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MULTIPLIERS AND TRANSLATION INVARIANT OPERATORS

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Abstract. This paper characterizes various spaces of translation operators and multipliers on Banach-valued function spaces. A relationship between invariant operators and multiplier operators is established. It explores a necessary and sufficient condition for each invariant operator to be a multiplier.

1. Introduction and preliminaries. Throughout we let G be a locally compact abelian group with Haar measure dt, A a commutative Banach algebra, Y and X Banach spaces. Denote by $L^1(G, A)$ the space of all Bochner integrable A-valued functions defined on G. It is a commutative Banach algebra under convolution. $L^p(G, X)$ is the space of all X-valued measurable functions defined on G whose X-norms are in usual L^p space. It is a Banach space for each p, $1 \leqq p \leqq \infty$.

A bounded linear operator T from a Banach function space $E(G, Y)$ to another $F(G, X)$ is *invariant* if T commutes with the translation operators τ_a $(a \in G)$. Throughout this paper, the space of all invariant operators from $E(G, Y)$ to $F(G, X)$ is denoted by

$$(E(G, Y), F(G, X)) .$$

Our purpose in this paper is to characterize the space of invariant operators under some appropriate conditions.

If X and Y are A-modules, it is known (see Rieffel [12] and also Lai [7], [8]) that

$$\mathrm{Hom}_A(X, Y^*) \cong (X \otimes_A Y)^* , \tag{1.1}$$

in which a linear operator $T \in \mathrm{Hom}_A(X, Y^*)$ corresponding to a continuous linear functional ψ on $X \otimes_A Y$ is given by

$$(Tx)(y) = \psi(x \otimes y) \qquad \text{for all} \quad x \in X , \quad y \in Y .$$

Here $\mathrm{Hom}_A(X, Y^*)$ is the space of all A-module homomorphisms from X to Y^*, the topological dual space of Y, that is, each $T \in \mathrm{Hom}_A(X, Y^*)$ satisfies

$$T(ax) = aT(x) \qquad \text{for all} \quad a \in A , \quad x \in X ,$$

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where T is a continuous linear operator from X to Y^*; $X \otimes_A Y$ denotes the A-module tensor product space of X and Y.

If X is a Banach A-module, then $L^p(G, X)$, $1 \leqq p \leqq \infty$, is a Banach $L^1(G, A)$-module. In [7] and [8], Lai has characterized various spaces of module homomorphisms for Banach-valued function spaces defined on a locally compact Abelian group G under certain appropriate conditions. The module homomorphism space is generally called the *multiplier space*. It is well-known that, in the scalar-valued function spaces over G, a bounded linear operator is a multiplier if and only if it is an invariant operator. For example,

$$\operatorname{Hom}_{L^1(G)}(L^1(G), L^1(G)) \cong (L^1(G), L^1(G)) \cong M(G), \tag{1.2}$$

where $M(G)$ denotes the space of bounded regular measures on G, that is, if T is a bounded linear operator on $L^1(G)$, then the following statements are equivalent:

(a) $T(f * g) = Tf * g = f * Tg$ for all $f, g \in L^1(G)$.

(b) $\tau_s T = T\tau_s$ for $s \in G$, where $\tau_s f(t) = f(ts^{-1}) = f(t-s)$.

(c) There is a unique measure $\mu \in M(G)$ such that

$$Tf = \mu * f \qquad \text{for any} \quad f \in L^1(G).$$

Moreover, it is also known that

$$\operatorname{Hom}_{L^1(G)}(L^1(G), F(G)) \cong (L^1(G), F(G)) \cong F(G), \tag{1.3}$$

where $F(G) = L^p(G)$ $(1 < p < \infty)$ or $C_0(G)$, the space of continuous functions on G vanishing at infinity, and the relationship between both sides of $\cong$ is given by the following equivalent statements: Let T be a bounded linear operator of $L^1(G)$ to $F(G)$. Then $\tau_s T = T\tau_s$ for all $s \in G$ if and only if there exists a function $g \in F(G)$ such that $Tf = f * g$ for all $f \in L^1(G)$. For all of these properties, one can consult Larsen [10]. However, in the Banach-valued function spaces, an invariant operator need not be a multiplier. In [13], Tewari, Dutta and Vaiya proved the following theorem.

