-n/2}}{s-(m-\mu)/2} \cdot \xi_\mu(Q^{-1}; n/2) - \frac{1}{s-\mu/2} \cdot \xi_\mu(Q; n/2)\right),$$

where we set

$$I_n(Q; s) = \int_{\substack{\mathfrak{R}_n \\ \det(Y) \geq 1}} \{\det(Y)^s \Theta_{n,n}(Q; Y) + \det(Q)^{-n/2} \det(Y)^{m/2-s} \Theta_{n,n}(Q^{-1}; Y)\} dv_n(Y),$$

$I_n(Q; s)$ *being an entire function of* s.

The proofs of Theorems 2.1 and 2.2 are similar to that of Proposition 2 in [AR]. We prove only Theorem 2.1 by induction on n by using Theorem 1.1 in §1, since Theorem 2.2 is quite similarly verified.

PROOF OF THEOREM 2.1. If $n=1$, the assertion of Theorem 2.1 is immediately verified. Suppose $n>1$. We assume that the assertion of Theorem 2.1 is valid for any modular forms of $M_k(\Gamma_0^{(\mu)}(N), \varepsilon)$ with degree μ less than n. We see easily from the integral expression (2.1) that, if $\mathrm{Re}(s)>k+(n+1)/2$,

$$(2.6)\quad \xi_n(f, s)=I_n(f, s)+\int_{\substack{\mathfrak{R}_n \\ \det(Y)\geq 1}}\left\{f^*\left(\frac{iY^{-1}}{\sqrt{N}}\right)-i^{nk}\det(Y)^k g^*\left(\frac{iY}{\sqrt{N}}\right)\right\}\det(Y)^{-s}dv_n(Y).$$

The integral $I_n(f, s)$ is absolutely convergent for any $s\in \boldsymbol{C}$ and indicates an entire function of s. Substituting $Y^{-1}/\sqrt{N}$ for Y in (1.7), we have

$$(2.7)\quad P_n\left(f, \frac{Y^{-1}}{\sqrt{N}}\right)=f^*\left(\frac{iY^{-1}}{\sqrt{N}}\right)-i^{nk}\det(Y)^k$$
$$\times\left\{a_g(0)+\sum_{\mu=1}^{n-1}\sum_{U\in\Gamma_n/\Gamma_{\mu,n-\mu}^{\infty}}P_\mu\left(\Phi^{n-\mu}g, \frac{1}{\sqrt{N}}(Y[U])\begin{bmatrix}E_\mu\\0\end{bmatrix}\right)\right\}.$$

Replacing f by $g=f|\omega_N^{(n)}$ and Y by $Y/\sqrt{N}$ in (1.7), we get

$$(2.8)\quad P_n\left(g, \frac{Y}{\sqrt{N}}\right)=g^*\left(\frac{iY}{\sqrt{N}}\right)-(-i)^{nk}\det(Y)^{-k}$$
$$\times\left\{a_f(0)+\sum_{\mu=1}^{n-1}\sum_{U\in\Gamma_n/\Gamma_{n-\mu,\mu}^{\infty}}P_\mu\left(\Phi^{n-\mu}f, \frac{1}{\sqrt{N}}(Y[U])^{-1}\begin{bmatrix}0\\E_\mu\end{bmatrix}\right)\right\}.$$

By virtue of Theorem 1.1, the infinite series on the right hand sides of (2.7), (2.8) are absolutely convergent under the assumption $k>n-1$. The identity (1.9) in Theorem 1.1 implies that

$$(2.9)\quad P_n\left(f, \frac{Y^{-1}}{\sqrt{N}}\right)=i^{nk}\det(Y)^k P_n\left(g, \frac{Y}{\sqrt{N}}\right).$$

With the help of (2.7), (2.8) and (2.9), the function $\xi_n(f, s)$ in (2.6) turns out to be of the following form if $\mathrm{Re}(s)>k+(n+1)/2$:

$$\xi_n(f, s)=I_n(f, s)+\int_{\substack{\mathfrak{R}_n \\ \det(Y)\geq 1}}\det(Y)^{-s}$$
$$\times\left[i^{nk}\det(Y)^k\left\{a_g(0)+\sum_{\mu=1}^{n-1}\sum_{U\in\Gamma_n/\Gamma_{\mu,n-\mu}^{\infty}}P_\mu\left(\Phi^{n-\mu}g, \frac{1}{\sqrt{N}}(Y[U])\begin{bmatrix}E_\mu\\0\end{bmatrix}\right)\right\}\right.$$
$$\left.-\left\{a_f(0)+\sum_{\mu=1}^{n-1}\sum_{U\in\Gamma_n/\Gamma_{n-\mu,\mu}^{\infty}}P_\mu\left(\Phi^{n-\mu}f, \frac{1}{\sqrt{N}}(Y[U])^{-1}\begin{bmatrix}0\\E_\mu\end{bmatrix}\right)\right\}\right]dv_n(Y).$$

By the inductive assumption, the integral

$$\int_{\mathfrak{R}_\mu} \det(W)^s P_\mu\left(\Phi^{n-\mu}f, \frac{W}{\sqrt{N}}\right) dv_\mu(W)$$

for each integer μ $(1 \leq \mu \leq n-1)$ is absolutely convergent for $(\mu-1)/2 < \mathrm{Re}(s) < k-(\mu-1)/2$ and coincides with $\xi_\mu(\Phi^{n-\mu}f, s)$. An easy calculation as in Lemma 5 of [AR] shows that, if $\mathrm{Re}(s) > k+(n+1)/2$,

$$\begin{aligned}(2.10)\qquad \xi_n(f, s) &= I_n(f, s)\\ &+ i^{nk}\left[\frac{v(n)}{s-k}\cdot a_g(0) + \sum_{\mu=1}^{n-1} \frac{v(n-\mu)}{2(s-k+\mu/2)}\cdot \xi_\mu(\Phi^{n-\mu}g, n/2)\right]\\ &- \left[\frac{v(n)}{s}\cdot a_f(0) + \sum_{\mu=1}^{n-1} \frac{v(n-\mu)}{2(s-\mu/2)}\cdot \xi_\mu(\Phi^{n-\mu}f, n/2)\right],\end{aligned}$$

which gives the meromorphic continuation of $\xi_n(f, s)$ to the whole s-plane. On the other hand, again with the use of (2.9), we have

$$\begin{aligned}(2.11)\qquad &\int_{\mathfrak{R}_n} \det(Y)^s P_n\left(f, \frac{Y}{\sqrt{N}}\right) dv_n(Y)\\ &= \int_{\substack{\mathfrak{R}_n\\ \det(Y)\geq 1}} \left\{\det(Y)^s P_n\left(f, \frac{Y}{\sqrt{N}}\right) + i^{nk}\det(Y)^{k-s} P_n\left(g, \frac{Y}{\sqrt{N}}\right)\right\} dv_n(Y).\end{aligned}$$

Thus in a manner similar to that in the proof of Proposition 3 of [AR] (especially (2.6), (2.7) in [AR]), we see easily that the integral on the left hand side of (2.11) is absolutely convergent for $(n-1)/2 < \mathrm{Re}(s) < k-(n-1)/2$ and coincides with the right hand side of the identity (2.10). Hence the identity (2.3) holds for $(n-1)/2 < \mathrm{Re}(s) < k-(n-1)/2$. The proof of Theorem 2.1 is now completed.

As an application of Theorems 2.1 and 2.2 we obtain the functional equations of the zeta functions $D_n(f, s)$ and $D_n(Q; s)$.

COROLLARY 2.3. *Let the notation and the assumption be the same as in Theorems* 2.1 *and* 2.2. *Then we have*

(i) $\xi_n(f\,|\,\omega_N^{(n)}, k-s) = i^{-nk}\xi_n(f, s)$,

(ii) $\xi_n(Q^{-1}; m/2-s) = \det(Q)^{n/2}\xi_n(Q; s)$.

The proof is immediate from (2.4) and (2.5) in Theorems 2.1 and 2.2.

REMARK 2.1. The functional equation for $\xi_n(f, s)$ was obtained by Maass [MA, §15] in the case of $N=1$ and by Weissauer in the case of f being a cusp form with level $N>1$. The functional equation for $\xi_n(Q; s)$ has been known by Maass [MA, §17, p. 284, p. 285].

