Tôhoku Math. J.
41 (1989), 689–710

ASYMPTOTIC CONSTANCY FOR LINEAR NEUTRAL VOLTERRA INTEGRODIFFERENTIAL EQUATIONS

JOHN R. HADDOCK[1], M. N. NKASHAMA[2] AND JIANHONG WU[3]

(Received September 9, 1988)

1. Introduction. An interesting problem that has been investigated over the past several years has been the study of asymptotic constancy of solutions of differential equations for which each constant function is a solution itself. For ordinary differential equations the only such equation is $x'=0$. A list of certain results and references for retarded functional differential equations (RFDEs) with finite or infinite delay can be found in Haddock [3].

The simplest neutral functional differential equation (NFDE) for which each constant function is a solution is the scalar linear homogeneous case

$$\frac{d}{dt}[x(t)-cx(t-r)]=-ax(t)+ax(t-r)\,. \tag{1.1}$$

Using a Liapunov functional technique adapted from Hale [6, p. 120] for the case $c=0$, the first two authors proved (classroom notes) that each solution of (1.1) tends to a constant as $t\to\infty$ provided $a\geq 0$, $r\geq 0$, $|c|<1$. Likewise, Wu [7] proved recently that each solution of the (nonlinear) scalar equation

$$\frac{d}{dt}[x(t)-cx(t-r)]=-F(x(t))+F(x(t-r)) \tag{1.2}$$

tends to a constant as $t\to\infty$, whenever $0\leq c<1$ and F: $R\to R$ is continuous and increasing. However, the problem often becomes more complicated when an integral or infinite delay (or both) is involved. Along these lines it appears that very little has been accomplished regarding asymptotic constancy of solutions of scalar NFDEs more general than (1.1) and (1.2).

The primary purpose of this paper is to provide conditions for the asymptotic constancy of solutions of linear neutral Volterra integrodifferential equations

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[1] This work was supported in part by the US National Science Foundation under grant DMS 8520577.

[2] This work was supported in part by a Memphis State University Faculty Research Grant.

[3] This author is deeply indebted to Prof. P. K. Lin for his help and to Memphis State University for its support during the writing of this paper.

1980 *Mathematics subject classifications* (1985 *Revision*): 34K15, 34K25.

Keywords. Neutral equations, unbounded delay equation and asymptotic constancy.

(1.3)
$$\frac{d}{dt}\left[x(t)-\sum_{i=1}^{\infty} b_i x(t-r_i)-\int_{-\infty}^{t} f(t-s)x(s)ds\right]$$
$$=-ax(t)+\sum_{i=1}^{\infty} a_i x(t-r_i)+\int_{-\infty}^{t} h(t-s)x(s)ds$$

for which $a\geq 0$, a_i and b_i are real numbers with $\sum_{i=1}^{\infty}(|a_i|+|b_i|)<\infty$, $\{r_i\}$ is an increasing unbounded positive real number sequence, $f, h\colon [0,\infty)\rightarrow R$ are continuous and $\int_0^\infty[|f(t)|+|h(t)|]dt<+\infty$. In particular, we combine properties of orbits through limit sets, Liapunov-Razumikhin techniques, limiting equation theory, invariance principles and precompactness of bounded orbits to give conditions for which each solution of (1.3) tends to a constant as $t\rightarrow\infty$.

In Section 2, we see how certain C_g phase spaces often arise in a natural way for (1.3) and obtain sufficient conditions for bounded positive orbits to be precompact with respect to these spaces. In Section 3, we establish that all solutions of (1.3) are bounded (in the future) and that the zero solution of (1.3) is uniformly stable. This is accomplished by employing Liapunov-Razumikhin techniques developed for NFDEs by Haddock and Wu [5]. Section 4 contains a result which shows that, for any solution z through limit sets of bounded positive orbits of (1.3), the corresponding D functional defined by

$$D(\phi)=\phi(0)-\sum_{i=1}^{\infty} b_i\phi(-r_i)-\int_{-\infty}^{0} f(-s)\phi(s)ds$$

satisfies a certain infinite delay retarded FDE. This result in turn has an immediate application regarding the asymptotic constancy of $D(z_t)$, where $z_t(s)=z(t+s)(s\leq 0, t\geq 0)$ is the usual FDE notation. For a solution z of (1.3), a result concerned with the equivalence of the asymptotic constancy of $z(t)$ and the corresponding $D(z_t)$ is the main content of Section 5. Finally, in Section 6 we combine the results of Sections 1–5 to prove under general conditions that each solution of (1.3) tends to a constant as $t\rightarrow\infty$.

Although we are interested in asymptotic constancy of solutions of NFDEs, some of the main merits lie within the supplementary results themselves. For instance, we establish a relationship between solutions of neutral and retarded FDEs in Section 4, and an equivalence of asymptotic constancy of $D(x_t)$ and $x(t)$ is given in Section 5. In addition, the importance of uniform continuity of various functions related to solutions is displayed throughout the paper.

2. C_g phase spaces and precompactness of bounded orbits. In this section we discuss the construction and ramifications of certain phase spaces for NFDEs. In particular, we examine C_g spaces, where $g\colon(-\infty,0]\rightarrow[1,\infty)$ is continuous and nonincreasing and C_g consists of the space of continuous functions ϕ on $(-\infty,0]$ with

$$\sup_{s\le 0} \frac{|\phi(s)|}{g(s)} < \infty .$$

The norm on C_g is defined by

$$|\phi|_{C_g} = \sup_{s\le 0} \frac{|\phi(s)|}{g(s)} .$$

It is important to note that we let conditions with respect to the equation at hand determine the phase space(s) as opposed to allowing the space to dictate the conditions. The techniques used to determine the spaces are an adaptation of those used by Burton [2, Chapter 4], Atkinson and Haddock [1], and Haddock and Hornor [4].

LEMMA 2.1. *Suppose*

$$\int_{-\infty}^{0} |f(-s)|ds + \sum_{i=1}^{\infty} |b_i| < \alpha \quad \text{and} \quad \int_{-\infty}^{0} |h(-s)|ds + \sum_{i=1}^{\infty} |a_i| < \beta$$

for some constants α and β, $0 \le \alpha, \beta < \infty$. Then for each $r \ge 0$ there is a function $g:(-\infty, 0] \to [1, \infty)$ satisfying

(g1) $g:(-\infty,0]\to[1,\infty)$ *is a continuous nonincreasing function such that $g(s)=1$ on $[-r,0]$;*

(g2) $g(s+u)/g(s)\to 1$ *uniformly on $(-\infty,0]$ as $u\to 0^-$;*

(g3) $g(s)\to\infty$ *as $s\to-\infty$; and*

(g4)

$$\int_{-\infty}^{0} |f(-s)|g(s)ds + \sum_{i=1}^{\infty} |b_i| g(-r_i) < \alpha , \qquad \int_{-\infty}^{0} |h(-s)|g(s)ds + \sum_{i=1}^{\infty} |a_i| g(-r_i) < \beta .$$

PROOF. Let

$$\alpha_1 = \int_{-\infty}^{0} |f(-s)|ds + \sum_{i=1}^{\infty} |b_i| , \qquad \beta_1 = \int_{-\infty}^{0} |h(-s)|ds + \sum_{i=1}^{\infty} |a_i| ,$$

and for each $i \geqq 1$, let $\varepsilon_i = (\alpha-\alpha_1)/2^{i+1}(i+2)$, $\delta_i = (\beta-\beta_1)/2^{i+1}(i+2)$. Owing to the absolute convergence of the series $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ and the boundedness of $\int_{-\infty}^{0} [|f(-s)|+|h(-s)|]ds$ there exists a subsequence $\{r_{n_i}\}$ of $\{r_i\}$ such that

(i) $r_{n_{i+1}} \ge r_{n_i} + 1$;

(ii)

$$\int_{-\infty}^{-r_{n_i}} |f(-s)|ds < \frac{\varepsilon_i}{2} , \qquad \sum_{k=n_i}^{\infty} |a_k| < \frac{\varepsilon_i}{2} ,$$

$$\int_{-\infty}^{-r_{n_i}} |h(-s)|ds < \frac{\delta_i}{2} , \qquad \sum_{k=n_i}^{\infty} |b_k| < \frac{\delta_i}{2} \quad \text{and}$$

(iii) $r_{n_1} \geq r$.

Now, define $g: (-\infty, 0] \to [1, \infty)$ as follows:

(a) g is continuous and piecewise linear (linear on intervals $[-r_{n_{i+1}}, -r_{n_i}]$),

(b) $g(s) = 1$ on $[-r_{n_1}, 0]$,

(c) $g(-r_{n_i}) = i+1$.