THEOREM A ([13; Theorem 3]). *If* $\dim A > 1$ *and* A *has unit of norm* 1, *then there is a bounded linear invariant operator* T *of* $L^1(G, A)$ *such that*

$$T \notin \operatorname{Hom}_A(L^1(G, A), L^1(G, A)).$$

Using this result, they disprove Akinyele's results about the equivalence between the multiplier and invariant operator on $L^1(G, A)$. In [13], they proved that

$$\operatorname{Hom}_{L^1(G,A)}(L^1(G, A), L^1(G, A)) \cong M(G, A) \tag{1.4}$$

provided A has an identity of norm 1. This result is extended by Lai [8] as in the following theorems.

THEOREM B ([8; Theorem 9]). *If* A *has an identity of norm* 1 *and* X *is a Banach* A-*module, then the following statements are equivalent*:

(a) $T \in \mathrm{Hom}_{L^1(G,A)}(L^1(G, A), L^1(G, X))$.

(b) *There exists a unique* $\mu \in M(G, X)$ *such that*

$$Tf = f * \mu \qquad \textit{for all} \quad f \in L^1(G, A).$$

Moreover,

$$\mathrm{Hom}_{L^1(G,A)}(L^1(G, A), L^1(G, X)) \cong M(G, X). \tag{1.5}$$

Evidently, if $X = A$, (1.5) is reduced to (1.4).

THEOREM C ([8; Theorem 6]). *Let X be a Banach A-module and A have an identity of norm* 1. *If the topological dual and bidual spaces* X^* *and* X^{**} *of X have the Radon-Nikodym property in the wide sense with respect to G, then the following statements are equivalent*:

(a) $T \in \mathrm{Hom}_{L^1(G,A)}(L^1(G, A), L^p(G, X))$, $1 < p < \infty$.

(b) *There exists a unique* $g \in L^p(G, X)$ *such that*

$$Tf = f * g \qquad \textit{for all} \quad f \in L^1(G, A).$$

Moreover,

$$\mathrm{Hom}_{L^1(G,A)}(L^1(G, A), L^p(G, X)) \cong L^p(G, X), \qquad 1 < p < \infty. \tag{1.6}$$

Recently, Quek [11, Theorem 9] proved that if X has the Radon-Nikodym property in the wide sense, then the isometric isomorphism of (1.6) in Theorem C holds. Quek's result improves Theorem C, since if X is embedded as a closed subspace of X^{**} in the norm topology, Theorem 2 of Diestel and Uhl [1; p. 81] implies that X has the Radon-Nikodym property in the wide sense whenever X^{**} does.

As the remark in [13; p. 229] indicated, it would be interesting to characterize the set of all bounded linear invariant operators on various Banach-valued function spaces over G. In this paper, we shall characterize various spaces of translation operators and multipliers under some appropriate conditions. Moreover, we establish a relationship between invariant operators and multipliers, and reduce a necessary and sufficient condition for each invariant operator to be a multiplier. Finally, we summarize our results as follows. Throughout we let X and Y be Banach spaces and A a commutative Banach algebra. Then we have:

(i) A bounded linear operator from $L^1(G)$ to $F(G, X)$ $(= L^p(G, X)$ $(1 \leqq p \leqq \infty)$ or $C_0(G, X))$ is invariant if and only if it is a multiplier.

(ii) The space $(L^1(G, Y), L^p(G, X))$ $(1 < p < \infty)$ is isometrically isomorphic to $\mathscr{L}(Y, L^p(G, X))$, the space of bounded linear operators of Y to $L^p(G, X)$.

(iii) If $p = 1$ in (ii), then

$$(L^1(G, Y), L^1(G, X)) \cong \mathscr{L}(Y, M(G, X)),$$

where $M(G, X)$ is the space of X-valued bounded regular measures on G.

