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AREA-MINIMIZING HYPERSURFACES DEFINED BY HOMOTOPY CLASSES OF MAPPINGS OF 1-ESSENTIAL MANIFOLDS

Dedicated to Professor Shingo Murakami on his sixtieth brithday

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0. Introduction. We consider a homotopy class Ξ of C^1-mappings of a compact 1-essential manifold M^n into a compact Riemannian manifold N^{n+1}. For $f\in\Xi$, $f_\# v(M)$ is an n-dimensional integral varifold in N. Then $\Omega=\{f_\# v(M):f\in\Xi\}$ is a subset of the space $V_n(N)$ of n-dimensional varifolds in N. In this paper, we will show that there exists an n-dimensional integral varifold V_0 in the closure of Ω such that $\|V_0\|(N)=\inf\{\|V\|(N):V\in\Omega\}$, and study the regularity of the weight $\|V_0\|$.

Let M be a smooth compact 1-essential manifold (§1, (A) and [6]) of dimension n with the following property:

(0.1) If any loop in a connected open set U is contractible in M, then U is contained in a coordinate neighborhood in M.

Let (N, g) be a smooth compact Riemannian manifold of dimension $n+1$. Then we assume the following condition:

(0.2) There exists a continuous mapping $f_0: M\to N$ such that the induced map $f_{0\#}:\pi_1(M,p)\to\pi_1(N,f_0(p))$ is injective,

where $\pi_1(M,p)$ denotes the fundamental group of M. (In case $n=2$, we can assume a condition weaker than (0.2). See §1, (B) (2).)

Let $G_n(N)$ be a fibre bundle over N associated with the bundle of linear frames $L(N)$ with fibre $G_n(N)_p$ over p, where $G_n(N)_p$ is the Grassmann manifold of n-dimensional subspaces of the tangent space $T_p(N)$. We denote by $\pi: G_n(N)\to N$ the canonical projection. Then an n-dimensional varifold $V\in V_n(N)$ is a Radon measure on $G_n(N)$. The weight $\|V\|$ of V is a Radon measure on N. In particular, we have $f_\# v(M)(G_n(N))=\|f_\# v(M)\|(N)=\mathrm{Volume}(M,f^*g)$ for C^1-mappings f (see §2, (B) and [1]). Put

$$\Xi=\{f: M\to N: f \text{ is a } C^1\text{-mapping and homotopic to } f_0\}\,,$$
$$\Omega=\{f_\# v(M): f\in\Xi\}\quad(\subset V_n(N))\,.$$

Let spt $\|V\|$ denote the support of the Radon measure $\|V\|$ on N, and $A \sim B$ denote the subtraction for subsets of N. For a hypersurface S in N, $v(S)$ is an n-dimensional varifold in N. For $V \in \bar{\Omega}$, put

$$\operatorname{sing}\|V\| = \operatorname{spt}\|V\| \sim \{p \in N : p \text{ satisfies the condition } (*) \text{ below}\}.$$

Condition $(*)$: There exist finite smooth hypersurfaces S_i $(i=1, \cdots, k)$ imbedded in N for some $r>0$ such that

$$V \llcorner \pi^{-1} U(p, r) = \sum_{i=1}^{k} v(S_i),$$

where $U(p, r) = N \cap \{q : d_g(p, q) < r\}$. $p \in \operatorname{spt}\|V\|$ is called a singular point if $p \in \operatorname{sing}\|V\|$. We say that p is a regular point of spt $\|V\|$ if $p \in \operatorname{reg}\|V\|$, where $\operatorname{reg}\|V\| = \operatorname{spt}\|V\| \sim \operatorname{sing}\|V\|$.

Our therorem is as follows.

THEOREM. Let M, *N and Ω be as above. Then there exists an n-dimensional integral varifold V_0 in N such that*

(1) $V_0 \in \bar{\Omega}$,

(2) $\|V_0\|(N) = \inf\{\|V\|(N) : V \in \Omega\} > 0$,

(3) $\mathscr{H}^k(\operatorname{sing}\|V_0\|) = 0$ *for* $k > n-7$,

where $\mathscr{H}^k$ is the k-dimensional Hausdorff measure on N.

We emphasize that this theorem is independent of the choice of a Riemannian metric of N. In case $n=1$, the result is well-known due to the existence of minimizing closed geodesics.

The contents of this paper are as follows.

§1. Examples and notation.

§2. Fundamental lemmas and preliminaries on geometric measure theory.

§3. Normalization of area-minimizing sequence.

§4. Regularity.

White [11] gave the following result related to our theorem.

THEOREM ([11, §5 Cor. 1]). *Suppose*

(1) *M and N are smooth compact manifolds,*

(2) $3 \leqq \dim M = \dim N - 1 \leqq 7$,

(3) *$f_{0\#}(\pi_1(M))$ is of finite index in $\pi_1(N)$,*

(4) *the integral current $f_{0\#}M$ is not homologous to* 0 *in some sense* (*see* [11], *Theorem* 3–6).

Then there exists a mapping $F: M \to N$ of least mapping area in the homotopy class of f_0, and the image of F is a smooth submanifold of N together with a singularity set of dimension $\leqq \dim M - 1$.

White's theorem says, under the conditions (2), (3) and (4), that $V_0 \in \Omega$ (to be exact, $V_0 \in \{f_{\#}v(M) \mid f$ is a Lipschitz map and homotopic to $f_0\}$) and $\mathscr{H}^k(\operatorname{sing}\|V_0\|) = 0$ for

$k>n-1$. Our theorem, of course, does not assume these conditions. In particular, our theorem also gives a condition for $\| V_0 \|(N)>0$.

1. Examples and notation. (A) Examples of compact 1-essential manifolds with property (0.1) are as follows:

(1) Compact manifolds which admit a metric with non-positive sectional curvature.

(2) The real projective space $\boldsymbol{P}^n\boldsymbol{R}$.

(B) Examples of mappings $f_0: M \to N$ are as follows:

(1) Let M be the n-torus T^n and let N be a compact manifold which has an $(n+1)$-torus as a covering space. Then there exists a continuous mapping $f_0: M \to N$ such that $f_{0\#}$ maps $\pi_1(M, p)$ injectively into a part of infinite abelian subgroup of $\pi_1(N, f_0(p))$.

(2) We denote by $L_g(\gamma)$ the length of the curve γ in N with respect to a metric g. Our assumption (0.2) is, in general, necessary for $\inf\{L_g(f(\gamma)): f \in \Xi,\ \gamma$ is a non-contractible loop in $M\}>0$. But, in case $n=2$, Theorem holds good under a weaker assumption. For example, we take $M=T^2$ and $N=\boldsymbol{P}^2\boldsymbol{R}\times S$, where S is a circle. We choose generators of $\pi_1(T^2, p)$ and $\pi_1(N, q)$ so that $\pi_1(T^2, p)=\langle \alpha_1, \alpha_2 \rangle$ and $\pi_1(N, q)=\langle \beta_1, \beta_2 \rangle$, where $2\beta_1=0$. Then we can construct a mapping $f_0: T^2 \to N$ such that $f_{0\#}(\alpha_1)=\beta_1$ anf $f_{0\#}(\alpha_2)=k\beta_2$ for any integer $k(\neq 0)$. In this case, $f_{0\#}$ is not injective, but $\inf\{L_g(f(\gamma)): f \in \Xi,\ \gamma$ is a non-contractible loop in $T^2\}>0$. Indeed, by $n=2$ every loop $\gamma \in [2m\alpha_1]$ for a non-zero integer m contains a subloop γ_1 such that $\gamma_1 \in [\alpha_1]$, where $\gamma \in [2m\alpha_1]$ implies that γ is free homotopic to a loop $\gamma_0 \in 2m\alpha_1$. Therefore, we have $L_g(f(\gamma)) \geqq L_g(f(\gamma_1)) \geqq a(>0)$ for $f \in \Xi$. (see §2, Lemma 2.1 and §3, Lemma 3.1).

