

## LINEAR DIFFERENTIAL EQUATIONS MODELED AFTER HYPERQUADRICS

Dedicated to Professor Ichiro Satake on his sixtieth birthday

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**0. Introduction.** In this paper, we study systems of linear partial differential equations in  $n$  ( $\geq 3$ ) variables of rank (= the dimension of the solution space)  $n+2$ . The case  $n=2$  is treated in [SY1] and [SY2].

Here we would like to mention our motivation. Let  $D$  be the symmetric domain of type IV of dimension  $n$  ( $\geq 3$ ),  $\Gamma$  be a transformation group acting properly discontinuously on  $D$ ,  $X$  be a quotient variety of  $D$  under  $\Gamma$  naturally equipped with the structure of orbifold,  $\pi$  be the projection of  $D$  onto  $X$  and finally let  $\varphi$  be the inverse map  $\pi^{-1}: X \rightarrow D$ , which is called the *developing map of the orbifold  $X$* . We think there should be a system of linear differential equations (E) defined on  $X$  such that the solution of the system gives rise to the map  $\varphi$ . It is called the *uniformizing differential equation of the orbifold  $X$* . Since  $D$  can be thought of as a part of a non-degenerate quadric hypersurface  $Q$  in  $P^{n+1}$  and since we have the following inclusion relations

$$\text{Aut}(D) \subset \text{Aut}(Q) \subset \text{Aut}(P^{n+1}) \cong PGL(n+2)$$

of the groups of complex analytic automorphisms, the system (E) must be of rank  $n+2$  and the mapping defined on  $X$  by the ratio of  $n+2$  linearly independent solutions of (E) has its image in the hyperquadric  $Q$ . In this way we encounter equations in  $n$  variables of rank  $n+2$ . Making a linear change of independent variables  $x=(x^1, \dots, x^n)$  if necessary, we may assume that any system in  $n$  variables of rank  $n+2$  with the unknown  $w$  has the form

$$(EQ) \quad \frac{\partial^2 w}{\partial x^i \partial x^j} = g_{ij} \frac{\partial^2 w}{\partial x^1 \partial x^n} + \sum_{k=1}^n A_{ij}^k \frac{\partial w}{\partial x^k} + A_{ij}^0 w \quad (1 \leq i, j \leq n)$$

where

$$g_{ij} = g_{ji}, \quad A_{ij}^k = A_{ji}^k, \quad A_{ij}^0 = A_{ji}^0, \quad g_{1n} = 1, \quad A_{1n}^k = A_{1n}^0 = 0.$$

This system is the key to connecting the theory of conformal connections, the projective

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theory of hypersurfaces and the theory of uniformizing differential equations of orbifolds uniformized by symmetric domains of type IV. The ratio of  $n+2$  linearly independent solutions is called a *projective solution*.

Let a hypersurface  $M$  in  $P^{n+1}$  be the image of the projective solution of (EQ). We study in §1, as a preparation, the induced conformal metric II on  $M$  and the cubic invariant form III of the embedding  $M \subset P^{n+1}$ , and formulate the fundamental theorem of projective hypersurfaces (Theorem 1.3). We show that the coefficients  $g_{ij}$  represent the induced conformal metric II and that the coefficients  $A_{ij}^k$  and  $A_{ij}^0$  are expressed in terms of II and the cubic invariant III (Theorem 2.1). When  $M$  is a quadric hypersurface, we show that the coefficients  $A_{ij}^k$  and  $A_{ij}^0$  are expressed in terms of the  $g_{ij}$ 's (Theorem 2.3). Conversely, for a given conformally flat quadratic form  $g_{ij}$ , we can associate a system of the form (EQ) with the principal part  $g_{ij}$  such that the projective solution has its image in a hyperquadric in  $P^{n+1}$  (Theorem 2.4).

Let  $X$  be an  $n$ -dimensional orbifold (or simply a manifold) which has a conformally flat structure. As Kuiper ([Kui]) pointed out, there is a conformal map, called the developing map, from the universal cover of  $X$  into the model space, hyperquadric in  $P^{n+1}$ . Applying Theorem 2.4, we can answer the following question: "How can we get the developing map?" Let  $g_{Uij}dx_U^i dx_U^j$  be the conformal structure for coordinate neighborhoods  $(U, x_U)$ . We consider the system (EQ) $_U$  of the form (EQ) with the  $g_{Uij}$ 's as the principal part such that the image of the projective solution  $\varphi_U$  is a part of a nondegenerate quadric hypersurface in  $P^{n+1}$ . If  $V$  is another chart such that  $V \cap U \neq \emptyset$  then  $\varphi_U$  and  $\varphi_V$  are projectively related. The developing map of  $X$  is given by the collection  $\{\varphi_U\}_U$ .

Let  $M=H_2$  be the Siegel upper half space of degree 2 and  $\Gamma(2)$  be the Siegel modular group of level 2. The regular orbit of  $H_2$  modulo  $\Gamma(2)$  is known to be the space

$$A = \{(\lambda^1, \lambda^2, \lambda^3) \in \mathbb{C}^3 \mid \lambda^i \neq 0, 1, \lambda^j \quad (i \neq j)\}$$

Let  $\pi: H_2 \rightarrow A$  be the natural projection. The space  $A$  can be thought of as the parameter space of a family of curves of genus 2:

$$C(\lambda): w^3 v^2 = u(u-w)(u-\lambda^1 w)(u-\lambda^2 w)(u-\lambda^3 w)$$

in the projective plane. The periods of  $C(\lambda)$  gives a (multi-valued) inverse of  $\pi$  and they satisfy a system of linear differential equations which is sometimes called the Gauss-Manin connection of the fiber space  $\bigcup_{\lambda} C(\lambda) \rightarrow A$ . In §3, we explicitly write down the system of differential equations, which turns out to be of the form (EQ).

### 1. Review of the projective theory of hypersurfaces.

1.0. Summary. In this section we recall the fundamental formulation of the intrinsic conformal geometry and the projective theory of hypersurfaces, which are necessary in the discussion of systems of linear differential equations in the following sections. Although the fact stated in this section is already known by [Sas], our present

version is made in order to clarify and to show up the story of the theory, which may not be easy to grasp in reading [Sas].

To have a better understanding of the theory, we first recall the gist of the intrinsic Riemannian geometry, that of hypersurfaces in the Euclidean spaces and the fundamental theory connecting them.

*Intrinsic Riemannian geometry:* Let  $M$  be an  $n$ -dimensional manifold equipped with a Riemannian metric. Then there is a unique affine connection compatible with the metric (Levi-Civita connection). The Riemannian curvature tensor is defined by the Levi-Civita connection.

*Hypersurfaces:* Let  $\iota: M \subset \mathbf{R}^{n+1}$  be an embedding of a manifold  $M$ . The induced metric and the second fundamental form are defined on  $M$ . The Levi-Civita connection and the Riemannian curvature tensor of the induced metric are defined as above. They are related as follows:

*Gauss equation:* The Riemannian curvature tensor is expressed in terms of the second fundamental form.

*Codazzi-Minardi equation:* The covariant derivatives of the second fundamental form and the induced metric are related.

*Fundamental theorem:* Let  $M$  be a manifold equipped with a Riemannian metric and a quadratic form. They are the induced metric and the second fundamental form defined by some embedding  $\iota: M \subset \mathbf{R}^{n+1}$  if they satisfy the Gauss equation and the Codazzi-Minardi equation. The embedding  $\iota$  is unique up to rigid motions of  $\mathbf{R}^{n+1}$ .

Now we summarize the gist of the intrinsic conformal geometry, that of hypersurfaces in the projective space and the fundamental theorem connecting them.

*Intrinsic conformal geometry:* Let  $M$  be a manifold equipped with a conformal metric  $h$ . Then there is a unique conformal connection  $\pi$  compatible with the conformal metric (the normal conformal connection). The conformal curvature tensor  $C$  is defined by the normal conformal connection.

*Hypersurfaces:* Let  $\iota: M \subset P^{n+1}$  be an embedding of an  $n$ -dimensional manifold  $M$ . The induced conformal metric  $h$  and the 1-form  $\tau$  (called the invariant of  $\iota$ ) are defined. The normal conformal connection  $\pi$  and the conformal curvature tensor  $C$  of the induced metric are defined intrinsically as above. They are related as follows.

*Gauss equation:* The conformal curvature tensor is expressed in terms of the invariant  $\tau$ .

*Codazzi-Minardi equation:* Covariant derivatives of  $\tau$  and the induced metric  $h$  are related.

*Fundamental theorem:* Let  $M$  be a manifold equipped with a conformal metric  $h$  and a 1-form  $\tau$ . They are the induced conformal metric and the invariant defined by some embedding  $\iota: M \subset P^{n+1}$  if they satisfy the Gauss equation and the Codazzi-Minardi equation. The embedding  $\iota$  is unique up to projective transformations of  $P^{n+1}$ .

1.1. Intrinsic conformal geometry. We recall some facts on the conformal

connection. A precise and detailed description can be found in the book [Kob].

Let  $M$  be an  $n$ -dimensional complex manifold and  $h=(h_{ij})$  be a non-singular symmetric matrix. Define the group of conformal transformations for  $h$  to be

$$CO(h)=\{\lambda a \mid a \in GL(n, C), ah^t a=h, \lambda \in C^*\}.$$

Let  $L(M)$  be the bundle of complex linear frames on  $M$ . A holomorphic principal subbundle  $P$  of  $L(M)$  with structure group  $CO(h)$  is called a *holomorphic  $CO(h)$ -structure*. Such subbundles on  $M$  are in a natural one-to-one correspondence with the sections  $M \rightarrow L(M)/CO(h)$ . In other words, for such a structure, we associate a conformal covariant tensor field  $g=(g_{ij})$  called a *conformal metric* written locally as

$$g_{ij}(x)dx^i dx^j, \quad \det g_{ij} \neq 0,$$

with respect to a local coordinate system  $(x^i)$ . (Throughout this paper, we follow Einstein's convention.)

We consider a non-singular hyperquadric  $Q^n$  in  $P^{n+1}$  defined in terms of the homogeneous coordinate system  $(z^0, \dots, z^{n+1})$  by the equation

$$-2z^0 z^{n+1} + h_{ij} z^i z^j = 0.$$

Let  $Q$  be the symmetric matrix of degree  $n+2$  corresponding to this quadratic form:

$$Q = \begin{pmatrix} 0 & 0 & -1 \\ 0 & h & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

The group

$$O(Q) = \{g \in GL(n+2) \mid gQ^t g = Q\}$$

acts transitively on the hyperquadric. Let  $H$  be the isotropy subgroup at  $(0, \dots, 0, 1)$ . It consists of matrices of the form

$$(1.1) \quad \begin{pmatrix} \lambda & 0 & 0 \\ b & a & 0 \\ \mu & c & v \end{pmatrix} \quad \lambda v = 1, ah^t a = h, b = \lambda ah^t c, \mu = \lambda ch^t c/2.$$

We have a principal bundle  $O(Q)$  over  $Q^n = O(Q)/H$  with structure group  $H$ . The linear isotropy representation of the group  $H$  at  $(0, \dots, 0, 1)$  has a non-trivial kernel consisting of matrices of the form

$$\begin{pmatrix} \pm 1 & 0 & 0 \\ b & \pm I_n & 0 \\ \mu & c & \pm 1 \end{pmatrix}.$$

Denote this kernel by  $N$ . Then  $H/N$  is isomorphic to  $\text{CO}(h)$ . Thus we have a principal bundle  $O(Q)/N$  over the hyperquadric  $Q^n = O(Q)/H$  with structure group  $\text{CO}(h)$ :

$$\begin{array}{ccc} O(Q) & & O(Q)/N \\ \downarrow H & & \downarrow H/N \cong \text{CO}(h) \\ Q^n = O(Q)/H & = & O(Q)/H \end{array}$$

This is called *the canonical conformal structure of the quadric*. The associated conformal metric is given as follows. Let  $\varphi = -2dz^0 dz^{n+1} + h_{ij} dz^i dz^j$  be the tensor field on  $\mathbb{C}^{n+2} - \{0\}$ . Let  $s$  be a local section of the bundle  $\mathbb{C}^{n+2} - \{0\}$  over  $P^{n+1}$ . Although the pull-back  $s^*\varphi$  depends on the section  $s$ , its restriction to  $Q^n$  is defined independently of  $s$  up to a multiplicative factor of non-vanishing holomorphic functions. Thus the conformal metric of  $s^*\varphi|_{Q^n}$  is uniquely defined.

