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EULER FACTORS ATTACHED TO UNRAMIFIED PRINCIPAL SERIES REPRESENTATIONS

Dedicated to Professor Ichiro Satake on his sixtieth birthday

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Introduction. Let G be a connected reductive unramified group defined over a non-archimedean local field F, T a maximal torus in G and B a Borel subgroup of G containing T. We denote by $G(F)$, $B(F), \cdots$, the locally compact and totally disconnected groups of F-rational points of G, $B, \cdots$, respectively. Let $X_{\mathrm{reg}}(T)$ be the set of regular unramified characters of $T(F)$ and $I(\chi) = \mathrm{Ind}(G(F), B(F); \chi)$ the admissible representation of $G(F)$ induced by $\chi \in X_{\mathrm{reg}}(T)$. In this paper, we first give the irreducible decomposition of $I(\chi)$ and then construct an Eular factor attached to some irreducible component of $I(\chi)$. This study stems from the problems mentioned in [2, Section 12]. For $G = GL_n$ or GSp_4, the same subject was treated by Jacquet, Godement, Shalika, Piatetski-Shapiro and Rodier. In [22], Rodier investigated the case of split classical type groups G and gave a construction of Euler factors associated to the standard representation of the L-group ${}^LG$ of G. Our result in the first part is a generalization of these results of Rodier to the case of unramified groups G. In the second part, we give a complete classification of irreducible rational representations of ${}^LG$ and construct an Euler factor for a pair $(r, \rho(D_\chi))$ of an (almost arbitrary) irreducible representation r of ${}^LG$ and a certain constituent $\rho(D_\chi)$ of $I(\chi)$ for $\chi \in X_{\mathrm{reg}}(T)$. Furthermore, we give a precise relation between our Euler factors and those defined by Langlands.

Now we give a summary of this paper. Let S be the maximal F-split torus contained in T, $X^*(S)$ the character group of S and $W_G(S)$ the relative Weyl group of G. For $\chi \in X_{\mathrm{reg}}(T)$, the transform of χ by $w \in W_G(S)$ is denoted by χ^w. Let $V = X^*(S) \otimes \boldsymbol{R}$ and C^+ the Weyl chamber in V corresponding to B. First, for $\chi \in X_{\mathrm{reg}}(T)$, we define a subset $H(\chi)$ of the coroot system $\Psi^\vee(G, S)$ of G with respect to S (see Section 2). The set $H(\chi)$ plays an important role in the irreducible decomposition of $I(\chi)$. Since it is known that $I(\chi)$ is irreducible if and only if $H(\chi)$ is empty, we are interested in the case where $H(\chi)$ is not empty. Put

$$JH(\chi) = \{\text{constituents of } I(\chi)\}$$

and

$$C(\chi) = \{\text{connected components of } V - \bigcup_{a^\vee \in H(\chi)} \mathrm{Ker}(a^\vee)\} .$$

For $D \in C(\chi)$, we choose an element $w \in W_G(S)$ such that $w^{-1}C^+ \subset D$. Let $\rho(D)$ be the unique irreducible subrepresentation of $I(\chi^w)$. Then it is shown that $\rho(D)$ depends only on D and is contained in $JH(\chi)$. Hence one has a map $\rho: C(\chi) \to JH(\chi)$. Our first main result is the following:

THEOREM. *Let $\chi \in X_{\mathrm{reg}}(T)$.*

(1) *The map $\rho: C(\chi) \to JH(\chi)$ is bijective.*

(2) *Let $\langle H(\chi)\rangle$ be the set of coroots which are represented by an integral linear combination of elements of $H(\chi)$. Then $\langle H(\chi)\rangle$ is a root system and $H(\chi)$ is a basis of $\langle H(\chi)\rangle$. In particular, the elements of $H(\chi)$ are linearly independent. Thus, combining with* (1), *one sees that the length of a composition series of $I(\chi)$ is equal to $2^{|H(\chi)|}$ and $|H(\chi)|$ is bounded by the semisimple F-rank of G.*

(3) *Let $D_\chi = \bigcap_{a^\vee \in H(\chi)} (a^\vee)^{-1}(\boldsymbol{R}_+)$ and φ a non-degenerate character of $U(F)$, where U is the unipotent radical of B. Then, for $D \in C(\chi)$, $\rho(D)$ has a Whittaker model with respect to φ if and only if $D = D_\chi$.*

These are proved in Section 3. Let $\mathscr{W}\mathscr{H}(\chi, \varphi)$ denote the Whittaker model of $\rho(D_\chi)$. In the rest of this paper (from Section 4 to the end), we will assume that the characteristic of F is equal to zero. In Section 4, we give an explicit form of the restriction to $S(F)$ of a Whittaker function $f \in \mathscr{W}\mathscr{H}(\chi, \varphi)$. This is used for calculations of the "zeta integral" (see below). In order to define the "zeta integral", we need a classification of finite dimensional irreducible rational representations of the L-group ${}^LG$ of G. Let $\mathscr{R}({}^LG)$ be the set of equivalence classes of irreducible rational representations of ${}^LG$. In Section 5, we give a parametrization of elements of $\mathscr{R}({}^LG)$. In terms of this parametrization, we introduce the coweight ξ_r of S, $e(r) \in \boldsymbol{N}$ and $c(r) \in \boldsymbol{C}$ for each $r \in \mathscr{R}({}^LG)$ and define a subset $\mathscr{R}_+({}^LG)$ of $\mathscr{R}({}^LG)$ (see (5.6)). Now, for $f \in \mathscr{W}\mathscr{H}(\chi, \varphi)$ and $r \in \mathscr{R}({}^LG)$, the "zeta integral" is defined by

$$(\#) \qquad Z(s, r, f) = \int_{F^*} f(\xi_r(t))|t|_F^s \cdot \delta_B^{-1}(\xi_r(t))dt ,$$

where δ_B^2 is the modulus character of $B(F)$. Note that (#) coincides with the definition given by Jacquet—Langlands [13] when $G = GL_2$ and r is the natural representation of ${}^LG = GL_2(\boldsymbol{C})$. The second main result is the following:

THEOREM. (1) *Let* $r \in \mathscr{R}_+(^LG)$. *Then for any* $f \in \mathscr{W}\mathscr{H}(\chi, \varphi)$, *the zeta integral* $Z(s, r, f)$ *is absolutely convergent for* $\mathrm{Re}(s) \gg 0$.

(2) *Let* $P(r, \chi)$ *be the set of polynomials* $P(X) \in \boldsymbol{C}[X]$ *such that* $P(q_F^{-s})Z(s, r, f)$ *is an entire function of* s *for every* $f \in \mathscr{W}\mathscr{H}(\chi, \varphi)$, *where* q_F *is the cardinality of the residual field of* F. *Then, for any* $(r, \chi) \in \mathscr{R}_+(^LG) \times X_{\mathrm{reg}}(T)$, $P(r, \chi)$ *is a non-trivial principal ideal of* $\boldsymbol{C}[X]$ *and has the generator* $P_{r,\chi}(X) \in \boldsymbol{C}[X]$ *with* $P_{r,\chi}(0) = 1$.

These are proved in Section 6. The generator $P_{r,\chi}(X)$ of $P(r, \chi)$ is uniquely determined by the pair (r, χ) and is independent of the choice of φ. The Euler factor attached to (r, χ) is defined to be $L(s, r, \chi) = P_{r,\chi}(q_F^{-s})^{-1}$.

Finally, for $r \in \mathscr{R}_+(^LG)$ and $\chi \in X_{\mathrm{reg}}(T)$, let $L(s, r, Sp(\chi))$ be the Euler factor defined by Langlands (see [2]). In Section 7, we compare $L(s, r, Sp(\chi))$ with $L(s, r, \chi)$. Then we have the following:

THEOREM. *For any* $(r, \chi) \in \mathscr{R}_+(^LG) \times X_{\mathrm{reg}}(T)$, $L(e(r)(s-c(r)), r, \chi)^{-1}$ *is a factor of* $L(s, r, Sp(\chi))^{-1}$ *as a polynomial in* q_F^{-s}.

With more conditions on (r, χ), one has $L(e(r)(s - c(r)), r, \chi) = L(s, r, Sp(\chi))$, but pairs satisfying these extra conditions are few and far between (see (7.3) and (7.4)).

ADDENDUM (December 22, 1987). I received from Professor F. Rodier the following paper:

V. A. Dũng, Décomposition de la série principale du sous-groupe des points k-rationnels d'un groupe algébrique affine réductif quasidéployé sur un coups k p-adique de caractéristique 0, Thèse, Université Paris VII (1985).

Dũng investigates the irreducible decompositions of principal series representations $\mathrm{Ind}(G(F), B(F); \chi)$ for any connected reductive quasi-split group G and any regular quasicharacter χ. Therefore, when F is of characteristic zero, our Theorems (3.2) and (3.3) are special cases of Dũng's results. Further, by Chapter VI, Propositions 2 and 3 of Dũng's paper, we know the irreducible constituents of $I(\chi)$ which are square integrable or tempered.

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NOTATION. Let F be a non-archimedean local field. Let G be a connected reductive algebraic group defined over F. Throughout this paper, we assume G is unramified, that is, G is quasi-split over F and split over an unramified extension of F. Let E be the minimal splitting field of G. We fix the notation as follows:

$|\ |_F$ (resp. $|\ |_E$) = the normalized absolute value of F (resp. E)

ϖ_F = a prime element of F

q_F (resp. q_E) = the cardinality of the residue field of F (resp. E)

$\mathcal{O}_F$ = the maximal compact subring of F

$\mathcal{P}_F$ = the maximal ideal of $\mathcal{O}_F$

S = a maximal F-split torus in G defined over F

T = the centralizer of S in G, (which is a maximal torus of G defined over F)

B = a Borel subgroup of G defined over F containing T

U = the unipotent radical of B

$X^*(T)$ (resp. $X^*(S)$) = the character group of T (resp. S)

$X_*(T)$ (resp. $X_*(S)$) = the cocharacter group of T (resp. S)

$V = X^*(S) \otimes \boldsymbol{R}$

Φ = the relative root system of G with respect to S

Φ_+ = the set of positive roots of Φ with respect to B

Δ = the set of simple roots of Φ

C^+ = the Weyl chamber in V corresponding to B

$G_E = G \times E$

Φ_E = the absolute root system of G with respect to T

Δ_E = the set of simple roots of Φ_E with respect to B

$N_G(S)$ (resp. $N_G(T)$) = the normalizer of S (resp. T) in G

$W_G(S)$ (resp. $W_G(T)$) = the relative (resp. absolute) Weyl group of G.

For each subset $\theta \subset \Delta$, let

P_θ = the standard parabolic subgroup corresponding to θ,

M_θ = the Levi subgroup of P_θ containing T,

and

U_θ = the unipotent radical of P_θ.

We denote by $G(F)$, $B(F), \cdots$, the locally compact and totally disconnected groups consisting of F-rational points of G, $B, \cdots$. Let δ_θ^2 be the modulus character of $P_\theta(F)$. We write δ_B instead of $\delta_\varnothing$. For each subfield L of E, let $\mathrm{Add}_L = \mathrm{Spec}(L([X])$ be the one dimensional additive group defined over L.

