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THE JACOBIANS AND THE DISCRIMINANTS OF FINITE REFLECTION GROUPS

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1. Introduction. Let $\boldsymbol{K}$ be a field with characteristic not equal to two. Let V be an l-dimensional vector space over $\boldsymbol{K}$. An invertible linear transformation g on V is called a *reflection* if $\ker(1-g)$ is of codimension one. In this note we study a finite subgroup $G \subseteq GL(V)$ generated by reflections, which is called a *finite reflection group*. Throughout this paper assume that *the order of G is not divisible by the characteristic of $\boldsymbol{K}$*. The aim of this paper is to do an algebraic study of the Jacobian J and the discriminant δ of a finite reflection group, especially of their relations with the derivations. When $\boldsymbol{K}$ has characteristic zero, one of the most powerful techniques to study finite reflection groups is the Molien series. Since the Molien series is not effective for positive characteristics, we have to find another way to get results. Sometimes we can simplify the proofs for characteristic zero by avoiding the Molien series as we will see in this paper.

Let $S=S(V^*)$ be the symmetric algebra of the dual space V^* of V. Then S can be regarded as the ring of polynomial functions on V. Identify S with $\boldsymbol{K}[x_1, \cdots, x_l]$ using a basis $\{x_1, \cdots, x_l\}$ for V^*. Agree that $\deg(x)=1$ for all $x \in V^*-\{0\}$. Since the reflection group G acts on V^* contragrediently, it also acts on $S=S(V^*)$. Let $R=S^G$ be the invariant subring of S under the action of G. By Chevalley's theorem [4], [3: Ch. 5, Sect. 5.5, Th. 4], we know that the invariant subring R is a polynomial graded $\boldsymbol{K}$-algebra, in other words, there exist algebraically independent homogeneous polynomials $f_1, \cdots, f_l \in S$ such that $R=\boldsymbol{K}[f_1, \cdots, f_l]$. The polynomials $f_1, \cdots, f_l$ are called *basic invariants* of G. Although the choice of basic invariants is not unique, their Jacobian $J=\det[\partial f_i/\partial x_j]_{1 \leqq i,j \leqq n}$ is unique up to a constant multiple. Let $\delta \in R$ be a generator of the ideal $JS \cap R$ (JS stands for the principal ideal of S generated by J). This δ is called the *discriminant* of G. The discriminant $\delta \in R$ is also unique up to a constant multiple.

In general, let A be an arbitrary $\boldsymbol{K}$-algebra. Let $\mathrm{Der}(A)$ be the module of $\boldsymbol{K}$-derivations of A:

$$\mathrm{Der}(A)=\{\theta\colon A \to A \mid \theta \text{ is } \boldsymbol{K}\text{-linear and } \theta(fg)=f\theta(g)+g\theta(f) \text{ for all } f, g \in S\} .$$

For $f \in A$, an A-submodule $D_A(f)$ of $\mathrm{Der}(A)$ is defined by

$$D_A(f)=\{\theta \in \mathrm{Der}(A) \mid \theta(f) \in fA\} .$$

This is an algebraic version of logarithmic vector fields along $\{f=0\}$ (cf. [8], [9]). The finite reflection group G naturally acts on $\mathrm{Der}(S)$. Let $\mathrm{Der}(S)^G$ be the R-module of G-

invariant derivations of Der(S). Our main result is the following:

(1.1) THEOREM. $\mathrm{Der}(S)^G \cong D_R(\delta)$ *(as R-modules).*

Let us briefly explain the geometric meaning of this. Temporarily suppose $\boldsymbol{K}=\boldsymbol{C}$. It is easy to show that the left hand-side $\mathrm{Der}(S)^G$ is characterized as the set of $\boldsymbol{K}$-derivations whose restrictions on R are derivations to R:

$$\mathrm{Der}(S)^G = \{\theta \in \mathrm{Der}(S) \mid \theta(R) \subseteq R\}\,.$$

So, in analytic geometry, it corresponds with the set of holomorphic vector fields on $\boldsymbol{C}^l$ obtained by lifting holomorphic vector fields on the orbit space $\boldsymbol{C}^l/G$. In this case, Theorem (1.1) asserts that a "liftable" vector field on $\boldsymbol{C}^l/G$ is characterized by being tangent to the discriminant variety $\{\delta=0\}$. In this context (1.1) was proved in [1] for Weyl groups G. When G is a compact Lie group, a smooth analogue of (1.1) was proved by [12]. Also the analytic method of [1] was used in [21] to prove a theorem of the same type for a general finite map. Our proof for a general field $\boldsymbol{K}$ given here is purely algebraic.