Consider again a  $\text{CO}(h)$ -structure  $P$  on a manifold  $M$ . Let  $P^2(M)$  be the bundle of 2-frames over  $M$  with structure group, elements of which are holomorphic 2-frames of  $\mathbb{C}^n$  at the origin ([Kob, Chapter 4, § 5]). The first prolongation of  $P$ , which is a principal subbundle of  $P^2(M)$  with structure group  $H$ , is denoted by  $P^{(1)}$ . The correspondence between  $P$  and  $P^{(1)}$  is known to be bijective ([Kob, Chapter 4, § 6]). In fact, we can recover  $P$  from  $P^{(1)}$  by putting  $P = P^{(1)}/N$ . For the hyperquadric  $Q^n$ , this bundle  $P^{(1)}$  is nothing but the bundle  $O(Q) \rightarrow O(Q)/H$ . The bundle  $P^{(1)}$  has Cartan connections ([Kob, Chapter 4]). Let  $\mathfrak{o}(Q)$  be the Lie algebra of  $O(Q)$ . Then a Cartan connection in question is a  $\mathfrak{o}(Q)$ -valued 1-form  $\pi$  on  $P^{(1)}$  considered as a set of 1-forms  $(\pi_i, \pi_i^j, \pi^j)$  by the identification

$$\pi = \begin{pmatrix} \pi^0 & \pi^j & 0 \\ \pi_i & \pi_i^j + \delta_i^j \pi^0 & h_{ik} \pi^k \\ 0 & h^{kj} \pi_k & -\pi^0 \end{pmatrix} \in \mathfrak{o}(Q)$$

where  $\pi^0 = -(1/n) \sum \pi_k^k$ . The forms  $\pi_i^j$  and  $\pi^j$  are the restriction to  $P^{(1)}$  of the components of the canonical form of  $P^{(1)}$ . They have the property  $d\pi^j = \sum \pi^k \wedge \pi_k^j$ . The curvature form  $\Pi$  of  $\pi$  is defined by  $\Pi = d\pi - \pi \wedge \pi$  which is written as

$$\Pi = \begin{pmatrix} \Pi^0 & 0 & 0 \\ \Pi_i^0 & \Pi_i^j & 0 \\ 0 & h^{jk} \Pi_k^0 & -\Pi^0 \end{pmatrix}.$$

There exists a unique Cartan connection, called the *normal conformal connection*, satisfying the (normalization) condition

$$C^j_{ijl} = 0$$

where

$$\Pi_i^j - \delta_i^j \Pi^0 = d\pi_i^j - \pi_i^k \wedge \pi_k^j - \pi_i \wedge \pi^j - h_{ik} h^{jl} \pi^k \wedge \pi_l - \delta_i^j \pi_k \wedge \pi^k =: \frac{1}{2} C^j_{ikl} \pi^k \wedge \pi^l.$$

In fact, this condition determines the forms  $\pi_i$  uniquely ([Kob, Chapter 4, Theorem 4.2]).

DEFINITION. A  $CO(h)$ -structure  $P$  (or a conformal metric  $g$ ) is said to be conformally flat if the normal conformal connection  $\pi$  is integrable, i.e.,  $\Pi = d\pi - \pi \wedge \pi = 0$ .

1.2. Projective theory of hypersurfaces. Suppose we are given a piece of an  $n$ -dimensional hypersurface  $M$  in the projective space  $P^{n+1}$ . Let  $i: M \rightarrow P^{n+1}$  be the embedding. We assume the map  $i$  to have a lift, denoted by  $e_0$ , to  $C^{n+2} - \{0\}$ , the natural covering of  $P^{n+1}$ . Let  $\{e_1, \dots, e_n\}$  be a set of independent tangent vector fields to  $M$  along  $e_0$  and choose another vector field  $e_{n+1}$  so that  $\det(e_0, e_1, \dots, e_n, e_{n+1}) = 1$  with respect to a fixed frame of  $C^{n+2}$ . Then the hypersurface  $M$  is described by the motion of the vectors  $e_\alpha$  ( $0 \leq \alpha \leq n+1$ ) which we call a projective moving frame field along  $M$ . We introduce the associated Maurer-Cartan form  $\omega$  by

$$de = \omega e.$$

Here we use abbreviations  $e = (e_0, e_1, \dots, e_{n+1})$  and  $\omega = (\omega_\alpha^\beta)$ , the indices  $\alpha, \beta, \dots$  ranging from 0 to  $n+1$ . When we use the indices  $i, j, \dots$ , these are understood to range from 1 to  $n$ . The 1-form  $\omega$  has values in  $sl(n+2, C)$ . It satisfies the Maurer-Cartan equation:

$$(1.2) \quad d\omega = \omega \wedge \omega, \quad \text{i.e.,} \quad d\omega_\alpha^\beta = \omega_\alpha^\gamma \wedge \omega_\gamma^\beta.$$

First notice that the above choice for a frame implies  $\omega_0^{n+1} = 0$ , and  $\{\omega_0^j \mid 1 \leq j \leq n\}$  are independent on  $M$ . In the rest of this paper, we write  $\omega^j = \omega_0^j$ . Then (1.2) implies  $0 = d\omega_0^{n+1} = \omega^k \wedge \omega_k^{n+1}$ , which allows us to put

$$(1.3) \quad \omega_i^{n+1} = h_{ik} \omega^k, \quad h_{ik} = h_{ki}.$$

We have

$$\omega = (\omega_\alpha^\beta) = \begin{pmatrix} \omega_0^0 & \omega^j & 0 \\ \omega_i^0 & \omega_i^j & h_{ik} \omega^k \\ \omega_{n+1}^0 & \omega_{n+1}^j & \omega_{n+1}^{n+1} \end{pmatrix}.$$

Let us define a symmetric quadratic form  $\Pi$  on  $M$  by

$$(1.4) \quad \Pi = h_{ij} \omega^i \omega^j.$$

An important property of this form is its invariance in the following sense. Let  $e'$  be another projective frame, which is easily seen to have the form

$$(1.5) \quad e' = ge \quad \text{with} \quad g = \begin{pmatrix} \lambda & 0 & 0 \\ b & a & 0 \\ \mu & c & v \end{pmatrix},$$

where  $\lambda, \mu$  and  $v$  are scalar functions,  $a$  is an  $n \times n$  matrix function, and  $b$  and  $c$  are  $n$ -

vector functions. Then the associated Maurer-Cartan form  $\omega'$  is given by

$$(1.6) \quad \omega' = (dg + g\omega)g^{-1},$$

and this leads to the identity

$$(1.7) \quad \lambda v h'_{ij} = a_i^k h_{kl} a_j^l.$$

In particular, the associate quadratic form  $\Pi'$  is given by

$$(1.8) \quad \Pi' = \frac{\lambda}{v} \Pi.$$

This implies that the conformal class of  $\Pi$  is intrinsic on the manifold  $M$ . Hence, especially, its rank is independent of the choice of frames. We now assume that the form  $\Pi$  is non-degenerate. Notice that the above process defining  $\Pi$  shows that it is determined by the second order derivatives of the embedding  $i$ . We next derive another invariant which depends on its third derivatives. In order to make the following formulae look simpler, we choose a frame so that

$$(1.9) \quad \det h_{ij} = 1, \quad \omega_0^0 + \omega_{n+1}^{n+1} = 0.$$

This is possible because of the non-degeneracy assumption for  $\Pi$  and the transformation rule (1.6). Then the exterior derivation of (1.2) gives

$$(dh_{ij} - h_{ik}\omega_j^k - h_{jk}\omega_i^k) \wedge \omega^j = 0,$$

which enables us to define a symmetric quantity  $h_{ijk}$  by

$$(1.10) \quad h_{ijk}\omega^k = dh_{ij} - h_{ik}\omega_j^k - h_{jk}\omega_i^k.$$

Let us define a symmetric cubic form  $\text{III}$  on  $M$  by

$$(1.11) \quad \text{III} = h_{ijk}\omega^i\omega^j\omega^k$$

and call this the (*Wilczynski-Fubini-Pick*) *cubic invariant form*. Indeed it has the invariance:

$$(1.12) \quad \text{III}' = \lambda^2 \text{III}$$

with respect to the frame change (1.5). The role of this form can be seen in:

**PROPOSITION 1.1.** *Let  $M$  be a connected piece of a hypersurface in  $P^{n+1}$ . Assume the quadratic form  $\Pi$  is non-degenerate and the cubic invariant form  $\text{III}$  vanishes everywhere. Then  $M$  is contained in a quadric hypersurface.*

The projective description of a hypersurface needs one more invariant. Take a derivation of  $\omega_0^0 + \omega_{n+1}^{n+1} = 0$ . Then we have

$$(h_{ij}\omega_{n+1}^j - \omega_i^0) \wedge \omega^i = 0,$$

which allows us to define a symmetric quantity  $L_{ij}$  by

$$(1.13) \quad h_{ij}\omega_{n+1}^j - \omega_i^0 = L_{ij}\omega^j.$$

It is possible to show the existence of a projective frame satisfying

$$(1.14) \quad \det h_{ij} = 1, \quad \omega_0^0 + \omega_{n+1}^{n+1} = 0 \quad \text{and} \quad \text{trace}_n L (= L_{ij}h^{ij}) = 0.$$

Now we fix a frame  $e$  with this property. Then, at every point  $p$  where the frame is defined, the matrix  $h = (h_{ij})$  defines a Lie group by (1.1) which we denote by  $H(p)$ . Analogously, the group  $O(Q(p))$  and its Lie algebra  $o(Q(p))$  are defined. Take another frame  $e'$  with the property  $h'_{ij} = h_{ij}$  and (1.14). A calculation shows that the frame change  $g$  from  $e$  into  $e'$  belongs to the group  $H(p)$  at each  $p$ .