REMARK. In the first half of this paper (from Section 1 until Section 3), the characteristic of F is arbitrary. In the second half of this paper (from Section 4 on), the characteristic of F is assumed to be zero.

1. The structure of G. Let F be a non-archimedean local field and G a connected unramified reductive algebraic group defined over F. In this section, we summarize known facts on the structure of G using the terminology in [8].

1.1. A "root ray" of G with respect to S is an open half line with starting point 0 in V containing at least one root relative to S. Let $\Psi = \Psi(G, S)$ be the set of root rays of G with respect to S. For $a \in \Psi$, let $\sigma(a)$ (resp. $\tau(a)$) be the non-divisible (resp. non-multipliable) root contained in a. If $\sigma(a) \neq \tau(a)$, a is called "plural". Let $\Psi_0 = \{a \in \Psi \mid \sigma(a) \in \Delta\}$. For $a \in \Psi$, denote by $-a$ the root ray containing the root $-\sigma(a)$.

1.2. Now we assume G is split over F. Thus one has $E = F$ and $T = S$. For $\alpha \in \Phi$, let $U_\alpha^\sim$ be the root subgroup corresponding to α. For a subfield F_1 of F, an F_1-isomorphism $\mathrm{Add}_{F_1} \to U_\alpha^\sim$ is called an "F_1-épinglage" of $U_\alpha^\sim$. For any F_1-épinglage $\tilde{x}_\alpha$ of $U_\alpha^\sim$, there exists a unique F_1-épinglage $\tilde{x}_{-\alpha}$ of $U_{-\alpha}^\sim$ satisfying the following conditions:

(1.2.1) $m_\alpha = \tilde{x}_\alpha(1)\tilde{x}_{-\alpha}(1)\tilde{x}_\alpha(1) \in N_G(T)$.

(1.2.2) $m_{-\alpha} = \tilde{x}_{-\alpha}(1)\tilde{x}_\alpha(1)\tilde{x}_{-\alpha}(1) \in N_G(T)$.

(1.2.3) There exists an F_1-homomorphism ζ_α of SL_2 into G such that for any $u \in \mathrm{Add}_{F_1}$

$$\tilde{x}_\alpha(u) = \zeta_\alpha\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) \quad \text{and} \quad \tilde{x}_{-\alpha}(u) = \zeta_\alpha\left(\begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix}\right).$$

This $\tilde{x}_{-\alpha}$ is called the épinglage opposite to $\tilde{x}_\alpha$.

A Chevalley F-system of G (with respect to T) is a family $\{\tilde{x}_\alpha\}_{\alpha \in \Phi}$ of F-épinglages of $U_\alpha^\sim$ satisfying the following conditions:

(1.2.4) For any $\alpha \in \Phi$, the épinglages $\tilde{x}_\alpha$ and $\tilde{x}_{-\alpha}$ are opposite to each other.

(1.2.5) For $\alpha, \beta \in \Phi$, there exists $\varepsilon(\alpha, \beta) \in \{\pm 1\}$ such that $\tilde{x}_{w_\alpha(\beta)}(u) = m_\alpha \cdot \tilde{x}_\beta(\varepsilon(\alpha, \beta)\cdot u)\cdot m_\alpha^{-1}$ for any $u \in \mathrm{Add}_F$, where w_α is the element of $W_G(T)$ defined by α.

1.3. Returning to the general case, let G be quasi-split over F. For $\alpha \in \Phi_E$, let $\Gamma_\alpha = \{\gamma \in \mathrm{Gal}(E/F) \mid \gamma(\alpha) = \alpha\}$ and L_α the invariant field of Γ_α in E. L_α is called the field of definition of α. Also, let $U_\alpha^\sim$ be the root subgroup of G_E corresponding to α. Note that $U_\alpha^\sim$ is defined over L_α. Let $\tilde{x}_\alpha$: $\mathrm{Add}_{L_\alpha} \to U_\alpha^\sim$ be an L_α-épinglage of $U_\alpha^\sim$. Then for $\gamma \in \mathrm{Gal}(E/F)$, one has $\gamma(L_\alpha) = L_{\gamma(\alpha)}$ and $\gamma\circ\tilde{x}_\alpha\circ\gamma^{-1}$ is an $L_{\gamma(\alpha)}$-épinglage of $U_{\gamma(\alpha)}^\sim$. Thus, one can take a family $\{\tilde{x}_\alpha\}_{\alpha \in \Delta_E}$ of E-épinglages satisfying the following properties:

(1.3.1) $\tilde{x}_\alpha$: $\mathrm{Add}_E \to U_\alpha^\sim$ is induced from an L_α-épinglage for each $\alpha \in \Delta_E$.

(1.3.2) $\tilde{x}_{\gamma(\alpha)} = \gamma\circ\tilde{x}_\alpha\circ\gamma^{-1}$ for any $\alpha \in \Delta_E$ and $\gamma \in \mathrm{Gal}(E/F)$.

Furthermore, it is known that this family $\{\tilde{x}_\alpha\}_{\alpha\in\Delta_E}$ is extended to a Chevalley E-system $\{\tilde{x}_\alpha\}_{\alpha\in\Phi_E}$ of G_E satisfying the following:

(1.3.3) If the restriction $\alpha|_S$ of $\alpha\in\Phi_E$ to S is non-divisible, then $\tilde{x}_\alpha$: $\mathrm{Add}_E\to U_\alpha^\sim$ is induced from an L_α-épinglage and one has $\tilde{x}_{\gamma(\alpha)}=\gamma\circ\tilde{x}_\alpha\circ\gamma^{-1}$ for any $\gamma\in\mathrm{Gal}(E/F)$.

(1.3.4) If $\alpha|_S$ is divisible, then there exist distinct roots β and β' in Φ_E such that $\beta|_S=\beta'|_S=(\alpha|_S)/2$ and $\alpha=\beta+\beta'$. The field L_β is a separable quadratic extension of L_α and $L_{\beta'}=L_\beta$. For any $\gamma\in\mathrm{Gal}(E/F)$, there exists $\varepsilon(\gamma)\in\{\pm1\}$ such that $\gamma\circ\tilde{x}_\alpha\circ\gamma^{-1}(u)=\tilde{x}_{\gamma(\alpha)}(\varepsilon(\gamma)u)$. When $\gamma\in\mathrm{Gal}(E/L_\alpha)$, one has $\varepsilon(\gamma)=-1$ if and only if γ induces the unique non-trivial automorphism on L_β.

$\{\tilde{x}_\alpha\}_{\alpha\in\Phi_E}$ is called a Chevalley—Steinberg system of G .

1.4. For a root ray $a\in\Psi$, let U_a be the root subgroup corresponding to a ([8] (1.1.3)). We assume $a\in\Psi_0$. Put $\Delta_a=\{\alpha\in\Delta_E\mid\alpha|_S\in a\}$. Δ_a is a single orbit of $\mathrm{Gal}(E/F)$ in Δ_E. Let μ: $G^a\to\langle U_a,U_{-a}\rangle$ be the universal covering of the semisimple group generated by U_a and U_{-a}. Then G^a is a semisimple and simply connected group of F-rank one which is quasi-split over F and split over E. The torus $\mu^{-1}(T)$ (resp. $\mu^{-1}(S)$) is a maximal torus (resp. a maximal F-split torus) in G^a defined over E. The orbit Δ_a is a base of the root system of G^a with respect to $\mu^{-1}(T)$. From the classification of Dynkin diagrams, one has only the following two types of G^a.

Type I. G^a_E is isomorphic to the direct product of SL_2 indexed by Δ_a. L_α is the field of definition of the simple factor of an index α and one has $G^a\simeq R_{L_\alpha/F}(SL_2)$, where $R_{L_\alpha/F}$ is Weil's scalar restriction functor.

Type II. Put $I=\{(\alpha,\alpha')\in\Delta_a\times\Delta_a\mid\alpha+\alpha'\in\Phi_E\}$. Then G^a_E is isomorphic to the direct product of SL_3 indexed by I. For $(\alpha,\alpha')\in I$, one has $L_\alpha=L_{\alpha'}$ and L_α is a separable quadratic extension of $L_2=L_{\alpha+\alpha'}$. Let SU_3 be the special unitary group defined over L_2 by the hermitian form on L^3_α with degree three and Witt index one. Then the simple factor of an index (α,α') is L_2-isomorphic to SU_3. Further G^a is F-isomorphic to $R_{L_2/F}(SU_3)$. Note that G^a is of Type II if and only if a is plural.

1.5. Assume G^a is of Type I. In this case, one has $(U_a)_E=\prod_{\alpha\in\Delta_a}U_\alpha^\sim$ and U_a is F-isomorphic to $R_{L_\alpha/F}(U_\alpha^\sim)$. We fix $\alpha\in\Delta_a$ and put $L=L_\alpha$. If $\tilde{x}_\alpha$ is an L-épinglage of $U_\alpha^\sim$, then $x_a=R_{L/F}(\tilde{x}_\alpha)$ is an F-isomorphism of $R_{L/F}(\mathrm{Add}_L)$ onto U_a. Such a couple (L,x_a) is called an épinglage of U_a. The épinglage of U_{-a} opposite to (L,x_a) is defined to be $(L,R_{L/F}(\tilde{x}_{-\alpha}))$, where $\tilde{x}_{-\alpha}$ is the épinglage opposite to $\tilde{x}_\alpha$.

1.6. Assume G^a is of Type II. We use the notation of (1.4). We fix $(\alpha, \alpha') \in I$ and put $L = L_\alpha$, $L_2 = L_{\alpha+\alpha'}$. Recall that L is a separable quadratic extension of L_2. Since L is an unramified extension of F, L_2 is uniquely determined by L and F. Let ι be the non-trivial element of $\mathrm{Gal}(L/L_2)$. We choose an L-épinglage $\tilde{x}_\alpha$ of $U_{\tilde{\alpha}}$ and put $\tilde{x}_{\alpha'} = \iota \circ \tilde{x}_\alpha \circ \iota^{-1}$, $\tilde{x}_{\alpha+\alpha'} = \mathrm{Int}(m_{\alpha'}^{-1}) \circ \tilde{x}_\alpha$, where $m_{\alpha'}$ is the element of $N_G(T)$ defined in (1.2.1). Let $\tilde{x}_{-\alpha}$, $\tilde{x}_{-\alpha'}$ and $\tilde{x}_{-(\alpha+\alpha')}$ be the épinglages opposite to $\tilde{x}_\alpha$, $\tilde{x}_{\alpha'}$ and $\tilde{x}_{\alpha+\alpha'}$, respectively. Let

$$H_0^L(L_2) = \{(u, v) \in L \times L \mid v + \iota(v) = u \cdot \iota(u)\} .$$

For (u, v), $(u', v') \in H_0^L(L_2)$, the composition law is defined by

$$(u, v) \cdot (u', v') = (u + u', v + v' + \iota(u) \cdot u') .$$

Then $H_0^L(L_2)$ is regarded as the group consisting of L_2-rational points of a unipotent algebraic group H_0^L defined over L_2. Furthermore, there exists an L_2-isomorphism j_α of H_0^L onto $U_{\widetilde{(\alpha,\alpha')}} = U_{\tilde{\alpha}} \cdot U_{\widetilde{\alpha+\alpha'}} \cdot U_{\tilde{\alpha}'}$ such that

$$j_\alpha(u, v) = \tilde{x}_\alpha(u) \cdot \tilde{x}_{\alpha+\alpha'}(-v) \cdot \tilde{x}_{\alpha'}(\iota(u))$$

for any $(u, v) \in H_0^L(L_2)$. Since U_a equals $R_{L_2/F}(U_{\widetilde{(\alpha,\alpha')}})$, $x_a = R_{L_2/F}(j_\alpha)$ gives an F-isomorphism of $H^L = R_{L_2/F}(H_0^L)$ onto U_a. We call such a couple (L, x_a) an épinglage of U_a. The épinglage of U_{-a} opposite to (L, x_a) is defined the same way as in (1.5).