It follows from (b) and the restrictions on α, α_1, β, and β_1 above that

$$\int_{-r_{n_1}}^{0} |f(-s)| g(s) ds + \sum_{i=1}^{n_1} |b_i| g(-r_i) < \frac{\alpha+\alpha_1}{2},$$

$$\int_{-r_{n_1}}^{0} |h(-s)| g(s) ds + \sum_{i=1}^{n_1} |a_i| g(-r_i) < \frac{\beta+\beta_1}{2}.$$

Hence,

$$\begin{aligned}
&\int_{-\infty}^{0} | f(-s) | g(s) ds + \sum_{i=1}^{\infty} |b_i| g(-r_i) \\
&= \int_{-r_{n_1}}^{0} |f(-s)| g(s) ds + \sum_{i=1}^{\infty} \int_{-r_{n_{i+1}}}^{-r_{n_i}} |f(-s)| g(s) ds + \sum_{i=1}^{n_1} |b_i| g(-r_i) + \sum_{i=1}^{\infty} \sum_{k=n_i}^{n_{i+1}} |b_k| g(-r_k) \\
&\leq \frac{\alpha+\alpha_1}{2} + \sum_{i=1}^{\infty} g(-r_{n_{i+1}}) \left[\int_{-r_{n_{i+1}}}^{-r_{n_i}} |f(-s)| ds + \sum_{k=n_i}^{n_{i+1}} |b_k| \right] \\
&< \frac{\alpha+\alpha_1}{2} + \sum_{i=1}^{\infty} (i+2) \left(\frac{\varepsilon_i}{2} + \frac{\varepsilon_i}{2} \right) \leq \alpha .
\end{aligned}$$

Likewise,

$$\int_{-\infty}^{0} |h(-s)| g(s) ds + \sum_{i=1}^{\infty} |a_i| g(-r_i) < \beta .$$

This completes the proof.

REMARK 2.1. It is clear that, in the proof of Lemma 2.1, we may replace condition (b) in the definition of the function g by the following

(b′) $g(0) = 1$, and $g(s) > 1$ for $s \in (-r_{n_1}, 0)$ is such that

$$\int_{-r_{n_1}}^{0} |f(-s)| g(s) ds + \sum_{i=1}^{n_1} |b_i| g(-r_{n_1}) < \frac{\alpha+\alpha_1}{2},$$

$$\int_{r_{n_1}}^{0} |h(-s)| g(s) ds + \sum_{i=1}^{n_1} |b_i| g(-r_{n_1}) < \frac{\beta+\beta_1}{2}.$$

This remark will be used in the proof of Theorem 4.2 in Section 4.

LEMMA 2.2. *Suppose there exists a continuous function* $g: (-\infty, 0] \to [1, \infty)$ *satisfying* (g1), (g2) *and* (g3), *such that*

$$\sum_{i=1}^{\infty} |b_i| g(-r_i) + \int_{-\infty}^{0} |f(-s)| g(s) ds = \alpha < 1 ,$$

and let $x: (-\infty, +\infty) \to R$ *be continuous with* $x_0 \in C_g$ *and* $|x_0(s)|/g(s) \to 0$ *as* $s \to -\infty$, (*where* $x_0(s) = x(s)$, $s \leq 0$). *If*

$$h(t) = x(t) - \sum_{i=1}^{\infty} b_i x(t-r_i) - \int_{-\infty}^{t} f(t-s) x(s) ds$$

is uniformly continuous on $[0, \infty)$, *then* $x: [0, \infty) \to R$ *also is uniformly continous.*

PROOF. Suppose the conclusion is not true for a continuous $x: (-\infty, +\infty) \to R$. Then there exists a constant $\varepsilon \geq 0$ such that, for every $\delta \geq 0$, there is a $t \geq 0$ so that $|x(t) - x(t-\delta)| \geq \varepsilon$. Let $\delta = \delta_1^* = 1$. Then there exists $t_1^* \geq 0$ so that $|x(t_1^*) - x(t_1^* - \delta_1^*)| \geq \varepsilon$. On the other hand, x is uniformly continuous on $[0, t_1^*]$, so we can find a positive constant $\delta_1 < \delta_1^*$ so that $|x(t) - x(t-\delta_1)| < \varepsilon$ for all $t \in [0, t_1^*]$. According to the definition of ε, there exists $t_1^{**} > t_1^*$ so that $|x(t_1^{**}) - x(t_1^{**} - \delta_1)| \geq \varepsilon$. Hence, there must be $t_1 \geq t_1^*$ so that $|x(t_1) - x(t_1 - \delta_1)| = \varepsilon$ and $|x(t) - x(t-\delta_1)| \leq \varepsilon$ for $t \in [0, t_1)$. Now, $x(t)$ is uniformly continuous on the closed interval $[0, t_1 + 1]$, so we can find a positive constant $\delta_2 \leq \delta_1/2$ so that $|x(t) - x(t-\delta_2)| \leq \varepsilon$ for all $t \in [0, t_1 + 1]$. Likewise, by the definition of ε, we can find $t_2^* \geq t_1 + 1$ so that $|x(t_2^*) - x(t_2^* - \delta_2)| \geq \varepsilon$, and, thus, there exists $t_2 \in [t_1 + 1, t_2^*]$ so that $|x(t_2) - x(t_2 - \delta_2)| = \varepsilon$ and $|x(t) - x(t-\delta_2)| \leq \varepsilon$ for all $t \in [0, t_2)$. Continuing the above argument, we can choose two sequences $\{t_k\}$ and $\{\delta_k\}$ so that

(a) $t_{k+1} \geq t_k + 1$ and thus $\lim_{k\to\infty} t_k = +\infty$,

(b) $0 < \delta_{k+1} < \delta_k/2$ and thus $\lim_{k\to\infty} \delta_k = 0$,

(c) $|x(t_k) - x(t_k - \delta_k)| = \varepsilon$ for all $k = 1, 2, \cdots$,

(d) $|x(t) - x(t-\delta_k)| < \varepsilon$ for all $k = 1, 2, \cdots$ and for $t \in [0, t_k)$.

Therefore,

$$\begin{aligned}
|x_{t_k} - x_{t_k - \delta_k}|_{C_g} &\leq \sup_{-(t_k-\delta_k) \leq \theta \leq 0} |x_{t_k}(\theta) - x_{t_k - \delta_k}(\theta)|/g(\theta) \\
&\quad + \sup_{\theta \leq -(t_k - \delta_k)} |x_{t_k}(\theta) - x_{t_k - \delta_k}(\theta)|/g(\theta) \\
&\leq \sup_{0 \leq \theta \leq t_k - \delta_k} |x(\delta_k + \theta) - x(\theta)| + \sup_{\theta \leq 0} \frac{|x(\theta + \delta_k) - x(\theta)|}{g(\theta - t_k + \delta_k)} \\
&\leq \sup_{0 \leq \theta \leq t_k - \delta_k} |x(\delta_k + \theta) - x(\theta)| + |x_{\delta_k} - x_0|_{C_g} \leq \varepsilon + |x_{\delta_k} - x_0|_{C_g}
\end{aligned}$$

and

$$
\begin{aligned}
&\left| \int_{-\infty}^{t_k} f(t_k - s)x(s)(ds) - \int_{-\infty}^{t_k-\delta_k} f(t_k-\delta_k-s)x(s)ds \right| \\
&\quad = \left| \int_{-\infty}^{0} f(-u)[x(u+t_k) - x(u+t_k-\delta_k]du \right| \\
&\quad = \left| \int_{-\infty}^{-t_k+\delta_k} f(-u)[x(u+t_k) - x(u+t_k-\delta_k)]du \right| \\
&\qquad + \left| \int_{-t_k+\delta_k}^{0} f(-u)[x(u+t_k) - x(u+t_k-\delta_k)]du \right| \\
&\quad \le \left| \int_{-\infty}^{-t_k+\delta_k} f(-s)g(s)\frac{x_{t_k}(s) - x_{t_k-\delta_k}(s)}{g(s)}\, ds \right| \\
&\qquad + \int_{-t_k+\delta_k}^{0} |f(-u)|\,|x(u+t_k) - x(u+t_k-\delta_k)|du \\
&\quad \le \int_{-\infty}^{-t_k+\delta_k} |f(-u)|g(u)du\, |x_{t_k} - x_{t_k-\delta_k}|_{C_g} + \varepsilon \int_{-t_k+\delta_k}^{0} |f(-u)|du \\
&\quad \le \int_{-\infty}^{-t_k+\delta_k} |f(-u)|g(u)du[\varepsilon + |x_{\delta_k} - x_0|_{C_g}] + \varepsilon \int_{-\infty}^{0} |f(-u)|du\,.
\end{aligned}
$$