(iv) If $L^1(G, X)$ is an order-free $L^1(G, A)$-module, that is, $\varphi \in L^1(G, X)$ and $L^1(G, A) * \varphi = \{0\}$ imply $\varphi = 0$, then

$$\mathrm{Hom}_{L^1(G,A)}(L^1(G, A), L^1(G, X)) \cong \mathrm{Hom}_A(A, M(G, X)).$$

(v) If X and X^* have the Radon-Nikodym property in the wide sense and $L^p(G, X)$, $1 < p < \infty$, is an order-free $L^1(G, A)$-module, then

$$\mathrm{Hom}_{L^1(G,A)}(L^1(G, A), L^p(G, X)) \cong \mathrm{Hom}_A(A, L^p(G, X)).$$

(vi) As in Theorem A, we can find a bounded linear invariant operator T of $L^1(G, A)$ to $F(G, X)$ $(= L^p(G, X)$ $(1 \leqq p \leqq \infty)$ or $C_0(G, X))$ such that $T \notin \mathrm{Hom}_A(L^1(G, A), F(G, X))$ provided that A has a faithful algebra representation on X, that is, $a \in A$ and $aX = \{0\}$ imply $a = 0$. However, if X is an order-free Banach A-module, then

$$\mathrm{Hom}_{L^1(G,A)}(L^1(G, A), F(G, X)) \subset \mathrm{Hom}_A(L^1(G, A), F(G, X)).$$

(vii) Let A have unit of norm 1 and X be a unit linked Banach A-module. Then each invariant operator $T: L^1(G, A) \to F(G, X)$ is a multiplier if and only if $A = \boldsymbol{C}$, where $F(G, X) = L^p(G, X)$ $(1 \leqq p \leqq \infty)$ or $C_0(G, X)$.

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2. Invariant operators. We will prove first that if $A = \boldsymbol{C}$ in (1.5) and (1.6) then every invariant operator of $L^1(G)$ to $L^p(G, X)$, $1 \leqq p < \infty$, will be a multiplier. Actually we have the following theorem.

THEOREM 1. *Let X be a Banach space. A bounded linear operator T: $L^1(G) \to F(G, X)$ is an invariant operator if and only if it is a multiplier, where $F(G, X) = L^p(G, X)$ $(1 \leqq p \leqq \infty)$ or $C_0(G, X)$.*

To prove this theorem we need the following result which was informed by Seiji Watanabe.

LEMMA 1. *Let (S, B, μ) be a positive measure space (not necessarily finite), X a Banach space and x_1, x_2 two strongly measurable X-valued functions on S. Suppose that $x^*(x_1(s)) = x^*(x_2(s))$ μ-almost everywhere for each bounded linear functional x^* on X. Then $x_1(s) = x_2(s)$ μ-almost everywhere.*

PROOF OF THEOREM 1. Let T be an invariant operator from $L^1(G)$ to $F(G, X)$. For any $x^* \in X^*$, define a mapping T_{x^*}: $L^1(G) \to F(G) = F(G, \boldsymbol{C})$ by

$$T_{x^*}f = x^* \circ Tf \qquad \text{for all} \quad f \in L^1(G).$$

Then T_{x^*} becomes a bounded linear invariant operator. Indeed, T_{x^*} is clearly bounded and linear. Now let τ_s, $s \in G$, be a translation operator. We have

$$\tau_s(T_{x^*}f)(t) = T_{x^*}f(ts^{-1}) = x^* \circ Tf(ts^{-1}) = x^* \circ \tau_s(Tf)(t) = x^* \circ T(\tau_s f)(t) = T_{x^*}(\tau_s f)(t)$$

for all $f \in L^1(G)$ and $t \in G$, that is,

$$\tau_s T_{x^*} = T_{x^*}\tau_s .$$

This shows that T_{x^*} is invariant whenever T is.