Let g be a reflection in a finite reflection group G. The hyperplane $H=\ker(1-g)$ is called a *reflecting hyperplane* of G. The family $\mathscr{A}(G)$ of all reflecting hyperplanes of G is called a *reflection arrangement*. In [9], [17] the class of free arrangements was studied (for the definition see Section 4), and turned out to have several nice properties. An important corollary of (1.1) is the following:

(1.2) COROLLARY. *The reflection arrangement $\mathscr{A}(G)$ is a free arrangement.*

This was proved for Weyl groups G in [1], [2], [8], [9]. For unitary reflection groups, it was proved in [18], [21] by means of analytical method. It is also possible, when $\boldsymbol{K}$ has characteristic zero, to prove (1.2) by using the Molien series (communicated by L. Solomon).

For a free arrangement $\mathscr{A}$, there is a multi-set $(d_1, \cdots, d_l)$ of nonnegative integers called the exponents of $\mathscr{A}$. (For the definition see Section 4.) The following is another corollary of (1.1).

(1.3) COROLLARY. *Let $\boldsymbol{K}$ be a finite field with q elements. Then the cardinality of the set of regular vectors (= vectors not on any reflecting hyperplane) is equal to $\prod_{i=1}^{l}(q-d_i)$, where $d_1, \cdots, d_l$ are the exponents of the reflection arrangement $\mathscr{A}(G)$.*

In Section 2 we study the concept of relative invariants associated with a character $G \to \boldsymbol{K}^\times := \boldsymbol{K} \setminus \{0\}$. The Jacobian J is an example of relative invariants. Stanley [14] studied relative invariants of finite reflection groups over $\boldsymbol{C}$. Here we prove Stanley's theorem [14; Th. 3.1] over a general field $\boldsymbol{K}$. In Section 3 we review Steinberg's theorem [16], [3; Ch. 5, Sect. 5.5, Prop. 6] concerning the expression for the Jacobian J. We shall give a new proof. The key for our proof is a theorem by Saito-Scheja-Storch [10], [11], which replaces Molien's formula used in Steinberg's original proof. In Section 4, we shall prove our main result (1.1) and its corollaries.

The author would like to thank Louis Solomon with whom he discovered Lemma (4.5) jointly.

2. Relative invariants. Adding to the terminology in the Introduction we adopt the following terminology for the remainder of this paper. For an element g of a finite reflection group $G \subseteq GL(V)$, denote the contragredient action of g on V^* by g^*. Then $g^* \in GL(V^*)$. Note that $\det(g^*) = \det(g)^{-1}$. For a reflecting hyperplane H (one-codimensional subspace) of V, let $\alpha_H \in V^*$ denote a linear form defining H, i.e., $\ker(\alpha_H) = H$. The set of all elements of G fixing H pointwise is denoted by G_H. Since G_H is a subgroup of G, the characteristic of $\boldsymbol{K}$ does not divide $|G_H|$. By Maschke's theorem there exists a one-dimensional G_H-stable subspace L_H with $V = H \oplus L_H$. Since the representation $G_H \to GL(L_H) \cong \boldsymbol{K}^\times := \boldsymbol{K} \setminus \{0\}$ is faithful, G_H is a cyclic group. Let $r_H = |G_H|$ and denote by s_H a fixed generator of G_H. Let $\chi\colon G \to \boldsymbol{K}^\times$ be a linear character. Define integers a_H by the condition that a_H is the least nonnegative integer satisfying $\chi(s_H) = (\det(s_H^*))^{a_H}$ (clearly a_H depends only on G_H, not on s_H) for each $H \in \mathscr{A}(G)$. Define $d_\chi \in S$ by

$$d_\chi = \prod_{H \in \mathscr{A}(G)} \alpha_H^{a_H}.$$

Finally define $S_\chi^G = \{f \in S \mid g(f) = \chi(g) f \text{ for all } g \in G\}$. Elements of S_χ^G are called *relative invariants.*

The purpose of this section is to prove (2.5) which was proved by Stanley [14; 3.1], by means of the Molien series, when $\boldsymbol{K} = \boldsymbol{C}$. The following lemma can be proved by the same technique as in [14; 2.2]:

(2.1) LEMMA. *If $f \in S_\chi^G$, then f is divisible by d_χ.*

(2.2) LEMMA. *If $g(H) = H'$ for $H, H' \in \mathscr{A}(G)$ and $g \in G$, then $a_H = a_{H'}$.*