By passing if necessary to a subsequence relabeled $t_k - \delta_k$, we may assume, without loss of generality, that, for every $k=1, 2, \cdots$, there exists an integer m_k so that $t_k - \delta_k - r_i \ge 0$ for $i=1, 2, \cdots m_k$, and $t_k - \delta_k - r_i < 0$ for $i = m_k+1, m_k+2, \cdots$. Thus,

$$
\begin{aligned}
&\left| \sum_{i=1}^{\infty} b_i x(t_k - r_i) - \sum_{i=1}^{\infty} b_i x(t_k - \delta_k - r_i) \right| \\
&\quad \le \sum_{i=1}^{m_k} |b_i|\,|x(t_k - r_i) - x(t_k-\delta_k-r_i)| + \sum_{i=m_k+1}^{\infty} |b_i|\,|x(t_k-r_i) - x(t_k-\delta_k-r_i)| \\
&\quad \le \sum_{i=1}^{\infty} |b_i|\varepsilon + \sum_{i=m_k+1}^{\infty} |b_i| g(-r_i) \frac{|x_{t_k}(-r_i) - x_{t_k-\delta_k}(-r_i)|}{g(-r_i)} \\
&\quad \le \sum_{i=1}^{\infty} |b_i|\varepsilon + \sum_{i=m_k+1}^{\infty} |b_i| g(-r_i)\,|x_{t_k} - x_{t_k-\delta_k}|_{C_g} \\
&\quad \le \sum_{i=1}^{\infty} |b_i|\varepsilon + \sum_{i=m_k+1}^{\infty} |b_i| g(-r_i)[\varepsilon + |x_{\delta_k} - x_0|_{C_g}]\,.
\end{aligned}
$$

It follows from the equality

$$
h(t) = x(t) - \sum_{i=1}^{\infty} b_i x(t-r_i) - \int_{-\infty}^{t} f(t-s)x(s)ds
$$

that

$$
\begin{aligned}
|x(t_k)-x(t_k-\delta_k)| &\le |h(t_k)-h(t_k-\delta_k)| + \left|\sum_{i=1}^{\infty} b_i[x(t_k-r_i)-x(t_k-\delta_k-r_i)]\right| \\
&\quad + \left|\int_{-\infty}^{t_k} f(t_k-s)x(s)ds - \int_{-\infty}^{t_k-\delta_k} f(t_k-\delta_k-s)x(s)ds\right| \\
&\le |h(t_k)-h(t_k-\delta_k)| + \sum_{i=1}^{\infty} |b_i|\varepsilon + \sum_{i=m_k+1}^{\infty} |b_i| g(-r_i)[\varepsilon + |x_{\delta_k}-x_0|_{C_g}] \\
&\quad + \varepsilon \int_{-\infty}^{0} |f(-u)|du + \int_{-\infty}^{-t_k+\delta_k} |f(-u)|\, g(u)du[\varepsilon + |x_{\delta_k}-x_0|_{C_g}] \\
&\le |h(t_k)-h(t_k-\delta_k)| + \left[\int_{-\infty}^{0} |f(-u)|\, du + \sum_{i=1}^{\infty} |b_i|\right]\varepsilon \\
&\quad + \left[\sum_{i=m_k+1}^{\infty} |b_i| g(-r_i) + \int_{-\infty}^{-t_k+\delta_k} |f(-u)| g(u)du\right][\varepsilon + |x_{\delta_k}-x_0|_{C_g}] .
\end{aligned}
$$

Now, $\sum_{i=1}^{\infty} |b_i| g(-r_i) + \int_{-\infty}^{0} |f(-u)| g(u)du < 1$ and $h(t)$ is uniformly continuous on $[0, \infty)$, so we can find a constant $K \ge 0$ with

$$
\begin{aligned}
&|h(t_k)-h(t_k-\delta_k)| + \sum_{i=m_k+1}^{\infty} |b_i| g(-r_i) + \int_{-\infty}^{-t_k+\delta_k} |f(-u)| g(u)du[\varepsilon + |x_{\delta_k}-x_0|_{C_g}] \\
&\quad < \left\{1 - \left[\int_{-\infty}^{0} |f(-u)|du + \sum_{i=1}^{\infty} |b_i|\right]\right\}\frac{\varepsilon}{2}
\end{aligned}
$$

for $k \ge K$. Now, (gl) and (g2) together with the assumption that $|x_0(s)|/g(s) \to 0$ as $s \to -\infty$ assures that the mapping $t \to x_t$ is continuous in C_g. Therefore, for $k \ge K$, we have

$$
|x(t_k)-x(t_k-\delta_k)| < \frac{1-[\int_{-\infty}^{0} |g(-u)du + \sum_{i=1}^{\infty} |b_i|]}{2}\varepsilon + \left[\int_{-\infty}^{0} |f(-u)|du + \sum_{i=1}^{\infty} |b_i|\right]\varepsilon .
$$

This is contrary to (c), and, thus, the proof is completed.

The following result, which provides conditions for a bounded positive orbit to be C_g-precompact, was established by Haddock and Hornor in [4].

LEMMA 2.3. *Suppose* $g\colon (-\infty, 0] \to [1, \infty)$ *satisfies* (gl), (g2) *and* (g3). *If* $x\colon R \to R$ *satisfies the following conditions*

(i) $x_0 \in C_g$ *and* x *is bounded and uniformly continuous on* $[0, \infty)$,

(ii) $|x_0(s)|/g(s) \to 0$ *as* $s \to -\infty$,

then the set (*i.e., positive orbit*) $\{x_t\colon t \ge 0\}$ *is precompact in* C_g.

Combining Lemmas 2.1, 2.2, and 2.3 with a brief argument, we obtain the main theorem of this section. Theorem 2.1 provides an important generalization of Theorem 3.1 in [1] and Theorem 3.2 in [4]. We illustrate its significance in subsequent sections.

THEOREM 2.1. *Suppose*

$$\sum_{i=1}^{\infty} |b_i| + \int_{-\infty}^{0} |f(-u)|du \leq \alpha < 1 \quad \textit{and} \quad \sum_{i=1}^{\infty} |a_i| + \int_{-\infty}^{0} |h(-u)|du \leq \beta < +\infty .$$

Then there exists a function $g: (-\infty, 0] \to [1, \infty)$ *satisfying* (g1)–(g4) *and*

(g5) *for any solution* $x(t)$ *of* (1.3) *defined on* $[0, \infty)$, *if* $x_0 \in C_g$, $|x_0(s)|/g(s) \to 0$ *as* $s \to -\infty$, *and* $x: [0, \infty) \to R$ *is bounded, then the set* $\{x_t: t \geq 0\}$ *is precompact in* C_g.

PROOF. By Lemma 2.1 we can find a function $g: (-\infty, 0] \to [1, \infty)$ satisfying (g1), (g2) and (g3) and such that

$$\int_{-\infty}^{0} |f(-s)|g(s)ds + \sum_{i=1}^{\infty} |b_i| g(-r_i) \leq M < 1 ,$$

$$\int_{-\infty}^{0} |h(-s)|g(s)ds + \sum_{i=1}^{\infty} |a_i| g(-r_i) \leq N < +\infty ,$$

where M and N are constants. Since $x(t)$ is a solution of (1.3) defined on $[0, \infty)$, we have

$$\left| \frac{d}{dt} \left[x(t) - \sum_{i=1}^{\infty} b_i x(t-r_i) - \int_{-\infty}^{t} f(t-s)x(s)ds \right] \right|$$

$$\leq a|x(t)| + \sum_{i=1}^{\infty} |a_i| |x(t-r_i)| + \int_{-\infty}^{t} |h(t-s)| |x(s)| ds .$$

Let K be an integer so that $r_K \leq t < r_{K+1}$. Then

$$\sum_{i=1}^{\infty} |a_i| |x(t-r_i)| \leq \sum_{i=K+1}^{\infty} |a_i| g(-r_i)| |x_0|_{C_g} + \sum_{i=1}^{\infty} |a_i| \sup_{0 \leq s \leq t} |x(s)|$$

and

$$\int_{-\infty}^{t} |h(t-s)| |x(s)| ds \leq \int_{-\infty}^{0} |h(-s)| g(s)ds\, |x_0|_{C_g} + \left(\int_{0}^{\infty} |h(s)| ds \right) \sup_{0 \leq s \leq t} |x(s)| ,$$

and, thus, $h(t) = x(t) - \sum_{i=1}^{\infty} b_i x(t-r_i) - \int_{-\infty}^{t} f(t-s)x(s)ds$ is uniformly continuous. Therefore $\{x_t: t \geq 0\}$ is precompact in C_g by Lemmas 2.2 and 2.3, thereby completing the proof.

3. Boundedness and stability.

In this section we examine boundedness and stability of solutions of (1.3). These properties coupled with the precompactness results of the previous section and the upcoming "reduction" results in Sections 4 and 5 will be instrumental in establishing the asymptotic constancy theorem of Section 6.