By (1.3) and [10, Theorem 3.1.1], we see that the invariant operators and multipliers are equivalent in the case of scalar-valued function spaces. It follows that for any $x^* \in X^*$,

$$x^* \circ T(f * g) = T_{x^*}(f * g) = f * T_{x^*}g = f * (x^* \circ Tg) = x^* \circ (f * Tg)$$

for all $f, g \in L^1(G)$. Hence by Lemma 1, $T(f * g) = f * Tg$ for all $f, g \in F(G)$. Note that every function in $F(G, X)$ is strongly measurable. Hence

$$T \in \mathrm{Hom}_{L^1(G)}(L^1(G), F(G, X)) .$$

The "if part" of the theorem is trivial. Indeed, for $f, g \in L^1(G)$, $t \in G$, and a multiplier T,

$$(T\tau_t)(f * g) = T(f * \tau_t g) = T(\tau_t g * f) = (\tau_t g) * Tf = \tau_t(g * Tf) = \tau_t(T(g * f)) = (\tau_t T)(f * g) .$$

Note also that $L^1(G) * L^1(G) = L^1(G)$ by Cohen's factorization theorem. Therefore every multiplier is invariant. q.e.d.

Applying Theorem 1, we can establish the following theorem for invariant operators.

THEOREM 2. *Let X and Y be Banach spaces. Then the following two statements are equivalent*:

(i) $T \in (L^1(G, Y), L^1(G, X))$.

(ii) *There exists a unique $L \in \mathscr{L}(Y, M(G, X))$, a bounded linear operator of Y to $M(G, X)$, such that*

$$T(f \otimes y) = f * Ly \qquad \textit{for all} \quad f \in L^1(G) , \quad y \in Y .$$

Moreover,

$$(2.1) \qquad (L^1(G, Y), L^1(G, X)) \cong \mathscr{L}(Y, M(G, X)) .$$

PROOF. (i) $\Rightarrow$ (ii). Let $T \in (L^1(G, Y), L^1(G, X))$. For each $y \in Y$, we define T_y: $L^1(G) \to L^1(G, X)$ by

$$T_y f = T(fy) \qquad \text{for all} \quad f \in L^1(G) .$$

Evidently, T_y is translation invariant whenever T is, so that $T_y \in (L^1(G), L^1(G, X))$. Applying Theorem 1, we see that T_y is a multiplier, that is,

$$T_y \in \mathrm{Hom}_{L^1(G)}(L^1(G), L^1(G, X)).$$

It follows from Theorem B, by taking $A=\boldsymbol{C}$, that there exists a $\mu_y \in M(G, X)$ such that

$$T_y f = f * \mu_y \qquad \text{for all} \quad f \in L^1(G)$$

and $\|T_y\| = \|\mu_y\|$. Note that $\|T_y\| \leq \|y\|_Y \|T\|$. Thus the mapping $Y \to M(G, X)$, defined by $L: y \to \mu_y$, is bounded linear such that

$$T(fy) = f * L(y) \qquad \text{with} \quad \|L\| \leq \|T\|.$$

(ii) $\Rightarrow$ (i). Conversely, if $L \in \mathscr{L}(Y, M(G, X))$, we define a mapping T_L^1: $L^1(G) \times Y \to L^1(G, X)$ by

$$T_L^1(f, y) = f * L(y) \qquad \text{for all} \quad f \in L^1(G), \quad y \in Y.$$

Then T_L^1 is a bilinear continuous operator, and by the universal property of tensor product, there exists a linear map

$$T_L: L^1(G) \hat{\otimes}_\gamma Y = L^1(G, Y) \to L^1(G, X)$$

such that

$$T_L(f \otimes y) = f * L(y) \qquad \text{for all} \quad f \in L^1(G), \quad y \in Y$$

and satisfying $\|T_L\| \leqq \|L\|$. This T_L is translation invariant since

$$\tau_s T_L(f \otimes y) = \tau_s(f * L(y)) = \tau_s f * L(y) = T_L(\tau_s f y) = T_L \tau_s(fy) = T_L \tau_s(f \otimes y)$$

for all $s \in G$, $y \in Y$, $f \in L^1(G)$. Hence $T_L \in (L^1(G, Y), L^1(G, X))$. By the first paragraph in the proof, we obtain $\|T_L\| = \|L\|$.