(C) We use the following notation in this paper:

For $p \in N$ and $r>0$, let

$$\begin{aligned} B(p, r) &= N \cap \{q: d_g(p, q) \leqq r\} \\ U(p, r) &= N \cap \{q: d_g(p, q) < r\} \\ \partial B(p, r) &= N \cap \{q: d_g(p, q) = r\} \\ B_p(0, r) &= T_p(N) \cap \{\boldsymbol{a}: |\boldsymbol{a}| \leqq r\} \\ U_p(0, r) &= T_p(N) \cap \{\boldsymbol{a}: |\boldsymbol{a}| < r\}, \end{aligned}$$

where $T_p(N)$ is the tangent space to N at p and $|\boldsymbol{a}|^2=g(\boldsymbol{a}, \boldsymbol{a})$. When $p \in N$ is fixed, we denote $B(p, r)$ (resp. $B_p(0, r), \cdots$) by $B(r)$ (resp. $B(0, r), \cdots$) for simplicity. When we say that $(y^1, \cdots, y^{n+1})$ is a normal coordinate of $U(p, r)$, we assume $g(\partial/\partial y^i, \partial/\partial y^j)(p)=\delta_{ij}$ and $y^i(p)=0$ $(i=1, \cdots, n+1)$. For a normal coordinate $(y^1, \cdots, y^{n+1})$ of $U(p, r)$, a polar coordinate $(a^1, \cdots, a^{n+1}; u)$ of $U(p, r)$ is defined by $y^i=a^i u$ and $(a^1)^2+\cdots+(a^{n+1})^2=1$. Let $\mathrm{Exp}_p: T_p(N) \to N$ be the exponential map. For $r>0$ and a fixed $p \in N$, let

$$\begin{aligned} \mu_r(\boldsymbol{a}) &= r\boldsymbol{a} && \text{for} \quad \boldsymbol{a} \in T_p(N) \\ \tilde{\mu}_r(q) &= \mu_r \,\mathrm{Exp}_p^{-1}(q) && \text{for} \quad q \in U(p, r). \end{aligned}$$

Let $G_n(N)_p$ be the Grassmann manifold of n-dimensional subspaces of $T_p(N)$. We denote by $G_n(N)$ a fibre bundle over N associated with the bundle of linear frames $L(N)$ with fibre $G_n(N)_p$ over p. We denote by $\pi: G_n(N) \to N$ the canonical projection. We fix a real number $s_0(>0)$ with the following property:

(1.1) For $0<r<2s_0$ and each $p \in N$, $U(p, r)$ is a convex normal neighborhood of p.

2. Fundmental lemmas and preliminaries on geometric measure theory.

(A) We put

$$a = \inf\{L_g(\gamma): \gamma \text{ is a non-contractible loop in } N\}\,. \tag{2.1}$$

For $f \in \Xi$, f^*g is a possibly degenerate Riemannian metric of M.

LEMMA 2.1. *There exists a constant $C_0(>0)$ depending only on the dimension n of M such that*

$$a \leqq C_0\{\mathrm{Volume}(M, f^*g)\}^{1/n} \qquad \textit{for} \quad f \in \Xi\,. \tag{2.2}$$

PROOF. Let γ be a non-contractible loop in M. By the assumption (0.2), $f(\gamma)$ is also non-contractible in N for $f \in \Xi$. Therefore we have $L_{f^*g}(\gamma) = L_g(f(\gamma)) \geqq a$ by (2.1). Thus, we have

$$\inf\{L_{f^*g}(\gamma): f \in \Xi,\ \gamma \text{ is a non-contractible loop in } M\} \geqq a\,. \tag{2.3}$$

Next, we fix a (non-degenerate) Riemannian metric h of M. $f^*g + \varepsilon h$ is a non-degenerate Riemannian metric of M for $\varepsilon > 0$. By (2.3) we have

$$L_{(f^*g+\varepsilon h)}(\gamma) \geqq L_{f^*g}(\gamma) \geqq a \tag{2.4}$$

for a non-contractible loop γ in M and $f \in \Xi$. By Gromov [6] and (2.4), there exists a constant $C_0(>0)$ depending only on the dimension n of M such that $a \leqq C_0\{\mathrm{Volume}(M, f^*g + \varepsilon h)\}^{1/n}$ for $f \in \Xi$ and $\varepsilon > 0$. Since $\lim_{\varepsilon \to 0} \mathrm{Volume}(M, f^*g + \varepsilon h) = \mathrm{Volume}(M, f^*g)$, we are done. q.e.d.

(B) We choose and fix an arbitrary Riemannian metric h of M. Then the n-dimensional integral varifold $v(M)$ in M is defined. Thus $f_\# v(M)$ is an n-dimensional integral varifold in N for $f \in \Xi$. The weight $\|f_\# v(M)\|$ is a Radon measure on N (cf. [1]). Then we have

$$f_\# v(M)(G_n(N)) = \|f_\# v(M)\|(N) = \mathrm{Volume}(M, f^*g) \tag{2.5}$$

for $f \in \Xi$. By Lemma 2.1 and (2.5), we have

$$b = \inf\{f_\# v(M)(G_n(N)): f \in \Xi\} \geqq (a/C_0)^n\,.$$

Since the set $\{\mu: \mu(G_n(N)) \leqq c\}$ of Radon measures on $G_n(N)$ is compact for any $c>0$, there exist a sequence $\{f_i\} \subset \Xi$ and an n-dimensional varifold V_0 in N such that $\lim_{i\to\infty} f_{i\#} v(M) = V_0$ and $\lim_{i\to\infty} f_{i\#} v(M)(G_n(N)) = b$. By Tychonoff's theorem (see also

[1, 2.6(2)]), we have $V_0(G_n(N))=b$. Thus we have the following:

LEMMA 2.2. *Put* $b=\inf\{f_{\#}v(M)(G_n(N)) : f\in \Xi\}$. *Then we have* $b>0$. *Moreover, there exist a sequence* $\{f_i\}\subset \Xi$ *and an n-dimensional varifold* V_0 *in* N *such that*

(1) $\lim_{i\to\infty} f_{i\#}v(M)=V_0$,

(2) $V_0(G_n(N))=\lim_{i\to\infty}f_{i\#}v(M)(G_n(N))=\lim_{i\to\infty} \|f_{i\#}v(M)\|(N)=\| V_0\|(N)=b$.