1.7. We fix a Chevalley—Steinberg system $\{\tilde{x}_\alpha\}_{\alpha \in \Phi_E}$ of G. For any $a \in \Psi$, we take a root $\alpha(a) \in \Phi_E$ whose restriction to S equals $\sigma(a)$. By (1.5) and (1.6), the épinglage $\tilde{x}_{\alpha(a)}$ of $U_{\widetilde{\alpha(a)}}$ induces the épinglage (L_a, x_a) of U_a, where $L_a = L_{\alpha(a)}$. Here, one can choose a set $\{\alpha(a)\}_{a \in \Psi}$ of roots such that the épinglage (L_{-a}, x_{-a}) of U_{-a} induced from $\tilde{x}_{\alpha(-a)}$ is that opposite to (L_a, x_a) for every $a \in \Psi$. Then the family $\{(L_a, x_a)\}_{a \in \Psi}$ of épinglages is called a coherent system of G induced from the Chevalley—Steinberg system $\{\tilde{x}_\alpha\}_{\alpha \in \Phi_E}$. This will be used in Section 4.

1.8. We consider the reduced root system $\tau(\Psi) = \{\tau(a) \mid a \in \Psi\}$. Let $\Psi^\vee$ be the coroot system attached to $\tau(\Psi)$. For $\tau(a) \in \tau(\Psi)$, denote by $a^\vee$ the coroot corresponding to $\tau(a)$. $a^\vee$ is an element of $X_*(S)$. Let $\langle\ ,\ \rangle : X^*(S) \times X_*(S) \to \boldsymbol{Z}$ be the perfect pairing, that is, for any $t \in F^*$, $\nu \in X^*(S)$ and $\xi \in X_*(S)$, $\nu(\xi(t)) = t^{\langle \nu, \xi \rangle}$. A coroot $a^\vee$ is considered as a linear functional on V by $v \mapsto \langle v, a^\vee \rangle$. Then by the definition, $C^+ = \bigcap_{a \in \Psi_0} (a^\vee)^{-1}(\boldsymbol{R}_+)$, where $\boldsymbol{R}_+$ is the set of positive real numbers.

2. Summary of the unramified principal series. In this section, we summarize known results on the unramified principal series of the unramified group G. For the general theory of admissible representations of p-adic groups, we refer the reader to Cartier [9].

2.1. Let H be a F-subgroup of G. For admissible representations π_1 and π_2 of $H(F)$, we denote by $\mathrm{Hom}_H(\pi_1, \pi_2)$ the space of $H(F)$-homomorphisms from π_1 into π_2.

2.2. Let P be a F-parabolic subgroup of G with a Levi-subgroup M and the unipotent radical N. Let δ_P be the square root of the modulus character of $P(F)$. Let (π, V_π) be an admissible representation of $M(F)$. Then π may be trivially extended to an admissible representation of $P(F) = M(F)N(F)$. The representation $I(G, P; \pi)$ of $G(F)$ induced by π is the right regular representation of $G(F)$ on the space of all locally constant functions $\phi: G(F) \to V_\pi$ such that $\phi(pg) = \delta_P(p)\pi(p)\phi(g)$ for any $p \in P(F)$ and $g \in G(F)$. The representation $I(G, P; \pi)$ is an admissible representation of $G(F)$.

2.3. We use the same notation as in (2.2). Let ψ be a character of $N(F)$. Put

$$Z_M(\psi) = \{m \in M(F) \mid \psi(m\cdot n\cdot m^{-1}) = \psi(n) \quad \text{for any} \quad n \in N(F)\} .$$

For an admissible representation (π, V_π) of $G(F)$, denote by $V_\pi^\psi(P, G)$ the subspace of V_π spanned by the vectors of the form $\pi(n)v - \psi(n)v$, $n \in N(F)$, $v \in V_\pi$. The quotient space $V_\pi / V_\pi^\psi(P, G)$ is called the ψ-localization of the space V_π and denoted by $V_\psi(P, G; \pi)$. Define the representation $(J_\psi(P, G; \pi),\ V_\psi(P, G; \pi))$ of $Z_M(\psi)$ by

$$J_\psi(P, G; \pi)(m)(v + V_\pi^\psi(P, G)) = \delta_P^{-1}(m)(\pi(m)v + V_\pi^\psi(P, G))$$

for $m \in Z_M(\psi)$ and $v \in V_\pi$. It is easily verified that $J_\psi(P, G; \pi)$ is well-defined. In particular, if ψ is trivial, then $J_\psi(P, G; \pi)$ is called the Jacquet representation (or Jacquet module) of π with respect to (P, M) and denoted by $J(P, G; \pi)$. The representation $J(P, G; \pi)$ of $M(F)$ is admissible.

2.4. By (2.2) and (2.3), $I(G, P; \cdot)$ (resp. $J(P, G; \cdot)$) defines a functor from the category of admissible representations of $M(F)$ (resp. $G(F)$) to that of $G(F)$ (resp. $M(F)$). Then the following facts are well-known.

(2.4.1) The functors $I(G, P; \cdot)$ and $J(P, G; \cdot)$ are exact.

(2.4.2) The functor $J(P, G; \cdot)$ is left adjoint to $I(G, P; \cdot)$. That is, for any admissible representation π (resp. σ) of $G(F)$ (resp. $M(F)$), there is a natural isomorphism

$$\mathrm{Hom}_M(J(P, G; \pi), \sigma) \simeq \mathrm{Hom}_G(\pi, I(G, P; \sigma))$$

(2.4.3) Let P' be an another F-parabolic subgroup of G with a Levi-subgroup M' and the unipotent radical N'. We assume $P' \subset P$, $M' \subset M$ and $N' \supset N$. Then one has

$$I(G, P; \cdot)\circ I(M, P'\cap M; \cdot) = I(G, P'; \cdot)$$

and

$$J(P'\cap M, M; \cdot)\circ J(P, G; \cdot) = J(P', G; \cdot)\,.$$

2.5. Next, we state Bernstein—Zelevinsky's Geometrical Lemma. This lemma plays an important role in Sections 3 and 4. Let θ_1 and θ_2 be two subsets of Δ. Put $P_i = P_{\theta_i}$, $M_i = M_{\theta_i}$ and $U_i = U_{\theta_i}$ for $i = 1, 2$. For $w \in W_G(S)$, let $^*w$ be a representative of w in $N_G(S)(F)$. Define

$$W(M_1, M_2) = \{w \in W_G(S) \mid \mathrm{Int}(^*w)(M_1\cap B)\subset B \quad \text{and} \quad \mathrm{Int}(^*w^{-1})(M_2\cap B)\subset B\}\,.$$

For $w \in W(M_1, M_2)$, note that $M_1\cap \mathrm{Int}(^*w^{-1})(M_2)$ (resp. $\mathrm{Int}(^*w)(M_1)\cap M_2$) is a Levi-subgroup of the standard F-parabolic subgroup $M_1\cap \mathrm{Int}(^*w^{-1})(P_2)$ (resp. $\mathrm{Int}(^*w)(P_1)\cap M_2$) of M_1 (resp. M_2). Now, for an admissible representation π of $M_1(F)$, let $F(\pi) = J(P_2, G; I(G, P_1; \pi))$ be the Jacquet module of $I(G, P_1; \pi)$ with respect to (P_2, M_2). Then Bernstein and Zelevinsky were proved the following lemma ([1]).

LEMMA (Geometrical Lemma). *There exists a numeration* $w_1, w_2, \cdots, w_k$ *of elements of* $W(M_1, M_2)$ *satisfying the following condition: for any admissible representation* π *of* $M_1(F)$, $F(\pi)$ *has a filtration* $0 = F_0\subset F_1\subset \cdots\subset F_k = F(\pi)$ *and a system of isomorphisms*

$$c_i\colon F_i/F_{i-1} \to I(M_2, \mathrm{Int}(^*w_i)(P_1)\cap M_2; J^{w_i}(M_1\cap \mathrm{Int}(^*w_i^{-1})(P_2), M_1; \pi))$$

functorially depending on π, *where* $J^{w_i}(M_1\cap \mathrm{Int}(^*w_i^{-1})(P_2), M_1; \pi)$ *is the admissible representation of* $\mathrm{Int}(^*w_i)(M_1)\cap M_2(F)$ *defined by* $J(M_1\cap \mathrm{Int}(^*w_i^{-1})(P_2), M_1; \pi)\circ\mathrm{Int}(^*w_i^{-1})$.

2.6. We recall fundamental results on unramified principal series representations. We start with the definition of these representations. Let T_0 (resp. S_0) be the maximal compact subgroup of $T(F)$ (resp. $S(F)$). Since G is unramified, the natural injection $S(F)\hookrightarrow T(F)$ gives rise to an isomorphism of $S(F)/S_0$ onto $T(F)T_0$. An element of $\mathrm{Hom}(T(F)/T_0, \boldsymbol{C}^*)$ is called an unramified character of $T(F)$. The relative Weyl group $W_G(S)$ acts on $\mathrm{Hom}(T(F)/T_0, \boldsymbol{C}^*)$ by $\chi^w(t) = \chi(^*w^{-1}\cdot t\cdot {}^*w)$, where $\chi \in \mathrm{Hom}(T(F)/T_0, \boldsymbol{C}^*)$, $t \in T(F)$ and $w \in W_G(S)$. We say χ is regular if $\chi^w \neq \chi$ for every element $w \neq 1$ of $W_G(S)$. Denote by $X_{\mathrm{reg}}(T)$ the set of unramified regular characters of $T(F)$. For $\chi \in \mathrm{Hom}(T(F)/T_0, \boldsymbol{C}^*)$, we consider the induced representation $I(G, B; \chi)$ of $G(F)$. For simplicity, we put $I(\chi) = I(G, B; \chi)$. This is called a representation of unramified principal series.

2.7. The following results are well known (see [9]). Let $\chi \in \mathrm{Hom}(T(F)/T_0, \boldsymbol{C}^*)$.

(2.7.1) $I(\chi)$ has a composition series.

(2.7.2) The contragredient representation of $I(\chi)$ is isomorphic to $I(\chi^{-1})$.

(2.7.3) For any admissible representation π of $G(F)$, one has $\mathrm{Hom}_G(\pi, I(\chi)) \simeq \mathrm{Hom}_T(J(B, G; \pi), \chi)$. (This is a special case of (2.4.2)).

(2.7.4) If χ is regular, then $J(B, G; I(\chi))$ is $T(F)$-isomorphic to $\bigoplus_{w \in W_G(S)} \chi^w$.