Now we consider the zeta function $\xi_n(Q; s)$ for $n \leq m \leq 2n-2$ following Böcherer's suggestion. He suggested that $\xi_n(Q; s)$ is related to $\xi_{m-n}(Q; s)$ in a simple manner and that one can get some information on the poles of $\xi_n(Q; s)$ from that on $\xi_{m-n}(Q; s)$. Suppose $n \leq m$. For $Q \in \mathfrak{P}_m$, we define the Eisenstein series $E_n(Q; s)$ by

$$E_n(Q; s) = \sum_{U \in \Gamma_m / \Gamma^{\infty}_{n,m-n}} \det\left(({}^tUQU)\begin{bmatrix} E_n \\ 0 \end{bmatrix} \right)^{-s}$$

which converges absolutely for $\mathrm{Re}(s) > m/2$. Then an easy calculation shows that

$$D_n(Q; s) = E_n(Q; s) \cdot \prod_{\nu=1}^{n} \zeta(2s - \nu + 1)\,.$$

Set

$$\eta(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)\,.$$

It follows from (0.5) that

$$\xi_n(Q; s) = 2 \prod_{\nu=1}^{n} \eta(2s - \nu + 1) \cdot E_n(Q; s)\,. \tag{2.12}$$

On the other hand, the easily verified identity

$$\det\left(({}^tUQU)\begin{bmatrix} E_n \\ 0 \end{bmatrix} \right)^{-s} = \det(Q)^{-s} \det\left((U^{-1}Q^{-1}\,{}^tU^{-1})\begin{bmatrix} 0 \\ E_{m-n} \end{bmatrix} \right)^{-s} \qquad (U \in \Gamma_m)$$

implies that

$$E_n(Q; s) = \det(Q)^{-s} E_{m-n}(Q^{-1}; s). \tag{2.13}$$

Now suppose that $n \leq m \leq 2n-2$. Therefore one can derive from (2.12) and (2.13) a simple relation between $\xi_n(Q; s)$ and $\xi_{m-n}(Q^{-1}; s)$:

$$\xi_n(Q; s) = \prod_{\nu=m-n+1}^{n} \eta(2s - \nu + 1) \cdot \det(Q)^{-s} \xi_{m-n}(Q^{-1}; s)\,, \tag{2.14}$$

where $\xi_0(Q^{-1}; s) = 1$ if $m = n$. We note here that $m > 2(m-n) - 2$ and $n > m - n + 1$.

Proposition 2.4. *Suppose $n \leq m \leq 2n-2$. The function $\xi_n(Q; s)$ has poles at $s = 0, 1/2, \cdots, m/2$, of which $s = 0, 1/2, \cdots, (m-n-1)/2$ and $s = (n+1)/2, \cdots, m/2$ are simple poles derived from the poles of $\xi_{m-n}(Q^{-1}; s)$. The residue of $\xi_n(Q; s)$ at the simple pole $s = (m-\mu)/2$ (resp. $s = \mu/2$) for each integer μ with $0 \leq \mu \leq m-n-1$ is given by*

$$\mathrm{Res}_{s=(m-\mu)/2}\xi_n(Q; s) = \varepsilon_\mu v(n-\mu) \det(Q)^{-n/2} \xi_\mu(Q; n/2) \tag{2.15}$$

$$(\text{resp. } \mathrm{Res}_{s=\mu/2}\xi_n(Q; s) = -\varepsilon_\mu v(n-\mu) \xi_\mu(Q; n/2))\,,$$

where ε_μ is 1 or 1/2 according as $\mu = 0$ or $\mu > 0$. The poles $s = (m-n)/2$, $(m-n+1)/2, \cdots, (n-1)/2$, $n/2$ of $\xi_n(Q; s)$ are derived from those of the product

$\prod_{\nu=m-n+1}^{n}\eta(2s-\nu+1)$. *Among those poles,* $s=(m-n+1)/2, \cdots, (n-1)/2$ *yield double poles of* $\xi_n(Q; s)$.

PROOF. We see from (2.14) and Theorem 2.2 that $\xi_n(Q; s)$ has simple poles at $s=0, 1/2, \cdots, (m-n-1)/2$; $s=(n+1)/2, \cdots, m/2$, and, moreover, that the residue of $\xi_n(Q; s)$ at $s=(m-\mu)/2$ for each μ $(0\leq\mu\leq m-n-1)$ is given by

$$\prod_{\nu=m-n+1}^{n} \eta(m-\mu-\nu+1)\cdot\varepsilon_\mu v(m-n-\mu)\xi_\mu(Q; (m-n)/2)\det(Q)^{-(n-\mu)/2}\,.$$

Thus the identity (2.15) follows from the functional equation of $\xi_\mu(Q; s)$ (Corollary 2.3) and (0.6). The residue of $\xi_n(Q; s)$ at $s=\mu/2$ $(0\leq\mu\leq m-n-1)$ is similarly calculated with the help of the functional equation $\eta(1-s)=\eta(s)$ of the Riemann zeta function. Since $\eta(s)$ has simple poles only at $s=0, 1$, the other assertions of Proposition 2.4 easily follows.

q.e.d.

3. Proof of Theorem 1.1. Let the notation be the same as in § 1 and § 2. Before giving the proof of Theorem 1.1, we describe some preparatory lemmas.

LEMMA 3.1. (i) *Let* $f\in M_k(\Gamma_0^{(n)}(N), \varepsilon)$ *and let* λ *be a positive constant. Then there exists a positive constant* C_2 *depending only on* f *and* λ *such that, if* $Y\in\mathfrak{P}_n$ *and* $\det(Y)\geq\lambda^n$,

$$|f^*(iY)|<C_2\det(Y)^{-k}\,.$$

(ii) *Suppose* $m\geq n$. *Let* $Q\in\mathfrak{P}_m$. *Then there exists a positive constant* C_3 *independent of* Y *such that, if* $Y\in\mathfrak{P}_n$ *and* $\det(Y)\geq 1$,

$$\Theta_{n,n}(Q; Y)<C_3\det(Y)^{-m/2}\,.$$

PROOF. Since $|f^*(iY[U])|=|f^*(iY)|$ for any $U\in\Gamma_n$, Y may be assumed to be a reduced matrix of size n. The estimates (1.2) for the Fourier coefficients $a_f(T)$ imply that

$$|F^*(iY)|<C_1\sum_{T>0}\det(T)^k\exp(-2\pi\,\mathrm{tr}\,(TY))\,.$$

It is not difficult to see that there exists a positive constant C_4 independent of Y with the inequality

$$\det(Y)^k\exp(-\pi\,\mathrm{tr}(Y))<C_4 \qquad \text{for any} \quad Y\in\mathfrak{P}_n\,.$$

By the reduction theory of positive quadratic forms (see for instance [MA, § 9]), there exists a positive constant C_5 such that, if $Y\in\mathfrak{R}_n$,

$$Y>C_5Y_0\,,$$

where $Y_0=(\delta_{\mu\nu}y_{\mu\nu})$ for $Y=(y_{\mu\nu})$. As is shown in [MA, p. 191], for $Y\in\mathfrak{R}_n$,

$$\mathrm{tr}(TY)>nC_5\cdot\det(Y)^{1/n}\left(\prod_{\nu=1}^{n}t_{\nu\nu}\right)^{1/n}\,,$$

where $T=(t_{\mu\nu})$. Thus if $\det(Y)\geq\lambda^n$ and $Y\in\mathfrak{R}_n$, then,

$$|f^*(iY)|<C_1C_4\det(Y)^{-k}\sum_{T>0}\exp(-\pi\,\mathrm{tr}(TY))$$

$$<C_1C_4\det(Y)^{-k}\sum_{T>0}\exp\left(-\pi n\lambda C_5\left(\prod_{\nu=1}^{n}t_{\nu\nu}\right)^{1/n}\right),$$

where the last infinite series is convergent (see [MA, p. 192]). The proof of the assertion (ii) is reduced to the case of $Q=\lambda E_m$ $(\lambda>0)$. For a positive definite integral matrix T of size n, the cardinality of the set $\{G\in M(m,n;\boldsymbol{Z})\,|\,{}^tGG=T\}$ is at most a constant multiple of $\det(T)^{m/2}$. Therefore the assertion (ii) is similarly verified. q.e.d

Let $h_1,\cdots,h_{r+1}$ $(r\geq 1)$ be positive integers with $h_1+\cdots h_{r+1}=n$ and set

$$\kappa(j)=\sum_{\nu=1}^{j}h_\nu.$$

Selberg [SE] introduced the following real analytic Eisenstein series for $\Gamma_n\diagdown GL_n(\boldsymbol{R})$ associated with complex numbers $s_1,\cdots,s_r$:

$$E(Y;h_1,\cdots,h_{r+1};s_1,\cdots,s_r)=\sum_{U\in\Gamma_n/\Gamma^{\infty}_{h_1,\dots,h_{r+1}}}\prod_{j=1}^{r}\det\left((Y[U])\begin{bmatrix}E_{\kappa(j)}\\0\end{bmatrix}\right)^{-s_j}\qquad(Y\in\mathfrak{P}_n).$$

LEMMA 3.2. *The Eisenstein series* $E(Y;h_1,\cdots,h_{r+1};s_1,\cdots,s_r)$ *is absolutely convergent if* $\mathrm{Re}(s_j)>(h_j+h_{j+1})/2$ $(1\leq j\leq r)$.