We next formulate the fundamental theorem by using the language of conformal geometry. Define a tensorial matrix-valued 1-form  $\tau$  by

$$(1.15) \quad \tau = \begin{pmatrix} 0 & 0 & 0 \\ (M_{ik} + L_{ik})\omega^k & \frac{1}{2} h_{ik}^j \omega^k & 0 \\ -\omega_{n+1}^0 & h^{jl} M_{lk} \omega & 0 \end{pmatrix}$$

where

$$M_{ik} = \frac{-1}{4(n-2)} K_{ik} + \frac{F}{8(n-2)(n-1)} h_{ik} - \frac{1}{2} L_{ik}$$

$$K_{ik} = h_{ipq} h^{pq}{}_k \quad \text{and} \quad F = h_{pqr} h^{pqr},$$

and put

$$(1.16) \quad \pi = \omega + \tau.$$

(Here the raising of indices relative to  $h_{ij}$  is used. e.g.  $h_{ij}{}^k = h_{ijp} h^{pk}$ .) Then a computation shows the invariance

$$(1.17) \quad \tau' = g\tau g^{-1}$$

under the frame changes belonging to the group  $H(p)$  for each point  $p$ ; and it is easy to see the form  $\pi$  has its value in the Lie algebra  $o(Q(p))$ . Let  $\Pi$  be the curvature tensor of  $\pi$ . It has the following expression with  $\Pi^0 = 0$ :

$$\Pi = d\pi - \pi \wedge \pi = \begin{pmatrix} 0 & 0 & 0 \\ \Pi_i^0 & \Pi_i^j & 0 \\ 0 & \Pi_{n+1}^j & 0 \end{pmatrix},$$



where  $\Pi_i^0 = h_{ij}\Pi_{n+1}^j$ . Since  $\Pi$  is a tensorial 2-form, we may put

$$(1.17) \quad \begin{aligned} \Pi_i^j &= \frac{1}{2} C^j_{ikl} \omega^k \wedge \omega^l & C^j_{ikl} + C^j_{ilk} &= 0 \\ \Pi_i^0 &= \frac{1}{2} C_{ikl} \omega^k \wedge \omega^l & C_{ikl} + C_{ilk} &= 0. \end{aligned}$$

The choice for  $\tau$  has been made under the normalization condition

$$(1.18) \quad C^j_{ijl} = 0.$$

In this notation, the following analogues of the Gauss and the Codazzi-Minardi equations hold:

(1.19) (The Gauss equation)

$$\begin{aligned} C_{ijkl} &= h_{im} C^m_{jkl} = \frac{1}{4} (h_{ilp} h_{kj}{}^p - h_{ikp} h_{jl}{}^p) + \frac{1}{4(n-2)} (h_{jk} K_{il} - h_{jl} K_{ik} + h_{il} K_{jk} - h_{ik} K_{jl}) \\ &\quad + \frac{1}{4(n-1)(n-2)} (h_{ik} h_{jl} - h_{il} h_{jk}) F \\ C_{ikl} &= f_{il,k} - f_{ik,l} + \frac{1}{4} (h_{ik}{}^j L_{jl} - h_{il}{}^j h_{jk}), \end{aligned}$$

where  $f_{il}$  is the projective analogue of the Schouten tensor defined by

$$f_{il} = -\frac{1}{4(n-2)} K_{il} + \frac{F}{8(n-1)(n-2)} h_{il}$$

and  $f_{il,k}$  is the covariant derivative of  $f_{il}$  with respect to  $\pi$ , i.e.,  $f_{il,k} \omega^k = df_{il} - f_{ik} \pi_i^k - f_{kl} \pi_i^k + 2f_{il} \pi^0$ .

(1.20) (The Codazzi-Minardi equation)

$$\begin{aligned} h_{ijk,l} - h_{ijl,k} &= L_{il} h_{jk} - L_{ik} h_{jl} + L_{jl} h_{ik} - L_{jk} h_{il} \\ L_{ij,k} - L_{ik,j} &= h_{ij}{}^l f_{lk} - h_{ik}{}^l f_{lj} + 2(h_{ik} \gamma_j - h_{ij} \gamma_k) \\ \gamma_{i,j} - \gamma_{j,i} &= L_{jl} f^l{}_i - L_{il} f^l{}_j, \end{aligned}$$

where  $\gamma_i$  is defined by  $\omega_{n+1}^0 = -\gamma_i \omega^i$  and  $h_{ijk,l}$ ,  $L_{ij,k}$  and  $\gamma_{i,j}$  are covariant derivatives of  $h_{ijk}$ ,  $L_{ij}$  and  $\gamma_i$  with respect to  $\pi$ .

Now we choose a frame  $e$  so that  $h$  is a constant matrix, which we denote by  ${}^0h$ . This is possible by (1.7). Then the set of projective frames satisfying (1.14) and  $h(p) = {}^0h$  becomes a principal bundle  $P$  with the group  $H$ , corresponding to  ${}^0h$ , as the structure group. The 1-forms  $\pi$  and  $\tau$  corresponding to  $ge(g \in H)$  can be thought of as 1-forms on  $P$  in view of (1.6) and (1.17). We denote these forms by  $\tilde{\pi}$  and  $\tilde{\tau}$ . These considerations then show:

PROPOSITION 1.2. *The pair  $(P, \tilde{\pi})$  defines a normal conformal connection defined in*

§ 1.1 on the hypersurface  $M$ . The form  $\bar{\tau}$  is the invariant satisfying the relations (1.19) and (1.20).

Conversely we have:

**THEOREM 1.3.** *Let  $M$  be an  $n (\geq 3)$ -dimensional complex manifold with a normal conformal connection  $\pi$ . Let  $\tau$  be a tensorial 1-form in the form (1.15). Assume that the covariant derivatives of  $\tau$  satisfy the relation (1.20) and that the curvature tensor of  $\pi$  is given by (1.19). Then, for a given point  $p$  of  $M$ , there exists a neighborhood of  $p$  which can be embedded as a non-degenerate hypersurface in a projective space of dimension  $n + 1$  so that  $\pi$  and  $\tau$  become the connection and the invariant induced by this embedding, respectively. This embedding is unique up to projective transformations.*

For the proof of this theorem and for the induction of the above formulae, see [Sas]. For use in the next section, we review the local expression for  $\pi_i^j$  and  $\pi_i^0$  in terms of the conformal structure tensor  $h_{ij}$ . Let  $(x^i)$  be a local coordinate system and choose a frame so that  $\omega^i = dx^i$ . (This frame is, in general, different from the frames defining the bundle  $P$ .) The definition of  $\pi$  in (1.16) is so made that  $dh_{ij} - h_{ik}\pi_j^k - h_{jk}\pi_i^k = 0$ . This leads, as usual, to the identity  $\pi_i^j = \Gamma_{ik}^j \omega^k$ , where  $\Gamma_{ik}^j$  is the Christoffel symbol of  $h_{ij}$ :

$$\Gamma_{ik}^j = \frac{1}{2} h^{jl} (h_{il,k} + h_{kl,i} - h_{ik,l}), \quad dh_{il} = h_{il,k} \omega^k.$$

Let  $R_{ikl}^j$  be the Riemannian curvature tensor:

$$d\pi_i^j - \pi_i^k \wedge \pi_k^j = \frac{1}{2} R_{ikl}^j \omega^k \wedge \omega^l.$$

The Ricci and the scalar curvatures are denoted by  $R_{ij}$  and by  $R$ , respectively:

$$R_{ij} = R_{ilj}^l, \quad R = h^{ij} R_{ij}.$$

If we put  $\pi_i^0 = -S_{ik} \omega^k$  then the definition (1.17) implies

$$C_{ikl}^j = R_{ikl}^j + S_{ik} \delta_l^j - S_{il} \delta_k^j + h_{ik} h^{jm} S_{ml} - h_{il} h^{jm} S_{mk}.$$

The requirement (1.18) easily shows

$$(1.21) \quad S_{ik} = \frac{1}{n-2} \left( R_{ik} - \frac{R}{2(n-1)} h_{ik} \right).$$

The tensor  $S_{ij}$  is called *the Schouten tensor* relative to the tensor  $h_{ij}$ .

**2. Local geometric theory of linear differential equations in  $n$  variables of rank  $n + 2$ .**

The purpose of this section is to give a geometric interpretation for the system of linear differential equations in  $n$  variables of rank  $n + 2$ , by using the projective study of hypersurfaces reviewed in § 1.

2.1. Geometry of hypersurfaces defined by linear differential equations. Let us first fix such a differential system.  $x=(x^1, \dots, x^n)$  will denote a coordinate system and subindices attached to functions mean derivatives with respect to these coordinates, e.g.,  $w_i=\partial w/\partial x^i$ ,  $w_{ij}=\partial^2 w/\partial x^i \partial x^j$ . Let us consider  $n+2$  linearly independent functions  $w^1, \dots, w^{n+2}$  in  $x$ . They are solutions of linear differential equations

$$\begin{vmatrix} w & w^1 & \dots & w^{n+2} \\ w_1 & w_1^1 & \dots & w_1^{n+2} \\ \cdot & \cdot & \dots & \cdot \\ w_n & w_n^1 & \dots & w_n^{n+2} \\ w_{ij} & w_{ij}^1 & \dots & w_{ij}^{n+2} \\ w_{kl} & w_{kl}^1 & \dots & w_{kl}^{n+2} \end{vmatrix} = 0$$

with the unknown  $w$ . Since the linear independence of  $w^i$  says that

$$\Delta_{ij} = \begin{vmatrix} w^1 & \dots & w^{n+2} \\ w_1^1 & \dots & w_1^{n+2} \\ \cdot & \dots & \cdot \\ w_n^1 & \dots & w_n^{n+2} \\ w_{ij}^1 & \dots & w_{ij}^{n+2} \end{vmatrix} \neq 0$$

for some pair  $(i, j)$ , we may assume  $\Delta_{1n} \neq 0$  without loss of generality. Then, dividing the equation by  $\Delta_{1n}$ , we get a system

(EQ) 
$$w_{ij} = g_{ij}w_{1n} + A_{ij}^k w_k + A_{ij}^0 w, \quad 1 \leq i, j \leq n,$$

where

(2.1) 
$$A_{ij}^k = A_{ji}^k, \quad A_{ij}^0 = A_{ji}^0, \quad g_{ij} = g_{ji}, \quad g_{1n} = 1, \quad A_{1n}^k = A_{1n}^0 = 0.$$

The functions  $w^i$  also satisfy equations

$$\begin{vmatrix} w & w^1 & \dots & w^{n+2} \\ w_1 & w_1^1 & \dots & w_1^{n+2} \\ \cdot & \cdot & \dots & \cdot \\ w_n & w_n^1 & \dots & w_n^{n+2} \\ w_{1n} & w_{1n}^1 & \dots & w_{1n}^{n+2} \\ w_{ijk} & w_{ijk}^1 & \dots & w_{ijk}^{n+2} \end{vmatrix} = 0.$$

A part of these equations will be written for short as

(2.2) 
$$w_{1jn} = G_j w_{1n} + B_j^k w_k + B_j^0 w, \quad 1 \leq j \leq n.$$

Notice that these equations are derived from (EQ) by differentiation.

The system (EQ) when  $n=2$  was first treated by Wilczynski in his memoirs [Wil] in the beginning of this century. The reformulation of this case was given by the authors in [SY 1] in terms of the moving frame method. In this paper we treat this system when  $n \geq 3$ , aiming at making the geometric meaning of the coefficients clear.