LEMMA 2.3. *The varifold* V_0 *is stationary.*

PROOF. We take a C^∞-isotopic deformation $h\colon \dot{R}\times N\to N$ with $h(0,p)=p$ for $p\in N$, and put $h_t(p)=h(t,p)$. Then we have $h_tf\in \Xi$ for $f\in \Xi$ and $t\in R$. Furthermore, by $\lim_{i\to\infty}f_{i\#}v(M)=V_0$, we have $\lim_{i\to\infty}(h_tf_i)_{\#}v(M)=h_{t\#}V_0$. Therefore we have

$$b\leqq \lim_{i\to\infty}(h_tf_i)_{\#}v(M)(G_n(N))=h_{t\#}V_0(G_n(N)).$$

q.e.d.

By Lemma 2.3 and Allard [1, 4.4, 5.1, 5.5], the varifold V_0 has the following properties (2.6), (2.7) and (2.8):

(2.6) There exists a real number $M(\geqq 0)$ such that $r^{-n}\| V_0\|\, B(p,r)\exp(Mr)$ is non-decreasing in $0<r<s_0$ for each $p\in N$, where the number s_0 is defined in (1.1). In particular, the n-dimensional density $\Theta^n(\| V_0\|,p)$ exists at each point $p\in N$, i.e.,

$$\Theta^n(\| V_0\|,p)=\lim_{r\to 0}\alpha(n)^{-1}r^{-n}\| V_0\|\, B(p,r)\in \boldsymbol{R}.$$

There exists a constant C_1 such that

$$\Theta^n(\| V_0\|,p)\leqq C_1 \qquad \text{for} \quad p\in N.$$

(2.7) $\Theta^n(\| V_0\|,p)$ is upper semi-continuous for $p\in N$. In particular, $\Theta^n(\| V_0\|,p)>0$ for $p\in \operatorname{spt}\| V_0\|$. Moreover, we have

$$\| V_0\|(u)=\int_N u(p)\Theta^n(\| V_0\|,p)d\mathscr{H}^n p \qquad \text{for} \quad u\in \mathscr{K}(N).$$

(2.8) V_0 is an n-dimensional rectifiable varifold in N.

For $0<r\leqq s_0$, $\tilde{\mu}_{1/r\#}(V_0 \llcorner \pi^{-1}B(p,r))$ is an n-dimensional varifold in $T_p(N)$ such that the support of $\|\tilde{\mu}_{1/r\#}(V_0 \llcorner \pi^{-1}B(p,r))\|$ is in $B_p(0,1)$. We have,

(2.9) $$\big|\, \|\tilde{\mu}_{1/r\#}(V_0 \llcorner \pi^{-1}B(p,r))\|\, B_p(0,1)-r^{-n}\| V_0\|\, B(p,r)\big|<o(r), \qquad \text{for} \quad 0<r\leqq s_0.$$

For each $p\in N$, there exist a sequence $\{t_k\}$ of real positive numbers, $t_k\to 0$ $(k\to\infty)$, and an n-dimensional varifold $C(p)$ in $T_p(N)$ with the following property:

(2.10)

(1) $\operatorname{spt}\| C(p)\|\subset B_p(0,1)$.

(2) $\lim_{k\to\infty}\tilde{\mu}_{1/t_k\#}(V_0 \llcorner \pi^{-1}B(p,t_k))=C(p)$.

(3) $\|\tilde{\mu}_{1/t_k\#}(V_0 \llcorner \pi^{-1}B(p,t_k))\|\, \partial B_p(0,1)=0$.

(4) $\| C(p)\|\, B_p(0,r)=\Theta^n(\| V_0\|,p)\alpha(n)r^n$.

(see Allard [1, 3.4]). Then, there exists a subsequence $\{f_{i(k)}\}$ of $\{f_i\}$ such that

(5) $\lim_{k\to\infty} \tilde{\mu}_{1/t_k\#}(f_{i(k)\#}v(M) \llcorner \pi^{-1}B(p, t_k)) = C(p)$,

(6) $\lim_{k\to\infty} \|\tilde{\mu}_{1/t_k\#}(f_{i(k)\#}v(M) \llcorner \pi^{-1}B(p, t_k))\| B_p(0, 1) = \| C(p) \| B_p(0, 1)$,

since $\lim_{i\to\infty} \tilde{\mu}_{1/t_k\#}(f_{i\#}v(M) \llcorner \pi^{-1}B(p, t_k)) = \tilde{\mu}_{1/t_k\#}(V_0 \llcorner \pi^{-1}B(p, t_k))$ for each k (cf. [1, 2.6 (2)]). Furthermore, since V_0 is stationary, we have the following:

(2.11) $C(p)$ is stationary. Namely, if h: $R\times T_p(N) \to T_p(N)$ is a C^∞-isotopic deformation with $h_t(q) = h(t, q) = q$ for $(t, q) \in R \times (T_p(N) \sim U_p(0, 1))$ and $h(0, q) = q$, then we have $\| C(p) \| B_p(0, 1) \leqq \| h_{t\#}C(p) \| B_p(0, 1)$.

By (2.10), (4) and (2.11), $C(p)$ has the properties in [1, 5.2, (2) Theorem]. In particular, $\| C(p) \|$ is a cone with vertex $0 \in T_p(N)$ (cf. [1, 5.2, (2) Theorem, (a) and (b)]).

(C) We consider the following condition on $f \in \Xi$:

(2.12) There exists no pair $\{D_1, D_2\}$ of $\mathscr{H}^n$-measurable sets in M such that $D_1 \cap D_2 = \varnothing$, $f_\# v(D_1) = f_\# v(D_2)$, and $\| f_\# v(D_1) \| (N) \neq 0$.

If f satisfies (2.12), the weight $\| f_\# v(M) \|$ of $f_\# v(M)$ coincides with the variation measure $\| f_\# M \|$ of the n-dimensional integral current $f_\# M$ on N, i.e.,

$$\| f_\# v(M) \| = \| f_\# M \| \; . \tag{2.13}$$

By a slight modification of f_i given in Lemma 2.2, we may assume that each f_i satisfies (2.12). Thus, we can replace the sequence $\{f_{i\#}v(M)\}$ by the sequence $\{f_{i\#}M\}$ of n-dimensional integral currents as far as the weight $\| f_\# v(M) \|$ is concerned. But note that $\lim_{i\to\infty} f_{i\#}M = T$ does not imply $\| T \| (N) = \| V_0 \| (N)$. For an integral current T on N (resp. $T_p(N)$), put $M(T) = \| T \| (N)$ (resp. $\| T \| (T_p(N))$).

3. Normalization of area-minimizing sequence. Let $\{f_i\} \subset \Xi$ be the sequence given in Lemma 2.2. Let s_0 be the number given in (1.1). In this section, we fix $p \in N$.

LEMMA 3.1. *We fix r with $0 < r \leqq s_0$. We can modify f_i to get F_i satisfying the following conditions*:

(1) $F_i \in \Xi$, $\| F_{i\#}v(M) \| (N) \leqq \| f_{i\#}v(M) \| (N)$.

(2) *Put*

$$F_i^{-1}U(r) = \textstyle\sum_k X_i^k \qquad (\textit{countable sum}),$$

where $F_i^{-1}U(r)$ is the inverse image of $U(r)$ by F_i and each X_i^k is a connected component of $F_i^{-1}U(r)$. Then each X_i^k is homeomorphic to $D^n = \boldsymbol{R}^n \cap \{x : |x| < 1\}$.

PROOF. Let $\Xi' = \{f: M \to N : f$ is a Lipschitz map and homotopic to $f_0\}$. Ξ is dense in Ξ'. Thus it suffices to construct a Lipschitz map F_i from f_i satisfying the properties (1) and (2).