PROOF. Since $g^{-1} G_{H'} g = G_H$, we have $|G_{H'}| = |G_H| = o(s_H)$. This shows that $g s_H g^{-1}$ is a generator of $G_{H'}$. We can choose $s_{H'} = g s_H g^{-1}$. Then $\det(s_H^*) = \det(s_{H'}^*)$ and $\chi(s_H) = \chi(s_{H'})$. Thus $a_H = a_{H'}$. □

(2.3) LEMMA. *Fix $H \in \mathscr{A}(G)$. Let $\mathscr{A}(G) = \{H_0, H_1, \cdots, H_n\}$ with $H = H_0$. Then $\prod_{i>0} \alpha_{H_i}^{a_{H_i}}$ is G_H-invariant.*

PROOF. For simplicity write $\alpha_i = \alpha_{H_i}$ and $a_i = a_{H_i}$. First note that we can replace α_i, if necessary, by its constant multiple. By considering the G_H-orbit decomposition of $\{H_1, \cdots, H_n\}$, we may assume that $\{H_1, \cdots, H_n\}$ itself is a G_H-orbit without loss of generality. Then the G_H-orbit of $\alpha_1 \in V^*$ is a set of linear forms defining $H_1, \cdots, H_n$. Therefore we may assume that $\{\alpha_1, \cdots, \alpha_n\}$ is a G_H-orbit. So the product $\prod_{i>0} \alpha_i$ is G_H-invariant. Since $a_i = a_j$ $(1 \leqq i < j \leqq n)$ by (2.2), one completes the proof. □

(2.4) PROPOSITION. $d_\chi \in S^G_\chi$.

PROOF. It is enough to show that

$$s_H(d_\chi) = \det(s_H^*)^{a_H} d_\chi$$

for each $H \in \mathscr{A}(G)$. Fix $H \in \mathscr{A}(G)$. Simply write $a = a_H$, $s = s_H$, and $\alpha = \alpha_H$. Then by (2.3) what we should prove is that

$$s(\alpha^a) = \det(s^*)^a \alpha^a ,$$

which is clear, because $s(\alpha) = \det(s^*)\alpha$. □

By (2.1) and (2.4) we obtain

(2.5) THEOREM. $S^G_\chi = S^G d_\chi$.

The following three examples of the d_χ are particularly important:

(2.6) We have $d_\chi = 1$, when χ is the trivial character. In this case $S^G_\chi = S^G = R$.

(2.7) We have $a_H = r_H - 1$ ($r_H = |G_H|$), when $\chi(g) = \det(g) = \det(g^*)^{-1}$. Thus $d_\chi = \prod_{H \in \mathscr{A}(G)} \alpha_H^{a_H - 1}$.

(2.8) We have $a_H = 1$, when $\chi(g) = \det(g^*) = \det(g)^{-1}$. Thus $d_\chi = \prod_{H \in \mathscr{A}(G)} \alpha_H$.

3. Jacobian and discriminant. We shall use the terminology in the previous sections. In this section we review explicit formulas for the Jacobian J and the discriminant δ. Although the formulas are well-known, we shall give a new (and simpler) proof of Steinberg's formula (see [16] when $\boldsymbol{K} = \boldsymbol{C}$, see [3; Ch. 5, Sect. 5.5, Prop. 6] for a general $\boldsymbol{K}$):

(3.1) STEINBERG'S FORMULA. $J \in \boldsymbol{K}^\times \prod_{H \in \mathscr{A}(G)} \alpha_H^{r_H - 1}$. ($\boldsymbol{K}^\times = \boldsymbol{K} \setminus \{0\}$).

PROOF. Consider the linear character $\chi(g) = \det(g)$ in (2.7). Then $d_\chi = \prod_{H \in \mathscr{A}(G)} \alpha_H^{r_H - 1}$. It is easy to see $g(J) = \det(g)J = \chi(J)$. So $J \in R^G_\chi$. By (2.5) one has $J \in Rd_\chi$. Note that J is homogeneous in S and that $R = \boldsymbol{K}[f_1, \cdots, f_l]$. If $J \notin \boldsymbol{K}^\times d_\chi$, then J belongs to the ideal $(f_1, \cdots, f_l)S$. This contradicts the following (3.2). □

(3.2) LEMMA (Saito [10; 3.4], Scheja-Storch [11; 1.2]). *Suppose $h_1, \cdots, h_l \in \boldsymbol{K}[x_1, \cdots, x_l] = S$ form a homogeneous regular sequence. Assume that $e_i = \deg(h_i)$ is invertible in $\boldsymbol{K}$ ($i = 1, \cdots, l$). Then the Jacobian $\partial(h_1, \cdots, h_l)/\partial(x_1, \cdots, x_l)$ does not belong to the ideal of S generated by $h_1, \cdots, h_l$.*