For the proof see, for instance, [MA, § 17].

We use the following notation later. Let $h_1,\cdots,h_{r+1}$ be the same as above. For $Y\in\mathfrak{P}_n$ and $U\in\Gamma_n$, the positive definite symmetric matrices $W_j^{\boldsymbol{h}}(U;Y)\in\mathfrak{P}_{h_j}$ $(1\leq j\leq r+1)$ are determined by the relation

$$Y[U]=D(W_1^{\boldsymbol{h}}(U;Y),\cdots,W_{r+1}^{\boldsymbol{h}}(U;Y))\left[\begin{pmatrix}E_{h_1}&&*\\&\ddots&\\0&&E_{h_{r+1}}\end{pmatrix}\right],\tag{3.1}$$

where $\boldsymbol{h}=(h_1,\cdots,h_{r+1})$, and $\det(W_j^{\boldsymbol{h}}(U;Y))$ for each j $(1\leq j\leq r+1)$ depends only on the coset $U\Gamma^{\infty}_{h_1,\cdots,h_{r+1}}$.

Now we start the proof of Theorem 1.1.

PROOF OF THEOREM 1.1. Assume that $k>n-1$, $\varepsilon(-1)=(-1)^k$ and that $m>2n-2$. We shall prove the theorem by induction on n. For $n=1$, the assertions of Theorem 1.1 are valid. Let $n>1$. We assume that, for any modular forms f_1 of $M_k(\Gamma_0^{(\nu)}(N),\varepsilon)$ of degree ν less than n, the functions $P_\nu(f_1,W)$ $(W\in\mathfrak{P}_\nu)$ are defined recurrently by (1.7) and satisfy the assertion (i) of Theorem 1.1 with n replaced by ν. Moreover, we assume that, for each positive integer ν less than n, the function $P_\nu(Q;W)$ $(W\in\mathfrak{P}_\nu)$ is defined recurrently by (1.8) and satisfies the assertion (ii) of Theorem 1.1 with n replaced by ν.

Let $f \in M_k(\Gamma_0^{(n)}(N), \varepsilon)$. Recall that $g = f|\omega_N^{(n)}$ and $c = N^{k/2}i^{-k}$. We calculate, at first in a formal manner, $P_n(g, (NY)^{-1})$ (resp. $P_n(Q^{-1}; Y^{-1})$) by the equality (1.7) (resp. (1.8)). Taking (1.1) into account, we have, formally,

$$P_n(g, (NY)^{-1}) = g^*(i(\mathrm{NY})^{-1}) - c^n \det(Y)^k \times \left\{ a_f(0) + \sum_{\mu=1}^{n-1} \sum_{U \in \Gamma_n/\Gamma_{\mu,n-\mu}^{\infty}} P_\mu\left(\Phi^{n-\mu}f, (Y[U])\begin{bmatrix} E_\mu \\ 0 \end{bmatrix}\right)\right\}.$$

Using (1.4) and the equality (1.7) for $P_\mu(\Phi^{n-\mu}f, \quad)$, we see that

$$\begin{aligned} P_n(g, (NY)^{-1}) &= g^*(i(NY)^{-1}) - c^n \det(Y)^k \cdot \sum_{\mu=0}^{n-1} f_\mu(iY) \\ &+ c^n \det(Y)^k \cdot \sum_{\mu=1}^{n-1} \sum_{U \in \Gamma_n/\Gamma_{\mu,n-\mu}^{\infty}} c^{-\mu} \det\left((Y[U])\begin{bmatrix} E_\mu \\ 0 \end{bmatrix}\right)^{-k} \cdot \left[\Phi^\mu(\Phi^{n-\mu}f|\omega_N^{(\mu)})\right. \\ &+ \left.\sum_{\nu=1}^{\mu-1} \sum_{V \in \Gamma_\mu/\Gamma_{\mu-\nu,\nu}^{\infty}} P_\nu\left(\Phi^{\mu-\nu}(\Phi^{n-\mu}f|\omega_N^{(\mu)}), \left(\left(N(Y[U])\begin{bmatrix} E_\mu \\ 0 \end{bmatrix}\right)[V]\right)^{-1}\begin{bmatrix} 0 \\ E_\nu \end{bmatrix}\right)\right]. \end{aligned}$$

If U runs over $\Gamma_n/\Gamma_{\mu,n-\mu}^{\infty}$ and V runs over $\Gamma_\mu/\Gamma_{\mu-\nu,\nu}^{\infty}$, then, $U\begin{pmatrix} V & 0 \\ 0 & E_{n-\mu} \end{pmatrix}$ runs over $\Gamma_n/\Gamma_{\mu-\nu,\nu,n-\mu}^{\infty}$. Thus the above identity becomes

$$\begin{aligned} P_n(g, (NY)^{-1}) &= g^*(i(NY)^{-1}) - c^n \det(Y)^k \cdot \sum_{\mu=0}^{n-1} f_\mu(iY) \\ &+ c^n \det(Y)^k \{E_n(f, Y) + M_n(f, Y)\}, \end{aligned} \tag{3.2}$$

where

$$E_n(f, Y) = \sum_{\mu=1}^{n-1} c^{-\mu} \sum_{U \in \Gamma_n/\Gamma_{\mu,n-\mu}^{\infty}} \det\left((Y[U])\begin{bmatrix} E_\mu \\ 0 \end{bmatrix}\right)^{-k} \cdot \Phi^\mu(\Phi^{n-\mu}f|\omega_N^{(\mu)})$$

and

$$\begin{aligned} M_n(f, Y) = \sum_{h_1+h_2+h_3=n} c^{-(h_1+h_2)} \sum_{U \in \Gamma_n/\Gamma_{h_1,h_2,h_3}^{\infty}} &\det\left((Y[U])\begin{bmatrix} E_{h_1+h_2} \\ 0 \end{bmatrix}\right)^{-k} \\ \times P_{h_2}&\left(\Phi^{h_1}(\Phi^{h_3}f|\omega_N^{(h_1+h_2)}), \left((NY[U])\begin{bmatrix} E_{h_1+h_2} \\ 0 \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 \\ E_{h_2} \end{bmatrix}\right), \end{aligned}$$

h_1, h_2, h_3 running over all positive integers with $h_1 + h_2 + h_3 = n$. With the help of the notation (3.1), we set, for a triple $\boldsymbol{h} = (h_1, h_2, h_3)$ of positive integers with $h_1 + h_2 + h_3 = n$,

$$\begin{aligned} M_n(h_1, h_2, h_3; f, Y) = (N^k/c)^{h_1+h_2} &\sum_{U \in \Gamma_n/\Gamma_{h_1,h_2,h_3}^{\infty}} \prod_{j=1}^{2} \det(W_j^{\boldsymbol{h}}(U; NY))^{-k} \\ &\times P_{h_2}(\Phi^{h_1}(\Phi^{h_3}f|\omega_N^{(h_1+h_2)}), W_2^{\boldsymbol{h}}(U; NY)^{-1}). \end{aligned} \tag{3.3}$$

Then,

$$M_n(f, Y)=\sum_{h_1+h_2+h_3=n} M_n(h_1, h_2, h_3; f, Y)\,. \tag{3.4}$$

If f is replaced by $f|\omega_N^{(n)}$, then, g is replaced by $(-1)^{nk}f$ (see (1.1)). We replace f by $f|\omega_N^{(n)}$ and Y by $(NY)^{-1}$ in (3.2). Then,

$$\begin{aligned} P_n(f, Y)&=f^*(iY)-c^{-n}\det(Y)^{-k}\cdot\sum_{\mu=0}^{n-1} g_\mu(i(NY)^{-1}) \\ &\quad +c^{-n}\det(Y)^{-k}\{E_n(g, (NY)^{-1})+M_n(g, (NY)^{-1})\}\,. \end{aligned} \tag{3.5}$$