(2.7.5) If χ is regular, then $\mathrm{Hom}_G(I(\chi), I(\chi^w))$ is of dimension one for every $w \in W_G(S)$.

2.8. From now on, we treat $I(\chi)$ for $\chi \in X_{\mathrm{reg}}(T)$. Let $JH(\chi)$ be the set of constituents of $I(\chi)$. By (2.7.1), $JH(\chi)$ is finite. From (2.7.1), (2.7.4) and the exactness of the functor $J(B; G; \cdot)$, it follows that the multiplicity one theorem holds for $I(\chi)$. (Generally, this fails if χ is non-regular ([16], [17])). Thus, for $\chi \in X_{\mathrm{reg}}(T)$ one can identify $JH(\chi)$ with the set of equivalence classes of constituents of $I(\chi)$. The following fact is a special case of a result of Bernstein and Zelevinsky [1, Theorem (2.9) and Remark (2.10)].

(2.8.1) $JH(\chi) = JH(\chi^w)$ for every $w \in W_G(S)$.

2.9. LEMMA. *Let $\chi \in X_{\mathrm{reg}}(T)$ and π an irreducible admissible representation of $G(F)$. Then π is isomorphic to an irreducible subrepresentation of $I(\chi)$ if and only if $J(B, G; \pi)$ contains χ as a subrepresentation of $T(F)$. In particular, there is a unique irreducible subrepresentation of $I(\chi)$. We call it "the irreducible subrepresentation" of $I(\chi)$.*

This is easily proved from (2.7.3) and (2.7.4).

2.10. Finally, we state Casselman's result on the irreducibility criterion of $I(\chi)$ (see [9]). We use the same notation as in Section 1. For a root ray $a \in \Psi$, we choose a root $\alpha \in \Phi_E$ such that $\alpha|_S = \sigma(a)$. Let $d(a)$ be the degree of L_α over F. $d(a)$ is independent of the choice of α. For $\chi \in \mathrm{Hom}(T(F)/T_0, C^*)$, denote by $H(\chi)$ the subset of $\Psi^\vee$ consisting $a^\vee$ ($a \in \Psi$) such that a is non-plural and $\chi \circ a^\vee = |\cdot|_F^{d(a)}$ or that a is plural and $\chi \circ a^\vee = |\cdot|_F^{d(a)}$ or $|\cdot|_F^{\varepsilon(a)}$, where $\varepsilon(a) = (d(a)/2) + \pi(\log(q_F))^{-1}\sqrt{-1}$ and $\pi = 3.141\cdots$.

THEOREM (Casselman). *Let $\chi \in X_{\mathrm{reg}}(T)$. Then $I(\chi)$ is irreducible if and only if $H(\chi)$ is empty.*

When G is of F-rank one, this result was also proved by Williams [28].

3. The irreducible decomposition of the unramified principal series. In this section, we give the irreducible decomposition of the unramified principal series representations $I(\chi)$ of the unramified group G for regular unramified characters $\chi \in X_{\mathrm{reg}}(T)$. The idea of proof can be found in Rodier [20] when G is split. Here we will use freely Rodier's techniques.

3.1. Throughout this section we fix $\chi \in X_{\mathrm{reg}}(T)$. Denote by $C(\chi)$ the set of connected components of $V - \bigcup_{a^\vee \in H(\chi)} \mathrm{Ker}(a^\vee)$. For $D \in C(\chi)$, let $W(D)$ be the set of $w \in W_G(S)$ such that $w^{-1}C^+ \subset D$. Also, let $D_\chi = \bigcap_{a^\vee \in H(\chi)} (a^\vee)^{-1}(\boldsymbol{R}_+)$ and $W(\chi) = W(D_\chi)$. For $D \in C(\chi)$ and $w \in W_G(S)$, let $\rho(D, w)$ be the irreducible subrepresentation of $I(\chi^w)$. It follows from (2.8.1) and Lemma (2.9) that $\rho(D, w)$ is uniquely determined by w and contained in $JH(\chi)$. The remainder of this section will be devoted to proving the following theorems.

3.2. THEOREM. *Let* $\chi \in X_{\mathrm{reg}}(T)$.

(1) *Let* $D \in C(\chi)$. *For any* $w_1, w_2 \in W(D)$, $\rho(D, w_1)$ *is* $G(F)$*-isomorphic to* $\rho(D, w_2)$. *That is, (the equivalence class of)* $\rho(D, w)$ *depends only on* D. *(Thus we denote it by* $\rho(D)$*).*

(2) *The correspondence* $\rho: C(\chi) \to JH(\chi): D \mapsto \rho(D)$ *is bijective.*

(3) *For every* $D \in C(\chi)$, $J(B, G; \rho(D))$ *is* $T(F)$*-isomorphic to* $\bigoplus_{w \in W(D)} \chi^w$.

3.3. THEOREM. *Let* $\chi \in X_{\mathrm{reg}}(T)$ *and* $\langle H(\chi)\rangle$ *the set of coroots which are represented by an integral linear combination of elements of* $H(\chi)$. *Then* $\langle H(\chi)\rangle$ *is a root system and* $H(\chi)$ *is a basis of* $\langle H(\chi)\rangle$. *Thus combining with Theorem* (3.2) (2), *one sees that the length of a composition series of* $I(\chi)$ *is equal to* $2^{|H(\chi)|}$ *and* $|H(\chi)|$ *is bounded by the semisimple* F*-rank of* G.

3.4. THEOREM. *Let* $\chi \in X_{\mathrm{reg}}(T)$ *and* φ *a non-degenerate character of* $U(F)$ (see Section 4). *Then, for* $D \in C(\chi)$, $\rho(D)$ *has a Whittaker model with respect to* φ *if and only if* $D = D_\chi$.

Here we recall the notion of Whittaker model. For a non-degenerate character φ of $U(F)$, we construct the induced representation $W(G, U; \varphi)$ of $G(F)$. The space of $W(G, U; \varphi)$ consists of all locally constant functions $f: G(F) \to \boldsymbol{C}$ such that $f(ug) = \varphi(u)f(g)$ for all $u \in U(F)$, $g \in G(F)$. $G(F)$ acts on this space by the right translation. Let π be an admissible representation of $G(F)$. A $G(F)$-embedding of π into $W(G, U; \varphi)$ is called a Whittaker functional of π and its image is called a Whittaker model of π (with respect to φ). It is known that if π is irreducible then π has at most one Whittaker model ([21]). When π has a Whittaker model, we call it "the Whittaker model" of π.

3.5. We define the notation. Let $a \in \Psi$ and $w_1, w_2 \in W_G(S)$. By the notation $w_1(a) \not\sim w_2(a)$ (resp. $w_1(a) \sim w_2(a)$), we means that the wall $\mathrm{Ker}(a^\vee)$ in V separates (resp. does not separate) two chambers $w_1^{-1}C^+$ and $w_2^{-1}C^+$. For a given subset H of Ψ (resp. $\Psi^\vee$), denote by $-H$ the set $\{-a \mid a \in H\}$ (resp. $\{-a^\vee \mid a^\vee \in H\}$). We start with the following lemma.

3.6. LEMMA. *Let $a \in \Psi_0 \cup (-\Psi_0)$ and w the reflection in V with respect to the wall* $\mathrm{Ker}(a^\vee)$. *Let A be a base of* $\mathrm{Hom}_G(I(\chi), I(\chi^w))$ *(see* (2.7.5)).

(1) *If $a^\vee \notin H(\chi) \cup (-H(\chi))$, then A is bijective.*

(2) *If $a^\vee \in H(\chi) \cup (-H(\chi))$, then one has*

$$J(B, G; \mathrm{Ker}(A)) \simeq \bigoplus_{\substack{w' \in W_G(S) \\ w'(a) \sim a}} \chi^{w'} \quad \text{and} \quad J(B, G; \mathrm{Im}(A)) \simeq \bigoplus_{\substack{w' \in W_G(S) \\ w'(a) \sim w(a)}} \chi^{w'} .$$

(3) *Let φ be a non-degenerate character of $U(F)$. If $a^\vee \in H(\chi)$ and $a^\vee(C^+) < 0$ or if $a^\vee \in -H(\chi)$ and $a^\vee(C^+) > 0$, then no constituents of* $\mathrm{Ker}(A)$ *has a Whittaker model with respect to φ.*

PROOF. We prove this Lemma in several steps. Clearly, it is enough to verify the assertions for $a \in \Psi_0$.

(Step 1) We remark the following facts. If G is of semisimple F-rank one, then one has $\Psi^\vee = \{\pm a^\vee\}$, $W_G(S) = \{1, w\}$ and $J(B, G; I(\chi)) \simeq \chi \oplus \chi^w$ ((2.7.4)). Thus the dimension of $J(B, G; \mathrm{Ker}(A))$ is always less than or equal to 1. Furthermore, by Lemma (2.9), if $\dim J(B, G; \mathrm{Ker}(A)) = 1$, then $J(B, G; \mathrm{Ker}(A))$ is isomorphic to χ. When G is of semisimple F-rank one, the assertions (1) and (2) in the Lemma are equivalent to

(3.6.1) If $H(\chi)$ is empty, then $\mathrm{Ker}(A) = \{0\}$

(3.6.2) If $H(\chi)$ is non-empty, then $\mathrm{Ker}(A) \neq \{0\}$.

By Theorem (2.10), the claim (3.6.1) is clearly true.

(Step 2) Assume G is a semisimple and simply connected group of F-rank one and a is non-plural. In this case, G is isomorphic to $R_{E/F}(SL_2)$ (see (1.4)). Thus one has $G(F) = SL_2(E)$ and the isomorphism $a^\vee \colon F^* \to S(F)$. As we mentioned earlier, $S(F)/S_0$ is isomorphic to $T(F)/T_0$. Hence $a^\vee$ induces an isomorphism of $F^*/\mathcal{O}^*$ onto $T(F)/T_0$. For a complex number z, we define the unramified character χ_z by $\chi_z \circ a^\vee = |\cdot|_E^z$. The correspondence $z \mapsto \chi_z$ gives rise to an isomorphism of $\boldsymbol{C}/\boldsymbol{Z}_E$ onto $\mathrm{Hom}(T(F)/T_0, \boldsymbol{C}^*)$, where let $\boldsymbol{Z}_E = 2\pi(\log(q_E))^{-1}\sqrt{-1}\boldsymbol{Z}$. By the definition, one has

$$H(\chi_z) = \begin{cases} \{a^\vee\} & \text{if} \quad z \equiv 1 \mod \boldsymbol{Z}_E \\ \{-a^\vee\} & \text{if} \quad z \equiv -1 \mod \boldsymbol{Z}_E \\ \varnothing & \text{otherwise} \end{cases} .$$

Thus the Lemma is a consequence of the following well known facts (see [12], [13]). Let z vary over all $\boldsymbol{C}/\boldsymbol{Z}_E$.

(3.6.3) $I(\chi_z)$ is reducible if and only if $z \equiv \pm 1 \mod \boldsymbol{Z}_E$.

(3.6.4) $I(\chi_1)$ has a composition series of the form $0=I_0\subset I_1\subset I_2=I(\chi_1)$. The representation I_1 is a special represntation of $G(F)$ with a Whittaker model and I_2/I_1 is a one-dimensional representation of $G(F)$ without a Whittaker model.