We put

$$f_i^{-1}U(r)=\textstyle\sum_k W_i^k \qquad \text{(countable sum)}\,, \tag{3.1}$$

where each W_i^k is a connected component of $f_i^{-1}U(r)$. The proof is divided into several steps.

(1) Suppose that there exist W_i^k and non-contractible loop γ in M such that $\gamma\subset W_i^k$. Then, $f_i(\gamma)$ is contractible in N by $f_i(\gamma)\subset U(r)$. This is a contradiction to our assumption (0.2). Therefore, each W_i^k is contained in a coordinate neighborhood in M by our assumption (0.1).

(2) Suppose that there exists W_i^k which is represented as $W_i^k=A\sim B$ for a connected open set A and a closed set B in M such that $A\supset B$. For simplicity, we denote f_i and W_i^k by f and W from now on. A is contained in a coordinate neighborhood in M. Let $(a^1,\cdots,a^{n+1};u)$ be a polar coordinate of $U(2s_0)$. We define a map $\beta\colon N\to B(r)$ by

$$\beta(q)=\begin{cases} p & \text{for} \quad q\in N\sim U(2r)\\ q & \text{for} \quad q\in U(r)\end{cases}$$

and $\beta[(a^1(q),\cdots,a^{n+1}(q)\colon u(q))]=\beta[(a^1(q),\cdots,a^{n+1}(q);\,2r-u(q))]$ for $q\in U(2r)\sim U(r)$. Then we define $\tilde F\colon M\to N$ by

$$\tilde F(x)=\begin{cases} \beta f(x) & \text{for} \quad x\in A\\ f(x) & \text{for} \quad x\in M\sim A.\end{cases}$$

We have $\tilde F(A)\subset B(r)$. Since no open set in $\partial B(r)$ is area-minimal in N, we can slightly modify this $\tilde F$ to get F with $F(A)\subset U(r)$ and $\|F_{i\#}v(M)\|(N)\leqq\|f_{i\#}v(M)\|(N)$. Furthermore, by $n\geqq 2$, we easily see $\tilde F\in\Xi$. Thus we may assume $\pi_{n-1}(W_i^k,*)=\{1\}$ in (3.1). In particular, in case $n=2$, we may assume that W_i^k is homeomorphic to D^2.

(3) Suppose that $n=3$ and there exists W_i^k such that $\pi_1(W_i^k,*)\neq\{1\}$. Then we will construct $F_i\in\Xi$ from f_i in such a way that $X_i^k\supset W_i^k$, $\pi_1(X_i^k,*)=\{1\}$, and $\|F_{i\#}v(M)\|(N)\leqq\|f_{i\#}v(M)\|(N)$.

For brevity, we denote W_i^k and f_i by W and f from now on. A typical case is shown in Figure 1:

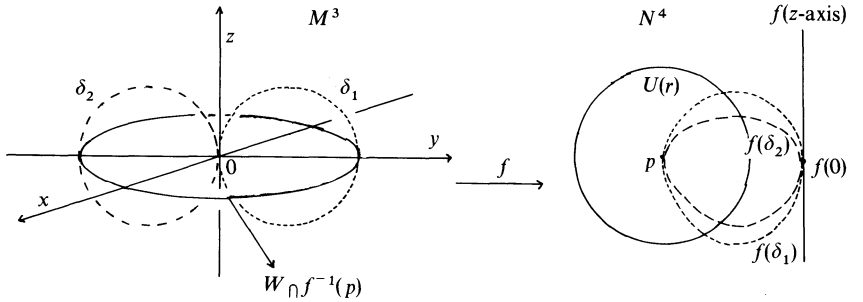

FIGURE 1

In Figure 1, $W \cap f^{-1}(p)$ is a circle in the xy-plane. W is a tubular neighborhood of this circle. δ_i $(i=1, 2)$ is a circle in the yz-plane through the origin and a point of $W \cap f^{-1}(p)$.

We consider the following cone Y in $\boldsymbol{R}^3$:

$$Y=\{(r\boldsymbol{a},\ t(2-r)) :\ -1 \leqq t \leqq 1,\ 0 \leqq r \leqq 2, \text{ and } \boldsymbol{a} \in \boldsymbol{R}^2 \text{ with } |\boldsymbol{a}|=1\}.$$

Then, $\partial Y=\{(r\boldsymbol{a},\ \pm(2-r)) : 0 \leqq r \leqq 2$, and $\boldsymbol{a} \in \boldsymbol{R}^2$ with $|\boldsymbol{a}|=1\}$.

Put $Y_0=\{(r\boldsymbol{a}, 0) : 0 \leqq r \leqq 2$, and $\boldsymbol{a} \in \boldsymbol{R}^2$ with $|\boldsymbol{a}|=1\}$.

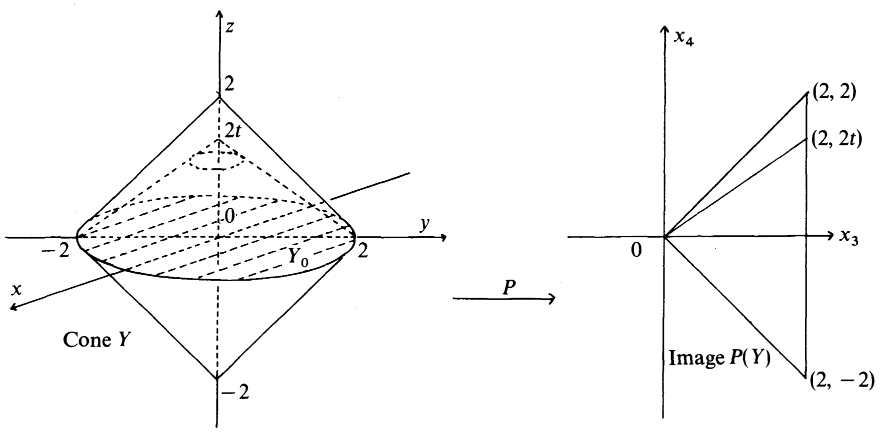

FIGURE 2

Then, by the identification of $\{(2\boldsymbol{a}, 0) : |\boldsymbol{a}|=1\} (\subset Y)$ with $W \cap f^{-1}(p)$ in Figure 1, we can regard Y to be contained in M. We will show that we can get a desirable F by a modification of f on $Y \sim \partial Y$.