The identity (1.1) implies that

$$g^*(i(NY)^{-1})+\sum_{\mu=0}^{n-1} g_\mu(i(NY)^{-1})=c^n \det(Y)^k\left\{f^*(iY)+\sum_{\mu=0}^{n-1} f_\mu(iY)\right\}. \tag{3.6}$$

Similarly, we have, in a formal manner,

$$\begin{aligned} P_n(Q^{-1}; Y^{-1})&=\Theta_{n,n}(Q^{-1}, Y^{-1})-\det(Q)^{n/2}\det(Y)^{m/2}\left\{1+\sum_{\mu=1}^{n-1}\Theta_{n,\mu}(Q; Y)\right\} \\ &\quad +\det(Q)^{n/2}\det(Y)^{m/2}\{E_n(Q; Y)+M_n(Q; Y)\}\,, \end{aligned} \tag{3.7}$$

where

$$E_n(Q; Y)=\sum_{\mu=1}^{n-1}\det(Q)^{-\mu/2}\sum_{U\in\Gamma_n/\Gamma_{\mu,n-\mu}^\infty}\det\left((Y[U])\begin{bmatrix} E_\mu \\ 0\end{bmatrix}\right)^{-m/2},$$

$$M_n(Q; Y)=\sum_{h_1+h_2+h_3=n} M_n(h_1, h_2, h_3; Q; Y)\,, \tag{3.8}$$

and

$$\begin{aligned} M_n(h_1, h_2, h_3; Q; Y)&=\det(Q)^{-h_1/2}\sum_{U\in\Gamma_n/\Gamma_{h_1,h_2,h_3}^\infty}\det(W_1^{\boldsymbol{h}}(U; Y))^{-m/2} \\ &\quad \times P_{h_2}(Q; W_2^{\boldsymbol{h}}(U; Y))\,. \end{aligned} \tag{3.9}$$

Replacing Q by Q^{-1} and Y by Y^{-1} in (3.7), we get

$$\begin{aligned} P_n(Q; Y)&=\Theta_{n,n}(Q; Y)-\det(Q)^{-n/2}\det(Y)^{-m/2}\left\{1+\sum_{\mu=1}^{n-1}\Theta_{n,\mu}(Q^{-1}; Y^{-1})\right\} \\ &\quad +\det(Q)^{-n/2}\det(Y)^{-m/2}\{E_n(Q^{-1}; Y^{-1})+M_n(Q^{-1}; Y^{-1})\}\,. \end{aligned} \tag{3.10}$$

LEMMA 3.3. *The Eisenstein series $E_n(f, Y)$ (resp. $E_n(Q; Y)$) is absolutely convergent and satisfies*

$$E_n(g, (NY)^{-1}) = c^n \det(Y)^k E_n(f, Y)$$
$$(resp.\ E_n(Q^{-1}; Y^{-1}) = \det(Q)^{n/2} \det(Y)^{m/2} E_n(Q; Y)).$$

The proof of Lemma 3.3 will be postponed until a little later. Now what we have to prove is the following.

PROPOSITION 3.4. (I) *The infinite series* $M_n(h_1, h_2, h_3; f, Y)$ *and* $M_n(h_1, h_2, h_3; Q; Y)$ *are absolutely convergent.*
(II) *The identities*

$$M_n(h_1, h_2, h_3; g, (NY)^{-1}) = c^n \det(Y)^k M_n(h_3, h_2, h_1; f, Y),$$
$$M_n(h_1, h_2, h_3; Q^{-1}; Y^{-1}) = \det(Q)^{n/2} \det(Y)^{m/2} M_n(h_3, h_2, h_1; Q; Y)$$

hold.

If we succeed in proving Lemma 3.3 and the assertions (I), (II) of Proposition 3.4 under the inductive assumption, we see immediately from (3.5) (resp. (3.10)) that the right hand side of (1.7) (resp. (1.8)) is absolutely convergent and, moreover from (3.2), (3.5), (3.6) (resp. (3.7), (3.10), (1.6)), that the transformation law (1.9) (resp. (1.11)) holds. The identities (1.10), (1.12) are easily seen from the definition of $P_n(f, Y)$, $P_n(Q; Y)$.

First we shall prove the assertion (II) of Proposition 3.4, assuming the validity of the assertion (I). Next we shall give a proof of Lemma 3.3. Our final goal is to prove the assertion (I) of Proposition 3.4 under the inductive assumption on n.

Set, for positive integers p, q with $p+q=n$,

$$J_{p,q} = \begin{pmatrix} E_p & 0 & 0 & 0 \\ 0 & 0 & 0 & -E_q \\ 0 & 0 & E_p & 0 \\ 0 & E_q & 0 & 0 \end{pmatrix} \qquad (\in Sp(n, \boldsymbol{R})).$$

LEMMA 3.5 *Let* h_1, h_2, h_3 *be positive integers with* $h_1+h_2+h_3=n$. *Then,*
(i) $\Phi^{h_1}(\Phi^{h_3}(f|\omega_N^{(n)})|\omega_N^{(h_1+h_2)}) = N^{h_3k/2}(-1)^{k(h_1+h_2)}\Phi^{h_1+h_3}(f|J_{h_1+h_2,h_3})$,
(ii) $\Phi^{h_3}(\Phi^{h_1}f|\omega_N^{(h_2+h_3)}) = N^{h_3k/2}\Phi^{h_1+h_3}(f|J_{h_1+h_2,h_3})|\omega_N^{(h_2)}$.

PROOF. It follows by definition that, for any $Z_1 \in \mathfrak{H}_{h_1+h_2}$,

$$\Phi^{h_3}(f|\omega_N^{(n)})(Z_1) = N^{nk/2}\det(NZ_1)^{-k}\lim_{\lambda\to+\infty}\det(i\lambda E_{h_3})^{-k} f\begin{pmatrix} -(NZ_1)^{-1} & 0 \\ 0 & -(i\lambda E_{h_3})^{-1}\end{pmatrix}.$$

Hence,

$$(\Phi^{h_3}(f|\omega_N^{(n)})|\omega_N^{(h_1+h_2)})(Z_1) = \det(NZ_1)^{-k} N^{k(h_1+h_2)/2}\Phi^{h_3}(f|\omega_N^{(n)})(-(NZ_1^{-1})).$$

Replacing Z_1 by $-(NZ_1)^{-1}$ in the above expression for $\Phi^{h_3}(f|\omega_N^{(n)})$, we get the identity (i). Similarly for $Z_2 \in \mathfrak{H}_{h_2+h_3}$,

$$(\Phi^{h_1}f) \,|\, \omega_N^{(h_2+h_3)}(Z_2) = N^{k(h_2+h_3)/2} \det(NZ_2)^{-k} \lim_{\lambda\to+\infty} f\begin{pmatrix} i\lambda E_{h_1} & 0 \\ 0 & -(NZ_2)^{-1} \end{pmatrix}.$$

Therefore we have, for $Z_3 \in \mathfrak{H}_{h_2}$,

$$\Phi^{h_3}(\Phi^{h_1}f \,|\, \omega_N^{(h_2+h_3)})(Z_3) = N^{k(h_2+h_3)/2} \det(NZ_3)^{-k} \lim_{\substack{\lambda\to+\infty \\ \mu\to+\infty}} \det(i\mu E_{h_3})^{-k}$$

$$\times f\begin{pmatrix} i\lambda E_{h_1} & 0 & 0 \\ 0 & -(NZ_3)^{-1} & 0 \\ 0 & 0 & -(i\mu E_{h_3})^{-1} \end{pmatrix}$$

$$= N^{kh_3/2}(\Phi^{h_1+h_3}(f|J_{h_1+h_2,h_3}) \,|\, \omega_N^{(h_2)})(Z_3)\,.$$

q.e.d.