(Step 3) Assume G is a semisimple and simply connected group of F-rank one and a is plural. In this case, one has $G = R_{L_2/F}(SU_3)$, where L_2 is the intermediate field between F and E such that E is the quadratic unramified extension of L_2 and SU_3 is the special unitary group defined over L_2 by the hermitian form on E^3 with degree three and Witt index one ((1.4)). Thus, one has $G(F) = SU_3(L_2)$ and the isomorphism $a^\vee\colon F^* \to S(F)$. (Note that $a^\vee$ is the coroot corresponding to $\tau(a)$). For $z \in \boldsymbol{C}$, we define the unramified character χ_z by $\chi_z\circ a^\vee = |\cdot|_E^z$. By the same reason as in (Step 2), the correspondence $z\mapsto\chi_z$ gives rise to an isomorphism of $\boldsymbol{C}/\boldsymbol{Z}_E$ onto $\mathrm{Hom}(T(F)/T_0, \boldsymbol{C}^*)$. By the definition,

$$H(\chi_z) = \begin{cases} \{a^\vee\} & \text{if} \quad z \equiv 1 \text{ or } (1/2) + \pi(\log(q_E))^{-1}\sqrt{-1} \bmod \boldsymbol{Z}_E \\ \{-a^\vee\} & \text{if} \quad z \equiv -1 \text{ or } -((1/2) + \pi(\log(q_E))^{-1}\sqrt{-1}) \bmod \boldsymbol{Z}_E\,. \\ \varnothing & \text{otherwise} \end{cases}$$

Hence the Lemma is a consequence of the following facts. Let z vary over all $\boldsymbol{C}/\boldsymbol{Z}_E$.

(3.6.5) $I(\chi_z)$ is reducible if and only if $z \equiv \pm 1$ or $\pm((1/2) + \pi(\log(q_E))^{-1}\sqrt{-1}) \bmod \boldsymbol{Z}_E$.

(3.6.6) For $z = 1$ or $(1/2) + \pi(\log(q_E))^{-1}\sqrt{-1}$, $I(\chi_z)$ has a composition series of the form $0 = I_0\subset I_1\subset I_2 = I(\chi_z)$ and I_1 is a special representation of $G(F)$.

(3.6.7) I_1 has a Whittaker model, but I_2/I_1 does not have a Whittaker model.

For (3.6.5) and (3.6.6), we refer the reader to Keys [15]. Here we give a proof of (3.6.7). Let $z = 1$ or $(1/2) + \pi(\log(q_E))^{-1}\sqrt{-1}$. We consider the contragredient representation $I(\chi_{-z})$ of $I(\chi_z)$. Let $0 = I'_0\subset I'_1\subset I'_2 = I(\chi_{-z})$ be a composition series of $I(\chi_{-z})$. Since $I_1 \simeq I'_2/I'_1$, I'_2/I'_1 is a special representation of $G(F)$. Let K be a hyperspecial maximal compact subgroup of $G(F)$. Then it is well known that $I(\chi_{-z})$ contains a unique K-spherical constituent $Sp(\chi_{-z})$ ([9]). Since $Sp(\chi_{-z})$ is not a special representation (see Borel [2, p. 45, Remark]), it follows that $Sp(\chi_{-z})$ is $G(F)$-isomorphic to I'_1. Hence I'_1 contains a K-invariant non-zero vector φ_K. Let Ω_{-z} be the Whittaker map of $I(\chi_{-z})$ constructed in [11]. Using the explicit formula of the "unramified Whittaker function" computed by Casselman and Shalika [11, Theorem 5.4], we obtain $\Omega_{-z}(\varphi_K) = 0$. This implies $I'_1 = \mathrm{Ker}(\Omega_{-z})$. Therefore $I_1 \simeq I'_2/I'_1$ has a Wittaker model. Further, from [11, Corollary 1.8] or [21, Theorem 7], it is known that I_2/I_1 does not have a Whittaker model.

We note that, in the case of (Step 2) and (Step 3), the Lemma remains true even if χ is non-regular.

(Step 4) Assume G is a semisimple group of F-rank one. Let $\mu\colon G^{\sim}\to G$ be the universal covering of G. $T^{\sim}=\mu^{-1}(T)$ (resp. $S^{\sim}=\mu^{-1}(S)$, $B^{\sim}=\mu^{-1}(B)$) is a maximal torus (resp. maximal F-split torus, Borel subgroup) of $G^{\sim}$ defined over F. One may identify the relative root system $\Phi(G^{\sim}, S^{\sim})$ of $G^{\sim}$ (with respect to $S^{\sim}$) with Φ. By the same way as in (Step 2) and (Step 3), each unramified character of $T^{\sim}(F)$ is denoted by χ_z for $z\in \boldsymbol{C}$. Let G^+ be the subgroup of $G(F)$ generated by $U(F)$ and $U^{\circ}(F)$, where U° is the unipotent radical of the opposite parabolic subgroup of B. G^+ is a normal closed subgroup of $G(F)$. Moreover, it is known from [4] that G^+ satisfies the following properties.

(3.6.8) $G(F)=T(F)G^+$.

(3.6.9) $\mu(G^{\sim}(F))=G^+$, that is, $\mu\colon G^{\sim}(F)/\mathrm{Ker}\,\mu(F)\to G^+$ is a topological group isomorphism.

Note that $\mathrm{Ker}\,\mu(F)$ is finite and central in $G^{\sim}(F)$. Let $T_0^{\sim}$ be the maximal compact subgroup of $T^{\sim}(F)$. Since $\mu(T_0^{\sim})\subset T_0$ and $\mathrm{Ker}\,\mu(F)\subset T_0^{\sim}$, we have an injection $T^{\sim}(F)/T_0^{\sim}\hookrightarrow T(F)/T_0$. Thus $\chi\circ\mu$ gives an unramified character of $T^{\sim}(F)$. Clearly $\chi\circ\mu=\chi_z$ if and only if $\chi\circ a^{\vee}=|\cdot|_E^z$. In particular, one has

(3.6.10) $H(\chi)=H(\chi\circ\mu)$.

Now, we denote by $I(\chi)|_{G^+}$ the restriction of $I(\chi)$ to G^+. On the other hand, by (3.6.9), $I(G^{\sim}, B^{\sim}; \chi\circ\mu)$ is considered as a representation of G^+. Then one has a G^+-isomorphism

$$(3.6.11)\qquad \begin{array}{c} I(\chi)\xrightarrow{\simeq} I(G^{\sim}, B^{\sim}; \chi\circ\mu)\,. \\ \phi\mapsto\phi\circ\mu \end{array}$$

According to this isomorphism, A transfers to a base $A^{\sim}$ of $\mathrm{Hom}_{G^{\sim}}(I(G^{\sim}, B^{\sim}; \chi\circ\mu), I(G^{\sim}, B^{\sim}; \chi^{w}\circ\mu))$. Clearly

(3.6.12) $\mathrm{Ker}(A)=\{0\}$ if and only if $\mathrm{Ker}(A^{\sim})=\{0\}$.

Since the Lemma on $G^{\sim}$, $\chi\circ\mu$, and $A^{\sim}$ has already proved in (Step 2) and (Step 3), the Lemma on G, χ and A is easily proved from (3.6.10), (3.6.11), (3.6.12) and (Step 1).

We remark that in this case the Lemma remains true even if χ is non-regular. In fact, for every non-regular unramified character χ, $I(\chi)$ is irreducible and $H(\chi)$ is empty.

(Step 5) Assume G is of semisimple F-rank one. Let G' be the derived group of G and C the central torus of G. Put $T'=T\cap G'$, $S'=$

$S \cap G'$ and $B' = B \cap G'$. We identify the relative root system $\Phi(G', S')$ of G' (with respect to S') with Φ. Let χ' (resp. $\chi|_C$) be the restriction of χ to $T'(F)$ (resp. $C(F)$). Note that χ' is not necessarily regular. Since $a^\vee(F) \subset S'(F)$, one has $\chi \circ a^\vee = \chi' \circ a^\vee$. Therefore we obtain

(3.6.13) $H(\chi) = H(\chi')$.

Let $I(\chi)|_{C\cdot G'}$ be the representation of $C(F)\cdot G'(F)$ obtained by restriction of $I(\chi)$ to $C(F)\cdot G'(F)$. Since $G(F) = T(F)\cdot G'(F)$, we have a $C(F)\cdot G'(F)$-isomorphism

$$\begin{aligned} I(\chi)|_{C\cdot G'} &\to \chi|_C \otimes I(G', B'; \chi') . \\ \phi &\mapsto \phi|_{G'(F)} \end{aligned} \tag{3.6.14}$$

According to this isomorphism, A transfers to a base A' of $\mathrm{Hom}_{G'}(I(G', B'; \chi'), I(G', B'; \chi'^w))$. Clearly

(3.6.15) $\mathrm{Ker}(A) = \{0\}$ if and only if $\mathrm{Ker}(A') = \{0\}$.

The Lemma on G', χ' and A' has already proved in (Step 4). Also we recall that if χ' is non-regular then $I(\chi')$ is irreducible and $H(\chi')$ is empty. Hence the Lemma on G, χ and A is easily proved from (3.6.13), (3.6.14), (3.6.15) and (Step 1).

(Step 6) We prove the Lemma for general G. For $\theta = \{\sigma(a)\}$, put $P = P_\theta$, $M = M_\theta$, $N = U_\theta$ and $\delta_P = \delta_\theta$. M has semisimple F-rank one. $M \cap B$ is a Borel subgroup of M containing T. The representation $I(P, B; \chi)$ of $P(F)$ induced by χ is the right regular representation of $P(F)$ on the space of all locally constant functions $\phi: P(F) \to \boldsymbol{C}$ such that $\phi(bp) = \delta_P^{-1}(b)\delta_B(b)\chi(b)\phi(p)$ for all $b \in B(F)$, $p \in P(F)$. Note that $N(F)$ acts trivially by $I(P, B; \chi)$. On the other hand, we have the representation $I(M, M \cap B; \chi)$ of $M(F)$. By the restriction map $\phi \mapsto \phi|_{M(F)}$ from $I(P, B; \chi)$ to $I(M, M\cap B; \chi)$, we obtain a $P(F)$-isomorphism from $I(P, B; \chi)$ onto $I(M, M \cap B; \chi) \otimes 1_N$, where 1_N denotes the trivial representation of $N(F)$. In particular, $\mathrm{Hom}_P(I(P, B; \chi), I(P, B; \chi^w))$ is isomorphic to $\mathrm{Hom}_M(I(M, M \cap B; \chi), I(M, M\cap B; \chi^w))$. Let A' be a base of $\mathrm{Hom}_P(I(P, B; \chi), I(P, B; \chi^w))$. Then, the $G(F)$-homomorphism A_1 of $I(\chi)$ to $I(\chi^w)$ is defined to be

$$\begin{aligned} I(\chi) \simeq I(G, P; I(P, B; \chi)) &\to I(G, P; I(P, B; \chi^w)) \simeq I(\chi^w) . \\ \phi &\mapsto A' \circ \phi \end{aligned}$$

By (2.7.5), there exists a non-zero $\lambda \in \boldsymbol{C}$ such that $A = \lambda \cdot A_1$. Clearly, we may assume $\lambda = 1$. Further, it is easy to check that if A' is an isomorphism then A is also an isomorphism. Since the Lemma on M, χ and A' has already proved in (Step 5), if $a^\vee \notin H(\chi) \cup (-H(\chi))$, then A' is an isomorphism and hence A is also an isomorphism. This proves the assertion (1) in the Lemma.