PROOF. Induction on l. When $l = 1$, it is obvious. Assume $l > 1$. Since $\mathrm{height}(h_2, \cdots, h_l) = l - 1$ and S is Cohen-Macaulay, each associated prime of $(h_2, \cdots, h_l)$ is of height $l - 1$. Thus $(x_1, \cdots, x_l)$, whose height is l, is not contained in any associated prime $\mathfrak{p}$ of $(h_2, \cdots, h_l)$. By a well-known theorem (e.g., [5; p. 81, Prop. 5]) in ideal theory, one has $(x_1, \cdots, x_l) \nsubseteq \bigcup \mathfrak{p}$, where the union is over the set of associated primes of $(h_2, \cdots, h_l)$. Thus we may assume that x_1 does not belong to any associated prime of $(h_2, \cdots, h_l)$. Therefore $(h_2, \cdots, h_l):(x_1)=(h_2, \cdots, h_l)$ and $h_2, \cdots, h_l, x_1$ form a regular sequence. Recall Euler's formula

$$e_j h_j = \sum_{i=1}^{l} x_i(\partial h_j/\partial x_i) \quad (j=1, \cdots, l).$$

Let $\Delta = \det[\partial h_i/\partial x_j]$. By Cramer's rule, one has

$$x_1 \Delta = \begin{vmatrix} e_1 h_1 & \partial h_1/\partial x_2 & \cdots & \partial h_1/\partial x_l \\ \vdots & \vdots & & \vdots \\ e_l h_l & \partial h_l/\partial x_2 & \cdots & \partial h_l/\partial x_l \end{vmatrix}.$$

Let $\Delta_2 = \partial(h_2, \cdots, h_l)/\partial(x_2, \cdots, x_l)$. In the remainder of the proof, the congruence $\equiv$ always is modulo the ideal $(h_2, \cdots, h_l)$. Thus $x_1\Delta \equiv e_1 h_1 \Delta_2$. Suppose that $\Delta \in (h_1, \cdots, h_l)$. Then $\Delta \equiv g_1 h_1$ for some $g_1 \in S$. Thus

$$x_1 g_1 h_1 \equiv e_1 h_1 \Delta_2 .$$

Since $h_1, \cdots, h_l$ form a regular sequence, one has

$$x_1 g_1 \equiv e_1 \Delta_2 .$$

Thus $\Delta_2 \in (x_1, h_2, \cdots, h_l)$. In general for $p \in S$, let $\bar{p}$ denote $p(0, x_2, \cdots, x_l) \in \bar{S} = \boldsymbol{K}[x_2, \cdots, x_l]$. Then $\bar{h}_2, \cdots, \bar{h}_l$ form a regular sequence in $\bar{S}$ because so do $x_1, h_2, \cdots, h_l$ in S. Note that $(\partial\bar{p}/\partial x_i) = \overline{(\partial p/\partial x_i)}$ $(i>1)$. Thus

$$\partial(\bar{h}_2, \cdots, \bar{h}_l)/\partial(x_2, \cdots, x_l) = \bar{\Delta}_2 \in (\bar{h}_2, \cdots, \bar{h}_l).$$

This contradicts the induction assumption, because $\bar{h}_2, \cdots, \bar{h}_l$ form a regular sequence in $\boldsymbol{K}[x_2, \cdots, x_l]$. □

Define the discriminant $\delta \in R$ to be a generator of the ideal $JS \cap R$. Then δ/J is a relative invariant in (2.8): $g(\delta/J) = \det(g)^{-1}(\delta/J)$ for $g \in G$. Thus $\delta/J \in (\prod_{H \in \mathscr{A}(G)} \alpha_H)R$, and δ is a non-zero multiple of $\prod_{H \in \mathscr{A}(G)} \alpha_H^{r_H}$. Thus we may let

$$J = \prod_{H \in \mathscr{A}(G)} \alpha_H^{r_H - 1} \quad \text{and} \quad \delta = \prod_{H \in \mathscr{A}(G)} \alpha_H^{r_H}$$

from now on.

4. Derivations. We keep the terminology in the preceding sections. A finite reflection group G acts on $\mathrm{Der}(S)$ by

$$[g(\theta)](f) = g[\theta(g^{-1}(f))]$$

$(g \in G,\ \theta \in \mathrm{Der}(S),\ f \in S)$. Then $g[\theta(f)] = [g(\theta)](g(f))$. Let $\mathrm{Der}(S)^G$ be the set of G-invariant derivations of S. Note that it is an R-module (not an S-module). Recall

$$D_S(f)=\{\theta\in \mathrm{Der}(S)\mid \theta(f)\in fS\}$$

for $f\in S$.