Let us consider the equation (EQ) with (2.1) of rank  $n+2$  which satisfies

$$(2.3) \quad \Delta = \det g_{ij} \neq 0.$$

We fix a vector  $w = (w^1, \dots, w^{n+2})$  made of linearly independent solutions, which defines a local embedding of the  $x$ -space into the projective space of dimension  $n+1$ . We call this embedding *the projective solution* of (EQ), which is unique up to  $PGL(n+2, \mathbb{C})$ . By abuse of language we sometimes regard  $w$  as the embedded hypersurface. Put

$$(2.4) \quad e^\theta = \det({}^t w, {}^t w_1, \dots, {}^t w_n, {}^t w_{1n}),$$

which we call *the normalization factor* of the system (EQ). The function  $\theta$  is independent of the choice of  $w$  up to additive constants. Define a set of vectors  $e = (e_0, \dots, e_{n+1})$  by

$$(2.5) \quad e_0 = w, \quad e_i = w_i, \quad e_{n+1} = e^{-\theta} w_{1n}.$$

Then this is a projective frame along  $w$  in the sense explained in § 1. The system (EQ) and the equations (2.2) can be written in a Pfaffian form

$$(2.6) \quad de = \omega e$$

where

$$\omega = (\omega^\beta_\alpha) = \begin{pmatrix} 0 & dx^j & 0 \\ A^0_{ik} dx^k & A^j_{ik} dx^k & e^\theta g_{ik} dx^k \\ e^{-\theta} B^0_k dx^k & e^{-\theta} B^j_k dx^k & (G_k - \theta_k) dx^k \end{pmatrix}.$$

is the Maurer-Cartan form of the frame  $e$ . The result of § 1 says that the tensor  $h_{ij} = e^\theta g_{ij}$  defines the induced conformal metric of the hypersurface. Then the process of normalization in § 1 can be applied to the above frame  $e$ . A suitable choice of a transformation  $g$  in the form

$$(2.7) \quad g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda I_n & 0 \\ \mu & c & \lambda^{-n} \end{pmatrix}$$

suffices for this normalization. Namely, writing  $\omega' = dg \cdot g^{-1} + g\omega g^{-1}$ , which is a coframe of the transformed frame  $e' = ge$ , the element  $g$  is determined so that

$$(2.8) \quad \det h'_{ij} = 1, \quad \omega'^0_0 + \omega'^{n+1}_{n+1} = 0 \quad \text{and} \quad \text{trace } L'_{ij} = 0,$$

(see Proposition 1.2). Then by (1.16)  $\omega'$  is decomposed into the sum of the connection

form  $\pi$  associated with  $h'_{ij}$  and the tensorial invariant form  $\tau$  of the embedding  $w$ . Reversing this process, we have

$$(2.9) \quad \omega = dh \cdot h^{-1} + h(\pi - \tau)h^{-1}$$

for  $h = g^{-1}$ . The point here is that the right hand side is known to have a geometrically invariant meaning. Consequently, the coefficients of the system (EQ) is written in terms of the invariants of the hypersurface  $w$ , which we now write down explicitly. Since  $h'_{ij} = \lambda^{n+2}h_{ij} = \lambda^{n+2}e^0g_{ij}$ , the component  $\lambda$  is determined by

$$(2.10) \quad \lambda = (e^{n0} \Delta)^{-1/n(n+2)}.$$

The other components  $c$  and  $\mu$  may be computed by the normalization process. In the present case, however, they can be determined also by the requirements  $A^k_{1n} = A^0_{1n} = 0$ . We prove:

**THEOREM 2.1.** *Let the equation (EQ) of rank  $n+2$  ( $n \geq 3$ ) with (2.1) and (2.3) be given. If the normalization factor satisfies*

$$(2.11) \quad \det(e^0g_{ij}) = 1,$$

then the coefficients  $A^j_{ik}$  are given by

$$A^j_{ik} = (\Gamma^j_{ik} - g_{ik}\Gamma^j_{1n}) - \frac{1}{2}(h^j_{ik} - g_{ik}h^j_{1n})$$

$$A^0_{ik} = (-S_{ik} + g_{ik}S_{1n}) - \{M_{ik} + L_{ik} - g_{ik}(M_{1n} + L_{1n})\}.$$

Here  $\Gamma^j_{ik}$  and  $S_{ik}$  are the Christoffel symbol and the Schouten tensor of the tensor  $e^0g_{ij}$ , defined in §1.2. The  $h^j_{ik}$ ,  $L_{ik}$  and  $M_{ik}$  are components of the form  $\tau$  defined in §1.2 with respect to the normalized frame  $ge$ .

**PROOF.** Put  $h_{ij} = e^0g_{ij}$ . Since  $\lambda$  is chosen to be 1, we have  $h'_{ij} = h_{ij}$ . By our discussion in §1,  $\pi$  and  $\tau$  have the following form

$$\pi = \begin{pmatrix} 0 & \pi^j & 0 \\ \pi^0_i & \pi^j_i & \pi^{n+1}_i \\ 0 & \pi^j_{n+1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & dx^j & 0 \\ -S_{ik}dx^k & \Gamma^j_{ik}dx^k & h_{ik}dx^k \\ 0 & -h^{jl}S_{ik}dx^k & 0 \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & 0 & 0 \\ \tau^0_i & \tau^j_i & 0 \\ \tau^0_{n+1} & \tau^j_{n+1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ (M_{ik} + L_{ik})dx^k & \frac{1}{2}h^j_{ik}dx^k & 0 \\ -\omega^0_{n+1} & h^{jl}M_{ik}dx^k & 0 \end{pmatrix}.$$

Note that  $\pi^0_0 = \pi^{n+1}_{n+1} = 0$  because  $\omega^0_0 = 0$ , which is deduced by  $\omega^0_0 = 0$  and by the choice of  $g(\lambda = 1)$ . Substituting these expressions into (2.9), we get

$$(2.12) \quad \omega = \begin{pmatrix} 0 & \pi^j & 0 \\ \pi_i^0 - \tau_i^0 + \mu\pi_i^{n+1} & \pi_i^j - \tau_i^j + c^j\pi_i^{n+1} & \pi_i^{n+1} \\ -\tau_{n+1}^0 - d\mu & \pi_{n+1}^j - \tau_{n+1}^j - dc^j - \mu\pi^j & c^i\pi_i^{n+1} \\ -c^i(\pi_i^0 - \tau_i^0 + \mu\pi_i^{n+1}) & -c^i(\pi_i^j - \tau_i^j + c^j\pi_i^{n+1}) & c^i\pi_i^{n+1} \end{pmatrix}.$$

Hence

$$(2.13) \quad \begin{aligned} A_{ik}^j &= \Gamma_{ik}^j - \frac{1}{2} h_{ik}^j + c^j h_{ik} \\ A_{ik}^0 &= -S_{ik} - (M_{ik} + L_{ik}) + \mu h_{ik}. \end{aligned}$$

The requirements  $A_{1n}^k = A_{1n}^0 = 0$  are satisfied when

$$(2.14) \quad c^j = -e^{-\theta}(\Gamma_{1n}^j - \frac{1}{2} h_{1n}^j), \quad \mu = e^{-\theta}(S_{1n} + M_{1n} + L_{1n}).$$

Substituting these equalities into (2.13), we have the formulae. □

REMARK. From (2.12) and (2.14) follow also the formulae for  $G_j$  and  $B_j$  in (2.2).

If the equation (EQ) does not satisfy the condition (2.11), then by multiplying a suitable function to the unknown  $w$ , one can transform (EQ), without changing the hypersurface  $w$  nor the coefficients  $g_{ij}$ , into one satisfying the condition. The other coefficients are obtained by the following lemma, the proof of which is a straightforward computation.

LEMMA 2.2 *Let the system (EQ) be given with the normalization factor  $e^\theta$ . If the unknown  $w$  is transformed into a new unknown  $u$  by  $w = e^{-\alpha}u$ , then the system is subject to the change*

$$(2.15) \quad u_{ik} = g_{ik}u_{ik} + P_{ik}^j u_j + P_{ik}^0 u,$$

where

$$(2.16) \quad \begin{aligned} P_{ik}^j &= A_{ik}^j + \alpha_i \delta_k^j + \alpha_k \delta_i^j - g_{ik}(\alpha_1 \delta_n^j + \alpha_n \delta_1^j) \\ P_{ik}^0 &= A_{ik}^0 + (\alpha_{ik} - \alpha_i \alpha_k) + A_{ik}^j \alpha_j - g_{ik}(\alpha_{1n} - \alpha_1 \alpha_n). \end{aligned}$$

The new normalization factor is  $e^{\theta + (n+2)\alpha}$

### 2.2. Linear differential equations defining maps into hyperquadrics.

DEFINITION. The system (EQ) is said to satisfy *the quadric condition* if the image of  $w$  is contained in a certain quadric hypersurface, i.e., if the cubic invariant form III vanishes identically (Proposition 1.1).

Since the quadric hypersurface is conformally flat, the invariant  $\tau$  vanishes under

the quadric condition; and the connection form  $\pi$  itself is flat. This fact can also be seen directly from the formulae (1.19) and (1.20). Therefore we get the following from Theorem 2.1.

**THEOREM 2.3.** *Let the equation (EQ) of rank  $n+2$  ( $n \geq 3$ ) satisfying (2.1), (2.3) and (2.11) be given. If it satisfies the quadric condition, then the coefficients  $A_{ik}^j$  are expressed as rational functions in  $g_{ij}$  and their derivatives:*

$$(2.17) \quad \begin{aligned} A_{ik}^j &= \Gamma_{ik}^j - g_{ik} \Gamma_{1n}^j \\ A_{ik}^0 &= -S_{ik} + g_{ik} S_{1n}. \end{aligned}$$

Here  $\Gamma_{ik}^j$  and  $S_{ik}$  are the Christoffel symbol and the Schouten tensor of  $e^0 g_{ij}$ .

Converse of this theorem holds.

**THEOREM 2.4.** *Assume  $n \geq 3$ . Let  $g_{ij} dx^i dx^j$  ( $g_{1n} = 1$ ) be a non-degenerate symmetric tensor which is conformally flat. Define  $\theta$  so that  $\det(e^\theta g_{ij}) = 1$ ; and define quantities  $A_{ij}^k$  and  $A_{ij}^0$  by (2.17) according to the tensor  $e^\theta g_{ij}$ . Then the equation*

$$w_{ij} = g_{ij} w_{1n} + A_{ij}^k w_k + A_{ij}^0 w$$

is of rank  $n+2$  and satisfies the quadric condition. Its normalization factor is  $e^\theta$ .

**PROOF.** Put  $h_{ij} = e^\theta g_{ij}$ . Since  $h_{ij}$  is conformally flat by assumption, the associated normal conformal connection  $\pi$  is integrable. Apply Theorem 1.3 by putting  $\tau = 0$ . The Gauss and the Codazzi-Minardi equations are trivially satisfied so that there is a unique embedding  $w = (w^1, \dots, w^{n+2})$  of the  $x$ -space into  $P^{n+1}$  such that the induced conformal metric is  $h_{ij}$  and the invariant form  $\tau$  is zero. Let

$$(\#) \quad w_{ij} = g'_{ij} w_{1n} + A'^k_{ij} w_k + A'^0_{ij} w$$

be the system with the projective solution  $w$  and with the normalization factor  $e^\theta$ . The argument in §2.1 tells us that the surface  $w$  has the induced conformal metric  $e^\theta g'_{ij}$ . Therefore we have  $g'_{ij} = g_{ij}$ . Since  $(\#)$  is of rank  $n+2$ , Theorem 2.2 asserts that  $A'^k_{ij} = A^k_{ij}$  and  $A'^0_{ij} = A^0_{ij}$ .  $\square$

We can formulate this in a more symmetric way:

**THEOREM 2.5.** *Assume  $n \geq 3$ . Let  $\sigma_{ij} dx^i dx^j$  be a non-degenerate symmetric tensor which is conformally flat. Then the system*

$$\sigma_{ij} \left( w_{kl} - \Gamma_{kl}^p w_p + \frac{1}{n-2} R_{kl} w \right) = \sigma_{kl} \left( w_{ij} - \Gamma_{ij}^p w_p + \frac{1}{n-2} R_{ij} w \right)$$

is of rank  $n+2$  and satisfies the quadric condition. Here  $\Gamma_{ij}^p$  and  $R_{ij}$  stand for the Christoffel symbol and the Ricci tensor with respect to  $\sigma_{ij}$ .