Now we assume $a^\vee \in H(\chi) \cup (-H(\chi))$. Then $I(M, M\cap B; \chi)$ has a composition series of the form $0 \subset \mathrm{Ker}(A') \subset I(M, M\cap B; \chi)$. One has $\mathrm{Ker}(A) \simeq I(G, P; \mathrm{Ker}(A'))$ as $G(F)$-modules. In order to determine the Jacquet module of $\mathrm{Ker}(A)$, we apply the Geometrical Lemma to $J(B, G; I(G, P; \mathrm{Ker}(A')))$. Then there exist a numeration $w_1, w_2, \cdots, w_k$ of elements of $W(M, T)$ and a filtration of $J(B, G; I(G, P; \mathrm{Ker}(A')))$ of the form $0 = J_0 \subset J_1 \subset \cdots \subset J_k = J(B, G; I(G, P; \mathrm{Ker}(A')))$ such that J_i/J_{i-1} is $T(F)$-isomorphic to

$$I(T, \mathrm{Int}(^*w_i)(M)\cap T; J^{w_i}(M\cap \mathrm{Int}(^*w_i^{-1})(T), M; \mathrm{Ker}(A')))$$

for $i = 1, 2, \cdots, k$. Here one has

$$\begin{aligned} &I(T, \mathrm{Int}(^*w_i)(M)\cap T; J^{w_i}(M\cap \mathrm{Int}(^*w_i^{-1})(T), M; \mathrm{Ker}(A'))) \\ &\quad = I(T, T; J^{w_i}(T, M; \mathrm{Ker}(A'))) \simeq J^{w_i}(T, M; \mathrm{Ker}(A')) \simeq \chi^{w_i} \end{aligned}$$

because $J(T, M; \mathrm{Ker}(A'))$ is $T(F)$-isomorphic to χ by (Step 5). Thus J_i/J_{i-1} is $T(F)$-isomorphic to χ^{w_i}. In particular, the filtration gives a composition series of $J(B, G; I(G, P; \mathrm{Ker}(A')))$. Since $W(M, T) = \{w' \in W_G(S) \mid w'(a) \sim a\}$, one has

$$J(B, G; \mathrm{Ker}(A)) \simeq J(B, G; I(G, P; \mathrm{Ker}(A'))) \simeq \bigoplus_{\substack{w' \in W_G(S) \\ w'(a)\sim a}} \chi^{w'} .$$

Further, the exactness of the functor $J(B, G; \cdot)$ implies the assertion on $J(B, G; \mathrm{Im}(A))$.

Finally we prove the assertion (3) in the Lemma. Since $\sigma(a)$ is positive, one has $a^\vee(C^+) \subset \boldsymbol{R}_+$. Thus we may assume $a^\vee \in -H(\chi)$. Then, by (Step 5), $\mathrm{Ker}(A')$ does not have a Whittaker model. By Casselman and Shalika [11, Corollary 1.7] or Rodier [21, Theorem 7], if σ has no Whittaker model, then neither does the representation of G induced by it. Hence, no constituents of $\mathrm{Ker}(A)$ have a Whittaker model. q.e.d.

3.7. COROLLARY. *We fix $a^\vee \in \Psi^\vee$ and take $w, w' \in W_G(S)$ such that $\mathrm{Cl}(w^{-1}C^+) \cap \mathrm{Cl}(w'^{-1}C^+) = \mathrm{Ker}(a^\vee)$, where for a subset D of V, $\mathrm{Cl}(D)$ denotes the closure of D in V. Let A be a base of $\mathrm{Hom}_G(I(\chi^w), I(\chi^{w'}))$.*

(1) *If $a^\vee \notin H(\chi) \cup (-H(\chi))$, then A is bijective.*

(2) *If $a^\vee \in H(\chi) \cup (-H(\chi))$, then one has*

$$J(B, G; \mathrm{Ker}(A)) \simeq \bigoplus_{\substack{w'' \in W_G(S) \\ w''(a)\sim w(a)}} \chi^{w''} \quad and \quad J(B, G; \mathrm{Im}(A)) \simeq \bigoplus_{\substack{w'' \in W_G(S) \\ w''(a)\sim w'(a)}} \chi^{w''} .$$

(3) *Let φ be a non-degenerate character of $U(F)$. If $a^\vee \in H(\chi)$ and $a^\vee(w^{-1}C^+) < 0$ or if $a^\vee \in -H(\chi)$ and $a^\vee(w^{-1}C^+) > 0$, then no constituents of $\mathrm{Ker}(A)$ have a Whittaker model with respect to φ.*

PROOF. Put $B' = \mathrm{Int}(^*w^{-1})(B)$ and $C' = w^{-1}C^+$. C' is the Weyl chamber corresponding to B'. Either $\sigma(a)$ or $-\sigma(a)$ is a simple root with respect to B'. For every function ϕ on $G(F)$, we define the function ϕ_w on $G(F)$ by $\phi_w(g)=\phi(^*wg)$ for $g\in G(F)$. $\phi\mapsto\phi_w$ gives rise to $G(F)$-isomorphisms $I(\chi^w)\xrightarrow{\simeq} I(G, B'; \chi)$ and $I(\chi^{w'})\xrightarrow{\simeq} I(G, B'; \chi^{w^{-1}w'})$. According to these isomorphisms, A transfers to a base A' of $\mathrm{Hom}_G(I(G, B'; \chi), I(G, B'; \chi^{w^{-1}w'}))$. It is easily seen that $\mathrm{Ker}(A)$ is $G(F)$-isomorphic to $\mathrm{Ker}(A')$ and $J(B, G; \mathrm{Ker}(A))$ is $T(F)$-isomorphic to $J^w(B', G; \mathrm{Ker}(A'))$. Applying Lemma (3.6) to $I(G, B'; \chi)$, $I(G, B'; \chi^{w^{-1}w'})$, a, C' and A', we obtain the assertion. q.e.d.

3.8. PROPOSITION. *For $w, w'\in W_G(S)$, let A be a base of $\mathrm{Hom}_G(I(\chi^w), I(\chi^{w'}))$. Let*

$$\begin{aligned} Y &= Y(\chi; w, w') \\ &= \{w''\in W_G(S) \mid w''(a)\sim w(a) \text{ and } w''(a)\not\sim w'(a) \text{ for some } a^\vee\in H(\chi)\}\,. \end{aligned}$$

Then one has

$$J(B, G; \mathrm{Ker}(A)) \simeq \bigoplus_{w''\in Y} \chi^{w''} \quad \text{and} \quad J(B, G; \mathrm{Im}(A)) \simeq \bigoplus_{w''\in W_G(S)-Y} \chi^{w''}\,.$$

PROOF. This proposition is proved by the same way as in [20]. For the sake of completeness, we give the proof. We take a minimal gallery in V between two chambers $w^{-1}C^+$ and $w'^{-1}C^+$ of the form $w^{-1}C^+ = C_0$, $C_1, \cdots, C_{n-1}, C_n = w'^{-1}C^+$. The Proposition is verified by the induction on n. For $n=1$, Corollary (3.7) implies the required assertion. Assume $n>1$. We take $w_{n-1}\in W_G(S)$ such that $C_{n-1} = w_{n-1}^{-1}C^+$. Let A_{n-1} (resp. A') be a base of $\mathrm{Hom}_G(I(\chi^w), I(\chi^{w_{n-1}}))$ (resp. $\mathrm{Hom}_G(I(\chi^{w_{n-1}}), I(\chi^{w'}))$). We denote by $J(A_{n-1})$ (resp. $J(A')$) the $T(F)$-homomorphism of $J(B, G; I(\chi^w))$ to $J(B, G; I(\chi^{w_{n-1}}))$ (resp. $J(B, G; I(\chi^{w_{n-1}}))$ to $J(B, G; I(\chi^{w'}))$) induced from A_{n-1} (resp. A'). Since an irreducible component of $J(B, G; I(\chi^w))$ is represented by $\chi^{w''}$ for $w''\in W_G(S)$, we can consider subsets

$$Y_1 = \{w''\in W_G(S) \mid J(A_{n-1})(\chi^{w''}) = 0\}$$

and

$$Y_2 = \{w''\in W_G(S) \mid J(A_{n-1})(\chi^{w''}) \neq 0 \quad \text{and} \quad J(A')\circ J(A_{n-1})(\chi^{w''}) = 0\}\,.$$

Then one has obviously

$$J(B, G; \mathrm{Ker}(A'\circ A_{n-1})) \simeq \mathrm{Ker}(J(A')\circ J(A_{n-1})) \simeq \bigoplus_{w''\in Y_1\cup Y_2} \chi^{w''}\,.$$

Now we show $Y_1\cup Y_2 = Y$. First, let $w''\in Y_1$. By the induction hypothesis, there exists $a^\vee\in H(\chi)$ such that $w''(a)\sim w(a)$ and $w''(a)\not\sim w_{n-1}(a)$. Since the gallery $\{C_i\}$ is minimal, one has necessarily $w''(a)\not\sim w'(a)$. Thus w'' is contained in Y. Next, let $w''\in Y_2$. There exists the coroot $a^\vee$ such that $\mathrm{Cl}(C_{n-1})\cap\mathrm{Cl}(C_n) = \mathrm{Ker}(a^\vee)$. Then, by Corollary (3.7), $a^\vee$ is an element in $H(\chi)\cup(-H(\chi))$. Further, one has $w''\sim w_{n-1}(a)$ and $w''(a)\not\sim w'(a)$. The minimality of the gallery $\{C_i\}$ implies $w''(a)\not\sim w(a)$. Thus w'' is contained in Y. On the other hand, let $w''\in Y$. We can take $a^\vee\in H(\chi)$ such that $w''(a)\sim w(a)$ and $w''(a)\not\sim w'(a)$. Then there exist adjacent chambers C_{i-1} and C_i in the gallery such that $\mathrm{Cl}(C_{i-1})\cap\mathrm{Cl}(C_i)=\mathrm{Ker}(a^\vee)$. From the induction hypothesis, the minimally of the gallery and Corollary (3.7), it follows that if $i<n$ then $w''\in Y_1$ and if $i=n$ then $w''\in Y_2$. This finishes the proof of $Y_1\cup Y_2=Y$.

In the result, $J(B, G; \mathrm{Ker}(A'\circ A_{n-1}))$ is $T(F)$-isomorphic to $\bigoplus_{w''\in Y}\chi^{w''}$. Since Y does not contain the element w_0 such that $w_0^{-1}C^+=-w^{-1}C^+$, $A'\circ A_{n-1}$ is non-trivial. Thus, $\mathrm{Ker}(A)$ coincides with $\mathrm{Ker}(A'\circ A_{n-1})$. This comletes the proof of the assertion on $\mathrm{Ker}(A)$. The assertion on $\mathrm{Im}(A)$ is derived from the exactness of the functor $I(B, G;\cdot)$. q.e.d.