Finally, the one-to-one correspondence between $(L^1(G, Y), L^1(G, X))$ and $\mathscr{L}(Y, M(G, X))$ is obvious. Therefore we obtain

$$(L^1(G, Y), L^1(G, X)) \cong \mathscr{L}(Y, M(G, X)). \qquad \text{q.e.d.}$$

According to Theorem C with $A=\boldsymbol{C}$ and Theorem 1, the invariant operators of $L^1(G, Y)$ to $L^p(G, X)$ for $1<p<\infty$ can be characterized as in the proof of Theorem 2.

THEOREM 3. *Let X and Y be Banach spaces. If X and X^* have the Radon-Nikodym property in the wide sense with respect to G, then the following two statements are equivalent*:

(i) $T \in (L^1(G, Y), L^p(G, X))$.

(ii) *There exists* $L \in \mathscr{L}(Y, L^p(G, X))$, $1<p<\infty$, *such that*

$$T(f \otimes y) = T(fy) = f * L(y) \qquad \textit{for all} \quad f \in L^1(G), \quad y \in Y.$$

Moreover,

$$(L^1(G, Y), L^p(G, X)) \cong \mathscr{L}(Y, L^p(G, X)).$$

REMARK 1. (i) Note that if $Y=\boldsymbol{C}$, then Theorems 2 and 3 reduce to Theorem 1.

(ii) If $Y=\boldsymbol{C}=X$, then the spaces in Theorems 2 and 3 coincide with the spaces of usual multipliers, that is,

$$(L^1(G), L^1(G)) \cong \mathrm{Hom}_{L^1(G)}(L^1(G), L^1(G)) \cong M(G),$$

$$(L^1(G), L^p(G)) \cong \mathrm{Hom}_{L^1(G)}(L^1(G), L^p(G)) \cong L^p(G).$$

3. Multipliers of vector-valued function spaces. Let $Y=A$ in Theorems 2 and 3 be a commutative Banach algebra. Then we have the following characterizations.

THEOREM 4. *Let A be a commutative Banach algebra (not necessarily with identity) and X a Banach A-module. Suppose that $L^1(G, X)$ is an order-free $L^1(G, A)$-module. Then*

$$\mathrm{Hom}_{L^1(G,A)}(L^1(G, A), L^1(G, X)) \cong \mathrm{Hom}_A(A, M(G, X)). \tag{3.1}$$

PROOF. Let $T \in \mathrm{Hom}_{L^1(G,A)}(L^1(G, A), L^1(G, X))$. Then for any $t \in G$ and $f, g \in L^1(G, A)$,

$$g*(T\tau_t)f = T(g*\tau_t(f)) = T(\tau_t(g)*f) = \tau_t(g)*Tf = \tau_t(g*Tf) = g*\tau_t(Tf),$$

and hence $L^1(G, A)*(T\tau_t - \tau_t T)f = \{0\}$ for all $t \in G$ and $f \in L^1(G, A)$. Since $L^1(G, X)$ is order-free, it follows that T is invariant, that is, $T \in (L^1(G, A), L^1(G, X))$. According to Theorem 2 with $Y=A$, there exists a unique $L \in \mathscr{L}(A, M(G, X))$ such that

(a) $T(fa) = f*L(a)$ for all $f \in L^1(G)$, $a \in A$.

Here $L(a) \in M(G, X)$ and $f*L(a)$ is an X-valued Bochner integrable function over G, since $L^1(G)$ acts on $M(G, X)$ under convolution and $f*L(a)$ vanishes on the singular part of $M(G, X)$. Hence it is an element of $L^1(G, X)$ and the relationship between T and L in (a) is well posed.