**PROOF.** Assume first  $e^n := \sigma_{1n} \neq 0$  and put  $g_{ij} = e^{-n} \sigma_{ij}$  and  $\det(g_{ij}) = e^{-2n\rho}$ . Define

$h_{ij} = e^{2\rho} g_{ij}$  so that  $\det(h_{ij}) = 1$ . We have only to combine Theorem 2.4 and Lemma 2.2 as well as the transformation formulae of the Christoffel symbol and the Ricci tensor for  $h_{ij}$  into those for  $\sigma_{ij}$ :

$$\begin{aligned} \Gamma_{ik}^j(\sigma) &= \Gamma_{ik}^j(h) + \alpha_i \delta_k^j + \alpha_k \delta_i^j - h_{ik} h^{jp} \alpha_p, \\ R_{ik}(\sigma) &= R_{ik}(h) - (n-2)(\alpha_{ik} - \alpha_i \alpha_k - \alpha_j \Gamma_{ik}^j(h)) - \{ \Delta_h \alpha + (n-2) h^{ip} \alpha_j \alpha_p \} h_{ik} \end{aligned}$$

where  $\alpha = (1/2)\eta + \rho$  and  $\Delta_h$  is the Laplacian for  $h_{ij}$  (see [Gol, p. 115]). Let us next turn to the case when  $\sigma_{ij} = 0$  for all  $i \neq j$ . Introduce new coordinates  $y = (y^i)$  by

$$(*) \quad y^1 = x^1 + x^n, \quad y^i = x^i \quad (2 \leq i \leq n),$$

and put

$$s_{ij} = \sigma_{pq} \frac{\partial x^p}{\partial y^i} \frac{\partial x^q}{\partial y^j}, \quad \text{i.e.,} \quad \sigma_{ij} dx^i dx^j = s_{ij} dy^i dy^j.$$

Then  $s_{ij}$  satisfies the above assumption  $s_{11} \neq 0$ . Denote by  $\gamma_{ij}^k$  and by  $r_{ij}$  the Christoffel symbol and the Ricci tensor with respect to the tensor  $\sigma_{ij}$ ; and put

$$W_{ij} = \frac{\partial^2 w}{\partial y^i \partial y^j} - \gamma_{ij}^k \frac{\partial w}{\partial y^k} + \frac{r_{ij}}{n-2} w.$$

Since  $(*)$  is linear we have

$$\gamma_{ij}^k = \Gamma_{pq}^r \frac{\partial x^p}{\partial y^i} \frac{\partial x^q}{\partial y^j} \frac{\partial y^k}{\partial x^r}, \quad r_{ij} = R_{pq} \frac{x^p}{y^i} \frac{\partial x^q}{\partial y^j},$$

and so

$$W_{ij} = Z_{pq} \frac{\partial x^p}{\partial y^i} \frac{\partial x^q}{\partial y^j}, \quad \text{where} \quad Z_{pq} = w_{pq} - \Gamma_{pq}^r w_r + \frac{1}{n-2} R_{pq} w.$$

Hence we have

$$s_{ij} W_{kl} - s_{kl} W_{ij} = \frac{\partial x^p}{\partial y^i} \frac{\partial x^q}{\partial y^j} \frac{\partial x^r}{\partial y^k} \frac{\partial x^s}{\partial y^l} (\sigma_{pq} Z_{rs} - \sigma_{rs} Z_{pq}). \quad \square$$

### 3. Uniformizing equation of a Siegel modular orbifold.

3.1. Statement of the result. The domain

$$D = \{ (z^1, \dots, z^n) \in \mathbb{C}^n \mid (\text{Im } z^1)(\text{Im } z^n) - \sum_{j=2}^{n-1} (\text{Im } z^j)^2 > 0, \text{Im } z^1 > 0 \}$$

is called the symmetric domain of type IV of dimension  $n (\geq 2)$ . If  $n = 2$ , then  $D$  is biholomorphically equivalent to the product  $H \times H$  of the upper half plane  $H = \{ \tau \in \mathbb{C} \mid \text{Im } \tau > 0 \}$ . If  $n = 3$ , then  $D$  is biholomorphically equivalent to the Siegel upper half space  $H_2$  of genus 2:



$$\left\{ \begin{pmatrix} \tau^1 & \tau^2 \\ \tau^2 & \tau^3 \end{pmatrix} \mid (\text{Im } \tau^1)(\text{Im } \tau^3) - (\text{Im } \tau^2)^2 > 0, \text{ Im } \tau^1 > 0 \right\}.$$

Let  $Q$  be an  $(n+2)$  by  $(n+2)$  symmetric matrix given by

$$(t^0, \dots, t^{n+1})Q'(t^0, \dots, t^{n+1}) = -t^0 t^{n+1} + t^1 t^n - \sum_{j=2}^{n-1} (t^j)^2$$

and let

$$Q^n = \{(t^0, \dots, t^{n+1}) \in P^{n+1} \mid (t^0, \dots, t^{n+1})Q'(t^0, \dots, t^{n+1}) = 0\}$$

be the quadric hypersurface of  $P^{n+1}$  defined by  $Q$ . Then the domain  $D$  can be regarded as a connected component of the open subset of  $Q^n$  through the embedding:

$$(z^1, \dots, z^n) \mapsto (t^0, \dots, t^{n+1}) = (1, z^1, \dots, z^n, z^1 z^n - \sum_{j=2}^{n-1} (z^j)^2).$$

The group  $\text{Aut}(D)$  of analytic automorphisms of  $D$  is a subgroup of  $\{X \in GL(n+2, \mathbf{R}) \mid XQ'X = Q'\} / \pm$  of index two via the embedding  $D \subset Q^n \subset P^{n+1}$ . The restriction of the canonical conformal structure of  $Q^n$  to  $D$  is represented by

$$\omega = dz^1 dz^n + dz^n dz^1 - 2 \sum_{j=2}^{n-1} (dz^j)^2.$$

Let  $\Gamma \subset \text{Aut}(D)$  be a properly discontinuous transformation group acting on  $D$ , and let  $D'$  be the maximal open subset of  $D$  on which  $\Gamma$  acts freely. Denote the quotient manifold  $D'/\Gamma$  by  $X$  and the natural projection  $D' \rightarrow X$  by  $\pi$ . Since  $\text{Aut}(D)$  acts conformally on  $D$ , there is a holomorphic conformal structure  $\pi_* \omega$  on  $X$ . Let  $g_{ij} dx^i dx^j$  ( $g_{1n} = 1$ ) be a conformal metric representing  $\pi_* \omega$  on a chart, with local coordinates  $x^i$ , of  $X$ . There is a linear differential equation (UDE) called the uniformizing equation of  $X$  in the form (EQ) with the principal part  $g_{ij}$  such that the projective solution gives the inverse of  $\pi$ . When  $n \geq 3$ , Theorem 2.2 tells us that  $g_{ij}$  determine the remaining coefficients of (UDE). Therefore if one knows  $g_{ij}$  as functions of  $x^i$ , then one knows the equation (UDE).

Indeed this is the case for the Siegel modular group  $\Gamma(2)$  of level 2 acting on the Siegel upper half space  $H_2$  of degree 2, equipped with the canonical conformal structure

$$\omega = d\tau^1 d\tau^3 + d\tau^3 d\tau^1 - 2(d\tau^2)^2.$$

**THEOREM 3.1.** *The regular orbit of  $H_2$  under  $\Gamma(2)$  is isomorphic to the space  $X = \{(x^1, x^2, x^3) \in \mathbf{C}^3 \mid x^i \neq 0, 1, x^j (i \neq j)\}$ . The image  $\pi_* \omega$  is a form on  $X$  conformal to*

$$(x^1 - x^2)x^3(x^3 - 1)(dx^1 dx^2 + dx^2 dx^1) + (x^2 - x^3)x^1(x^1 - 1)(dx^2 dx^3 + dx^3 dx^2) + (x^3 - x^1)x^2(x^2 - 1)(dx^3 dx^1 + dx^1 dx^3).$$

*The uniformizing equation (UDE) on  $X$  is given as follows:*

$$\begin{aligned}
 (3.1) \quad w_{ii} &+ \left\{ \frac{1}{x^i} + \frac{1}{x^i-1} + \frac{1}{2} \left( \frac{1}{x^i-x^j} + \frac{1}{x^i-x^k} \right) \right\} w_i \\
 &- \frac{x^j(x^j-1)}{2x^i(x^i-1)(x^i-x^j)} w_j - \frac{x^k(x^k-1)}{2x^i(x^i-1)(x^i-x^k)} w_k + \frac{1}{x^i(x^i-1)} w = 0, \\
 &(x^k-x^i)x^j(x^j-1) \left\{ 2w_{ij} + \left( \frac{1}{x^j-x^k} + \frac{1}{x^j-x^i} \right) w_i \right. \\
 &\quad \left. + \left( \frac{1}{x^i-x^k} + \frac{1}{x^i-x^j} \right) w_j + \frac{1}{(x^k-x^i)(x^k-x^j)} w \right\} \\
 &= (x^i-x^j)x^k(x^k-1) \left\{ 2w_{ik} + \left( \frac{1}{x^k-x^j} + \frac{1}{x^k-x^i} \right) w_i \right. \\
 &\quad \left. + \left( \frac{1}{x^i-x^k} + \frac{1}{x^i-x^j} \right) w_k + \frac{1}{(x^j-x^i)(x^j-x^k)} w \right\}
 \end{aligned}$$

where  $(i, j, k)$  is a cyclic permutation of  $(1, 2, 3)$ .

In the next subsection (Proposition 3.4), we shall express  $\pi_*\omega$  in terms of the  $x$ -coordinate. Once it is done, the equation (UDE) is derived as follows: We apply Theorem 2.5 to the tensor  $\sigma_{ij}$  of the form

$$(\sigma_{ij}) = \begin{pmatrix} 0 & E_3 & E_2 \\ E_3 & 0 & E_1 \\ E_2 & E_1 & 0 \end{pmatrix},$$

which is assumed to be conformally flat and  $E_1E_2E_3 \neq 0$ . Put

$$W_{ij} = w_{ij} - \Gamma_{ij}^k w_k + R_{ij} w.$$

Then the system is

$$\begin{aligned}
 (3.2) \quad W_{ii} &= 0 \\
 E_j W_{ij} - E_k W_{ik} &= 0,
 \end{aligned}$$

where  $(i, j, k)$  is a cyclic permutation of  $(1, 2, 3)$ .

The actual computation is now sketched. The inverse matrix of  $\sigma$  is given by

$$(2E_1E_2E_3)^{-1} \begin{pmatrix} -E_1^2 & E_1E_2 & E_1E_3 \\ E_2E_1 & -E_2^2 & E_2E_3 \\ E_3E_1 & E_3E_2 & -E_3^2 \end{pmatrix}.$$

LEMMA 3.2. *The Christoffel symbols of  $\sigma$  are given by*

$$\Gamma_{ii}^i = \frac{1}{2}(\log E_j E_k)_i, \quad \Gamma_{ii}^j = \frac{E_j}{2E_i} \left( \log \frac{E_j}{E_k} \right)_i,$$

$$\Gamma_{ij}^i = \frac{A_k}{4E_j}, \quad \Gamma_{ij}^k = \frac{-A_k E_k}{4E_i E_j},$$

where  $A_i = (E_j)_j + (E_k)_k - (E_i)_i$  and  $(i, j, k)$  is as above.