3.9. COROLLARY. *Let w and w' be elements in $W_G(S)$. Assume that we have $w(a)\sim w'(a)$ for any $a^\vee\in H(\chi)$. Then $I(\chi^w)$ is $G(F)$-isomorphic to $I(\chi^{w'})$.*

3.10. COROLLARY. *For $w\in W_G(S)$, let*

$$Y(\chi; w)=\{w''\in W_G(S)\mid w''(a)\sim w(a)\quad \textit{for any}\quad a^\vee\in H(\chi)\}\,.$$

Let π be the irreducible subrepresentation of $I(\chi^w)$. Then $J(B, G;\pi)$ is $T(F)$-isomorphic to $\bigoplus_{w''\in Y(\chi;w)}\chi^{w''}$.

PROOF. By Corollary (3.9), $I(\chi^w)$ is $G(F)$-isomorphic to $I(\chi^{w''})$ for every $w''\in Y(\chi; w)$. Thus π is also the irreducible subrepresentation of $I(\chi^{w''})$ for every $w''\in Y(\chi; w)$. Then, by Lemma (2.9), we obtain

(3.10.1) $\bigoplus_{w''\in Y(\chi;w)}\chi^{w''}\subset J(B, G;\pi)$.

Let w_0 be the element in $W_G(S)$ such that $w_0^{-1}C^+=-w^{-1}C^+$. Let A be a base of $\mathrm{Hom}_G(I(\chi^{w_0}), I(\chi^w))$. Clearly, π is the irreducible subrepresentation of $\mathrm{Im}(A)$. Furthermore, one has $Y(\chi; w_0, w)=\{w''\in W_G(S)\mid w''(a)\not\sim w(a)$ for some $a^\vee\in H(\chi)\}$, that is $Y(\chi; w)=W_G(S)-Y(\chi; w_0, w)$. By Proposition (3.8), one has

(3.10.2) $J(B, G;\pi)\subset J(B, G;\mathrm{Im}(A))=\bigoplus_{w''\in Y(\chi;w)}\chi^{w''}$.

(3.10.1) and (3.10.2) complete the proof.

3.11. PROOF OF THEOREM (3.2). Let $w\in W(D)$. Then $W(D)$ is equal to $Y(\chi; w)$. Thus the assertions (1) and (3) are consequences of Corollary (3.9) and (3.10). The assertion (2) follows from (2.7.4) and the assertion (3). q.e.d.

3.12. PROOF OF THEOREM (3.4). It is known from Casselman and Shalika [11, Corollary 1.8] or Rodier [21, Theorem 7] that there exists a unique constient of $I(\chi)$ which has a Whittaker model with respect to

φ. Now, we take a connected component $D\in C(\chi)$ which differs from D_χ. Then there exists a coroot $a^\vee\in H(\chi)$ such that $a^\vee(D)\subset \boldsymbol{R}_-$, where $\boldsymbol{R}_-$ is the set of negative real numbers. We can take a chamber $w^{-1}C^+\subset D$ with the wall $\mathrm{Ker}(a^\vee)$. By the definition, $\rho(D)$ is the irreducible subrepresentation of $I(\chi^w)$. Let w' be the reflection with respect to $\mathrm{Ker}(a^\vee)$. If A is a base of $\mathrm{Hom}_G(I(\chi^w), I(\chi^{w\cdot w'}))$, then $\mathrm{Ker}(A)$ is non-trivial by Corollary (3.7) (2). In particular, $\rho(D)$ is the irreducible subrepresentation of $\mathrm{Ker}(A)$. From $a^\vee(w^{-1}C^+)\subset a^\vee(D)\subset \boldsymbol{R}_-$ and Corollary (3.7) (3), it follows that no constituents of $\mathrm{Ker}(A)$ have a Whittaker model. q.e.d.

3.13. Proof of Theorem (3.3). $\langle H(\chi)\rangle$ is clearly a subsystem of $\Psi^\vee$. We show that $H(\chi)$ is a basis of $\langle H(\chi)\rangle$. Obviously, the proof is reduced to the case where the relative root system Φ is irreducible. We take an irreducible component $\Delta_{E,0}$ of Δ_E. Let $\Phi_{E,0}$ be the subsystem of Φ_E generated by $\Delta_{E,0}$ and $\Gamma_0=\{\gamma\in \mathrm{Gal}(E/F)\,|\,\gamma(\Delta_{E,0})=\Delta_{E,0}\}$. Then Φ_E has the irreducible decomposition of the form

$$\Phi_E=\coprod_{\gamma\in \mathrm{Gal}(E/F)/\Gamma_0}\gamma(\Phi_{E,0})\,.$$

It is easily seen that the proof is reduced to the case of $\Phi_E=\Phi_{E,0}$. Thus we assume Φ_E is irreducible. Let Σ be the automorphism group of Δ_E induced from $\mathrm{Gal}(E/F)$. Then we have the following four types (see, [8], [24]).

Type I. $|\Sigma|=1$. Then one has $E=F$ and $\Phi_E=\Phi$.

Type II. $|\Sigma|=2$ and Ψ has no plural root ray. In this case, E is the quadratic unramified extension of F. For $\alpha\in\Phi_E$, let L_α be the field of definition of α and $\alpha|_S$ the restriction of α to S. If $\alpha|_S$ is a short (resp. long) root, then one has $L_\alpha=E$ (resp. $L_\alpha=F$).

Type III. $|\Sigma|=2$ and Ψ has plural root rays. Then E is the quadratic unramified extension of F. For $\alpha\in\Phi_E$, if $\alpha|_S$ is a non-divisible (resp. divisible) root, then one has $L_\alpha=E$ (resp. $L_\alpha=F$).

Type IV. $|\Sigma|=3$. Then E is the cubic unramified extension of F. For $\alpha\in\Phi_E$, if $\alpha|_S$ is a short (resp. long) root, then one has $L_\alpha=E$ (resp. $L_\alpha=F$).

For $a\in\Psi$, by the definition, one has

$$(3.13.1)\quad d(a)=\begin{cases}1 & \text{if Type I, Tpye II and } \sigma(a) \text{ is long or}\\ & \text{Type IV and } \sigma(a) \text{ is long}\\ 2 & \text{if Type II and } \sigma(a) \text{ is short or Type III.}\\ 3 & \text{if Type IV and } \sigma(a) \text{ is short}\end{cases}$$

Put $\varepsilon=1+\pi(\log(q_F))^{-1}\sqrt{-1}$. For each $a^\vee\in\langle H(\chi)\rangle$, by the definition of $H(\chi)$, there exist $p(a^\vee)\in\boldsymbol{Z}$ and $q(a^\vee)\in\{0,1\}$ such that $\chi\circ a^\vee=|\cdot|_F^{p(a^\vee)+q(a^\vee)\varepsilon}$.

Since $|\cdot|_F^{2\varepsilon} = |\cdot|_F^2$, $p(a^\vee)$ and $q(a^\vee)$ is uniquely determined by $a^\vee$. Further we have the following:

3.14. LEMMA. (1) *For any $a^\vee \in \langle H(\chi)\rangle$, $p(a^\vee) + q(a^\vee)\varepsilon$ does not vanish.*

(2) *Assume we have Type* I, *Type* II *or Type* IV. *Then one has the following relations for any $a^\vee$, $b^\vee \in \langle H(\chi)\rangle$:*

$$q(a^\vee) = 0\ , \quad p(a^\vee + b^\vee) = p(a^\vee) + p(b^\vee)\ , \quad p(-a^\vee) = -p(a^\vee)\ .$$

(3) *Assume we have Type* III. *Then one has the following relations for any $a^\vee$, $b^\vee \in \langle H(\chi)\rangle$:*

$$p(a^\vee) \equiv 0 \mod 2\ ,$$

$$p(a^\vee + b^\vee) = \begin{cases} p(a^\vee) + p(b^\vee) + 2 & \textit{if} \quad q(a^\vee) = q(b^\vee) = 1 \\ p(a^\vee) + p(b^\vee) & \textit{otherwise}\ , \end{cases}$$

$$q(a^\vee + b^\vee) \equiv q(a^\vee) + q(b^\vee) \mod 2\ ,$$

$$p(-a^\vee) = \begin{cases} -p(a^\vee) - 2 & \textit{if} \quad q(a^\vee) = 1 \\ -p(a^\vee) & \textit{if} \quad q(a^\vee) = 0 \end{cases}, \quad q(-a^\vee) = q(a^\vee)\ .$$

PROOF. (1) For $a^\vee \in \langle H(\chi)\rangle$, suppose $p(a^\vee) + q(a^\vee)\varepsilon = 0$. Then $\chi\circ a^\vee$ is trivial. Let $w \in W_G(S)$ be the reflection in V with respect to $\mathrm{Ker}(a^\vee)$. Then, for any $\lambda \in X_*(S)$, one has

$$(\chi^w\circ\chi^{-1})\circ\lambda = \chi\circ(w(\lambda) - \lambda) = (\chi\circ a^\vee)^{-\langle\tau(a),\lambda\rangle} \equiv 1\ .$$

Thus χ^w equals χ. This contradicts to the regularity of χ.

(2) When we have Type I, Type II or Type IV, Ψ has no plural root ray. Hence $q(a^\vee)$ equals zero for any $a^\vee \in \langle H(\chi)\rangle$. Other relations are clear.

(3) It follows from (3.13.1) that $p(a^\vee)$ is even. Other relations are proved by simple calculations. q.e.d.

3.15. We continue the proof of Theorem (3.3). Let Ω be the closed cone in $\boldsymbol{C}$ generated by ε and 1, that is, $\Omega = \{x + y\varepsilon \mid x \geqq 0 \text{ and } y \geqq 0\}$. For $a^\vee \in \langle H(\chi)\rangle$, $a^\vee$ is called Ω-positive if $p(a^\vee) + q(a^\vee)\varepsilon \in \Omega$. We denote by $\langle H(\chi)\rangle_+$ the set of all Ω-positive elements in $\langle H(\chi)\rangle$. Also, an element $a^\vee$ in $\langle H(\chi)\rangle_+$ is called Ω-simple if it is not decomposed to the sum of two Ω-positive elements. It follows from Lemma (3.14) that $\langle H(\chi)\rangle_+$ satisfies the following properties (see, Bourbaki [5, Chapitre VI]).

(3.15.1) $\langle H(\chi)\rangle_+$ is closed.

(3.15.2) $\langle H(\chi)\rangle = \langle H(\chi)\rangle_+ \coprod (-\langle H(\chi)\rangle_+)$.

Then, by Bourbaki [5, Chapitre VI, Section 1, Corollaire 1 to Proposition 19 and Corollaire 1 to Proposition 20], the set $\Omega(\chi)$ of all Ω-simple elements in $\langle H(\chi)\rangle_+$ is a basis of $\langle H(\chi)\rangle$. Therefore, to verify that $H(\chi)$ is a basis of $\langle H(\chi)\rangle$, it is enough to show that $H(\chi)$ is contained in $\Omega(\chi)$, (then, automatically, one has $H(\chi)=\Omega(\chi)$ by Bourbaki [5, Chapitre VI Section 1, Corollaire 4 to Proposition 20]). First we obtain the following:

3.16. LEMMA. *Assume we have Type* I, *Type* II *or Type* III. *Then* $H(\chi)$ *is contained in* $\Omega(\chi)$.

PROOF. $H(\chi)$ is clearly contained in $\langle H(\chi)\rangle_+$. Thus it suffices to prove that each element of $H(\chi)$ is Ω-simple. We show it in every type.