We define a map $P: Y \to \boldsymbol{R}^4$ by

$$P[(r\boldsymbol{a},\ t(2-r))]=(2-r)(r\boldsymbol{a},\ 1,\ t)\,.$$

Then we have $P[(2\boldsymbol{a}, 0)]=(0, 0, 0)$ and $P[(0, 2t)]=(0, 2, 2t)$. Furthermore, P is injective for $r \neq 2$. Therefore, f: $Y \to N$ defines $G=fP^{-1}$: $P(Y) \to N$. Take a C^∞-function $v(s, t)$ on $[0, 1] \times [-1, 1]$ with $0 \leqq v(s, t) \leqq 1$, $v(1, t)=1$, $v(s, \pm 1)=1$ and $v(0, 0)=0$. Put

$$h_s[(2-r)(r\boldsymbol{a},\ 1,\ t)]=G[(2-r)v(s,\ t)(\{2-(2-r)v(s,\ t)\}\boldsymbol{a},\ 1,\ t)]\,.$$

Then, h_s maps $P(Y)$ into M and satisfies

$$h_1=G\,,\quad h_s(P(Y)) \subset G(P(Y)) \qquad \text{for}\quad s \in [0, 1],$$
$$h_s[(2-r)(r\boldsymbol{a},\ 1,\ \pm 1)]=G[(2-r)(r\boldsymbol{a},\ 1,\ \pm 1)] \qquad \text{for}\quad s \in [0, 1],$$
$$h_0[(2-r)(r\boldsymbol{a},\ 1, 0)]=p \qquad \text{for}\quad r \in [0, 2] \text{ and } \boldsymbol{a} \in \boldsymbol{R}^2 \text{ with } |\boldsymbol{a}|=1.$$

By $P[(r\boldsymbol{a},\ \pm(2-r))]=(2-r)(r\boldsymbol{a},\ 1,\ \pm 1)$ and $P[(r\boldsymbol{a}, 0)]=(2-r)(r\boldsymbol{a}, 1, 0)$, $\{h_s\}_{s \in I}$ defines a homotopy map $\{H_s\}_{s \in I}$, H_s: $Y \to N$, such that $H_1(q)=f(q)$ for $q \in Y$, $H_s(q)=f(q)$ for $q \in \partial Y$ and $s \in [0, 1]$, and $H_0(q)=p$ for $q \in Y_0$. Therefore, we take $F=H_0$, and then F is a desirable map on Y.

(4) In the same way as in (3), for the case $n \geqq 4$, we can construct F_i from f_i so that $\pi_j(X_i^k, *)=\{1\}$ $(j=1, 2, \cdots, n-2)$. q.e.d.

Let $\{t_k\}$ be the sequence of real numbers with the property (2.10). Then, we can further assume

$$s_0 \geqq t_1\,, \quad t_k \geqq 2t_{k+1} > 0 \qquad (k=1, 2, \cdots). \tag{3.2}$$

The following lemma will be used in §4.

LEMMA 3.2. *There exists a sequence $\{F_k\} \subset \Xi$ satisfying the following properties:*

(1) *$\{F_k\}$ satisfies Lemma 2.2 (with f_i replaced by F_k).*

(2) $\| F_{k\#}v(M) \| = \| F_{k\#}M \|$.

(3) *There exist constants C_2, C_3 and $r_k \in (t_k/2, t_k]$ for each k so that the following hold:*

(A) $M[\tilde{\mu}_{1/r_k\#}(F_{k\#}M \llcorner B(r_k))] \leqq C_2$ *for* $k=1, 2, \cdots$.

(B) $M[\partial\tilde{\mu}_{1/r_k\#}(F_{k\#}M \llcorner B(r_k))] \leqq C_3$ *for* $k=1, 2, \cdots$.

(C) *Put $F_k^{-1}U(r_k)=\sum_h X_k^h$ (countable sum), where each X_k^h is a connected component of $F_k^{-1}U(r_k)$. Then each X_k^h is homeomorphic to D^n.*

REMARK. (1) We will show in §4 that the sum on the right hand side of $F_k^{-1}U(r_k)$ in (3) (C) is *essentially* finite.

(2) By (3) (C), the rectifiable current $\tilde{\mu}_{1/r_k\#}(F_{k\#}M \llcorner B(r_k))$ is an integral current for each k (cf. [3, 4.2.16. (2)]). The sequence $\{(\tilde{\mu}_{1/r_k}F_k)_\# \sum_h v(\partial X_k^h)\}_{k=1}^{\infty}$ of $(n-1)$-dimensional integral varifolds in N has a convergent subsequence, because of

$$M[\partial\tilde{\mu}_{1/r_k\#}(F_{k\#}M \llcorner B(r_k))] = \|(\tilde{\mu}_{1/r_k}F_k)_\# \textstyle\sum_h v(\partial X_k^h)\| T_p(N)$$

and (3) (B) (see §2 (C)). Furthermore, there exists a sequence $\{P_k\}$ of integral polyhedral chains (cf. [3, 4.2.20 and 21]) such that, for $\varepsilon > 0$,

$$\begin{aligned}
&\operatorname{spt} P_k \subset U(0, 1+\varepsilon) \sim B(0, 1-\varepsilon)\,,\\
&\mathscr{F}(P_k - \partial\tilde{\mu}_{1/r_k\#}(F_{k\#}M \llcorner B(r_k))) \leqq \varepsilon/k\,,\\
&N(P_k) \leqq M[\partial\tilde{\mu}_{1/r_k\#}(F_{k\#}M \llcorner B(r_k))] + \varepsilon/k\,.
\end{aligned}$$

Therefore, we approximately replace $\partial\tilde{\mu}_{1/r_k\#}(F_{k\#}M \llcorner B(r_k))$ by P_k (or $\rho_\#(P_k)$, where $\rho\colon T_p(N) \sim \{0\} \to \partial B(0, 1)$ is the canonical projection). Then, we may assume, for each k and h, that the mapping $\tilde{\mu}_{1/r_k}F_k\colon \partial X_k^h \to \partial B(0, 1)$ is a Lipschitz map $G_k\colon \partial D^n \to \partial B(0, 1)$.

(3) By $1 \leqq t_k/r_k \leqq 2$ for $k=1, 2, \cdots$ in Lemma 3.2 and $\lim_{k\to\infty} \tilde{\mu}_{1/t_k\#}(V_0 \llcorner \pi^{-1}B(t_k)) = C(p)$ in (2.10), we have also $\lim_{k\to\infty} \tilde{\mu}_{1/r_k\#}(V_0 \llcorner \pi^{-1}B(r_k)) = C(p)$. So, taking a subsequence of $\{F_k\}$ if necessary, we also have

$\lim_{k\to\infty} \tilde{\mu}_{1/r_k\#}(F_{k\#}v(M) \llcorner \pi^{-1}B(r_k)) = C(p)$,

$\lim_{k\to\infty} \|\tilde{\mu}_{1/r_k\#}(F_{k\#}v(M) \llcorner \pi^{-1}B(r_k))\| B(0, 1) = \|C(p)\| B(0, 1) = \alpha(n)\Theta^n(\|V_0\|, p)$,

in the same way as in (2.10), (5) and (6).

PROOF OF LEMMA 3.2. We already showed (2) in §2, (C). Let $\{f_{i(k)}\}$ be the sequence given in (2.10), (5) and (6) for the point p. For brevity, we denote $\{f_{i(k)}\}$ by $\{f_k\}$ from now on. Since (3) (C) is shown by Lemma 3.1, we have only to fix C_2, C_3 and $r_k \in (t_k/2, t_k]$ for each k. Since we may assume $\|f_{k\#}v(M)\| = \|f_{k\#}M\|$, we have $\lim_{k\to\infty} M[\tilde{\mu}_{1/t_k\#}(f_{k\#}M \llcorner B(t_k))] = \alpha(n)\Theta^n(\|V_0\|, p)$ by (2.10). We take $C_2 = \alpha(n)\Theta^n(\|V_0\|, p) + 1$. Then we have $M[\tilde{\mu}_{1/t_k\#}(f_{k\#}M \llcorner B(t_k))] \leqq C_2$ $(k = 1, 2, \cdots)$.