PROOF. Here the summation rule is *not* applied to the indices  $i, j$  and  $k$ . By definition, we have

$$\Gamma_{ii}^i = \frac{1}{2} \sum_l \sigma^{il} (2\sigma_{il,i} - \sigma_{ii,l}) = \sum_l \sigma^{il} \sigma_{il,i} \quad (\sigma_{ii} = 0)$$

$$= \sigma^{ij} \sigma_{ij,i} + \sigma^{ik} \sigma_{ik,i} = \frac{1}{2E_k} (E_k)_i + \frac{1}{2E_j} (E_j)_i = \frac{1}{2} (\log E_j E_k)_i.$$

Hence the first equality. The others are similarly obtained. □

As for the Ricci tensor recall the definition:

$$R_{ij} = \sum_l R^l_{ij} = \sum_l \Gamma^l_{ij,l} - \sum_l \Gamma^l_{il,j} + \sum_{l,m} \Gamma^m_{ij} \Gamma^l_{ml} - \sum_{l,m} \Gamma^m_{il} \Gamma^l_{jm}.$$

LEMMA 3.3. *The Ricci tensor is given as follows*

$$R_{ii} = -\frac{1}{2} \left\{ (\log E_i)_{ii} - \frac{E_j}{E_i} \left( \log \frac{E_j}{E_k} \right)_{ij} - \frac{E_k}{E_i} \left( \log \frac{E_k}{E_j} \right)_{ik} \right\} - \frac{1}{4} (\log E_i)_i \left( \log \frac{E_i}{E_j E_k} \right)_i$$

$$- \frac{1}{4} \left( \log \frac{E_j}{E_k} \right)_i \left\{ \frac{E_j}{E_i} \left( \log \frac{E_i}{E_j E_k} \right)_j - \frac{E_k}{E_i} \left( \log \frac{E_i}{E_j E_k} \right)_k \right\}$$

$$E_k R_{ik} - E_j R_{ij} = \frac{1}{2} \{ E_j (\log E_k)_{ij} - E_k (\log E_j)_{ik} \}$$

$$- \frac{1}{4} (\log E_i)_i (E_j - E_k) + \frac{1}{4} \left( \log \frac{E_j}{E_k} \right)_i \left\{ E_j \left( \log \frac{E_i}{E_k} \right)_j + E_k \left( \log \frac{E_i}{E_j} \right)_k \right\}.$$

PROOF. We show the first identity only.  $R_{ii}$  is the sum of three parts:

$$R_{ii} = \sum_l \Gamma^l_{ii,l} - \sum_l \Gamma^l_{il,i} + \sum_{l,m} (\Gamma^m_{ii} \Gamma^l_{lm} - \Gamma^m_{il} \Gamma^l_{mi}).$$

Lemma 3.2 shows

$$\sum_l \Gamma^l_{ii} = \Gamma_{ii}^i + \Gamma_{ij}^j + \Gamma_{ik}^k = \frac{1}{2} (\log E_j E_k)_i + \frac{A_k}{4E_i} + \frac{A_j}{4E_i} = \frac{1}{2} (\log E_i E_j E_k)_i.$$

Here note that  $A_j + A_k = 2(E_i)_i$ . The second term is

$$\begin{aligned} \sum_l \Gamma_{ii,l}^l &= \Gamma_{ii,i}^i + \Gamma_{ii,j}^j + \Gamma_{ii,k}^k \\ &= \frac{1}{2} (\log E_j E_k)_{ii} + \left\{ \frac{E_j}{2E_i} \left( \log \frac{E_j}{E_k} \right)_i \right\}_j + \left\{ \frac{E_k}{2E_i} \left( \log \frac{E_k}{E_j} \right)_i \right\}_k. \end{aligned}$$

The third term is computed as follows:

$$\begin{aligned} \sum_{l,m} (\Gamma_{ii}^m \Gamma_{mi}^l - \Gamma_{ii}^m \Gamma_{lm}^l) &= \sum_m (\Gamma_{ij}^m \Gamma_{mi}^j - \Gamma_{ii}^m \Gamma_{jm}^j + \Gamma_{ik}^m \Gamma_{mi}^k - \Gamma_{ii}^m \Gamma_{km}^k) \\ &= \Gamma_{ii}^j (\Gamma_{ij}^i - \Gamma_{jj}^j - \Gamma_{kj}^k) + \Gamma_{ii}^k (\Gamma_{ik}^i - \Gamma_{jk}^j - \Gamma_{kk}^k) \\ &\quad - \Gamma_{ii}^i (\Gamma_{ji}^j + \Gamma_{ki}^k) + \{(\Gamma_{ij}^j)^2 + 2\Gamma_{ik}^j \Gamma_{ij}^k + (\Gamma_{ik}^k)^2\}. \end{aligned}$$

The first bracket is equal to  $(A_k - A_i)/4E_j - (1/2)(\log E_i E_k)_j$ . Notice the  $A_k - A_i = 2\{(E_i)_i - (E_k)_k\}$ . Hence the sum of the first two terms is

$$\frac{1}{4E_i} \left( \log \frac{E_j}{E_k} \right)_i \left\{ E_k \left( \log \frac{E_i E_j}{E_k} \right)_k - E_j \left( \log \frac{E_i E_k}{E_j} \right)_j \right\}.$$

The third term is, in view of  $\Gamma_{ji}^j + \Gamma_{ki}^k = (1/2)(\log E_i)_i$ , equal to

$$-\frac{1}{4} (\log E_j E_k)_i (\log E_i)_i.$$

The last term is

$$\left( \frac{A_k}{4E_i} \right)^2 + \left( \frac{A_j}{4E_i} \right)^2 + 2 \frac{-A_j E_j}{4E_i E_k} - \frac{A_k E_k}{4E_i E_j} = \frac{1}{16} E_i^{-2} (A_j + A_k)^2 = \frac{1}{4} \{(\log E_i)_i\}^2.$$

Summing up these, we have

$$\begin{aligned} R_{ii} &= -\frac{1}{2} (\log E_i E_j E_k)_{ii} + \frac{1}{2} (\log E_j E_k)_{ii} + \left\{ \frac{E_j}{2E_i} \left( \log \frac{E_j}{E_k} \right)_i \right\}_j + \left\{ \frac{E_k}{2E_i} \left( \log \frac{E_k}{E_j} \right)_i \right\}_k \\ &\quad - \frac{1}{4E_i} \left( \log \frac{E_j}{E_k} \right)_i \left\{ E_k \left( \log \frac{E_i E_j}{E_k} \right)_k - E_j \left( \log \frac{E_i E_k}{E_j} \right)_j \right\} \\ &\quad + \frac{1}{4} (\log E_j E_k)_i (\log E_i)_i + \frac{1}{4} \{(\log E_i)_i\}^2, \end{aligned}$$

which implies the first equality of the lemma. □

PROOF OF (3.1). We choose a conformal class  $\sigma$  given by

$$(3.3) \quad E_i^{-1} = -(x^i - x^j)(x^i - x^k)x^j(x^j - 1)x^k(x^k - 1),$$

which is conformal to  $\pi_* \omega$ . Substituting these into the identities in Lemmas 3.2 and 3.3, the system (3.2) becomes the system (3.1). □

3.2. The conformal structure on a modular variety. The real symplectic group  $Sp(2, \mathbf{R})$  is by definition

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(4, \mathbf{R}) \mid \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}^t \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \right\},$$

where  $I_k$  stands for the  $k$  by  $k$  identity matrix. The group of analytic automorphisms of  $H_2$  is given by  $Sp(2, \mathbf{R})/\pm$  with the action

$$\tau = \begin{pmatrix} \tau^1 & \tau^2 \\ \tau^2 & \tau^3 \end{pmatrix} \mapsto (A\tau + B)(C\tau + D)^{-1}.$$

Let us consider the following two discrete subgroups of  $Sp(2, \mathbf{R})$ :

$\Gamma = GL(4, \mathbf{Z}) \cap Sp(2, \mathbf{R})$ : the full modular group

$\Gamma(2) = \{X \in \Gamma \mid X \equiv I_4 \pmod{2}\}$ : the principal congruence subgroup of level two.

The group  $\Gamma(2)$  is a normal subgroup of  $\Gamma$  such that

$$\Gamma/\Gamma(2) \cong (\Gamma/\pm)/(\Gamma(2)/\pm) \cong S_6$$

where  $S_6$  is the symmetric group on six letters. The transformation

$$\iota: \begin{pmatrix} \tau^1 & \tau^2 \\ \tau^2 & \tau^3 \end{pmatrix} \mapsto \begin{pmatrix} \tau^1 & -\tau^2 \\ -\tau^2 & \tau^3 \end{pmatrix}$$

( $\iota \in \Gamma(2)$ ) fixes the hyperplane  $F_0 \subset H_2$  given by  $\tau^2 = 0$ . The set  $F$  of fixed points of  $\Gamma(2)$  on  $H_2$  is given by  $F = \Gamma F_0$ .

Consider the space

$$\Xi = \{ \xi = (\xi^1, \dots, \xi^6) \in (\mathbf{P}^1)^6 \mid \xi^i \neq \xi^j \quad (i \neq j) \}.$$

The group  $PGL(2, \mathbf{C})$  and the symmetric group  $S_6$  act on  $\Xi$  as follows:

$$\gamma: (\xi^1, \dots, \xi^6) \mapsto (\gamma\xi^1, \dots, \gamma\xi^6) \quad \gamma \in PGL(2, \mathbf{C})$$

$$\sigma: (\xi^1, \dots, \xi^6) \mapsto (\xi^{\sigma(1)}, \dots, \xi^{\sigma(6)}) \quad \sigma \in S_6.$$

For  $\xi \in \Xi$ , we consider a non-singular plane curve

$$C(\xi): w^4 v^2 = (u - \xi^1 w) \cdots (u - \xi^6 w)$$

of genus two in  $P^2$  with a homogeneous coordinate system  $(u, v, w)$ . Two curves  $C(\xi)$  and  $C(\xi')$  are biholomorphically equivalent if and only if  $\xi = g\xi'$  for some  $g \in S_6 \times PGL(2, \mathbf{C})$ . The space  $\Xi$  modulo  $PGL(2, \mathbf{C})$  is isomorphic to the space

$$A = \{ (\lambda^1, \lambda^2, \lambda^3) \in \mathbf{C}^3 \mid \lambda^i \neq 0, 1, \lambda^j \quad (i \neq j) \},$$

which parameterizes plane curves in the Rosenhein normal form

$$C(\lambda): w^3 v^2 = u(u - w)(u - \lambda^1 w)(u - \lambda^2 w)(u - \lambda^3 w).$$

Notice that the group  $\text{Aut}(A)$  of automorphisms is isomorphic to  $S_6$ . We consider the

curve  $C(\lambda)$  for a fixed  $\lambda \in \Lambda$ . We take a basis of the homology group  $H_1(C(\lambda), \mathbf{Z})$  so that the corresponding four by four intersection matrix takes the canonical form

$$\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

Then we take two linearly independent differentials of the first kind on  $C(\lambda)$  so that the period matrix takes the form  $(\tau, I_2)$ . This is always possible and we get a point  $\tau$  of  $H_2$ . Notice that the choice for the basis of  $H_1(C(\lambda), \mathbf{Z})$  is not unique but, once it is chosen, the choice of two differentials is unique. Notice also that  $\tau \in H_2 - F$ , since the Jacobian variety  $\mathbf{C}^2/(\tau, I_2)\mathbf{Z}^4$  of the curve  $C(\lambda)$  cannot be the product of two elliptic curves. Now we let  $\lambda \in \Lambda$  vary and let the basis of  $H_1(C(\lambda), \mathbf{Z})$  depend continuously on  $\lambda$ . Then the correspondence  $\lambda \rightarrow \tau(\lambda)$  gives a multi-valued map

$$\varphi: \Lambda \rightarrow H_2 - F,$$

which turns out to be an inverse map of the natural projection

$$\pi: H_2 - F \rightarrow (H_2 - F)/\Gamma(2) \simeq \Lambda.$$

Notice that  $(H_2 - F)/\Gamma \cong \Lambda/S_6$ . The isomorphism  $(H_2 - F)/\Gamma(2) \simeq \Lambda$  can be explicitly given as follows.