(4.1) PROPOSITION. *Let $\chi: G\to \boldsymbol{K}^\times$ be a linear character. Then* $\mathrm{Der}(S)^G\subseteq D_S(d_\chi)$.

PROOF. For $\theta\in \mathrm{Der}(S)^G$ and $g\in G$,

$$g[\theta(d_\chi)]=[g(\theta)](g(d_\chi))=\theta(\chi(g)d_\chi)=\chi(g)\theta(d_\chi)\,.$$

Thus $\theta(d_\chi)\in S_\chi^G=Rd_\chi$. □

The following lemma is well-known:

(4.2) LEMMA. *Let $A=\boldsymbol{K}[z_1,\cdots,z_l]$ be a polynomial graded $\boldsymbol{K}$-algebra. Assume that $h_1,\cdots,h_l\in A$ are homogeneous and algebraically independent. Define $B=\boldsymbol{K}[h_1,\cdots,h_l]$. Denote the quotient field of A and B by $F(A)$ and $F(B)$, respectively. Suppose that $F(A)/F(B)$ is a separable extension. Let Δ be the Jacobian $\partial(h_1,\cdots,h_l)/\partial(z_1,\cdots,z_l)$. Then any element θ of* $\mathrm{Der}(B)$ *is uniquely extended to a derivation $\theta_A: A\to F(A)$ such that $\Delta\theta_A\in \mathrm{Der}(S)$.*

PROOF. Since $F(A)/F(B)$ is a finite separable extension, θ is uniquely extendable to a derivation $\theta_{F(A)}: F(A)\to F(A)$. Let $\theta_A=\theta_{F(A)}|_A$. Because $\theta(h_j)=\sum_{i=1}^{l}(\partial h_j/\partial z_i)\theta_A(z_i)$, one has $\Delta\theta_A(z_j)\in A$ by Cramer's rule ($j=1,\cdots,l$). □

For example, the derivation $\partial/\partial f_i\in \mathrm{Der}(R)$, which is characterized by $(\partial/\partial f_i)(f_j)=\delta_{ij}$, is extendable to a derivation $(\partial/\partial f_i)_S: S\to F(S)$. For simplicity, let $\partial/\partial f_i$ denote $(\partial/\partial f_i)_S$ also. Let $\widetilde{\mathrm{Der}}(R)$ be the R-module of derivations of R extendable to a derivation $S\to S$:

$$\widetilde{\mathrm{Der}}(R)=\{\theta\in \mathrm{Der}(R)\mid \theta_S(S)\subseteq S\}\,.$$

(4.3) LEMMA. *The map $\theta\longmapsto\theta_S$ gives an R-isomorphism*

$$\widetilde{\mathrm{Der}}(R)\xrightarrow{\sim}\mathrm{Der}(S)^G\,.$$

PROOF. The correspondence is clearly injective and R-linear. Let us show the surjectivity. For $\eta\in \mathrm{Der}(S)^G$ and $f\in R=S^G$, $\eta(f)$ is G-invariant. Thus $\eta(R)\subseteq R$ and $\eta|_R\in\widetilde{\mathrm{Der}}(R)$. Then $(\eta|_R)_S=\eta$. □

We prove Theorem (1.1) in the following form:

(4.4) THEOREM. $\widetilde{\mathrm{Der}}(R)=D_R(\delta)$.

The inclusion $\widetilde{\mathrm{Der}}(R)\subseteq D_R(\delta)$ is not difficult: If $\theta\in\widetilde{\mathrm{Der}}(R)$, then $\theta_S\in \mathrm{Der}(S)^G$. Let $\chi(g)=\det(g^*)$ for $g\in G$. Then $J=d_\chi$ by (2.7). By (4.1), we have $\theta_S\in D_S(J)$. Write $\delta=Jp$. Then $\theta(\delta)=\theta_S(pJ)=p\theta_S(J)+J\theta_S(p)\in JS\cap R=\delta R$. This proves $\widetilde{\mathrm{Der}}(R)\subseteq D_R(\delta)$.