THEOREM 3.1. *Suppose there exists a function* $g: (-\infty, 0] \to [1, \infty)$ *which satisfies* (g1) *and* (g2) *such that*

$$\sum_{i=1}^{\infty} |b_i| g(-r_i) + \int_{-\infty}^{0} |f(-s)| g(s) ds < 1$$

and

$$\sum_{i=1}^{\infty} |a_i - ab_i| g(-r_i) + \int_{-\infty}^{0} |h(-s) - af(-s)| g(s) ds$$
$$\leq a\left[1 - \sum_{i=1}^{\infty} |b_i| g(-r_i) - \int_{-\infty}^{0} |f(-s)| g(s) ds\right].$$

Then, for any $\phi \in C_g$ *with* $|\phi(s)|/g(s) \to 0$ *as* $s \to -\infty$ *and any solution* $x(t)$ *of* (1.3) *through* $(0, \phi)$,

$$|x(t)| \leq 2\frac{1 + \int_{-\infty}^{0} |f(-s)| g(s) ds + \sum_{i=1}^{\infty} |b_i| g(-r_i)}{1 - \sum_{i=1}^{\infty} |b_i| - \int_{-\infty}^{0} |f(-s)| ds} |\phi|_{C_g}, \qquad t \geq 0.$$

PROOF. First, we prove that, for

$$D(t) = x(t) - \sum_{i=1}^{\infty} b_i x(t - r_i) - \int_{-\infty}^{t} f(t-s) x(s) ds,$$

we have

$$|D(t)| \leq M(\phi) = \left[1 + \int_{-\infty}^{0} |f(-s) g(s) ds + \sum_{i=1}^{\infty} |b_i| g(-r_i)\right] |\phi|_{C_g}. \tag{3.1}$$

Suppose not and notice that

$$|D(0)| \leq \left[1 + \sum_{i=1}^{\infty} |b_i| g(-r_i) + \int_{-\infty}^{0} |f(-s)| g(s) ds\right] |\phi|_{C_g}.$$

If (3.1) does not hold for some $\phi \in C_g - \{0\}$, then there exists a first $\tau > 0$ so that

$$D^2(\tau) = M^2(\phi)$$

and

$$\frac{d}{dt} D^2(t) > 0 \qquad \text{at} \quad t = \tau.$$

On the other hand, by the definition of τ, we have

$$|D(t)| \leq M(\phi)$$

for $t\in[0,\tau]$; that is,

$$\left|x(t)-\sum_{i=1}^{\infty} b_i x(t-r_i)-\int_{-\infty}^{0} f(-s)x(t+s)ds\right| M(\phi)\,. \tag{3.2}$$

Choose $\tau^*\in[0,\tau]$ so that

$$|x_{\gamma^*}|_{C_g}=\max_{u\in[0,\tau]}|x_u|_{C_g}\,.$$

(We note that this max exists due to (g1), (g2) and the choice of $\phi=x_0$, i.e. $u\to x_u$ is continuous with u restricted to $[0,\tau]$). In particular,

$$\sup_{s\le 0}\frac{|x(\tau^*+s)|}{g(s)}=\max_{u\in[0,\tau]}\sup_{s\le 0}\frac{|x(u+s)|}{g(s)}\,. \tag{3.3}$$

Note that if $s\le-\tau^*$, then

$$\begin{aligned}|x(\tau^*+s)|/g(s)&=[|x(\tau^*+s)|/g(\tau^*+s)][g(\tau^*+s)/g(s)]\\&\le|x(\tau^*+s)|/g(\tau^*+s)\le|x_0|_{C_g}\,.\end{aligned}$$

Therefore, (3.3) implies that there exists a $s^*\in[-\tau^*,0]$ so that

$$|x_{\tau^*}|_{C_g}=\sup_{s\in[-\tau^*,0]}|x(\tau^*+s)|/g(s)=|x(\tau^*+s^*)|/g(s^*)\,.$$

Let $r_K\le\tau^*<r_{K+1}$. Then by (3.2) we have

$$\begin{aligned}&|x(\tau^*+s^*)|/g(s^*)\\&\le\sum_{i=1}^{K}|b_i|\frac{|x(\tau^*-r_i+s^*)|}{g(s^*)}+\sum_{i=K+1}^{\infty}|b_i|g(-r_i)\frac{|x(\tau^*+s^*-r_i)|}{g(-r_i)g(s^*)}\\&\quad+\int_{-\infty}^{-\tau^*}|f(-s)|g(s)\frac{|x(\tau^*+s^*+s)|}{g(s)g(s^*)}ds+\int_{-\tau^*}^{0}|f(-s)|\frac{|x(\tau^*+s+s^*)}{g(s^*)}ds+M(\phi)/g(s^*)\\&\le\sum_{i=1}^{K}|b_i|g(-r_i)|x_{\tau^*-r_i}|_{C_g}+\sum_{i=K+1}^{\infty}|b_i|g(-r_i)|x_0|_{C_g}\\&\quad+\int_{-\infty}^{-\tau^*}|f(-s)|g(s)|x_0|_{C_g}ds+\int_{-\tau^*}^{0}|f(-s)|g(s)|x_{\tau^*+s}|_{C_g}ds+M(\phi)\\&\le\left[\sum_{i=1}^{\infty}|b_i|g(-r_i)+\int_{-\infty}^{0}|f(-s)|g(s)ds\right]|x_{\tau^*}|_{C_g}+M(\phi)\,.\end{aligned}$$

This implies

$$|x_t|_{C_g}\le|x_{\tau^*}|_{C_g}\le\frac{M(\phi)}{1-\sum_{i=1}^{\infty}|b_i|g(-r_i)-\int_{-\infty}^{0}|f(-s)|g(s)ds}$$

for $t\in[0,\tau]$. Thus, at $t=\tau$ we have

$$\frac{d}{dt}D^2(t)/2 = -aD^2(\tau) + \left[\sum_{i=1}^{\infty}(a_i - ab_i)x(\tau - r_i) + \int_{-\infty}^{0}(h(-s) - af(-s))x(\tau + s)ds\right]D(\tau)$$

$$\leq -aD^2(\tau) + \left[\sum_{i=1}^{\infty}|a_i - ab_i|g(-r_i) + \int_{-\infty}^{0}|h(-s) - af(-s)|g(s)ds\right]|D(\tau)|\,|x_\tau|_{C_g} \leq 0,$$

which leads to a contradiction. Hence, (3.1) is true for all $t \geq 0$. Suppose $|x(v)| = \max_{0 \leq s \leq v}|x(s)|$ and let $r_L \leq v < r_{L+1}$ for some integer L. Then by (3.1) we have

$$|x(v)| \leq \sum_{i=1}^{L}|b_i|\,|x(v - r_i)| + \sum_{i=L+1}^{\infty}|b_i|g(-r_i)\frac{|x(v-r_i)|}{g(v-r_i)}\frac{g(v-r_i)}{g(-r_i)}$$

$$+ \int_{-\infty}^{-v}|f(-s)|g(s)\frac{|x(v+s)|}{g(v+s)}\frac{g(v+s)}{g(s)}ds + \int_{-v}^{0}|f(-s)|\,|x(v+s)|ds + M(\phi)$$

$$\leq \left[\sum_{i=1}^{\infty}|b_i| + \int_{-\infty}^{0}|f(-s)|ds\right]|x(v)|$$

$$+ \left[\sum_{i=1}^{\infty}|b_i|g(-r_i) + \int_{-\infty}^{0}|f(-s)|g(s)ds\right]|\phi|_{C_g} + M(\phi).$$

It follows that

$$|x(v)| \leq \frac{[\sum_{i=1}^{\infty}|b_i|g(-r_i) + \int_{-\infty}^{0}|f(-s)|g(s)ds]\,|\phi|_{C_g} + M(\phi)}{1 - \sum_{i=1}^{\infty}|b_i| - \int_{-\infty}^{0}|f(-s)|ds}$$

$$\leq 2\frac{1 + [\sum_{i=1}^{\infty}|b_i|g(-r_i) + \int_{-\infty}^{0}|f(-s)|g(s)ds]}{1 - \sum_{i=1}^{\infty}|b_i| - \int_{-\infty}^{0}|f(-s)|ds}|\phi|_{C_g},$$

which completes the proof.

The final result stated in this section is an immediate consequence of Theorem 3.1.