Moreover, for $f, g \in L^1(G)$, $a, b \in A$,

$$T(fa*gb) = T((f*g)ab) = (f*g)*L(ab)$$

and

$$T(fa*gb) = fa*T(gb) = (f*g)*aL(b).$$

Note also that $L^1(G)*L^1(G) = L^1(G)$ by Cohen's factorization theorem and that $M(G, X)$ is an order-free $L^1(G)$-module by Theorem B with $A=\boldsymbol{C}$. It follows that

$$L(ab) = aL(b) \qquad \text{for all} \quad a, b \in A.$$

This shows that L is an A-module homomorphism, that is,

$$L \in \mathrm{Hom}_A(A, M(G, X)).$$

Conversely, for $L \in \mathrm{Hom}_A(A, M(G, X))$, we define

(b) $T_L(fa) = f * L(a)$ for all $f \in L^1(G)$, $a \in A$.

Then $fa \in L^1(G, A)$ and T_L is a bounded linear mapping from $L^1(G, A)$ to $L^1(G, X)$, since $L^1(G) * M(G, X)$ is contained in the space $L^1(G, X)$. We show that T_L is an $L^1(G, A)$-module homomorphism. Indeed, for any $f, g \in L^1(G)$ and $a, b \in A$, we have

$$T_L(gb * fa) = T_L((g*f)ba) = (g*f) * L(ba) = (g*f) * bL(a) = gb * (f * L(a)) = gb * T_L(fa) .$$

Since $\{gb : b \in A, g \in L^1(G)\}$ is total in $L^1(G, A)$, it follows that T_L is an $L^1(G, A)$-module homomorphism.

It is easy to show $\|T\| = \|L\|$ for T and L in the relations (a) and (b). Therefore the isometric isomorphism of (3.1) is proved. q.e.d.

By the same argument as in Theorem 4, we have the following theorem.

THEOREM 5. *Let A be a commutative Banach algebra and X a Banach A-module. Suppose that X and X^* have the Radon-Nikodym property in the wide sense with respect to G and that $L^p(G, X)$ is an order-free $L^1(G, A)$-module. Then*

$$\text{(3.2)} \quad \mathrm{Hom}_{L^1(G,A)}(L^1(G, A), L^p(G, X)) \cong \mathrm{Hom}_A(A, L^p(G, X)) \qquad \textit{for} \quad 1 < p < \infty .$$

If A has unit of norm 1 and X is a unit linked Banach A-module, that is, $ex = x$ for all $x \in X$, where e is a unit of A, then $M(G, X)$ and $L^p(G, X)$ become unit linked A-modules. Thus $M(G, X)$ and $L^p(G, X)$ are isometrically isomorphic to $\mathrm{Hom}_A(A, M(G, X))$ and $\mathrm{Hom}_A(A, L^p(G, X))$, respectively.

Thus we have the following:

REMARK 2. If A has an identity of norm 1 and if X is unit linked in Theorems 4 and 5, then

(3.1) is isometrically isomorphic to $M(G, X)$, and

(3.2) is isometrically isomorphic to $L^p(G, X)$.

4. Necessary condition for an invariant operator to be a multiplier. This section gives a main characterization for an invariant operator to be a multiplier in Banach function spaces. Although a multiplier is an invariant operator, the converse is not true. For example, one can consult Theorem A. We will prove the following theorem.

THEOREM 6. *Let A be a commutative Banach algebra of dimension greater than one with an identity of norm* 1, *and let X be a unit linked Banach A-module such that the corresponding representation is faithful* (*i.e.*, $a \in A$ and $aX = \{0\} \Rightarrow a = 0$). *Then there exists a bounded linear invariant operator T of $L^1(G, A)$ to $F(G, X)$ such that*

$$T \notin \mathrm{Hom}_A(L^1(G, A), F(G, X)) ,$$

where $F(G, X) = L^p(G, X)$ $(1 \leqq p \leqq \infty)$ or $C_0(G, X)$.