In (4.5) and (4.6) we shall give an interesting alternative proof for this inclusion:

(4.5) LEMMA. *For $\theta \in \mathrm{Der}(R)$, we have*

$$\theta_S(J) = J[\textstyle\sum_{i=1}^{l} (\partial\theta(f_i)/\partial f_i) - \sum_{i=1}^{l} (\partial\theta_S(x_i)/\partial x_i)]\,.$$

PROOF. Both sides are additive with respect to θ. If θ is replaced by $p\theta$ for $p \in S$, the left hand side is multiplied by p. The right hand side will be

$$\begin{aligned} &J[\textstyle\sum_{i=1}^{l} (\partial(p\theta(f_i))/\partial f_i) - \sum_{i=1}^{l} (\partial(p\theta_S(x_i))/\partial x_i)] \\ &\quad = pJ[\textstyle\sum_{i=1}^{l} (\partial\theta(f_i)/\partial f_i) - \sum_{i=1}^{l} (\partial\theta_S(x_i)/\partial x_i)] \\ &\quad\quad + J[\textstyle\sum_{i=1}^{l} \theta(f_i)(\partial p/\partial f_i) - \sum_{i=1}^{l} \theta_S(x_i)(\partial p/\partial x_i)]\,. \end{aligned}$$

Note that $\sum_{i=1}^{l} \theta(f_i)(\partial p/\partial f_i) = \theta_S(p) = \sum_{i=1}^{l} \theta_S(x_i)(\partial p/\partial x_i)$. Thus we know that both sides are S-linear with respect to θ. Therefore one may assume $\theta = \partial/\partial x_1$. Put $p_i = \partial f_i/\partial x_1$. By the chain rule we have

$$\partial p_i/\partial x_j = \textstyle\sum_{k=1}^{l} (\partial p_i/\partial f_k)(\partial f_k/\partial x_j) \quad (i, j = 1, \cdots, l)\,.$$

By Cramer's rule one has

$$\begin{aligned} J(\partial p_i/\partial f_i) &= \begin{vmatrix} \partial f_1/\partial x_1 & \cdots & \partial p_i/\partial x_1 & \cdots & \partial f_1/\partial x_l \\ \vdots & & \vdots & & \vdots \\ \partial f_l/\partial x_1 & \cdots & \partial p_i/\partial x_l & \cdots & \partial f_l/\partial x_l \end{vmatrix} \\ &= \begin{vmatrix} \partial f_1/\partial x_1 & \cdots & \partial/\partial x_1(\partial f_i/\partial x_1) & \cdots & \partial f_1/\partial x_l \\ \vdots & & \vdots & & \vdots \\ \partial f_l/\partial x_1 & \cdots & \partial/\partial x_1(\partial f_i/\partial x_l) & \cdots & \partial f_l/\partial x_l \end{vmatrix} \end{aligned}$$

for $i = 1, \cdots, l$. By summing these, one obtains

$$J\textstyle\sum_{i=1}^{l} \partial p_i/\partial f_i = \partial J/\partial x_1\,.$$

This proves the lemma when $\theta = \partial/\partial x_1$. □

(4.6) PROPOSITION. $\widetilde{\mathrm{Der}}(S) \subseteq D_R(\delta)$.

PROOF. For $\theta \in \widetilde{\mathrm{Der}}(S)$, one has $\theta_S(J) \in JS$ by (4.5). Write $\delta = Jp$, and $\theta(\delta) = p\theta_S(J) + J\theta_S(p) \in JS \cap R = \delta R$. □

REMARK. (4.6) remains valid in the situation $S = \boldsymbol{K}[x_1, \cdots, x_l] \supseteq R = \boldsymbol{K}[f_1, \cdots, f_l]$ for a homogeneous regular sequence $f_1, \cdots, f_l$ (we need not assume that R is the invariant subring under some group), because Lemma (4.5) is valid in this more general situation. Note that we did not use the existence of a group G in (4.5) or (4.6).

PROOF OF (4.4). Let us prove the inclusion $\widetilde{\mathrm{Der}}(R) \supseteq D_R(\delta)$. Fix $H \in \mathscr{A}(G)$ and recall $G_H = \{g \in G \mid g \text{ fixes } H \text{ pointwise}\}$. Let $R_1 = S^{G_H}$. Since G_H is a finite reflection group, by Chevalley's theorem, R_1 is a polynomial graded $\boldsymbol{K}$-algebra with $S \supseteq R_1 \supseteq R$.