We define sixteen theta constants  $\theta_{g'g''h'h''}(\tau)$  for  $g', g'', h', h'' = 0, 1$  by

$$\begin{aligned} \theta_{g'g''h'h''}(\tau) = \sum_{p', p'' \in \mathbf{Z}} \exp \pi i \left\{ \left( p' + \frac{g'}{2} \right)^2 \tau^1 + 2 \left( p' + \frac{g'}{2} \right) \left( p'' + \frac{g''}{2} \right) \tau^2 \right. \\ \left. + \left( p'' + \frac{g''}{2} \right)^2 \tau^3 + \left( p' + \frac{g'}{2} \right) h' + \left( p'' + \frac{g''}{2} \right) h'' \right\} \end{aligned}$$

which are holomorphic functions in

$$\tau = \begin{pmatrix} \tau^1 & \tau^2 \\ \tau^2 & \tau^3 \end{pmatrix} \in H_2.$$

In terms of theta constants, the natural map  $\pi: \tau \rightarrow (\lambda^1, \lambda^2, \lambda^3)$  can be expressed (cf. [Igu]) by

$$\begin{aligned} \lambda^1 &= \left( \frac{\theta_{1100}(\tau)}{\theta_{0100}(\tau)} \right)^2 \left( \frac{\theta_{1000}(\tau)}{\theta_{0000}(\tau)} \right)^2 \\ \lambda^2 &= \left( \frac{\theta_{1100}(\tau)}{\theta_{0100}(\tau)} \right)^2 \left( \frac{\theta_{1001}(\tau)}{\theta_{0001}(\tau)} \right)^2 \\ \lambda^3 &= \left( \frac{\theta_{1000}(\tau)}{\theta_{0000}(\tau)} \right)^2 \left( \frac{\theta_{1001}(\tau)}{\theta_{0001}(\tau)} \right)^2. \end{aligned}$$

We have chosen the above expression from  $6! = \# \text{Aut}(\Lambda)$  possibilities. We want an

explicit expression of the form  $\pi_*(d\tau^1 d\tau^3 + d\tau^3 d\tau^1 - 2(d\tau^2)^2)$  in terms of coordinates  $\lambda^1, \lambda^2$  and  $\lambda^3$ .

**PROPOSITION 3.4.** *The quadratic form  $\pi_*(d\tau^1 d\tau^3 + d\tau^3 d\tau^1 - 2(d\tau^2)^2)$  is a form on  $\Lambda$  conformal to*

$$(\lambda^1 - \lambda^2)\lambda^3(\lambda^3 - 1)(d\lambda^1 d\lambda^2 + d\lambda^2 d\lambda^1) + (\lambda^2 - \lambda^3)\lambda^1(\lambda^1 - 1)(d\lambda^2 d\lambda^3 + d\lambda^3 d\lambda^2) + (\lambda^3 - \lambda^1)\lambda^2(\lambda^2 - 1)(d\lambda^3 d\lambda^1 + d\lambda^1 d\lambda^3),$$

whose discriminant is

$$\lambda^1 \lambda^2 \lambda^3 (\lambda^1 - 1)(\lambda^2 - 1)(\lambda^3 - 1)(\lambda^1 - \lambda^2)(\lambda^2 - \lambda^3)(\lambda^3 - \lambda^1).$$

The rest of this section is devoted to the proof of the proposition. We put  $q = \exp \pi i \tau^3$  and study expansions of three lambdas  $\lambda^1, \lambda^2$  and  $\lambda^3$  in  $q$ . Put

$$\lambda^1 = \lambda_0^1 + \lambda_1^1 q + O(q^2), \quad \lambda^2 = \lambda_0^2 + \lambda_1^2 q + O(q^2), \quad \lambda^3 = \lambda_0^3 + \lambda_1^3 q + O(q^2),$$

where  $O(q^k)$  stands for a holomorphic function or a form divisible by  $q^k$ .

**LEMMA 3.5.**

$$\lambda_0^1 = \lambda_0^2 = \left( \frac{\sum_{p \in \mathbf{Z}} \exp \pi i \left\{ \left( p + \frac{1}{2} \right)^2 \tau^1 + \left( p + \frac{1}{2} \right) \tau^2 \right\}}{\sum_{p \in \mathbf{Z}} \exp \pi i (p^2 \tau^1 + p \tau^2)} \right)^2 \left( \frac{\theta_{10}(\tau^1)}{\theta_{00}(\tau^1)} \right)^2,$$

$$\lambda_1^1 = -\lambda_1^2 = 4\lambda_0^1 \left\{ \frac{\sum_{p \in \mathbf{Z}} \exp \pi i \left\{ \left( p + \frac{1}{2} \right)^2 \tau^1 + 2 \left( p + \frac{1}{2} \right) \tau^2 \right\}}{\theta_{10}(\tau^1)} - \frac{\sum_{p \in \mathbf{Z}} \exp \pi i (p^2 \tau^1 + 2p \tau^2)}{\theta_{00}(\tau^1)} \right\},$$

$$\lambda_0^3 = \lambda(\tau^1), \quad \lambda_1^3 = 0,$$

where  $\theta_{gh}(\omega)(g, h=0, 1)$  are elliptic theta constants:

$$\theta_{gh}(\omega) = \sum_{p \in \mathbf{Z}} \exp \pi i \left\{ \left( p + \frac{g}{2} \right)^2 \omega + \left( p + \frac{g}{2} \right) h \right\} \quad (\omega \in H)$$

and  $\lambda(\omega)$  is the lambda function defined by

$$\lambda(\omega) = \left( \frac{\theta_{10}(\omega)}{\theta_{00}(\omega)} \right)^4.$$

**PROOF.** We have

$$\theta_{0000}(\tau) = \sum_{p, n \in \mathbf{Z}} \exp \pi i (p^2 \tau^1 + 2pn\tau^2) q^{n^2} = \theta_{00}(\tau^1) + \theta_{0000}^{(1)} q + O(q^2)$$

$$\theta_{0001}(\tau) = \sum_{p, n \in \mathbf{Z}} \exp \pi i (p^2 \tau^1 + 2pn\tau^2 + n) q^{n^2} = \theta_{00}(\tau^1) + \theta_{0001}^{(1)} q + O(q^2)$$

$$\theta_{1000}(\tau) = \sum_{p, n \in \mathbf{Z}} \exp \pi i \left\{ \left( p + \frac{1}{2} \right)^2 \tau^1 + 2 \left( p + \frac{1}{2} \right) n \tau^2 \right\} q^{n^2} = \theta_{10}(\tau^1) + \theta_{1000}^{(1)} q + O(q^2)$$

$$\begin{aligned} \theta_{1001}(\tau) &= \sum_{p, n \in \mathbf{Z}} \exp \pi i \left\{ \left( p + \frac{1}{2} \right)^2 \tau^1 + 2 \left( p + \frac{1}{2} \right) n \tau^2 + n \right\} q^{n^2} \\ &= \theta_{10}(\tau^1) + \theta_{1001}^{(1)} q + O(q^2) \end{aligned}$$

$$\theta_{0100}(\tau) = \sum_{p, n \in \mathbf{Z}} \exp \pi i \left\{ p^2 \tau^1 + 2p \left( n + \frac{1}{2} \right) \tau^2 \right\} q^{(n+1/2)^2} = q^{1/4} \{ \theta_{0100}^{(0)} + O(q^2) \}$$

$$\begin{aligned} \theta_{1100}(\tau) &= \sum_{p, n \in \mathbf{Z}} \exp \pi i \left\{ \left( p + \frac{1}{2} \right)^2 \tau^1 + 2 \left( p + \frac{1}{2} \right) \left( n + \frac{1}{2} \right) \tau^2 \right\} q^{(n+1/2)^2} \\ &= q^{1/4} \{ \theta_{1100}^{(0)} + O(q^2) \}, \end{aligned}$$

where

$$\theta_{0000}^{(1)} = -\theta_{0001}^{(1)} = 2 \sum_{p \in \mathbf{Z}} \exp \pi i (p^2 \tau^1 + 2p \tau^2)$$

$$\theta_{1000}^{(1)} = -\theta_{1001}^{(1)} = 2 \sum_{p \in \mathbf{Z}} \exp \pi i \left\{ \left( p + \frac{1}{2} \right)^2 \tau^1 + 2 \left( p + \frac{1}{2} \right) \tau^2 \right\}$$

$$\theta_{0100}^{(0)} = 2 \sum_{p \in \mathbf{Z}} \exp \pi i (p^2 \tau^1 + p \tau^2)$$

$$\theta_{1000}^{(0)} = 2 \sum_{p \in \mathbf{Z}} \exp \pi i \left\{ \left( p + \frac{1}{2} \right)^2 \tau^1 + \left( p + \frac{1}{2} \right) \tau^2 \right\}.$$

The following identity

$$\begin{aligned} & \left( \frac{a_0 + a_1 q + O(q^2)}{b_0 + b_1 q + O(q^2)} \right)^2 \left( \frac{c_0 + c_1 q + O(q^2)}{d_0 + d_1 q + O(q^2)} \right)^2 \\ &= \left( \frac{a_0 c_0}{b_0 d_0} \right)^2 + 2 \frac{a_0 c_0}{(b_0 d_0)^2} \left\{ \frac{a_0}{d_0} (c_1 d_0 - c_0 d_1) + \frac{c_0}{b_0} (a_1 b_0 - a_0 b_1) \right\} q + O(q^2) \end{aligned}$$

leads to

$$\begin{aligned} \lambda^1 &= \left( \frac{\theta_{1100}^{(0)} \theta_{10}(\tau^1)}{\theta_{0100}^{(0)} \theta_{00}(\tau^1)} \right)^2 + 2 \frac{\theta_{1100}^{(0)} \theta_{10}(\tau^1)}{\{ \theta_{0100}^{(0)} \theta_{00}(\tau^1) \}^2} \\ &\quad \times \left\{ \frac{\theta_{1100}^{(0)}}{\theta_{00}(\tau^1)} (\theta_{1000}^{(1)} \theta_{00}(\tau^1) - \theta_{10}(\tau^1) \theta_{0000}^{(1)}) \right\} q + O(q^2) \end{aligned}$$



$$\begin{aligned} \lambda^2 &= \left( \frac{\theta_{1100}^{(0)} \theta_{10}(\tau^1)}{\theta_{0100}^{(0)} \theta_{00}(\tau^1)} \right)^2 + 2 \frac{\theta_{1100}^{(0)} \theta_{10}(\tau^1)}{\{\theta_{0100}^{(0)} \theta_{00}(\tau^1)\}^2} \\ &\quad \times \left\{ \frac{\theta_{1100}^{(0)}}{\theta_{00}(\tau^1)} (\theta_{1001}^{(1)} \theta_{00}(\tau^1) - \theta_{10}(\tau^1) \theta_{0001}^{(1)}) \right\} q + O(q^2) \\ \lambda^3 &= \left( \frac{\theta_{10}(\tau^1)}{\theta_{00}(\tau^1)} \right)^4 + 2 \frac{\theta_{10}(\tau^1)^2}{\theta_{00}(\tau^1)^4} \frac{\theta_{10}(\tau^1)}{\theta_{00}(\tau^1)} \{ \theta_{1001}^{(1)} \theta_{00}(\tau^1) - \theta_{10}(\tau^1) \theta_{0001}^{(1)} \\ &\quad + \theta_{1000}^{(1)} \theta_{00}(\tau^1) - \theta_{10}(\tau^1) \theta_{0000}^{(1)} \} q + O(q^2). \end{aligned}$$

Since we have  $\theta_{1000}^{(1)} = -\theta_{1001}^{(1)}$  and  $\theta_{0000}^{(1)} = -\theta_{0001}^{(1)}$ , the lemma is proved. □

**COROLLARY 3.6.** *We have  $\lambda^1 - \lambda^2 = qh(\tau^1, \tau^2, q)$  and*

$$\det \left( \frac{\partial(\lambda^1, \lambda^2, \lambda^3)}{\partial(\tau^1, \tau^2, \tau^3)} \right) = qf(\tau^1, \tau^2, q),$$

where  $h$  and  $f$  are holomorphic functions in  $\tau^1, \tau^2$  and  $q$  which are not divisible by  $q$ .