PROOF. Since A has an identity, say e, there exists a nonzero multiplicative linear functional χ on A. Define $\psi: A\to A$ by $\psi(a)=\chi(a)e$ for all $a\in A$. Then ψ is a bounded linear operator on A. Since $\dim A>1$, we have $\{\chi(a)e: a\in A\}\subsetneqq A$. Thus there is an element $b\in A$ such that $\psi(b)\neq b$. For any $x\in X$ and $\phi\in F(G)=F(G, \boldsymbol{C})$, set

$$\mu_{x,\phi}(a)=\phi\psi(a)x\,,\qquad a\in A\,.$$

Then $\mu_{x,\phi}$ is a bounded linear operator of A to $F(G, X)$. It is easy to see that the mapping

$$L^1(G)\times A\ni(f, a)\to f*\mu_{x,\phi}(a)\in F(G, X)$$

is bounded linear, so it follows from the universal property of tensor product that there exists a bounded linear map $T_{x,\phi}: L^1(G)\hat{\otimes}_\gamma A\to F(G, X)$ such that $T_{x,\phi}(f\otimes a)=f*\mu_{x,\phi}(a)$ for all $f\in L^1(G)$ and $a\in A$. Let us identify $L^1(G, A)$ with $L^1(G)\hat{\otimes}_\gamma A$. Then $T_{x,\phi}\in(L^1(G, A), F(G, X))$. Indeed, let $t\in G$. For any $f\in L^1(G)$ and $a\in A$,

$$\tau_t(T_{x,\phi}(fa))=\tau_t(f*(\phi\psi(a)x))=(\tau_t f)*(\phi\psi(a)x)=T_{x,\phi}((\tau_t f)a)=T_{x,\phi}(\tau_t(fa))\,.$$

Since $\{fa: a\in A, f\in L^1(G)\}$ is total in $L^1(G, A)$, it follows that $T_{x,\phi}$ is invariant.

Suppose now that $T_{x,\phi}\in\mathrm{Hom}_A(L^1(G, A), F(G, X))$ for all $x\in X$ and $\phi\in F(G)$. Then

$$\begin{aligned}(f*\phi)(b-\psi(b))x&=(f*\phi)bx-(f*\phi)\psi(b)x=b((f*\phi)x)-f*(\phi\psi(b)x)\\&=b(f*(\phi\psi(e)x))-T_{x,\phi}(fb)=bT_{x,\phi}(fe)-T_{x,\phi}(fb)\\&=T_{x,\phi}(fb)-T_{x,\phi}(fb)=0\,,\end{aligned}$$

and hence

$$\|f*\phi\|_{F(G)}\|(b-\psi(b))x\|_X=0$$

for all $f\in L^1(G)$, $\phi\in F(G)$ and $x\in X$. However, note that $L^1(G)*F(G)\neq\{0\}$. Therefore $(b-\psi(b))x=0$ for all $x\in X$. Since the corresponding representation is faithful, it follows that $b-\psi(b)=0$, a contradiction. Thus $T_{x_0,\phi_0}\notin\mathrm{Hom}_A(L^1(G, A), F(G, X))$ for some $x_0\in X$ and $\phi_0\in F(G)$. Then $T=T_{x_0,\phi_0}$ is a desired operator. q.e.d.

REMARK 3. Note that

$$\mathrm{Hom}_A(L^1(G, A), F(G, X))\neq\mathrm{Hom}_{L^1(G,A)}(L^1(G, A), F(G, X))$$

in general. If $X=A$, $p=1$, then Theorem 6 is reduced to Theorem A. Also under the same condition as in Theorem 6, we can show, by the same method, that there is a bounded linear invariant operator T of $L^1(G, A)$ to $F(G, X)$ such that $T\notin\mathrm{Hom}_{L^1(G,A)}(L^1(G, A), F(G, X))$ where $F(G, X)=L^p(G, X)$ $(1\leqq p\leqq\infty)$ or $C_0(G, X)$.

However we have the following inclusion relation.

THEOREM 7. *Let A be a commutative Banach algebra with identity e of norm* 1 *and X an order-free Banach A-module. Then*

$$\mathrm{Hom}_{L^1(G,A)}(L^1(G, A), F(G, X))\subset\mathrm{Hom}_A(L^1(G, A), F(G, X))\,,$$

where $F(G, X) = L^p(G, X)$ $(1 \leqq p \leqq \infty)$ *or* $C_0(G, X)$.