Since $M[f_{k\#}M \llcorner B(t)]$ is non-decreasing in t, it is differentiable for $\mathscr{L}^1$-almost all t. Let

$$A_k = (t_k/2, t_k] \cap \{t : dM[f_{k\#}M \llcorner B(t)]/dt \text{ exists}\}.$$

We have, for $t \in A_k$,

$$M[\partial\tilde{\mu}_{1/t\#}(f_{k\#}M \llcorner B(t))] \leqq dM[\tilde{\mu}_{1/t\#}(f_{k\#}M \llcorner B(s))]/ds|_{s=t}, \tag{3.3}$$

$$\begin{aligned} d[\tilde{\mu}_{1/t\#}(f_{k\#}M \llcorner B(t))]/dt = &-nt^{-1}M[\tilde{\mu}_{1/t\#}(f_{k\#}M \llcorner B(t))] \\ &+ t^{-1}dM[\tilde{\mu}_{1/t\#}(f_{k\#}M \llcorner B(s))]/ds|_{s=t}. \end{aligned} \tag{3.4}$$

Put

$$D_k(t, u) = M[\tilde{\mu}_{1/t\#}(f_{k\#}M \llcorner B(t))] - M[\tilde{\mu}_{1/u\#}(f_{k\#}M \llcorner B(u))],$$
$$\delta_k(t) = \liminf_{u\to t-} D_k(t, u)/(t-u).$$

We have

$$\liminf_{u\to t-} D_k(t, u) \geqq 0. \tag{3.5}$$

Put $B_k = (t_k/2, t_k] \cap \{t : \delta_k(t) \leqq 0\}$.

(1) The case $A_k \cap B_k = \varnothing$. Then, $\delta_k(t) \geqq 0$ for $t \in A_k$. By (3.4) and (3.5) there exists $r_k \in (t_k/2, t_k]$ such that

$$\begin{aligned} 2^{-1}t_k d[\tilde{\mu}_{1/r_k\#}(f_{k\#}M \llcorner B(s))]/ds|_{t=r_k} \leqq\ & t_k\{M[\tilde{\mu}_{1/t_k\#}(f_{k\#}M \llcorner B(t_k))] \\ &- M[\tilde{\mu}_{2/t_k\#}(f_{k\#}M \llcorner B(t_k/2))]\} + 2^{-1}nt_k M[\tilde{\mu}_{1/t_k\#}(f_{k\#}M \llcorner B(t_k))]. \end{aligned} \tag{3.6}$$

By (3.3) and (3.6), we have

$$M[\partial\tilde{\mu}_{1/r_k\#}(f_{k\#}M \llcorner B(r_k))] \leqq (n+2)M[\tilde{\mu}_{1/t_k\#}(f_{k\#}M \llcorner B(t_k))].$$

Furthermore, by $\delta_k(t) \geqq 0$ and (3.5) we have

$$M[\tilde{\mu}_{1/r_k\#}(f_{k\#}M \llcorner B(r_k))] \leqq M[\tilde{\mu}_{1/t_k\#}(f_{k\#}M \llcorner B(t_k))].$$

Therefore, we can take $C_3 = (n+2)C_2$ in this case.

(2) The case $A_k \cap B_k \neq \varnothing$. Put $s = \sup A_k \cap B_k$. If $s \in A_k \cap B_k$, then we have

$$M[\partial \tilde{\mu}_{1/s\#}(f_{k\#}M \llcorner B(s))] \leqq nM[\tilde{\mu}_{1/s\#}(f_{k\#}M \llcorner B(s))]\,, \tag{3.7}$$

by (3.3) and (3.4). By $\delta_k(t) \geqq 0$ for $t \in A_k \cap (s, t_k]$ and (3.5), we have

$$M[\tilde{\mu}_{1/s\#}(f_{k\#}M \llcorner B(s))] \leqq M[\tilde{\mu}_{1/t_k\#}(f_{k\#}M \llcorner B(t_k))]\,. \tag{3.8}$$

In this case, we can take $r_k = s$ and $C_3 = nC_2$.

If $s \notin A_k$ or $t \notin B_k$, we take $r_k \in A_k \cap B_k \cap (s, t_k]$ sufficiently close to s. For this r_k, we can show (3.7) and (3.8) (with s replaced by r_k). q.e.d.

Now, we may further assume the following in Lemma 3.2:

(1) $\lim_{k \to \infty} \|F_{k\#}v(M) \llcorner \pi^{-1}B(r_1)\| B(r_1) = \|V_0 \llcorner \pi^{-1}B(r_1)\| B(r_1)$.

(2) $M[\partial F_{k\#}M \llcorner B(r_1)] \leqq C_4$ for $k = 1, 2, \cdots$.

(3) Put $F_k^{-1}U(r_1) = \sum_h Z_k^h$ (countable sum), where each Z_k^h is a connected component of $F_k^{-1}U(r_1)$. Then each Z_k^h is homeomorphic to D^n.

The reason is the same as in (2.10), (5) and (6), Lemma 3.2 and the above remark (3).

4. **Regularity.**

In this section, we will prove the following theorems.

THEOREM 1. *We have*

(1) $\|C(p)\| B_p(0, 1) \geqq \alpha(n)$ *for* $p \in \operatorname{spt}\|V_0\|$,

(2) $\mathscr{H}^m(\operatorname{sing}\|C(p)\| \sim \partial B_p(0, 1)) = 0$ *for* $p \in \operatorname{spt}\|V_0\|$ *and* $m > n-7$.

THEOREM 2. *We have*

$$\mathscr{H}^m(\operatorname{sing}\|V_0\|) = 0 \qquad \textit{for} \quad m > n-7.$$

THEOREM 3. V_0 *is an n-dimensional integral varifold.*

We fix $p \in \operatorname{spt}\|V_0\|$. By (2.7) we have $\Theta^n(\|V_0\|, p) > 0$. Here we use Lemma 3.2 and the remark after it. We take an element $X_k^{h(k)}$ of $\{X_k^h\}$ for each k, where $\{X_k^h\}$ is as given in Lemma 3.2, (3) (C). By Lemma 3.2, (3) and $\Theta^n(\|V_0\|, p) > 0$, we may assume that the sequence

$$\{(\tilde{\mu}_{1/r_k}F_k)_\# v(X_k^{h(k)})\}_{k=1}^{\infty} \tag{4.1}$$

has a subsequence, which converges to a non-zero varifold. For simplicity, we write the subsequence in the same notation as in (4.1). Thus, there exists a varifold Y in $T_p(N)$ such that $\operatorname{spt}\|Y\| \subset B(0, 1)$ and

$$\begin{aligned} &\lim_{k \to \infty} (\tilde{\mu}_{1/r_k}F_k)_\# v(X_k^{h(k)}) = Y \\ &\lim_{k \to \infty} \|(\tilde{\mu}_{1/r_k}F_k)_\# v(X_k^{h(k)})\| B(0, 1) = \|Y\| B(0, 1)\,. \end{aligned} \tag{4.2}$$

$\|Y\|$ is a cone and Y is stationary under any isotopic deformation h of $T_p(N)$ with $h(t, q) = q$ for $(t, q) \in R \times (T_p(N) \sim U(0, 1))$ and $h(0, q) = q$, by (2.11).