COROLLARY 3.1. *Suppose there exists a function* $g\colon(-\infty, 0] \to [1, \infty)$ *satisfying* (g1) *and* (g2) *such that*

$$\sum_{i=1}^{\infty}|b_i|g(-r_i) + \int_{-\infty}^{0}|f(-s)|g(s)ds < 1 \tag{3.4}$$

and

$$\sum_{i=1}^{\infty}|a_i - ab_i|g(-r_i) + \int_{-\infty}^{0}|h(-s) - af(-s)|g(s)ds \tag{3.5}$$

$$\leq a\left[1 - \sum_{i=1}^{\infty}|b_i|g(-r_i) - \int_{-\infty}^{0}|f(-s)|g(s)ds\right],$$

then all solutions of (1.3) *are bounded* (*in the future*). *Moreover, for any* $\varepsilon > 0$, *there exists*

$\delta>0$ so that $|\phi-\psi|_{C_g}<\delta$ implies $|x(t;\phi)-x(t;\psi)|<\varepsilon$ for $t\geq 0$ and any ϕ and ψ for which $|\phi(s)|/g(s)\to 0$ and $|\psi(s)|/g(s)\to 0$ as $s\to-\infty$. (That is, each solution of (1.3) *is stable. Moreover, δ depends on ε alone and not on ϕ and ψ.)*

4. Reduction to retarded equations. In this section we establish a relationship between solutions through ω-limit points of (1.3) and solutions of a certain retarded FDE. According to Theorems 2.1 and 3.1 and Corollary 3.1, if (3.4) and (3.5) hold, then $\{x_t: t\geq 0\}$ is precompact in C_g if $x_0=\phi\in C_g$ and $|x_0(s)|/g(s)\to 0$ as $s\to-\infty$. Let $\omega(\phi)$ be the positive limit set of the orbit $\{x_t: t\geq 0\}$ in C_g, ψ an element in $\omega(\phi)$ and $z(t)$ the solution of (1.3) through ψ. Then $z(t)$ satisfies the following equation

$$\frac{d}{dt}\left[z(t)-\sum_{i=1}^{\infty} b_i z(t-r_i)-\int_{-\infty}^{t} f(t-s)z(s)ds\right] = -az(t)+\sum_{i=1}^{\infty} a_i z(t-r_i)+\int_{-\infty}^{t} h(t-s)z(s)ds \tag{4.1}$$

for all $t\in R$, and the boundedness of the solution $x(t)$ implies the existence of a constant $M>0$ such that

$$|z(t)|\leq M \tag{4.2}$$

for $t\in R$. Let

$$y(t)=z(t)-\sum_{i=1}^{\infty} b_i z(t-r_i)-\int_{-\infty}^{t} f(t-s)z(s)ds\,. \tag{4.3}$$

The following theorem shows that $y(t)$ satisfies a retarded equation with infinite delay.

THEOREM 4.1. *Suppose* (3.4) *and* (3.5) *hold. Then $y(t)$ defined above satisfies the retarded equation*

$$\begin{aligned}\frac{d}{dt}y(t)= &-ay(t)+\sum_{j=1}^{\infty}\sum_{i_1=1}^{\infty}\cdots\sum_{i_j=1}^{\infty}(a_{i_1}-ab_{i_1})b_{i_2}\cdots b_{i_j}y(t-r_{i_1}-\cdots-r_{i_j})\\ &+\int_{-\infty}^{t} h_1(t-s)y(s)ds\\ &+\sum_{j=1}^{\infty}\sum_{i_1=1}^{\infty}\cdots\sum_{i_j=1}^{\infty} b_{i_2}\cdots b_{i_j}\int_{-\infty}^{t-r_{i_1}-\cdots-r_{i_j}} h_{i_1 j}(t-r_{i_1}-\cdots-r_{i_j}-s)y(s)ds\,,\end{aligned} \tag{4.4}$$

where, for any $u, v\in L^1[0,\infty)$,

$$(u*v)(t)=\int_0^t u(t-s)v(s)ds$$

and

$$k(t)=\sum_{j=1}^{\infty} f_j^*(t)\,,$$

$$f_j^*(t)=f*f*\cdots*f(t) \quad j \text{ times, with } f_1^*=f\,.$$

$$h_1(t)=h(t)-af(t)+[h-af]*k$$

$$h_{i_1 1}(t)=(a_{i_1}-ab_{i_1})k(t)+b_{i_1}h_1(t)+b_{i_1}h_1*k(t)$$

$$h_{i_1 j+1}(t)=(a_{i_1}-ab_{i_1})k(t)+h_{i_1 j}(t)+h_{i_1 j}*k(t)\,, \qquad j\geq 1\,. \tag{4.5}$$

PROOF. Obviously

$$z(t)=y(t)+\sum_{i=1}^{\infty} b_i z(t-r_i)+\int_{-\infty}^{t} f(t-s)z(s)ds \tag{4.6}$$

for all $t\in R$, and thus

$$\begin{aligned} z(t)&=y(t)+\sum_{i=1}^{\infty} b_i z(t-r_i)+\int_{-\infty}^{t} f(t-s)z(s)ds \\ &=y(t)+\sum_{i=1}^{\infty} b_i z(t-r_i)+\int_{-\infty}^{t} f(t-s_1)\left[y(s_1)+\sum_{i=1}^{\infty} b_i z(s_1-r_i)\right. \\ &\quad\left.+\int_{-\infty}^{s_1} f(s_1-s)z(s)ds\right]ds_1=y(t)+\sum_{i=1}^{\infty} b_i z(t-r_i)+\int_{-\infty}^{t} f(t-s_1)y(s_1)ds_1 \\ &\quad+\sum_{i=1}^{\infty} b_i\int_{-\infty}^{t-r_i} f(t-r_i-s_1)z(s_1)ds_1+\int_{-\infty}^{t} f_2^*(t-s)z(s)ds\,. \end{aligned}$$

Using (4.6) at s, we get

$$\begin{aligned} z(t)&=y(t)+\sum_{i=1}^{\infty} b_i z(t-r_i)+\int_{-\infty}^{t} f(t-s_1)y(s_1)ds_1 \\ &\quad+\sum_{i=1}^{\infty} b_i\int_{-\infty}^{t-r_i} f(t-r_i-s)z(s)ds+\int_{-\infty}^{t} f_2^*(t-s)\left[y(s)+\sum_{i=1}^{\infty} b_i z(s-r_i)\right. \\ &\quad\left.+\int_{-\infty}^{s} f(s-u)z(u)du\right]ds=y(t)+\sum_{i=1}^{\infty} b_i z(t-r_i)+\int_{-\infty}^{t} f(t-s)y(s)ds \\ &\quad+\int_{-\infty}^{t} f_2^*(t-s)y(s)ds+\sum_{i=1}^{\infty} b_i\int_{-\infty}^{t-r_i} f(t-r_i-s)z(s)ds \\ &\quad+\sum_{i=1}^{\infty} b_i\int_{-\infty}^{t-r_i} f_2^*(t-r_i-s)z(s)ds+\int_{-\infty}^{t} f_3^*(t-s)z(s)ds\,. \end{aligned}$$

By repeating (4.6), we get

$$z(t)=y(t)+\sum_{i=1}^{\infty} b_i z(t-r_i)+\int_{-\infty}^{t} \sum_{j=1}^{k} f_j^*(t-s)y(s)ds$$

$$+\sum_{i=1}^{\infty} b_i \int_{-\infty}^{t-r_i} \sum_{j=1}^{k} f_j^*(t-r_i-s)z(s)ds+\int_{-\infty}^{t} f_{k+1}^*(t-s)z(s)ds. \tag{4.7}$$

It is easy to prove that

$$\int_{-\infty}^{t} |f_{k+1}^*(t-s)|ds \leq \left[\int_0^{\infty} |f(s)|ds\right]^{k+1}.$$

By $\int_0^{\infty} |f(s)|ds<1$ and from (4.2), we know that

$$k(t)=\sum_{j=1}^{\infty} f_j^*(t) \tag{4.8}$$

is well defined and

$$\left|\int_{-\infty}^{t} f_{k+1}^*(t-s)z(s)ds\right| \leq \left[\int_0^{\infty} |f(t)|dt\right]^{k+1} M \to 0$$

as $k\to\infty$. Passing to the limit as $k\to\infty$ in (4.7) we get

$$z(t)=y(t)+\sum_{i=1}^{\infty} b_i z(t-r_i)+\int_{-\infty}^{t} k(t-s)y(s)ds+\sum_{i=1}^{\infty} b_i \int_{-\infty}^{t-r_i} k(t-r_i-s)z(s)ds. \tag{4.9}$$

Therefore,

$$\int_{-\infty}^{t} h(t-s)z(s)ds=\int_{-\infty}^{t} h(t-s)y(s)ds+\sum_{i=1}^{\infty}\int_{-\infty}^{t-r_i} b_i h(t-r_i-s)z(s)ds$$
$$+\int_{-\infty}^{t}(h*k)(t-s)y(s)ds+\sum_{i=1}^{\infty} b_i \int_{-\infty}^{t-r_i}(h*k)(t-r_i-s)z(s)ds.$$