PROOF. Let $T \in \mathrm{Hom}_{L^1(G,A)}(L^1(G, A), F(G, X))$. Fix $a \in A$ and $\phi \in L^1(G, A)$, and set $\xi = T(a\phi) - aT\phi$. We have only to show that $\xi = 0$. To do so, let f be any element of $L^1(G)$. Then

$$f * (e\xi) = (fe) * \xi = (fe) * T(a\phi) - (fa) * T\phi = T((fe) * (a\phi)) - T((fa) * \phi) = 0\,,$$

and hence for any $x^* \in X^*$, we have

$$f * (x^* \circ e\xi)(t) = \int_G f(t-s)x^*(e\xi(s))ds = x^* \left\{ \int_G f(t-s)e\xi(s)ds \right\} = x^*(f * (e\xi)(t)) = 0$$

almost everywhere. Since $L^1(G)$ is faithful, it follows that $x^* \circ e\xi = 0$ for all $x^* \in X^*$. By Lemma 1, $e\xi(t) = 0$ a.e. and hence $A\xi(t) = \{0\}$ a.e. Since X is order-free, it follows that $\xi(t) = 0$ a.e., that is, $\xi = 0$. q.e.d.

REMARK 4. If $X = A$, $p = 1$, then Theorem 7 is reduced to Corollary 5.2 in [13].

In view of Remark 3, we ask under what conditions

$$(L^1(G, A), F(G, X)) = \mathrm{Hom}_{L^1(G,A)}(L^1(G, A), F(G, X))\,.$$

The answer is that A must be isometrically isomorphic to the complex field $\boldsymbol{C}$.

THEOREM 8. *Let A be a commutative Banach algebra with identity of norm* 1, *X be a unit linked, order-free, Banach A-module and A a faithful representation on X. Then each invariant operator T: $L^1(G, A) \to F(G, X)$ is a multiplier if and only if $A \cong \boldsymbol{C}$, where $F(G, X) = L^p(G, X)$ $(1 \leqq p \leqq \infty)$ or $C_0(G, X)$.*

PROOF. The proof of this theorem follows immediately from Theorem 1 and Remark 3.

REFERENCES

[1] J. DIESTEL AND J. J. UHL, JR., Vector Measures, Math. Surveys 15, Amer. Math. Soc., 1977.

[2] N. DINCULEANU, Vector Measures, Pergamon, Oxford, 1967.

[3] N. DINCULEANU, Integration on Locally Compact Spaces, Noordhoff International Publishing, 1974.

[4] G. P. JOHNSON, Spaces of functions with values in a Banach algebra, Trans. Amer. Math. Soc. 92 (1959), 411–429.

[5] R. KHALIL, Multipliers for some spaces of vector-valued functions, J. Univ. Kuwait (Sci.) 8 (1981), 1–7.

[6] H. C. LAI, Multipliers of a Banach algebra in the second conjugate algebra as an idealizer, Tohoku Math. J. 26 (1974), 431–452.

[7] H. C. LAI, Multipliers for some spaces of Banach algebra valued functions, Rocky Mountain J. Math. 15 (1985), 157–166.

[8] H. C. LAI, Multipliers of Banach-valued function spaces, J. Austral. Math. Soc. 39 (Ser. A) (1985), 51–62.

[9] H. C. LAI, Duality of Banach function spaces and the Radon-Nikodym property, Acta Math. Hung. 47

(1–2) (1986), 45–52.

[10] R. LARSEN, An Introduction to the Theory of Multipliers, Springer-Verlag, New York, Heidelberg, 1971.

[11] T. S. QUEK, Multipliers of certain vector valued function spaces, preprint.

[12] M. A. RIEFFEL, Multipliers and tensor products on L^p-spaces of locally compact groups, Studia Math. 33 (1969), 71–82.

[13] U. TEWARI, M. DUTTA AND D. P. VAIDYA, Multipliers of group algebras of vector valued functions, Proc. Amer. Math. Soc. 81 (1981), 223–229.

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