Proof of the Assertion (II) of Proposition 3.4. Substituting $f|\omega_N^{(n)}$ for f and $(NY)^{-1}$ for Y in (3.3), we have

$$(3.11) \qquad M_n(h_1, h_2, h_3; g, (NY)^{-1}) = (N^k/c)^{h_1+h_2} \sum_{U\in\Gamma_n/\Gamma^\infty_{h_1,h_2,h_3}} \prod_{j=1}^{2} \det(W_j^{\boldsymbol{h}}(U;\, Y^{-1}))^{-k}$$
$$\times P_{h_2}(\Phi^{h_1}(\Phi^{h_3}(f|\omega_N^{(n)}) \,|\, \omega_N^{(h_1+h_2)}),\, W_2^{\boldsymbol{h}}(U,\, Y^{-1})^{-1})\,.$$

It is not difficult to see from the notation (3.1) that, for $Y \in \mathfrak{P}_n$ and $U \in \Gamma_n$,

$$W_j^{(h_3,h_2,h_1)}(U^*;\, Y) = W_{4-j}^{(h_1,h_2,h_3)}(U,\, Y^{-1})^{-1} \qquad (j=1, 2, 3)\,,$$

where we put

$$U^* = ({}^tU^{-1})\begin{pmatrix} 0 & 0 & E_{h_1} \\ 0 & E_{h_2} & 0 \\ E_{h_3} & 0 & 0 \end{pmatrix}.$$

We note here that if U runs over $\Gamma_n/\Gamma^\infty_{h_1,h_2,h_3}$, then U^* runs over $\Gamma_n/\Gamma^\infty_{h_3,h_2,h_1}$. Thus the identity (3.11) with the help of (i) of Lemma 3.5 becomes

$$(3.12) \qquad M_n(h_1, h_2, h_3; g, (NY)^{-1}) = (N^k/c)^{h_1+h_2} N^{h_3k/2}(-1)^{k(h_1+h_2)} \det(Y)^k$$

$$\times \sum_{U^*\in\Gamma_n/\Gamma^\infty_{h_3,h_2,h_1}} \det(W_1^{\boldsymbol{h}'}(U^*,\, Y))^{-k} P_{h_2}(\Phi^{h_1+h_3}(f|\, J_{h_1+h_2,h_3}),\, W_2^{\boldsymbol{h}'}(U^*, Y))\,,$$

where we put $\boldsymbol{h}' = (h_3, h_2, h_1)$. On the other hand, by using (3.3) and (ii) of Lemma 3.5, we get

$$M_n(h_3, h_2, h_1; f, Y) = N^{h_3 k/2} c^{-h_2-h_3} \sum_{U \in \Gamma_n/\Gamma^{\infty}_{h_3,h_2,h_1}} \prod_{j=1}^{2} \det(W_j^{\boldsymbol{h}'}(U, Y))^{-k}$$
$$\times P_{h_2}(\Phi^{h_1+h_3}(f | J_{h_1+h_2,h_3}) | \omega_N^{(h_2)}, (NW_2^{\boldsymbol{h}'}(U, Y))^{-1}).$$

Thus,

$$M_n(h_3, h_2, h_1; f, Y) = N^{h_3 k/2} c^{-h_3} \sum_{U \in \Gamma_n/\Gamma^{\infty}_{h_3,h_2,h_1}} \det(W_1^{\boldsymbol{h}'}(U, Y))^{-k}$$
$$\times P_{h_2}(\Phi^{h_1+h_3}(f | J_{h_1+h_2,h_3}), (W_2^{\boldsymbol{h}'}(U, Y)). \tag{3.13}$$

The first identity in the assertion (II) follows from (3.12), (3.13). The second one is also proved in a similar manner. Thus we have completed the proof of the assertion (II) of Proposition 3.4 assuming the validity of (I).

Proof of Lemma 3.3. Since $k > n-1 \geq n/2$ (resp. $m/2 > n-1 \geq n/2$), the abolute convergence of $E_n(f, Y)$ (resp. $E_n(Q; Y)$) follows from Lemma 3.2. It is immediate to see from the definition of $E_n(f, Y)$ that

$$E_n(f, Y) = \sum_{h_1+h_3=n} c^{-h_1} \sum_{U \in \Gamma_n/\Gamma^{\infty}_{h_1,h_3}} \det(W_1^{\boldsymbol{h}}(U; Y))^{-k} \cdot \Phi^{h_1}(\Phi^{h_3} f | \omega_N^{(h_1)}),$$

where $\boldsymbol{h} = (h_1, h_3)$ runs over all pairs of positive integers with $h_1 + h_3 = n$. Since Lemma 3.5 holds also for $h_2 = 0$, we have, in a manner similar to that for the proof of the assertion (II) of Proposition 3.4,

$$E_n(f | \omega_N^{(n)}, (NY)^{-1}) = \sum_{h_1+h_3=n} c^{-h_1} \sum_{U \in \Gamma_n/\Gamma^{\infty}_{h_1,h_3}} N^{h_1 k} \det(W_1^{\boldsymbol{h}}(U; Y^{-1}))^{-k}$$
$$\times N^{h_3 k/2}(-1)^{h_1 k} \Phi^n(f | J_{h_1,h_3})$$
$$= \det(Y)^k \sum_{h_1+h_3=n} c^{h_1} \sum_{U^* \in \Gamma_n/\Gamma^{\infty}_{h_3,h_1}} \det(W_1^{(h_3,h_1)}(U^*; Y))^{-k} \cdot \Phi^{h_3}(\Phi^{h_1} f | \omega_N^{(h_3)})$$
$$= c^n \det(Y)^k E_n(f, Y).$$

Another identity for $E_n(Q; Y)$ is similarly verified. q.e.d.

Finally we shall prove the assertion (I) of Proposition 3.4 under the inductive assumption.

Proof of the Assertion (I). Suppose $n \geq 3$ (if $n = 2$, the infinite series $M_n(f, Y)$, $M_n(Q; Y)$ do not occur). For a positive integer r with $r \geq 2$, we consider the following equation with indeterminates $p_1, \cdots, p_d, q_1, \cdots, q_d$, with d also being regarded as an indeterminate,

$$\text{(EQ)} \quad \sum_{j=1}^{d} p_j + \sum_{j=1}^{d} q_j = r, \qquad p_1, \cdots, p_d \in \boldsymbol{N},\ q_1, \cdots, q_{d-1} \in \boldsymbol{N}, \ \text{ and } \ q_d \in \boldsymbol{Z},\ q_d \geq 0.$$

The number of the solutions $(p_1, \cdots, p_d; q_1, \cdots, q_d)$ of the equation (EQ) is 2^{r-1}. Let $h_1, \cdots, h_{r+1}$ be $r+1$ positive integers with $h_1+\cdots+h_{r+1}=n$ and $(p_1, \cdots, p_d; q_1, \cdots, q_d)$ a solution of the equation (EQ). Set

$$\lambda(j)=p_1+\cdots+p_j, \qquad \rho(j)=q_1+\cdots+q_j \quad (1\le j\le d). \tag{3.14}$$

We write, for simplicity,

$$\boldsymbol{h}=(h_1, \cdots, h_{r+1}), \quad \boldsymbol{p}=(p_1, \cdots, p_d), \quad \boldsymbol{q}=(q_1, \cdots, q_d). \tag{3.15}$$

We set, for $Y\in\mathfrak{P}_n$ and $t>0$,

$$I^{\boldsymbol{h}}_{(\boldsymbol{p};\boldsymbol{q})}(Y; t)=\sum_U\prod_{j=1}^{\lambda(d)}\det(W^{\boldsymbol{h}}_j(U; Y))^{-t}, \tag{3.16}$$

where U runs over a complete set of respresentatives of $\Gamma_n/\Gamma^{\infty}_{h_1,\cdots,h_{r+1}}$ satisfying the following condition $A(\boldsymbol{p}, \boldsymbol{q}; \boldsymbol{h}; Y)$:

Condition $A(\boldsymbol{p}, \boldsymbol{q}; \boldsymbol{h}; Y)$

$$\prod_{j=v}^{r}\det(W^{\boldsymbol{h}}_j(U; Y))\ge 1 \quad \text{for every } v \text{ with } 2\le v\le p_1,$$

(if $p_1=1$, we begin with the next condition)

$$\prod_{j=p_1+1}^{r+1-\mu}\det(W^{\boldsymbol{h}}_j(U; Y))\le 1 \quad \text{for every } \mu \text{ with } 1\le\mu\le q_1,$$

$\cdots\cdots$

for each i $(1\le i\le d-1)$

$$\prod_{j=v}^{r-\rho(i)}\det(W^{\boldsymbol{h}}_j(U; Y)\ge 1 \quad \text{for every } v \text{ with } \lambda(i)+1\le v\le\lambda(i+1),$$

$$\prod_{j=\lambda(i+1)+1}^{r+1-\mu}\det(W^{\boldsymbol{h}}_j(U; Y))\le 1 \quad \text{for every } \mu \text{ with } \rho(i)+1\le\mu\le\rho(i+1).$$

The condition $A(\boldsymbol{p}, \boldsymbol{q}; \boldsymbol{h}; Y)$ depends only on the coset $U\Gamma^{\infty}_{h_1,\cdots,h_{r+1}}$. In the case of $q_d=0$, the last condition for $i=d-1$ does not occur, since $\rho(d-1)=\rho(d)$. Set

$$H=\sum_{j=1}^{r}(h_j+h_{j+1}), \qquad z_j=(h_j+h_{j+1})/H \quad (1\le j\le r).$$

We note that, if $t>n-1$, then, since $H\le 2n-2$,

$$tz_j>(h_j+h_{j+1})/2 \qquad (1\le j\le r).$$

The following lemma plays a key role in the proof of the assertion (I) of Proposition 3.4.