Furthermore, by $\|V_0\|(N)=\inf\{\|f_{\#}v(M)\|(N): f\in \Xi\}$, we have:

(4.3) For each k, take a Lipschitz mapping $G_k\colon \bar{X}_k^{h(k)}\to B(0, 1)$ *such that* $G_k(q)=\tilde{\mu}_{1/r_k}F_k(q)$ for $q\in\partial X_k^{h(k)}$ and $G_k(X_k^{h(k)})\subset U(0, 1)$. Then, we have

$$\liminf_{k\to\infty}\|G_{k\#}v(X_k^{h(k)})\|B(0, 1)\geqq\|Y\|B(0, 1)\,.$$

LEMMA 4.1. *We have* $\|Y\|B(0, 1)\geqq\alpha(n)$.

PROOF. We prove this by induction on n. Let $D^k=\boldsymbol{R}^k\cap\{x\colon |x|<1\}$. We reformulate the conditions on Y as follows: There exists a sequence $\{F_k^{(n)}\}_{k=1}^{\infty}$, $F_k^{(n)}\colon \bar{D}^n\to\bar{D}^{n+1}$, such that the following hold:

(4.4)
(1) $F_k^{(n)}$ is a Lipschitz map satisfying $F_k^{(n)}(\partial D^n)\subset\partial D^{n+1}$ and $F_k^{(n)}(D^n)\subset D^{n+1}$.
(2) $\lim_{k\to\infty}F_k^{(n)}{}_{\#}v(D^n)=Y^{(n)}$.
(3) $0<\lim_{k\to\infty}\|F_k^{(n)}{}_{\#}v(D^n)\|\boldsymbol{R}^{n+1}=\|Y^{(n)}\|\boldsymbol{R}^{n+1}<\infty$.
(4) $\|Y^{(n)}\|$ is a cone with vertex 0.
(5) For each k, take a Lipschitz mapping $G_k\colon\bar{D}^n\to\bar{D}^{n+1}$ such that $G_k(q)=F_k^{(n)}(q)$ for $q\in\partial D^n$ and $G_k(D^n)\subset D^{n+1}$. Then, we have $\liminf_{k\to\infty}\|G_{k\#}v(D^n)\|\boldsymbol{R}^{n+1}\geqq\|Y^{(n)}\|\boldsymbol{R}^{n+1}$.

Under this condition (4.4), we must prove $\|Y^{(n)}\|\boldsymbol{R}^{n+1}\geqq\alpha(n)$.

If $n=1$, then $\|Y^{(1)}\|$ is an $\mathscr{L}^1$-measure on a line in D^2 through 0. Thus, we have the assertion in this case. We assume that the assertion is true for $n\leqq m-1$. Let $n=m$. We take a vector $\boldsymbol{a}$ such that $\boldsymbol{a}\in\operatorname{spt}\|Y^{(m)}\|\cap\{x\colon |x|=1/2\}$. We defined $\mu_r\colon\boldsymbol{R}^{m+1}\to\boldsymbol{R}^{m+1}$ by $\mu_r(x)=r(x-\boldsymbol{a})$. Let $U(\boldsymbol{a}, r)=\boldsymbol{R}^{m+1}\cap\{x\colon |x-\boldsymbol{a}|<r\}$. Applying Lemma 3.2 and the remark after it (with F_k and M replaced by $F_k^{(m)}$ and D^m, respectively), we have a sequence $\{r_k\}$, $r_k>0$ and $r_k\to 0$, so that the following hold:

(4.5)
(1) $\lim_{k\to\infty}\|\mu_{1/r_k\#}(F_k^{(m)}{}_{\#}v(D^m)\llcorner\pi^{-1}U(\boldsymbol{a}, r_k))\|\boldsymbol{R}^{m+1}=\alpha(m)\Theta^n(\|Y^{(m)}\|, \boldsymbol{a})\,.$
(2) Let $F_k^{(m)-1}U(\boldsymbol{a}, r_k)=\sum W_k^j$, where W_k^j is a connected component of $F_k^{(m)-1}U(\boldsymbol{a}, r_k)$. Then each W_k^j is homeomorphic to D^m.

Thus, in the same way as in the argument before this lemma, we may assume that there exist a sequence $\{W_k^{j(k)}\}$ and a varifold $D(\neq 0)$ satisfying the following:

(3) $\lim_{k\to\infty}(\mu_{1/r_k}F_k^{(m)})_{\#}v(W_k^{j(k)})=D\,.$
(4) $\lim_{k\to\infty}\|(\mu_{1/r_k}F_k^{(m)})_{\#}v(W_k^{j(k)})\|\boldsymbol{R}^{m+1}=\|D\|\boldsymbol{R}^{m+1}$.
(5) $\|D\|$ is a cone with vertex 0.
(6) For each k, take a Lipschitz mapping $G_k\colon\bar{W}_k^{j(k)}\to\bar{D}^{m+1}$ such that $G_k(q)=\mu_{1/r_k}F_k^{(m)}(q)$ for $q\in\partial W_k^{j(k)}$ and $G_k(W_k^{j(k)})\subset D^{m+1}$. Then we have

$$\liminf_{k\to\infty}\|G_{k\#}v(W_k^{j(k)})\|\boldsymbol{R}^{m+1}\geqq\|D\|\boldsymbol{R}^{m+1}\,.$$

Furthermore, we have by (4.4), (4):

(7) $\|D\|$ is a cylinder with direction $\boldsymbol{a}/|\boldsymbol{a}|$. (We say that measure μ on $\boldsymbol{R}^{m+1}$ is a

cylinder with direction $\boldsymbol{c}$ if $\mu(A+t\boldsymbol{c})=\mu(A)$ for $A\subset\boldsymbol{R}^{m+1}$ and $t\in\boldsymbol{R}$.)

Therefore, we may assume that each $\mu_{1/r_k}F_k^{(m)}\colon \bar{W}_k^{j(k)}\to\bar{D}^{m+1}$ is also a cylinder map. Thus, by the induction assumption used in the same way as in [3, Proof of 5.4.15], we have $\|D\|\boldsymbol{R}^{m+1}\geqq\alpha(m)$. By (2.7) (with V_0 replaced by $Y^{(m)}$), we have $\|Y^{(m)}\|\boldsymbol{R}^{m+1}\geqq\|D\|\boldsymbol{R}^{m+1}$. q.e.d.

If necessary, we take a subsequence of $\{F_k\}$, and rearrange $\{X_k^h\}$ for each k. Then we have the following:

LEMMA 4.2. *There exists an integer $N(p)$ with the following properties for some $\varepsilon>0$:*

(1) *$F_k^{-1}U(r_k)$ is represented as*

$$F_k^{-1}U(r_k)=\sum_{h\in H}X_k^h+\sum_{h\in H'}X_k^h,$$

where the cardinality of H is $N(p)$,

$$\|(\tilde{\mu}_{1/r_k}F_k)_\sharp v(X_k^h)\|B(0,1)\geqq\alpha(n)/2 \qquad \text{for} \quad h\in H$$

and $\sum_{h\in H'}\|(\tilde{\mu}_{1/r_k}F_k)_\sharp v(X_k^h)\|B(0,1)<\varepsilon/k$.

(2) *$F_k^{-1}U(r_1)$ is represented as*

$$F_k^{-1}U(r_1)=\sum_{h\in H}Z_k^h+\sum_{h\in H'}Z_k^h$$

where $\|F_{k\sharp}v(X_k^h)\|B(r_1)\geqq r_1^n\alpha(n)/2$ *for* $h\in H$ *and* $\sum_{h\in H'}\|F_{k\sharp}v(X_k^h)\|B(r_1)<\varepsilon r_1^n\alpha(n)/k$.