According to (4.1), we get

$$\frac{d}{dt}y(t)=-ay(t)+\sum_{i=1}^{\infty}(a_i-ab_i)z(t-r_i)+\int_{-\infty}^{t}[h(t-s)-ak(t-s)+(h*k)(t-s)]y(s)ds$$
$$+\sum_{i=1}^{\infty} b_i \int_{-\infty}^{t-r_i}\left[h(t-r_i-s)-ak(t-r_i-s)+(h*k)(t-r_i-s)\right]z(s)ds$$
$$=-ay(t)+\sum_{i=1}^{\infty}(a_i-ab_i)z(t-r_i)+\int_{-\infty}^{t} h_1(t-s)y(s)ds+\sum_{i=1}^{\infty} b_i \int_{-\infty}^{t-r_i} h_1(t-r_i-s)z(s)ds.$$

Using the equality (4.9) at $t-r_{i_1}$ we get

$$\frac{d}{dt}y(t) = -ay(t) + \int_{-\infty}^{t} h_1(t-s)y(s)ds + \sum_{i_1=1}^{\infty}(a_{i_1}-ab_{i_1})\Bigg[y(t-r_{i_1}) + \sum_{i_2=1}^{\infty} b_{i_2}z(t-r_{i_1}-r_{i_2})$$
$$+\int_{-\infty}^{t-r_{i_1}} k(t-r_{i_1}-s)y(s)ds + \sum_{i_2=1}^{\infty}\int_{-\infty}^{t-r_{i_1}-r_{i_2}} b_{i_2}k(t-r_{i_1}-r_{i_2}-s)z(s)ds\Bigg]$$
$$+\sum_{i_1=1}^{\infty} b_{i_1}\int_{-\infty}^{t-r_{i_1}} h_1(t-r_{i_1}-u)\Bigg[y(u) + \sum_{i_2=1}^{\infty} b_{i_2}z(u-r_{i_2})$$
$$+\int_{-\infty}^{u} k(u-s)y(s)ds + \sum_{i_2=1}^{\infty} b_{i_2}\int_{-\infty}^{u-r_{i_2}} k(u-r_{i_2}-s)z(s)ds\Bigg]du$$
$$= -ay(t) + \int_{-\infty}^{t} h_1(t-s)y(s)ds + \sum_{i_1=1}^{\infty}\int_{-\infty}^{t-r_{i_1}} h_{i_1 1}(t-r_{i_1}-s)y(s)ds$$
$$+\sum_{i_1=1}^{\infty}(a_{i_1}-ab_{i_1})y(t-r_{i_1}) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty} b_{i_2}\int_{-\infty}^{t-r_{i_1}-r_{i_2}} h_{i_1 1}(t-r_{i_1}-r_{i_2}-s)z(s)ds$$
$$+\sum_{i_1=1}^{\infty}(a_{i_1}-ab_{i_1})\sum_{i_2=1}^{\infty} b_{i_2}z(t-r_{i_1}-r_{i_2}).$$

Again, by using (4.9) at $t-r_{i_1}-r_{i_2}-\cdots-r_{i_k}$ and $s \le t-r_{i_1}-\cdots-r_{i_k}$ we get the following iterative formula.

$$\frac{d}{dt}y(t) = -ay(t) + \int_{-\infty}^{t} h_1(t-s)y(s)ds$$
$$+\sum_{j=1}^{\infty}\sum_{i_1=1}^{\infty}\cdots\sum_{i_j=1}^{\infty}(a_{i_1}-ab_{i_1})b_{i_2}\cdots b_{i_j}y(t-r_{i_1}-\cdots-r_{i_j})$$
$$+\sum_{j=1}^{k}\sum_{i_1=1}^{\infty}\cdots\sum_{i_j=1}^{\infty}\int_{-\infty}^{t-r_{i_1}-\cdots-r_{i_j}} b_{i_2}\cdots b_{i_j}h_{i_1 j}(t-r_{i_1}-\cdots-r_{i_j}-s)y(s)ds \tag{4.10}$$
$$+\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_{k+1}=1}^{\infty} b_{i_2}\cdots b_{i_{k+1}}\int_{-\infty}^{t-r_{i_1}-\cdots-r_{i_{k+1}}} h_{i_1 k}(t-r_{i_1}-\cdots-r_{i_{k+1}}-s)z(s)ds$$
$$+\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_{k+1}=1}^{\infty}(a_{i_1}-ab_{i_1})b_{i_2}\cdots b_{i_{k+1}}z(t-r_{i_1}-\cdots-r_{i_{k+1}}).$$

Obviously,

$$\left|\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_{k+1}=1}^{\infty}(a_{i_1}-ab_{i_1})b_{i_2}\cdots b_{i_{k+1}}z(t-r_{i_1}\cdots-r_{i_{k+1}})\right| \tag{4.11}$$
$$\le \sum_{i_1=1}^{\infty}|a_{i_1}-ab_{i_1}|\left[\sum_{i=1}^{\infty}|b_i|\right]^k M \to 0$$

as $k \to \infty$. On the other hand,

$$\int_0^\infty |k(t)|\,dt \le \sum_{j=1}^\infty |f_j^*(t)|\,dt \le \sum_{j=1}^\infty \left[\int_0^\infty |f(t)|\,dt\right]^j = \frac{\int_0^\infty |f(t)|\,dt}{1-\int_0^\infty |f(t)|\,dt}. \tag{4.12}$$

Therefore,

$$\begin{aligned}
\int_0^\infty |h_1(t)|\,dt &\le \int_0^\infty |h(t)-af(t)|\,dt + \int_0^\infty |h(t)-af(t)|\,dt \int_0^\infty |k(t)|\,dt \\
&\le \frac{\int_0^\infty |h(t)-af(t)|\,dt}{1-\int_0^\infty |f(t)|\,dt}.
\end{aligned}$$

$$\begin{aligned}
\int_0^\infty |h_{i_1 1}(t)|\,dt &\le |a_{i_1}-ab_{i_1}| \int_0^\infty |k(t)|\,dt + |b_{i_1}| \int_0^\infty |h_1(t)|\,dt\left[1+\int_0^\infty |k(t)|\,dt\right] \\
&\le |a_{i_1}-ab_{i_1}| \frac{\int_0^\infty |f(t)|\,dt}{1-\int_0^\infty |f(t)|\,dt} + |b_{i_1}| \frac{\int_0^\infty |h_1(t)|\,dt}{1-\int_0^\infty |f(t)|\,dt} \\
&\le |a_{i_1}-b_{i_1}| \frac{\int_0^\infty |f(t)|\,dt}{1-\int_0^\infty |f(t)|\,dt} + |b_{i_1}| \frac{\int_0^\infty |h(t)-af(t)|\,dt}{[1-\int_0^\infty |f(t)|\,dt][1-\int_0^\infty |f(t)|\,dt]}.
\end{aligned}$$

$$\begin{aligned}
\int_0^\infty |h_{i_1 j+1}(t)|\,dt &\le |a_{i_1}-ab_{i_1}| \int_0^\infty |k(t)|\,dt + \int_0^\infty |h_{ij}(t)+(h_{ij}*k)(t)|\,dt \\
&\le |a_{i_1}-ab_{i_1}| \int_0^\infty |k(t)|\,dt + \int_0^\infty |h_{i_1 j}(t)|\,dt\left[1+\int_0^\infty |k(t)|\,dt\right].
\end{aligned}$$

Using the above iterative inequality, we get the following estimate.

$$\begin{aligned}
\int_0^\infty |h_{i_1 j}(t)|\,dt &\le |a_{i_1}-ab_{i_1}| \int_0^\infty |k(t)|\,dt \left\{1+\left[1+\int_0^\infty |k(t)|\,dt\right]+\cdots\right. \\
&\quad \left. +\left[1+\int_0^\infty |k(t)|\,dt\right]^{j-2}\right\} + \left[1+\int_0^\infty |k(t)|\,dt\right]^{j-1} \int_0^\infty |h_{i_1 1}(t)|\,dt \\
&\le |a_{i_1}-ab_{i_1}| \left\{\left[1+\int_0^\infty |k(t)|\,dt\right]^{j-1}-1\right\} + \left[1+\int_0^\infty |k(t)|\,dt\right]^{j-1} \int_0^\infty |h_{i_1}(t)|\,dt \\
&\le |a_{i_1}-ab_{i_1}| \left\{\left[\frac{1}{1-\int_0^\infty |f(t)|\,dt}\right]^{j-1}-1\right\} \\
&\quad +\left[\frac{1}{1-\int_0^\infty |f(t)|\,dt}\right]^{j-1} \left\{|a_{i_1}-ab_{i_1}| \frac{\int_0^\infty |f(t)|\,dt}{1-\int_0^\infty |f(t)|\,dt} + |b_i| \frac{\int_0^\infty |h(t)-af(t)|\,dt}{[1-\int_0^\infty |f(t)|\,dt]^2}\right\} \\
&\le \frac{|a_{i_1}-ab_{i_1}|}{[1-\int_0^\infty |f(t)|\,dt]^{j-1}} - |a_{i_1}-ab_{i_1}| + |a_{i_1}-ab_{i_1}| \frac{\int_0^\infty |f(t)|\,dt}{[1-\int_0^\infty |f(t)|\,dt]^j}
\end{aligned} \tag{4.13}$$

$$+|b_{i_1}|\frac{\int_0^\infty |h(t)-af(t)|dt}{[1-\int_0^\infty |f(t)|dt]^{j+1}}.$$