LEMMA 3.6. *Suppose $t>n-1$. Then,*

$$I^{\boldsymbol{h}}_{(\boldsymbol{p};\boldsymbol{q})}(Y; t)\le E(Y; h_1, \cdots, h_{r+1}; tz_1, \cdots, tz_r).$$

Hence the infinite series $I^{\boldsymbol{h}}_{(\boldsymbol{p};\boldsymbol{q})}(Y;t)$ *is convergent.*

PROOF OF LEMMA 3.6. By virtue of Lemma 3.2, we have only to verify the following inequality.

LEMMA 3.7. *Let* $Y\in\mathfrak{P}_n$ *and let* $U\in\Gamma_n$ *satisfy the condition* $A(\boldsymbol{p},\boldsymbol{q};\boldsymbol{h};Y)$. *Then, for* $t>0$,

$$\prod_{j=1}^{\lambda(d)}\det(W_j^{\boldsymbol{h}}(U;Y))^{-t}\le\prod_{j=1}^{r}\det\left((Y[U])\begin{bmatrix}E_{\kappa(j)}\\0\end{bmatrix}\right)^{-tz_j}.$$

PROOF. Let $U\in\Gamma_n$ satisfy the condition $A(\boldsymbol{p},\boldsymbol{q};\boldsymbol{h};Y)$. First we observe that

$$\prod_{j=\nu}^{r-\rho(i)}\det(W_j^{\boldsymbol{h}}(U;Y))\ge 1\qquad\text{for}\quad 0\le i\le d-1\quad\text{and}\quad \lambda(i)+1\le\nu\le\lambda(i+1)\,,\tag{3.17}$$

and that

$$\prod_{j=\lambda(i+1)+1}^{r-\rho(i)}\det(W_j^{\boldsymbol{h}}(U;Y))\le 1\qquad\text{for}\quad 0\le i\le d-1\,,\tag{3.18}$$

where we may assume that $\rho(0)=0$ and $\lambda(0)=1$. If $q_d=0$, the inequality (3.18) for $i=d-1$ does not occur since $\lambda(d)+\rho(d-1)=r$. Dividing (3.17) by (3.18), we get the inequalities

$$\prod_{j=\nu}^{\lambda(i+1)}\det(W_j^{\boldsymbol{h}}(U;Y))\ge 1\qquad\text{for}\quad 0\le i\le d-1\quad\text{and}\quad \lambda(i)+1\le\nu\le\lambda(i+1),$$

which further imply that

$$\prod_{j=\nu}^{\lambda(d)}\det(W_j^{\boldsymbol{h}}(U;Y))\ge 1\qquad\text{for}\quad 2\le\nu\le\lambda(d)\,.\tag{3.19}$$

Suppose $d>1$. We see from the condition $A(\boldsymbol{p},\boldsymbol{q};\boldsymbol{h};Y)$ that

$$1\le\prod_{i=0}^{d-1}\prod_{\mu=\rho(i)+1}^{\rho(i+1)}\left\{\prod_{j=\lambda(i+1)+1}^{r+1-\mu}\det(W_j^{\boldsymbol{h}}(U;Y))\right\}^{-tz_{r+1-\mu}},$$

where, if $q_d=0$, the index i in the first product actually ranges from 0 to $d-2$, since $\rho(d-1)=\rho(d)$. Thus with the help of (3.19),

$$\begin{aligned}1&\le\prod_{\mu=1}^{\rho(d)}\left\{\prod_{j=1}^{r+1-\mu}\det(W_j^{\boldsymbol{h}}(U;Y))\right\}^{-tz_{r+1-\mu}}\times\prod_{i=0}^{d-1}\prod_{\mu=\rho(i)+1}^{\rho(i+1)}\left\{\prod_{j=1}^{\lambda(i+1)}\det(W_j^{\boldsymbol{h}}(U;Y))\right\}^{tz_{r+1-\mu}}\\&\le\prod_{\nu=1}^{\lambda(d)-1}\left\{\prod_{j=\nu+1}^{\lambda(d)}\det(W_j^{\boldsymbol{h}}(U;Y))\right\}^{tz_\nu}\times\prod_{\mu=1}^{\rho(d)}\left\{\prod_{j=1}^{r+1-\mu}\det(W_j^{\boldsymbol{h}}(U;Y))\right\}^{-tz_{r+1-\mu}}\\&\quad\times\prod_{i=0}^{d-1}\prod_{\mu=\rho(i)+1}^{\rho(i+1)}\left\{\prod_{j=1}^{\lambda(d)}\det(W_j^{\boldsymbol{h}}(U;Y))\right\}^{tz_{r+1-\mu}}.\end{aligned}$$

Since $\sum_{\nu=1}^{\lambda(d)} z_\nu + \sum_{\mu=1}^{\rho(d)} z_{r+1-\mu} = 1$, we have

$$1 \leq \prod_{\nu=1}^{\lambda(d)-1} \left\{ \prod_{j=1}^{\nu} \det(W_j^{\boldsymbol{h}}(U;\, Y)) \right\}^{-tz_\nu} \times \prod_{\mu=1}^{\rho(d)} \left\{ \prod_{j=1}^{r+1-\mu} \det(W_j^{\boldsymbol{h}}(U;\, Y)) \right\}^{-tz_{r+1-\mu}}$$

$$\times \left\{ \prod_{j=1}^{\lambda(d)} \det(W_j^{\boldsymbol{h}}(U;\, Y)) \right\}^{t - tz_{\lambda(d)}} .$$

Consequently,

$$\prod_{j=1}^{\lambda(d)} \det(W_j^{\boldsymbol{h}}(U;\, Y))^{-t} \leq \prod_{\nu=1}^{r} \left\{ \prod_{j=1}^{\nu} \det(W_j^{\boldsymbol{h}}(U;\, Y)) \right\}^{-tz_\nu} = \prod_{j=1}^{r} \det\left((Y[U]) \begin{bmatrix} E_{\kappa(j)} \\ 0 \end{bmatrix} \right)^{-tz_j} .$$

The above argument is applicable to the case of $d=1$, $q_1>0$. If $d=1$ and $q_1=0$, then $p_1=r$ and $\lambda(1)=r$. It is easy to see from (3.19) that

$$\prod_{j=1}^{\lambda(1)} \det(W_j^{\boldsymbol{h}}(U;\, Y))^{-t} \leq \prod_{\nu=1}^{r-1} \left\{ \prod_{j=\nu+1}^{r} \det(W_j^{\boldsymbol{h}}(U;\, Y)) \right\}^{tz_\nu} \times \prod_{j=1}^{r} \det(W_j^{\boldsymbol{h}}(U;\, Y))^{-t}$$

$$= \prod_{\nu=1}^{r} \left\{ \prod_{j=1}^{\nu} \det(W_j^{\boldsymbol{h}}(U;\, Y)) \right\}^{-tz_\nu} .$$

q.e.d.

We define certain infinite series to reduce the proof of the assertion (I) to Lemma 3.6. Let $h_1, \cdots, h_{r+1}$ $(r \geq 2)$ be positive integers with $h_1 + \cdots + h_{r+1} = n$ and let $(p_1, \cdots, p_d;\, q_1, \cdots, q_d)$ be a solution of the equation (EQ). Let $\lambda(j)$, $\rho(j)$, $\boldsymbol{h}$, $\boldsymbol{p}$, $\boldsymbol{q}$ be the same as in (3.14) and (3.15). We simply denote the group $\Gamma_{h_1,\cdots,h_{r+1}}^{\infty}$ by $\Gamma_{\boldsymbol{h}}^{\infty}$. We divide into two cases according as whether $q_d=0$ or $q_d>0$.