Let $R_1 = \boldsymbol{K}[h_1(\boldsymbol{x}), \cdots, h_l(\boldsymbol{x})]$, where $\boldsymbol{x} = (x_1, \cdots, x_l)$. Put $J_1 = \partial(h_1, \cdots, h_l)/\partial(x_1, \cdots, x_l)$. There exist polynomials $F_1(y), \cdots, F_l(y) \in \boldsymbol{K}[y_1, \cdots, y_l]$ satisfying

$$f_i(\boldsymbol{x}) = F_i(h_1(\boldsymbol{x}), \cdots, h_l(\boldsymbol{x})) \quad (i = 1, \cdots, l).$$

Let

$$J_H = [\partial(F_1, \cdots, F_l)/\partial(y_1, \cdots, y_l)](h_1(\boldsymbol{x}), \cdots, h_l(\boldsymbol{x})) \in R_1.$$

By the chain rule for Jacobians, one has

$$J = \partial(f_1, \cdots, f_l)/\partial(x_1, \cdots, x_l) = J_H J_1.$$

We may assume that $\alpha_H = x_1$ and that $x_2, \cdots, x_l \in R_1$. Simply write $r = r_H = |G_H|$. Then by Steinberg's formula (3.1) one may assume that $J_1 = x_1^{r-1}$. Let $\theta \in D_R(\delta)$. It is not difficult to see that $F(S)/F(R)$ is a Galois extension with Galois group G (e.g., [3; Ch. 5, Sect. 5.2]). So $F(R_1)/F(R)$ is finite separable. By (4.2), θ is extendable to $\theta_{R_1}: R_1 \to F(R_1)$ in such a way that $\psi := J_H \theta_{R_1} \in \mathrm{Der}(R_1)$. Let δ_1 be a discriminant of G_H, and $\delta_1 = x_1^r$. Let $\delta_2 = \delta/\delta_1$. Then one has

$$\delta_1 \psi(\delta_2) + \delta_2 \psi(\delta_1) = \psi(\delta) \in \delta R_1 = \delta_1 \delta_2 R_1.$$

Since $(\delta_1, \delta_2) = 1$, one has $\psi(\delta_1) \in \delta_1 R_1$. Thus we have

$$r x_1^{r-1} \psi_S(x_1) = \psi(\delta_1) \in \delta_1 R_1 = x_1^r R_1.$$

Since r is invertible in $\boldsymbol{K}$, one has $\psi_S(x_1) \in x_1 R_1$. Since $\psi_S(x_i) = \psi(x_i) \in R_1$ $(i > 1)$, we have $J_H \theta_S = \psi_S \in \mathrm{Der}(S)$. This is the case for all $H \in \mathscr{A}(G)$. Since the polynomials $J_H = J/\alpha_H^{r-1}$ for all $H \in \mathscr{A}(G)$ have no common factors, we obtain $\theta_S \in \mathrm{Der}(S)$. This completes the proof of (4.4). □

(4.7) COROLLARY. (1) *$D_R(\delta)$ is a free R-module.*

(2) *Let $\{\theta_1, \cdots, \theta_l\}$ be a free basis for $D_R(\delta)$. Then* $\det[\theta_i(f_j)] \in \boldsymbol{K}^\times \delta$.

(3) *The 0-th Fitting ideal of the R-module* $\mathrm{Der}(R)/\widetilde{\mathrm{Der}}(R)$ *is δR.*

PROOF. (1) There exists a G-stable vector subspace U such that $R \otimes_{\boldsymbol{K}} U \cong S$ (e.g., [3; Ch. 5, Sect. 5.2, Th. 2]). Thus

$$D_R(\delta) \cong \mathrm{Der}(S)^G \cong (R \otimes U \otimes V)^G = R \otimes_{\boldsymbol{K}} (U \otimes V)^G.$$

(2) Follows from (1) and Saito's criterion [9; 3.3].

(3) Since the exact sequence

$$0 \to \widetilde{\mathrm{Der}}(R) \to \mathrm{Der}(R) \to \mathrm{Der}(R)/\widetilde{\mathrm{Der}}(R) \to 0$$

is an R-free resolution of $\mathrm{Der}(R)/\widetilde{\mathrm{Der}}(R)$, the 0-th Fitting ideal is the determinant of the matrix $[\theta_i(f_j)] \in \boldsymbol{K}^\times \delta$. □

(4.8) COROLLARY. *Define* $Q(\mathscr{A}(G))=\prod_{H\in\mathscr{A}(G)}\alpha_H$. *Then* $D_S(Q(\mathscr{A}(G)))=\mathrm{Der}(S)^G\otimes_R S$.