This implies

$$\left|\sum_{i_1=1}^\infty \sum_{i_2=1}^\infty \cdots \sum_{i_{k+1}=1}^\infty b_{i_2}\cdots b_{i_{k+1}} \int_{-\infty}^{t-r_{i_1}-\cdots-r_{i_{k+1}}} h_{i_1k}(t-r_{i_1}-\cdots-r_{i_{k+1}}-s)z(s)ds\right|$$

$$\le \sum_{i_1=1}^\infty \left(\sum_{i=1}^\infty |b_i|^k\right)\left\{\frac{|a_{i_1}-ab_{i_1}|}{[1-\int_0^\infty |f(t)|dt]^k}-|a_{i_1}-ab_{i_1}|\right.$$

$$\left.+|a_{i_1}-ab_{i_1}|\frac{\int_0^\infty |f(t)|dt}{[1-\int_0^\infty |f(t)|dt]^{k+1}}+|b_{i_1}|\frac{\int_0^\infty |h(t)-af(t)|dt}{[1-\int_0^\infty |f(t)|dt]^{k+1}}\right\}M$$

$$\le M\left\{\sum_{i_1=1}^\infty |a_{i_1}-b_{i_1}|\left(\frac{1}{1-\int_0^\infty |f(t)|dt}\right)\left(\frac{\sum_{i=1}^\infty |b_i|}{1-\int_0^\infty |f(t)|dt}\right)^k\right.$$

$$-\sum_{i_1=1}^\infty |a_{i_1}-ab_{i_1}|\left(\sum_{i=1}^\infty |b_i|\right)^k$$

$$\left.+\sum_{i_1=1}^\infty |b_{i_1}|\frac{\int_0^\infty |h(t)-af(t)|dt}{1-\int_0^\infty |f(t)|dt}\left(\frac{\sum_{i=1}^\infty |b_i|}{1-\int_0^\infty |f(t)|dt}\right)^k\right\}\to 0 \quad \text{as} \quad k\to\infty.$$

Computing the limit as $k\to\infty$ in (4.10), we get (4.4). This completes the proof.

THEOREM 4.2. *Suppose* (3.4) *and* (3.5) *hold. Then* $y(t)$ *defined by* (4.3) *tends to some constant as* $t\to\infty$.

PROOF. Let

$$V(t)=\sup\left\{\left[\frac{y(t-r_{i_1}-\cdots r_{i_l}-s)}{g(-r_{i_1})\cdots g(-r_{i_l})g(-s)}\right]^2;\, s\ge 0,\, i_1,\, \cdots,\, i_l=1,2,\cdots;\, r_{i_1}+\cdots r_{i_1}+s\le t\right\}.$$

For any $t\in R$ with $y^2(t)=V(t)$, replacing b_i by $b_ig(-r_i)$ and $f(s)$ by $f(s)g(-s)$ in the estimates (4.11), (4.12) and (4.13) obtained in the proof of Theorem 4.1, we get

$$\frac{1}{2}\frac{d}{dt}y^2(t)\le -ay^2(t)\sum_{j=1}^\infty \sum_{i_1=1}^\infty \cdots \sum_{i_j=1}^\infty |a_{i_1}-ab_{i_1}|g(-r_{i_1})|b_{i_2}|g(-r_{i_2})\cdots|b_{i_j}|g(-r_{i_j})y^2(t)$$

$$+\int_0^\infty |h_1(t)|g(-t)dty^2(t)$$

$$+\sum_{j=1}^\infty \sum_{i_1=1}^\infty \cdots \sum_{i_j=1}^\infty b_{i_2}g(-r_{i_2})\cdots b_{i_j}g(-r_{i_j})\int_0^\infty |h_{i_1j}(t)|g(-t)dty^2(t)$$

$$\le -ay^2(t)+y^2(t)\left\{\frac{\sum_{i_1=1}^\infty |a_{i_1}-ab_{i_1}|g(-r_{i_1})}{1-\sum_{j=1}^\infty |b_j|g(-r_j)}+\frac{\int_0^\infty |h(t)-af(t)|g(-t)dt}{1-\int_0^\infty |f(t)|g(-t)dt}\right.$$

$$+\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}|b_i|g(-r_i)\right)^{j-1}\left[\frac{\sum_{i_1=1}^{\infty}|a_{i_1}-ab_{i_1}|g(-r_{i_1})}{[1-\int_0^\infty|f(t)|g(-t)dt]^{j-1}}-\sum_{i_1=1}^{\infty}|a_{i_1}-ab_{i_1}|g(-r_{i_1})\right.$$

$$\left.+\frac{\sum_{i_1=1}^{\infty}|a_{i_1}-ab_{i_1}|g(-r_{i_1})\int_0^\infty|f(t)|g(-t)dt}{[1-\int_0^\infty|f(t)|g(-t)dt]^{j}}+\frac{\sum_{i_1=1}^{\infty}|h(t)-af(t)|g(-t)dt}{[1-\int_0^\infty|f(t)|g(-t)dt]^{j+1}}\right]\Bigg\}$$

$$=-ay^2(t)+y^2(t)\left\{\frac{\int_0^\infty|h(t)-af(t)|g(-t)dt}{1-\int_0^\infty|f(t)|g(-t)dt}+\frac{\sum_{i_1=1}^{\infty}|a_{i_1}-ab_{i_1}|g(-r_{i_1})}{1-\dfrac{\sum_{i=1}^{\infty}|b_i|g(-r_i)}{1-\int_0^\infty|f(t)|g(-t)dt}}\right.$$

$$+\frac{\sum_{i=1}^{\infty}|a_{i_1}-ab_{i_1}|g(-r_{i_1})\int_0^\infty|f(t)|g(-t)dt}{1-\sum_{i=1}^{\infty}|b_i|g(-r_i)-\int_0^\infty|f(t)|g(-t)dt}$$

$$\left.+\frac{\sum_{i=1}^{\infty}|b_i|g(-r_i)}{[1-\int_0^\infty|f(t)|g(-t)dt]^2}\frac{\int_0^\infty|h(t)-af(t)|g(-t)dt}{1-\dfrac{\sum_{i=1}^{\infty}|b_i|g(-r_i)}{1-\int_0^\infty|f(t)|g(-t)dt}}\right\}$$

$$=-\left[a-\frac{\int_0^\infty|h(t)-af(t)|g(-t)dt+\sum_{i=1}^{\infty}|a_i-ab_i|g(-r_i)}{1-\sum_{i=1}^{\infty}|b_i|g(-r_i)-\int_0^\infty|f(t)|g(-t)dt}\right]y^2(t)\le 0\,.$$

Therefore $\dot V(t)\le 0$. This shows that $V(t)$ is a bounded nonincreasing function, and thus $\lim_{t\to\infty}V(t)=C^2$ exists, where C is a nonnegative constant.

By Lemma 2.3, $\omega(y_0)$ is nonempty. Let ψ be a given element in $\omega(y_0)$ and $z(t)$ be the solution of (4.1) through $(0,\psi)$. Then we have the following identity.

$$W(t)=\sup\left\{\left[\frac{z(t-r_{i_1}-\cdots r_{i_l}-s)}{g(-r_{i_1})\cdots g(-r_{i_l})g(-s)}\right]^2;\ s\ge 0,\ i_1,\ \cdots,\ i_l=1,2,\ \cdots;\ s+r_{i_1}+\cdots r_{i_l}\le t\right\}$$
$$=C^2$$

for all $t\in R$. Therefore $z^2(t)\le C^2$ for all $t\in R$.

If $z^2(t)<C^2$ for some $t=\tau$, then $z^2(t-s)<C^2$ for all $s\in[0,\delta_0]$, where $\delta_0<r_1$ is some positive constant depending on z and τ, and thus, by Remark 2.1,

$$W(t)\le\max\left\{\sup_{s\in[0,\delta_0]}z^2(t-s),\ \frac{C^2}{g^2(-\delta_0)}\right\}<C^2\,,$$

which is contrary to the identity $W(t)=C^2$.