(a) The case of $q_d=0$: Let F be an element of $M_k(\Gamma_0^{(h_{\lambda(d)})}(N), \varepsilon^*)$, ε^* being ε or $\bar{\varepsilon}$. We define the infinite series $J_{(\boldsymbol{p};\boldsymbol{q})}^{\boldsymbol{h}}(F, Y)$ and $J_{(\boldsymbol{p};\boldsymbol{q})}^{\boldsymbol{h}}(Q;\, Y)$ $(Y \in \mathfrak{P}_n)$ by

$$J_{(\boldsymbol{p};\boldsymbol{q})}^{\boldsymbol{h}}(F, Y) = \sum_U \prod_{j=1}^{\lambda(d)-1} \det(W_j^{\boldsymbol{h}}(U;\, Y))^{-k} \left| P_{h_{\lambda(d)}}\left(F, \frac{1}{\sqrt{N}} W_{\lambda(d)}^{\boldsymbol{h}}(U;\, Y) \right) \right|$$

and

$$J_{(\boldsymbol{p};\boldsymbol{q})}^{\boldsymbol{h}}(Q;\, Y) = \sum_U \prod_{j=1}^{\lambda(d)-1} \det(W_j^{\boldsymbol{h}}(U;\, Y))^{-m/2} \,|\, P_{h_{\lambda(d)}}(Q;\, W_{\lambda(d)}^{\boldsymbol{h}}(U;\, Y)) \,| ,$$

where in the summations U runs over $\Gamma_n / \Gamma_{\boldsymbol{h}}^{\infty}$ satisfying the condition $A(\boldsymbol{p}, \boldsymbol{q};\, \boldsymbol{h};\, Y)$. In the case of $q_d=0$, putting $i=d-1$ and $\nu=\lambda(d)$ in the condition $A(\boldsymbol{p}, \boldsymbol{q};\, \boldsymbol{h};\, Y)$, we get the inequality

$$\det(W_{\lambda(d)}^{\boldsymbol{h}}(U;\, Y)) \geq 1 . \tag{3.20}$$

(b) The case of $q_d>0$: Let F be an element of $M_k(\Gamma_0^{(h_{\lambda(d)+1})}(N), \varepsilon^*)$, ε^* being ε or $\bar{\varepsilon}$. We set

$$J^{\boldsymbol{h}}_{(\boldsymbol{p};\boldsymbol{q})}(F, Y)=\sum_U \prod_{j=1}^{\lambda(d)+1} \det(W^{\boldsymbol{h}}_j(U; Y))^{-k} \times \left| P_{h_{\lambda(d)+1}}\left(F, \frac{1}{\sqrt{N}}(W^{\boldsymbol{h}}_{\lambda(d)+1}(U; Y))^{-1}\right)\right|$$

and

$$J^{\boldsymbol{h}}_{(\boldsymbol{p};\boldsymbol{q})}(Q; Y)=\sum_U \prod_{j=1}^{\lambda(d)+1} \det(W^{\boldsymbol{h}}_j(U; Y))^{-m/2} \times \left| P_{h_{\lambda(d)+1}}\left(Q^{-1}; (W^{\boldsymbol{h}}_{\lambda(d)+1}(U; Y))^{-1}\right)\right| ,$$

where in the summations U runs over $\Gamma_n/\Gamma^\infty_{\boldsymbol{h}}$ satisfying the condition $A(\boldsymbol{p}, \boldsymbol{q}; \boldsymbol{h}; Y)$. In this case $(q_d>0)$, by putting $i=d-1$ and $\mu=\rho(d)$ in the condition $A(\boldsymbol{p}, \boldsymbol{q}; \boldsymbol{h}; Y)$, we get

$$\det(W^{\boldsymbol{h}}_{\lambda(d)+1}(U; Y))\leq 1 .$$

It is not difficult to see from the identity (3.3) that

$$\left| M_n\left(h_1, h_2, h_3; f, \frac{Y}{\sqrt{N}}\right)\right| \leq J^{\boldsymbol{h}}_{(1;1)}(f_{\boldsymbol{h}}, Y)+J^{\boldsymbol{h}}_{(2;0)}(f_{\boldsymbol{h}}\,|\,\omega^{(h_2)}_N, Y) , \tag{3.21}$$

where $\boldsymbol{h}=(h_1, h_2, h_3)$ and $f_{\boldsymbol{h}}=\Phi^{h_1}(\Phi^{h_3}f\,|\,\omega^{(h_1+h_2)}_N)\in M_k(\Gamma^{(h_2)}_0(N), \bar{\varepsilon})$. Moreover the infinite series on the right hand side of (3.21) gives a dominant series for $M_n(h_1, h_2, h_3; f, Y/\sqrt{N})$. To obtain (3.21) from (3.3) we note that $|c|=N^{k/2}$. We see from (3.9) in a similar manner that

$$|M_n(h_1, h_2, h_3; Q; Y)|\leq C_6\{J^{\boldsymbol{h}}_{(1;1)}(Q; Y)+J^{\boldsymbol{h}}_{(2;0)}(Q; Y)\} \tag{3.22}$$

with a certain positive constant C_6 independent of $Y\in\mathfrak{P}_n$. In view of (3.21), (3.22), the final task we have to do is to estimate the infinite series $J^{\boldsymbol{h}}_{(\boldsymbol{p};\boldsymbol{q})}(F, Y)$, $J^{\boldsymbol{h}}_{(\boldsymbol{p};\boldsymbol{q})}(Q; Y)$ from the above.

LEMMA 3.8. *Let $\boldsymbol{h}$, $\boldsymbol{p}$, $\boldsymbol{q}$ be the same as in* (3.15).
(a) *The case of $q_d=0$: Let F be an element of $M_k(\Gamma^{h_{\lambda(d)}}_0(N), \varepsilon^*)$. Set, for simplicity, $h^*=h_{\lambda(d)}$ and $G=F\,|\,\omega^{(h^*)}_N$. Then there exist positive constants C_7, C_8 independent of $Y\in\mathfrak{P}_n$ such that*

$$J^{\boldsymbol{h}}_{(\boldsymbol{p};\boldsymbol{q})}(F, Y)\leq C_7\left(I^{\boldsymbol{h}}_{(\boldsymbol{p};\boldsymbol{q})}(Y; k)+\sum_{\mu=1}^{h^*-1}\{J^{\boldsymbol{h}(\mu)}_{(\boldsymbol{p};q_1,\dots,q_{d-1},1)}(\Phi^{h^*-\mu}G, Y) + J^{\boldsymbol{h}(\mu)}_{(p_1,\dots,p_{d-1},p_d+1;\boldsymbol{q})}(\Phi^{h^*-\mu}G\,|\,\omega^{(\mu)}_N, Y)\}\right)$$

and that

$$J^{\boldsymbol{h}}_{(\boldsymbol{p};\boldsymbol{q})}(Q; Y)\leq C_8\left(I^{\boldsymbol{h}}_{(\boldsymbol{p};\boldsymbol{q})}(Y; m/2)+\sum_{\mu=1}^{h^*-1}\{J^{\boldsymbol{h}(\mu)}_{(\boldsymbol{p};q_1,\dots,q_{d-1},1)}(Q; Y) + J^{\boldsymbol{h}(\mu)}_{(p_1,\dots,p_{d-1},p_d+1;\boldsymbol{q})}(Q; Y)\}\right),$$

where we put, for each $\mu (1 \leq \mu \leq h^* - 1)$,

$$\boldsymbol{h}(\mu) = (h_1, \cdots, h_{\lambda(d)-1}, h^* - \mu, \mu, h_{\lambda(d)+1}, \cdots, h_{r+1}) .$$