**PROOF.** The first assertion is obvious. The second follows from the calculations below.

$$\begin{aligned} \frac{\partial \lambda}{\partial \tau} &= \begin{pmatrix} \frac{\partial \lambda_0^1}{\partial \tau^1} + \frac{\partial \lambda_1^1}{\partial \tau^1} q & \frac{\partial \lambda_0^1}{\partial \tau^2} + \frac{\partial \lambda_1^1}{\partial \tau^2} q & \pi i \lambda_1^1 q \\ \frac{\partial \lambda_0^2}{\partial \tau^1} + \frac{\partial \lambda_1^2}{\partial \tau^1} q & \frac{\partial \lambda_0^2}{\partial \tau^2} + \frac{\partial \lambda_1^2}{\partial \tau^2} q & \pi i \lambda_1^2 q \\ \frac{\partial \lambda_0^3}{\partial \tau^1} & 0 & 0 \end{pmatrix} + O(q^2) \\ \det \left( \frac{\partial \lambda}{\partial \tau} \right) &= \pi i \frac{\partial \lambda_0^3}{\partial \tau^1} \left( -\lambda_1^1 \frac{\partial \lambda_0^2}{\partial \tau^2} + \lambda_1^2 \frac{\partial \lambda_0^1}{\partial \tau^2} \right) q + O(q^2) = -2\pi i \lambda_1^1 \frac{\partial \lambda_0^3}{\partial \tau^1} \frac{\partial \lambda_0^1}{\partial \tau^2} q + O(q^2). \quad \square \end{aligned}$$

**LEMMA 3.7.**

$$\begin{aligned} qf(\tau^1, \tau^2, q) d\tau^1 &= 2\pi i \lambda_1^1 \frac{\partial \lambda_0^1}{\partial \tau^3} q d\lambda^3 + O(q^2) \\ qf(\tau^1, \tau^2, q) d\tau^2 &= O(q) \\ qf(\tau^1, \tau^2, q) d\tau^3 &= \frac{\partial \lambda_0^3}{\partial \tau^1} \frac{\partial \lambda_0^2}{\partial \tau^2} (d\lambda^1 - d\lambda^2) + O(q^2). \end{aligned}$$

**PROOF.** This follows from the expression for  $qf(\tau^1, \tau^2, q) \partial \tau / \partial \lambda$ :

$$\left( \begin{array}{ccc} O(q^2) & O(q^2) & 2\pi i \lambda_1^1 \frac{\partial \lambda_0^1}{\partial \tau^3} q + O(q^2) \\ O(q) & O(q) & O(q) \\ \frac{\partial \lambda_0^3}{\partial \tau^1} \frac{\partial \lambda_0^2}{\partial \tau^3} + O(q) & -\frac{\partial \lambda_0^3}{\partial \tau^1} \frac{\partial \lambda_0^1}{\partial \tau^3} + O(q) & -\frac{\partial \lambda_0^1}{\partial \tau^3} \frac{\partial \lambda_0^2}{\partial \tau^1} + \frac{\partial \lambda_0^2}{\partial \tau^3} \frac{\partial \lambda_0^1}{\partial \tau^1} + O(q) \end{array} \right) \quad \square$$

Let  $U \subset A$  be an open subset of  $C^3$  such that the closure  $\bar{U}$  in  $C^3$  has the property:  $U \cap \{(\lambda^1, \lambda^2, \lambda^3) \in C^3 \mid \lambda^1 \neq \lambda^2\} = U \cap A$ .

**COROLLARY 3.8.** *The quadratic form  $\pi_*(d\tau^1 d\tau^2 + d\tau^2 d\tau^1 - 2(d\tau^3)^2)$  is conformally equivalent on  $U$  to a quadratic form with the following local expression around  $U \cap V$ :*

$$\frac{1}{\lambda^1 - \lambda^2} (d\lambda^2 d\lambda^3 + d\lambda^3 d\lambda^2) - \frac{1}{\lambda^1 - \lambda^2} (d\lambda^1 d\lambda^3 + d\lambda^3 d\lambda^1) + (\text{holomorphic quadratic form in } \lambda),$$

whose discriminant has double pole along  $\{\lambda^1 = \lambda^2\}$ .

This implies that the holomorphic conformal structure  $\pi_*(d\tau^1 d\tau^2 + d\tau^2 d\tau^1 - 2(d\tau^3)^2)$  on  $A$  can be extended to a meromorphic conformal structure  $\eta$  on  $P^3 \supset A$ . We can put

$$\begin{aligned} \eta &= \sum_{i,j=1}^3 a_{ij}(\lambda) d\lambda^i d\lambda^j \quad a_{ij} = a_{ji} \\ a_{ij}(\lambda) &= \frac{p_{ij}(\lambda)}{D(\lambda)} \quad p_{ij} \in C[\lambda^1, \lambda^2, \lambda^3] \\ D(\lambda) &= \lambda^1 \lambda^2 \lambda^3 (\lambda^1 - 1)(\lambda^2 - 1)(\lambda^3 - 1)(\lambda^1 - \lambda^2)(\lambda^2 - \lambda^3)(\lambda^3 - \lambda^1). \end{aligned}$$

We can assume  $\det(a_{ij}(\lambda)) = D(\lambda)^{-2}$  since  $\eta$  should be a holomorphic non-degenerate quadratic form on  $A$ . Since  $\Gamma/\Gamma(2) \cong S_6$  acts on  $A$  as the group of automorphisms of  $A$ , the conformal structure of  $A$  represented by  $\eta$  is invariant under the action of  $S_6$ . In particular, it is invariant under the transformation:

$$\sigma: \mu^1 = \frac{\lambda^1}{\lambda^3}, \quad \mu^2 = \frac{\lambda^2}{\lambda^3}, \quad \mu^3 = \frac{1}{\lambda^3}.$$

Put

$$\sigma^* \eta = \sum b_{ij}(z) d\mu^i d\mu^j \quad b_{ij} = b_{ji}.$$

Then we have

$$\begin{aligned} b_{ij}(\mu) &= \frac{a_{ij}(\lambda)}{(\mu^3)^2} \quad (i, j = 1, 2) \\ b_{i3}(\mu) &= -\frac{\mu^i a_{ij}(\lambda) + a_{i3}(\lambda) + \mu^i a_{ii}(\lambda)}{(\mu^3)^3} \quad \{i, j\} = \{1, 2\} \end{aligned}$$

$$b_{33}(\mu) = \frac{\mu^1 \mu^2 a_{12}(\lambda) + \sum_{k=1}^2 \{ \mu^k a_{k3}(\lambda) + (\mu^k)^2 a_{kk}(\lambda) \} + a_{33}(\lambda)}{(\mu^3)^4} .$$

Since

$$D(\lambda) = -(\mu^3)^{-10} D(\mu) \quad \text{and} \quad \det \left( \frac{\partial \lambda}{\partial \mu} \right) = -(\mu^3)^{-4} ,$$

we have

$$\det(b_{ij}(\mu)) = \det(a_{ij}(\lambda)) (\mu^3)^{-8} = D(\lambda)^{-2} (\mu^3)^{-8} = (\mu^3)^{12} D(\mu)^{-2} .$$

Therefore multiplying a conformal factor  $(\mu^3)^{-4}$  to  $g^* \eta$ , we should have

$$a_{ij}(\mu) = (\mu^3)^{-4} b_{ij}(\mu) , \quad (i, j = 1, 2, 3) .$$

In particular if  $i, j = 1, 2$ , then

$$a_{ij}(\mu) = (\mu^3)^{-4} b_{ij}(\mu) = (\mu^3)^{-6} a_{ij}(\lambda) = (\mu^3)^{-6} p_{ij}(\lambda) D(\lambda)^{-1} ;$$

so

$$(3.4) \quad p_{ij}(\mu^1, \mu^2, \mu^3) = -(\mu^3)^4 p_{ij} \left( \frac{\mu^1}{\mu^3}, \frac{\mu^2}{\mu^3}, \frac{1}{\mu^3} \right) .$$

This implies, in particular, that the total degree  $\deg(p_{ij})$  of  $p_{ij}$  ( $i, j = 1, 2$ ) is at most four. Since the form  $\eta$  is invariant under permutations of  $\lambda^1, \lambda^2$  and  $\lambda^3$ , we conclude that  $\deg(p_{ij}) \leq 4$  ( $i, j = 1, 2, 3$ ). On the other hand, by Corollary 3.8,  $p_{12}(\lambda)$  and  $p_{kk}(\lambda)$  ( $k = 1, 2, 3$ ) are divisible by  $\lambda^1 - \lambda^2$ . By using symmetry with respect to  $\lambda^1, \lambda^2$  and  $\lambda^3$  again, we have the following expressions

$$\begin{aligned} p_{kk}(\lambda) &= (\lambda^1 - \lambda^2)(\lambda^2 - \lambda^3)(\lambda^3 - \lambda^1) q_{kk} \\ p_{ij}(\lambda) &= (\lambda^i - \lambda^j) r_{ij} \quad (i \neq j) , \end{aligned}$$

where  $q_{kk}$  and  $r_{ij}$  are polynomials with  $\deg(q_{kk}) \leq 1$  and  $\deg(r_{ij}) \leq 3$ . The first expression satisfies (3.4) if and only if  $q_{kk}$  is identically zero. Thus we have

$$a_{kk}(\lambda) = 0 \quad (k = 1, 2, 3)$$

and that the determinant of the matrix  $(a_{ij}(\lambda))$  can be computed as follows:

$$\det(a_{ij}(\lambda)) = 2 \frac{p_{12}(\lambda) p_{23}(\lambda) p_{31}(\lambda)}{D(\lambda)^3} .$$

This expression together with the identity:  $\det(a_{ij}(\lambda)) = D(\lambda)^{-2}$  implies that  $\deg(p_{ij}) = 3$  ( $1 \leq i \neq j \leq 3$ ) and  $p_{12}(\lambda) p_{23}(\lambda) p_{31}(\lambda) = D(\lambda)/2$ . Substituting the expression

$$p_{12}(\lambda) = (\lambda^1 - \lambda^2) r_{12} , \quad \deg(r_{12}) = 2 ,$$

into (3.2), we have

$$p_{12}(\lambda) = (\text{const.})(\lambda^1 - \lambda^2)\lambda^3(\lambda^3 - 1).$$

By using symmetry of  $\eta$  with respect to the  $\lambda$ 's and the identity  $D(\lambda^1, \lambda^3, \lambda^2) = -D(\lambda^1, \lambda^2, \lambda^3)$ , we conclude that  $\eta$  is expressed, up to a multiplicative constant, by

$$\begin{aligned} & \frac{(\lambda^1 - \lambda^2)\lambda^3(\lambda^3 - 1)}{D(\lambda)}(d\lambda^1 d\lambda^2 + d\lambda^2 d\lambda^1) + \frac{(\lambda^2 - \lambda^3)\lambda^1(\lambda^1 - 1)}{D(\lambda)}(d\lambda^2 d\lambda^3 + d\lambda^3 d\lambda^2) \\ & + \frac{(\lambda^3 - \lambda^1)\lambda^2(\lambda^2 - 1)}{D(\lambda)}(d\lambda^3 d\lambda^1 + d\lambda^1 d\lambda^3). \end{aligned}$$

This completes the proof of Proposition 3.4.

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