Furthermore, there exists an integer N such that $N(p)\leqq N$ for each $p\in\operatorname{spt}\|V_0\|$.

PROOF. This lemma follows from (2.6), (2.10), the remark after the proof of Lemma 3.2, and Lemma 4.1. In particular, if necessary, we replace r_1 by a sufficiently smaller number. q.e.d.

We will call $\sum_{h\in H}X_k^h$ (resp. $\sum_{h\in H}Z_k^h$) the essential part of $F_k^{-1}U(r_k)$ (resp. $F_k^{-1}U(r_1)$). Thus, we have essentially the following conditions:

(4.6)
(1) $F_k^{-1}U(r_1)=\sum_{h\in H}Z_k^h$ and $F_k^{-1}U(r_k)=\sum_{h\in H}X_k^h$, where $Z_k^h\supset X_k^h$, and Z_k^h and X_k^h are homeomorphic to D^n.
(2) $\lim_{k\to\infty}F_{k\sharp}v(Z_k^h)=W^h$, $\lim_{k\to\infty}(\tilde{\mu}_{1/r_k}F_k)_\sharp v(X_k^h)=Y^h$, $\lim_{k\to\infty}\|F_{k\sharp}v(Z_k^h)\|B(r_1)=\|W^h\|B(r_1)$, $\lim_{k\to\infty}\|(\tilde{\mu}_{1/r_k}F_k)_\sharp v(X_k^h)\|B(0,1)=\|Y^h\|B(0,1)$.
(3) $\|Y^h\|B(0,1)\geqq\alpha(n)$ for $h\in H$.
(4) $\sum_{h\in H}\|W^h\|=\|V_0 \llcorner \pi^{-1}U(r_1)\|$, $\sum_{h\in H}\|Y^h\|=\|C(p)\|$.

LEMMA 4.3. *There exists a number $\Upsilon(>1)$ with the following property: If $\|Y^h\|B(0,1)<\Upsilon\alpha(n)$, then there exists an open neighborhood U around p such that $\operatorname{spt}\|W^h\|\cap U$ is a C^∞-hypersurface in N.*

Furthermore, in this case we have $\| Y^h \| = \| \boldsymbol{R}^n \llcorner U(0, 1)\|$, *where* $\|\boldsymbol{R}^n \llcorner U(0, 1)\|$ *is the* $\mathscr{L}^n$*-measure on a hyperplane through* 0 *of* $T_p(N)$ *restricted to* $U(0, 1)$.

PROOF. If $\| Y^h\| B(0, 1) < C\alpha(n)$, we have $\| W^h\| U(r) < C\alpha(n) r^n \exp(Mr)$ for $0 < r < r_1$, by $\Theta^n(\| W^h\|, p) = \alpha(n)^{-1} \| Y^h\| B(0, 1)$ and (2.6). And we have $\Theta^n(\| W^h\|, q) \geqq 1$ for $q \in \operatorname{spt}\| W^h\| \cap U(r_1)$ by Lemma 4.1. Furthermore, W^h is stationary. Thus, by Allard's regularity theorem, there exists a number $\Upsilon(>1)$ with the following property:
If $\| Y^h\| B(0, 1) < \Upsilon\alpha(n)$, there exists an open neighborhood U around p such that $\operatorname{spt}\| W^h\| \cap U$ is a C^1-hypersurface in N.

The C^∞-differentiability of $\operatorname{spt}\| W^h\| \cap U$ follows from Schoen-Simon-Almgren [9, Lemma 2.3]. Since Y^h is a tangent varifold of W^h at p, $\| Y^h\| = \| \boldsymbol{R}^n \llcorner U(0, 1)\|$ holds in this case. q.e.d.

By Lemma 4.3 and (2.8), if $\| Y^h\| B(0, 1) < \Upsilon\alpha(n)$, then there exists a smooth hypersurface S imbedded in N such that $W^h \llcorner \pi^{-1}(U) = v(S)$. Therefore, if $p \in \operatorname{spt}\| V_0\|$ is a singular point, there exists Y^h in (4.6) such that $\| Y^h\| B(0, 1) \geqq \Upsilon\alpha(n)$. The following lemma can be proved in the same way as Federer [4, Lemma 2]. Therefore, the regularity around p depends only on one of $\operatorname{spt}\| Y^h\| \sim \partial B(0, 1)$.

LEMMA 4.4. *If* $\Theta^{*k}[\phi^k_\infty \llcorner \operatorname{sing}(\| W^h\| \sim \partial B(r_1)), p] > 0$, *then there exists a sequence* $\{r_k\}_{k=2}^\infty$, *which satisfies* (4.6) *and* $\mathscr{H}^k(\operatorname{sing}\| Y^h\| \sim \partial B(0, 1)) > 0$.

LEMMA 4.5. *We have* $\mathscr{H}^m(\operatorname{sing}\| Y^h\| \sim \partial B(0, 1)) = 0$ *for* $m > n-7$.

PROOF. Let $n=2$. We take a vector $\boldsymbol{a} \in \operatorname{spt}\| Y^h\| \cap \{x : |x| = 1/2\}$. We denote $(\tilde{\mu}_{1/r_k} F_k)_\# v(X^h_k)$ by $F^{(2)}_{k\#} v(D^2)$. Then, in the same way as in the proof of Lemma 4.1, there exists a sequence $\{r_k\}$, $r_k \to 0$ $(k \to \infty)$ satisfying (4.5) (with $F^{(m)}_k$ replaced by $F^{(2)}_k$). Thus $\operatorname{spt}\| Y^h\| \cap \partial B(0, 1/2)$ is a smooth closed immersed curve in $\partial B(0, 1/2)$ by Lemma 4.3 [see also the proof of Lemma 4.1]. Then, by the non-existence theorem of branch points (cf. [7], [8]) we have $\| Y^h\| = \| \boldsymbol{R}^2 \llcorner U(0, 1)\|$. So, we are done in the case $n=2$. Let $n=3$. Then, first of all, $\operatorname{spt}\| Y^h\| \cap \partial B(0, 1/2)$ is a smooth closed surface immersed in $\partial B(0, 1/2)$ by the result for $n=2$. Furthermore, $\operatorname{spt}\| Y^h\| \cap \partial B(0, 1/2)$ is totally geodesic by (4.3) and Simons [10]. Then we have $\| Y^h\| B(0, 1) = k\alpha(3)$ for an integer $k > 0$. Next, from the non-existence of branch points in the case $n=2$ and the simple connectedness of S^2, we have $\| Y^h\| B(0, 1) = \alpha(3)$. Thus, we have $\| Y^h\| = \| \boldsymbol{R}^3 \llcorner U(0, 1)\|$, and we are done in this case.

By induction on n, we have $\| Y^h\| = \| \boldsymbol{R}^n \llcorner U(0, 1)\|$ for $n \leqq 6$ by Simons [10], and $\mathscr{H}^k(\operatorname{sing}\| Y^h\| \sim \partial B(0, 1)) = 0$ for $k > n-7$ (cf. [4, Proof of Theorem 1]). q.e.d.

We have completed the proofs of Therems 1, 2 and 3.

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