Therefore, $z^2(t)=C^2$ and, thus, $z(t)=C$ or $z(t)=-C$ for all $t\in R$, which implies that $\omega(y_0)$ is a singleton consisting of a constant function. Our conclusion trivially follows from the attractivity of $\omega(y_0)$. This completes the proof.

5. Equivalence of asymptotic constancy of $D(z(t))$ and $z(t)$. In this section we will establish an equivalence result with respect to asymptotic constancy of $D(z(t))$ and $z(t)$.

THEOREM 5.1. *Suppose*

$$\sum_{i=1}^{\infty} |b_i| + \int_0^{\infty} |f(t)| dt < 1 .$$

Let $z(t)$ be a bounded continuous function defined on R such that the function

$$y(t) = z(t) - \sum_{i=1}^{\infty} b_i z(t-r_i) - \int_{-\infty}^{t} f(t-s) z(s) ds$$

tends to a constant c as $t \to \infty$. Then

$$\lim_{t\to\infty} z(t) = \left[1 + \int_0^{\infty} k(t)dt + \sum_{j=1}^{\infty} \sum_{i_1=1}^{\infty} \cdots \sum_{i_j=1}^{\infty} b_{i_1} \cdots b_{i_j} \int_0^{\infty} h_j(t)dt \right.$$
$$\left. + \sum_{j=1}^{\infty} \sum_{i_1=1}^{\infty} \cdots \sum_{i_j=1}^{\infty} b_{i_1} \cdots b_{i_j} \right] c ,$$

where

$$k(t) = \sum_{j=1}^{\infty} f_j^*(t)$$

$$h_1(t) = 2k(t) + (k * k)(t)$$

$$h_{j+1}(t) = h_j(t) + k(t) + (h_j * k)(t) .$$

PROOF. By (4.9) we have

$$z(t) = y(t) + \sum_{i=1}^{\infty} b_i z(t-r_i) + \int_{-\infty}^{t} k(t-s) y(s) ds + \sum_{i=1}^{\infty} \int_{-\infty}^{t-r_i} k(t-r_i-s) z(s) ds . \tag{5.1}$$

Using this equality at $t-r_i$, we have

$$z(t) = y(t) + \int_{-\infty}^{t} k(t-s)y(s)ds + \sum_{i_1=1}^{\infty} b_{i_1} \left[y(t-r_{i_1}) + \int_{-\infty}^{t-r_{i_1}} k(t-r_{i_1}-s)y(s)ds \right.$$
$$\left. + \sum_{i_2=1}^{\infty} b_{i_2} z(t-r_{i_1}-r_{i_2}) + \sum_{i_2=1}^{\infty} b_{i_2} \int_{-\infty}^{t-r_{i_1}-r_{i_2}} z(s)ds \right]$$
$$+ \sum_{i_1=1}^{\infty} b_{i_1} \int_{-\infty}^{t-r_{i_1}} k(t-r_{i_1}-u) \left[y(u) + \int_{-\infty}^{u} k(u-s)y(s)ds \right.$$
$$\left. + \sum_{i_2=1}^{\infty} b_{i_2} z(t-r_{i_1}-r_{i_2}) + \sum_{i_2=1}^{\infty} \int_{-\infty}^{u-r_{i_2}} k(u-r_{i_2}-s)z(s)ds \right] du$$
$$= y(t) + \int_{-\infty}^{t} k(t-s)y(s)ds + \sum_{i_1=1}^{\infty} b_{i_1} \int_{-\infty}^{t-r_{i_1}} h_1(t-r_{i_1}-s)y(s)ds$$

$$+\sum_{i_1=1}^{\infty} b_{i_1} y(t-r_{i_1})+\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty} b_{i_1}b_{i_2} z(t-r_{i_1}-r_{i_2})$$

$$+\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty} b_{i_1}b_{i_2}\int_{-\infty}^{t-r_{i_1}-r_{i_2}} h_1(t-r_{i_1}-r_{i_2})z(s)ds\,.$$

By continuing to use (5.1) at $t-r_{i_1}-r_{i_2}, t-r_{i_1}-r_{i_2}-r_{i_3}, \cdots$, we can establish the expression:

$$z(t)=y(t)+\int_{-\infty}^{t} k(t-s)y(s)ds$$

$$+\sum_{j=1}^{\infty}\sum_{i_1=1}^{\infty}\cdots\sum_{i_j=1}^{\infty} b_{i_1}\cdots b_{i_j}\int_{-\infty}^{t-r_{i_1}-\cdots-r_{i_j}} h_j(t-r_{i_1}-\cdots-r_{i_j}-s)y(s)ds$$

$$+\sum_{j=1}^{\infty}\sum_{i_1=1}^{\infty}\cdots\sum_{i_j=1}^{\infty} b_{i_1}\cdots b_{i_j}\int_{-\infty}^{t-r_{i_1}-\cdots-r_{i_j}} h_j(t-r_{i_1}-\cdots-r_{i_j}-s)z(s)ds$$

from which (by a standard ε-δ argument) we can prove that

$$\lim_{t\to\infty} z(t)=\left[1+\int_0^{\infty} k(t)dt+\sum_{j=1}^{\infty}\sum_{i_1=1}^{\infty}\cdots\sum_{i_j=1}^{\infty} b_{i_1}\cdots b_{i_j}\right.$$
$$\left.+\sum_{j=1}^{\infty}\sum_{i_1=1}^{\infty}\cdots\sum_{i_j=1}^{\infty} b_{i_1}\cdots b_{i_j}\int_0^{\infty} h_j(t)dt\right]c\,.$$

This completes the proof.

6. Conclusion. In this section, we will prove an asymptotic constancy result by combining all results in previous sections.

THEOREM 6.1. *Suppose there exists a function* $g:(-\infty,0]\to[1,\infty)$ *satisfying* (g1)–(g3) *such that*

(i)

$$\sum_{i=1}^{\infty}|b_i|g(-r_i)+\int_{-\infty}^{0}|f(-s)|g(s)ds<1$$

and

(ii)

$$\sum_{i=1}^{\infty}|a_i-ab_i|g(-r_i)+\int_{-\infty}^{0}|h(-s)-af(-s)|g(s)ds$$
$$\le a\left[1-\sum_{i=1}^{\infty}|b_i|g(-r_i)-\int_{-\infty}^{0}|f(-s)|g(s)ds\right].$$

Then each solution $x(t)$ *of* (1.3) *through* $\phi \in C_g$ *with* $\lim_{s\to-\infty} |x_0(s)|/g(s)=0$ *tends to a constant as* $t\to\infty$.

PROOF. Let $x(t)$ be a solution of (1.3) through $\phi \in C_g$ with $\lim_{s\to-\infty} |x_0(s)|/g(s)=0$. By Corollary 3.1 there exists a constant $M>0$ so that $|x(t)|\leq M$ for all $t\geq 0$. By Theorem 2.1, the set $\{x_t : t\geq 0\}$ is precompact in C_g.

Now choose a sequence $t_n\to\infty$ so that $|x_{t_n}-\psi|_{C_g}\to 0$ as $n\to\infty$, where $\psi\in\omega(\phi)$. Let $z(t)$ be the solution of (1.3) through ψ, then the function $y(t)$ defined by

$$y(t)=z(t)-\sum_{i=1}^{\infty} b_i z(t-r_i)-\int_{-\infty}^{t} f(t-s)z(s)ds$$

tends to a constant c as $t\to\infty$ according to Theorem 4.2. Thus $z(t)$ tends to a constant c^* by Theorem 5.1.

For any $\varepsilon>0$, by Corollary 3.1 there exists $\delta>0$ so that $|\phi-\psi|_{C_g}<\delta$ implies that $|x(t;\phi)-x(t;\psi)|<\varepsilon/2$ for $t\geq 0$. For this given $\delta>0$ find $N>0$ so that $|x_{t_n}-\psi|_{C_g}<\delta$ for all $n\geq N$. Then

$$|x(t;x_{t_n})-x(t;\psi)|=|x(t+t_n)-z(t)|<\frac{\varepsilon}{2}$$

for all $t\geq 0$ and for all $n\geq N$. On the other hand,

$$|z(t)-c^*|<\frac{\varepsilon}{2}$$

for all $t\geq T_1$. This implies

$$|x(t+t_n)-c^*|<\varepsilon$$

for all $t\geq T_1$ and $n\geq N$. Therefore

$$|x(t)-c^*|<\varepsilon$$

for all $t\geq T_1+t_N$. This shows $\lim_{t\to\infty} x(t)=c^*$, which completes the proof. ■

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DEPARTMENT OF MATHEMATICAL SCIENCES
MEMPHIS STATE UNIVERSITY
MEMPHIS, TN 38152
U.S.A.

AND

INSTITUTE OF APPLIED MATHEMATICS
HUNAN UNIVERSITY
CHANGSHA, HUNAN
PEOPLE'S REPUBLIC OF CHINA