PROOF. By the chain rule we have

$$\theta(f)=\sum_{i=1}^{l}(\partial f/\partial x_i)\theta(x_i)\quad(f\in S)\,.$$

Thus, for a basis $\{\theta_1,\cdots,\theta_l\}$ for $\mathrm{Der}(S)^G$,

$$\det[\theta_i(f_j)]=\det[\partial f_i/\partial x_j]\det[\theta_i(x_j)]\,.$$

Define a linear character χ by $\chi(g)=\det(g^*)$. By (2.8) $d_\chi=Q(\mathscr{A}(G))$. By (4.1) $\theta_1,\cdots,\theta_l$ belong to $D_S(Q(\mathscr{A}(G)))$. By Saito's criterion [9; 3.3] $\theta_1,\cdots,\theta_l$ also form a basis for $D_S(Q(\mathscr{A}(G)))$. This shows (4.8). □

Before stating further corollaries to (4.4), we need more terminology. In general, we say that $\mathscr{A}$ is an *arrangement* (of hyperplanes) if $\mathscr{A}$ is a finite collection of one-codimensional subspaces of V. Let $\mathscr{A}$ be an arrangement. Define

$$Q=Q(\mathscr{A})=\prod_{H\in\mathscr{A}}\alpha_H\in S\,.$$

Then $Q(\mathscr{A})$ is determined up to a constant multiple. We call $Q(\mathscr{A})$ a defining equation for $\mathscr{A}$. An arrangement $\mathscr{A}$ of hyperplanes is called a *free arrangement* if $D_S(Q(\mathscr{A}))$ is a free S-module.

The following is obvious from (3.7):

(1.2) COROLLARY. *The reflection arrangement* $\mathscr{A}(G)$ *is free.*

Let $\mathscr{A}$ be an arrangement. An element $\theta\in D_S(Q(\mathscr{A}))$ is said to be *homogeneous of degree d* if $\deg\theta(x)=d$ for all $x\in V^*\setminus\{0\}$. When $\mathscr{A}$ is a free arrangement, it is easy to see that there exists a homogeneous basis $\{\theta_1,\cdots,\theta_l\}$. The *exponents* of $\mathscr{A}$ is defined to be $(\deg\theta_1,\cdots,\deg\theta_l)$. Let $L(\mathscr{A})$ be the collection of all intersections of elements of $\mathscr{A}$. We agree that $\bigcap_{H\in\varnothing}H=V\in L(\mathscr{A})$. Introduce a partial order $\leqq$ on $L(\mathscr{A})$ by $X\leqq Y\Leftrightarrow X\supseteq Y$. Then $L(\mathscr{A})$ is a geometric lattice. Write $L=L(\mathscr{A})$. Let $\mu: L\times L\to\boldsymbol{Z}$ be the Möbius function [7], which is characterized by

$$\sum_{\substack{Y\in L\\ W\leqq Y\leqq X}}\mu(Y,X)=\begin{cases}1 & \text{if}\quad W=X\,,\\ 0 & \text{otherwise}\,,\end{cases}$$

for $W, X\in L$ with $W\leqq X$. The characteristic polynomial χ is defined by

$$\chi(\mathscr{A};t)=\sum_{X\in L}\mu(V,X)t^{\dim X}\,.$$

It is an important combinatorial invariant for $\mathscr{A}$. If $\mathscr{A}$ is free with exponents $(e_1,\cdots,e_l)$, then the Factorization Theorem in [19], [20] asserts $\chi(\mathscr{A};t)=\prod_{i=1}^{l}(t-e_i)$. Thus by (1.2) we have:

(4.9) COROLLARY. *Let $d_1, \cdots, d_l$ be the exponents of $\mathscr{A}(G)$. Then $\chi(\mathscr{A}(G); t) = \prod_{i=1}^{l}(t-d_i)$.*

This was proved in [6] when G is a finite unitary reflection group.

Finally we consider the case where the field $\boldsymbol{K}$ is a finite field. Let $M(\mathscr{A}) = \{v \in V \mid v$ is not on any hyperplane belonging to $\mathscr{A}\}$. By using the Möbius inversion (e.g., [7], [15; 3.7.2]) it is not difficult to show:

(4.10) PROPOSITION. *For an arrangement $\mathscr{A}$ in V over a finite field $\boldsymbol{K}$ with q elements, we have $|M(\mathscr{A})| = \chi(\mathscr{A}; q)$.*

By (4.9) and (4.10), we have:

(1.3) COROLLARY. *Let $\boldsymbol{K}$ be a finite field with q elements. Then the cardinality of the set of regular vectors (i.e., vectors not on any reflecting hyperplane) is equal to $\prod_{i=1}^{l}(q-d_i)$, where $d_1, \cdots, d_l$ are the exponents of the reflection arrangement $\mathscr{A}(G)$.*

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