(b) *The case of* $q_d > 0$: *Let* F *be an element of* $M_k(\Gamma_0^{(h_{\lambda(d)+1})}(N), \varepsilon^*)$. *Set* $h^* = h_{\lambda(d)+1}$ *and* $G = F \mid \omega_N^{(h^*)}$. *Then there exist positive constants* C_9, C_{10} *independent of* $Y \in \mathfrak{P}_n$ *such that*

$$J^{\boldsymbol{h}}_{(\boldsymbol{p};\boldsymbol{q})}(F, Y) \leq C_9 \Bigg(I^{\boldsymbol{h}}_{(\boldsymbol{p};\boldsymbol{q})}(Y; k) + \sum_{\mu=1}^{h^*-1} \{ J^{\boldsymbol{h}'(\mu)}_{(p_1, \cdots, p_d, 1; (\boldsymbol{q}, 0))}(\Phi^{h^*-\mu} G, Y)$$

$$+ J^{\boldsymbol{h}'(\mu)}_{(\boldsymbol{p}; q_1, \cdots, q_{d-1}, q_d+1)}(\Phi^{h^*-\mu} G \mid \omega_N^{(\mu)}, Y) \} \Bigg)$$

and that

$$J^{\boldsymbol{h}}_{(\boldsymbol{p};\boldsymbol{q})}(Q; Y) \leq C_{10} \Bigg(I^{\boldsymbol{h}}_{(\boldsymbol{p};\boldsymbol{q})}(Y; m/2) + \sum_{\mu=1}^{h^*-1} \{ J^{\boldsymbol{h}'(\mu)}_{(p_1, \cdots, p_d, 1; (\boldsymbol{q}, 0))}(Q; Y)$$

$$+ J^{\boldsymbol{h}'(\mu)}_{(\boldsymbol{p}; q_1, \cdots, q_{d-1}, q_d+1)}(Q; Y) \} \Bigg) ,$$

where we put, for each μ $(1 \leq \mu \leq h^* - 1)$,

$$\boldsymbol{h}'(\mu) = (h_1, \cdots, h_{\lambda(d)}, \mu, h^* - \mu, h_{\lambda(d)+2}, \cdots, h_{r+1}) .$$

REMARK 3.1. If $h^* = 1$ ($h_{\lambda(d)} = 1$ or $h_{\lambda(d)+1} = 1$ according as $q_d = 0$ *or* $q_d > 0$), then the summation $\sum_{\mu=1}^{h^*-1}$ does not occur in (a), (b) of Lemma 3.8.

PROOF OF LEMMA 3.8. Only the assertion (a) will be proved. The proof of (b) is quite similar to that of (a).

Suppose $q_d = 0$. Let $F \in M_k(\Gamma_0^{(h^*)}(N), \varepsilon^*)$. We can replace n, f, and Y by h^*, F, and $W^{\boldsymbol{h}}_{\lambda(d)}(U; Y) / \sqrt{N}$, respectively, in the equality (1.7) under the inductive assumption. Taking the inequality (3.20) into account, we see easily from Lemma 3.1 and the definition of $J^{\boldsymbol{h}}_{(\boldsymbol{p};\boldsymbol{q})}(F, Y)$ that, with a certain positive constant C'_7 independent of $Y \in \mathfrak{P}_n$,

$$(3.23) \qquad J^{\boldsymbol{h}}_{(\boldsymbol{p};\boldsymbol{q})}(F, Y) \leq C'_7 \Bigg(I^{\boldsymbol{h}}_{(\boldsymbol{p};\boldsymbol{q})}(Y; k) + \sum_{U} \prod_{j=1}^{\lambda(d)} \det(W^{\boldsymbol{h}}_j(U; Y))^{-k}$$

$$\times \sum_{\mu=1}^{h^*-1} \sum_{V \in \Gamma_{h^*} / \Gamma^{\infty}_{h^*-\mu, \mu}} \left| P_\mu \left(\Phi^{h^*-\mu} G, \frac{1}{\sqrt{N}} (W^{\boldsymbol{h}}_{\lambda(d)}(U; Y)[V])^{-1} \begin{bmatrix} 0 \\ E_\mu \end{bmatrix} \right) \right| \Bigg) ,$$

where in the first summation U runs over $\Gamma_n / \Gamma^{\infty}_{\boldsymbol{h}}$ satisfying the condition $A(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{h}; Y)$. Similarly, there exists a positive constant C'_8 independent of Y such that

$$(3.24)\qquad J^{\boldsymbol{h}}_{(\boldsymbol{p};\boldsymbol{q})}(Q;Y)\le C'_8\left(I^{\boldsymbol{h}}_{(\boldsymbol{p};\boldsymbol{q})}(Y;m/2)+\sum_{U}\prod_{j=1}^{\lambda(d)}\det(W^{\boldsymbol{h}}_j(U;Y))^{-m/2}\right.$$
$$\left.\times\sum_{\mu=1}^{h^*-1}\sum_{V\in\Gamma_{h^*}/\Gamma^{\infty}_{h^*-\mu,\mu}}\left|P_\mu\left(Q^{-1};(W^{\boldsymbol{h}}_{\lambda(d)}(U;Y)[V])^{-1}\begin{bmatrix}0\\E_\mu\end{bmatrix}\right)\right|\right),$$

where in the first summation U also runs over $\Gamma_n/\Gamma^{\infty}_{\boldsymbol{h}}$ satisfying the condition $A(\boldsymbol{p},\boldsymbol{q};\boldsymbol{h};Y)$. For each $U\in\Gamma_n$ and $V\in\Gamma_{h^*}$, let $U^*(V)$ denote the matrix

$$U\cdot D(E_{h_1},\cdots,E_{h_{\lambda(d)-1}},V,E_{h_{\lambda(d)+1}},\cdots,E_{h_{r+1}}).$$

Let $h^*>1$ and μ an integer with $1\le\mu\le h^*-1$. If U runs over $\Gamma_n/\Gamma^{\infty}_{\boldsymbol{h}}$ and V over $\Gamma_{h^*}/\Gamma^{\infty}_{h^*-\mu,\mu}$, then, $U^*=U^*(V)$ runs over a complete set of representatives of $\Gamma_n/\Gamma^{\infty}_{\boldsymbol{h}(\mu)}$. It follows from the notation (3.1) that

$$W^{\boldsymbol{h}}_j(U;Y)=W^{\boldsymbol{h}(\mu)}_j(U^*;Y)\qquad(1\le j\le\lambda(d)-1),$$

$$W^{\boldsymbol{h}}_{\lambda(d)}(U;Y)[V]=D(W^{\boldsymbol{h}(\mu)}_{\lambda(d)}(U^*;Y),W^{\boldsymbol{h}(\mu)}_{\lambda(d)+1}(U^*;Y))\left[\begin{pmatrix}E_{h^*-\mu}&*\\0&E_\mu\end{pmatrix}\right],$$

and

$$W^{\boldsymbol{h}}_j(U;Y)=W^{\boldsymbol{h}(\mu)}_{j+1}(U^*;Y)\qquad(\lambda(d)+1\le j\le r+1).$$

Then

$$(W^{\boldsymbol{h}}_{\lambda(d)}(U;Y)[V])^{-1}\begin{bmatrix}0\\E_\mu\end{bmatrix}=(W^{\boldsymbol{h}(\mu)}_{\lambda(d)+1}(U^*;Y))^{-1}.$$

If U satisfies the condition $A(\boldsymbol{p},\boldsymbol{q};\boldsymbol{h};Y)$ and V satisfies

$$\det\left((W^{\boldsymbol{h}}_{\lambda(d)}(U;Y)[V])^{-1}\begin{bmatrix}0\\E_\mu\end{bmatrix}\right)\le 1$$

$$\left(\text{resp. }\det\left((W^{\boldsymbol{h}}_{\lambda(d)}(U;Y)[V])^{-1}\begin{bmatrix}0\\E_\mu\end{bmatrix}\right)\ge 1\right),$$

then $U^*=U^*(V)$ satisfies the condition

$$A((p_1,\cdots,p_{d-1},p_d+1),\boldsymbol{q};\boldsymbol{h}(\mu);Y)$$
$$(\text{resp. }A(\boldsymbol{p},(q_1,\cdots,q_{d-1},1);\boldsymbol{h}(\mu);Y)).$$

Therefore if we observe (3.23) and (3.24) carefully and choose positive constants C_7 and C_8 suitably large compared with C'_7 and C'_8, then we obtain the assertion (a) (actually we may take $C_7=C'_7$). q.e.d.

Let h_1,h_2,h_3 be positive integers with $h_1+h_2+h_3=n$. Let $\boldsymbol{h}$ and $f_{\boldsymbol{h}}$ be the same as in (3.21). By virtue of Lemma 3.6 and recurrent use of Lemma 3.8, the infinite series

$$J^{h}_{(1;1)}(f_h, Y) + J^{h}_{(2;0)}(f_h \mid \omega_N^{(h_2)}),$$
$$J^{h}_{(1;1)}(Q; Y) + J^{h}_{(2;0)}(Q; Y)$$

are convergent. Therefore in view of the inequalities (3.21) and (3.22), the infinite series $M_n(h_1, h_2, h_3; f, Y)$ and $M_n(h_1, h_2, h_3; Q; Y)$ are absolutely convergent. Thus we obtain the assertion (I) of Proposition 3.4.

Now we have completed the proof of Theorem 1.1.

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