Type I. For $a^\vee \in H(\chi)$, one has $p(a^\vee)=d(a)=1$. Thus $a^\vee$ is Ω-simple.

Type II. In this case, $\Psi^\vee$ is a root system of type B_n, C_n or F_4. Let $a^\vee \in H(\chi)$. When $\sigma(a)$ is long, one has $p(a^\vee)=d(a)=1$. Thus $a^\vee$ is Ω-simple. We assume $\sigma(a)$ is short. Suppose that $a^\vee$ is not Ω-simple. Then one can take Ω-positive elements $a_1^\vee$ and $a_2^\vee$ such that $a^\vee=a_1^\vee+a_2^\vee$. By (3.13.1) and Lemma (3.14) (2), one has $2=p(a^\vee)=p(a_1^\vee)+p(a_2^\vee)$. Since $p(a_1^\vee)>0$ and $p(a_2^\vee)>0$, both $p(a_1^\vee)$ and $p(a_2^\vee)$ equals 1. Hence both $a_1^\vee$ and $a_2^\vee$ are contained in $H(\chi)$ and both $\sigma(a_1)$ and $\sigma(a_2)$ are long, in other words, both $a_1^\vee$ and $a_2^\vee$ are short. Then, by the properties of root systems of type B_n, C_n and F_4, one knows $a_1^\vee-a_2^\vee \in \langle H(\chi)\rangle$. However one has $p(a_1^\vee-a_2^\vee)=p(a_1^\vee)-p(a_2^\vee)=0$. This contradicts to Lemma (3.14) (1). Hence $a^\vee$ is Ω-simple.

Type III. In this case, $\Psi^\vee$ is the B_n-type root system. We recall that $p(a^\vee)$ is even for every $a^\vee \in \langle H(\chi)\rangle$. Let $a^\vee \in H(\chi)$. Then $p(a^\vee)$ equals either 0 or 2. First we assume $p(a^\vee)=2$. Suppose that $a^\vee$ is not Ω-simple. Then there exist Ω-simple roots $a_1^\vee$, $a_2^\vee \in \langle H(\chi)\rangle_+$ such that $a^\vee=a_1^\vee+a_2^\vee$. By Lemma (3.14) (3), one has $p(a_1^\vee)=p(a_2^\vee)=0$ and $q(a_1^\vee)=q(a_2^\vee)=1$. Thus both $a_1^\vee$ and $a_2^\vee$ are contained in $H(\chi)$ and both a_1 and a_2 are plural, in other words, both $a_1^\vee$ and $a_2^\vee$ are short. Then by the properties of the B_n-type root system, $a_1^\vee-a_2^\vee$ is contained in $\langle H(\chi)\rangle$. However one has

$$p(a_1^\vee-a_2^\vee)=p(a_1^\vee)+p(-a_2^\vee)+2=0$$

and

$$q(a_1^\vee-a_2^\vee)\equiv q(a_1^\vee)+q(-a_2^\vee) \bmod 2 \equiv 0 \bmod 2 .$$

This contradicts to Lemma (3.14) (1). Therefore $a^\vee$ is Ω-simple. Second, when $p(a^\vee)$ equals 0, one has $q(a^\vee)=1$. This implies that $a^\vee$ is Ω-simple. q.e.d.

Next, for Type IV, we obtain the following:

3.17. LEMMA. *Assume we have Type* IV. *If* $H(\chi)$ *is not contained in* $\Omega(\chi)$, *then one has* $\langle H(\chi)\rangle = \Psi^\vee$ *and* $H(\chi)$ *is a set of simple roots of* $\Psi^\vee$.

PROOF. $\Psi^\vee$ is the G_2-type root system. Under the assumption, we can take a coroot $a^\vee \in H(\chi)$ which is not Ω-simple. Then $\sigma(a)$ is necessarily a short root. There exist Ω-positive roots $a_1^\vee$, $a_2^\vee \in \langle H(\chi)\rangle_+$ such that $a^\vee = a_1^\vee + a_2^\vee$. By (3.13.1) and Lemma (3.14) (2), we obtain $3 = p(a^\vee) = p(a_1^\vee) + p(a_2^\vee)$. One may put $p(a_1^\vee) = 1$ and $p(a_2^\vee) = 2$. Thus $a_1^\vee$ is contained in $H(\chi)$ and $\sigma(a_1)$ is long. Obviously, $a^\vee$, $a_1^\vee$ and $a_2^\vee$ are distinct each other and contained in $\langle H(\chi)\rangle_+$. Furthermore, the length of $a^\vee$ is different from that of $a_1^\vee$. Then $\langle H(\chi)\rangle$ coincides necessarily with $\Psi^\vee$ by the properties of the G_2-type root system.

Now, we show $H(\chi) = \{a^\vee, a_1^\vee\}$. Suppose that there exists $b^\vee \in H(\chi)$ such that $b^\vee \notin \{a^\vee, a_1^\vee\}$. If $b^\vee$ is short, then $b^\vee - a_1^\vee$ is contained in $\Psi^\vee$ because $a_1^\vee$ is short. However, one has $p(b^\vee - a_1^\vee) = p(b^\vee) - p(a_1^\vee) = 0$. This contradicts to Lemma (3.14) (1). Thus $b^\vee$ must be long. If $b^\vee - a^\vee$ is contained in $\Psi^\vee$, then we have also a contradiction by $p(b^\vee - a^\vee) = 0$. Thus $b^\vee - a^\vee$ is not contained in $\Psi^\vee$. Nevertheless, by the properties of the G_2-type root system, we can take the short root $c^\vee \in \Psi^\vee$ such that $b^\vee - a^\vee = 3c^\vee$ in $X_*(S)$. Then one has $p(c^\vee) = 0$. This is a contradiction. Consequently, $H(\chi)$ equals $\{a^\vee, a_1^\vee\}$. Since $a^\vee$ is long, $a_1^\vee$ is short and $H(\chi)$ generates $\Psi^\vee$, $H(\chi)$ is a basis of $\Psi^\vee$. This completes the proof of the Lemma and hence Theorem (3.3).

4. **Explicit form of Whittaker functions.** In this section, we give an explicit form of Whittaker functions restricted to $S(F)$. Here, we consider only the Whittaker model attached to the constituent $\rho(D_\chi)$ of $I(\chi)$ for $\chi \in X_{\mathrm{reg}}(T)$. This result is used for calculations of zeta integrals in Section 6.

From now on, we assume the characteristic of F is equal to zero.

4.1. First we consider the group $U^\wedge$ consisting of all (unitary) characters of $U(F)$. We fix a Chevalley—Steinberg system $\{\tilde{x}_\alpha\}_{\alpha\in\Phi_E}$ of G ((1.3)) and take a coherent system $\{(L_a, x_a)\}_{a\in\Psi}$ of G induced from $\{\tilde{x}_\alpha\}_{\alpha\in\Phi_E}$ ((1.7)). For $a \in \Psi$, let $U_{\sigma(a)}$ (resp. $U_{2\sigma(a)}$) be the root subgroup of G corresponding to the root $\sigma(a)$ (resp. $2\sigma(a)$). When a is non-plural, we put $U_{2\sigma(a)} = \{1\}$. Let $N_a = U_{\sigma(a)}(F)/U_{2\sigma(a)}(F)$ for $a \in \Psi$. Then the coherent system $\{(L_a, x_a)\}_{a\in\Psi}$ induces the isomorphisms $z_a\colon L_a \xrightarrow{\simeq} N_a$ (see, (1.5), (1.6)). Since the derived group $U'(F)$ of $U(F)$ has the form $\prod_{\beta\in\Phi_+-\Delta} U_\beta(F)$, the quotient group $U(F)/U'(F)$ is equal to $\prod_{a\in\Psi_0} N_a$. Thus, we obtain an isomorphism

$$z = \prod_{a \in \Psi_0} z_a \colon \prod_{a \in \Psi_0} L_a \xrightarrow{\simeq} U(F)/U'(F) .$$

By this isomorphism, $U^\wedge$ is isomorphic to the Pontrjagin dual $\prod_a L_a^\wedge$ of $\prod_a L_a$. For $\psi \in U^\wedge$, ψ is called a non-degenerate character if $\psi \circ z_a$ is non-trivial for any $a \in \Psi_0$. $T(F)$ acts on $U^\wedge$ by $\psi^t(u) = \psi(t \cdot u \cdot t^{-1})$ for $\psi \in U^\wedge$, $t \in T(F)$ and $u \in U(F)$.

Next, we construct an isomorphism of $\prod_{a \in \Psi_0} L_a$ to $U^\wedge$. We fix a non-trivial character ψ_F of F. For each $a \in \Psi_0$, let $\psi_a = \psi_F \circ \mathrm{tr}_{L_a/F}$ be a non-trivial character of L_a, where $\mathrm{tr}_{L_a/F}$ is the trace of L_a over F. Further, for each $a \in \Psi_0$, we define the homomorphism p_a from $U(F)$ to L_a by the composition of three homomorphisms, the natural homomorphism $U(F) \to U(F)/U'(F)$, $z^{-1} \colon U(F)/U'(F) \to \prod_a L_a$ and the projection $\prod_a L_a \to L_a$. Then $\varphi = \prod_{a \in \Psi_0} (\psi_a \circ p_a)$ is a non-degenerate character of $U(F)$. Notice that φ depends on the coherent system $\{(L_a, x_a)\}_{a \in \Psi_0}$. Using this character, we define the isomorphism λ_F of $\prod_{a \in \Psi_0} L_a$ onto $U^\wedge$ by

$$\lambda_F((r_a))(u) = \varphi(\prod_{a \in \Psi_0} z_a(r_a \cdot p_a(u)))$$

for $(r_a) \in \prod_{a \in \Psi_0} L_a$, $u \in U(F)$. Let $U_F^\wedge$ be the image of $\prod_{a \in \Psi_0} F$ by λ_F, that is,

$$U_F^\wedge = \{\varphi\,(\prod_a z_a(r_a \cdot p_a(\cdot))) \mid r_a \in F \quad \text{for any} \quad a \in \Psi_0\} .$$

$U_F^\wedge$ is a closed subgroup of $U^\wedge$ and depends on the coherent system $\{(L_a, x_a)\}_{a \in \Psi_0}$. The following property is easily seen.

(4.1.1) $U_F^\wedge$ is $S(F)$-invariant.

Let $U_0^\wedge = \{\varphi^s \mid s \in S(F)\}$. By (4.1.1), $U_0^\wedge$ is a subset of $U_F^\wedge$. Note that one has $\varphi^s = \lambda_F((\sigma(a)(s))_{a \in \Psi_0})$ for $s \in S(F)$.

Let $C_0^\infty(U^\wedge)$ (resp. $C_0^\infty(U_F^\wedge)$) denote the set of all locally constant functions on $U^\wedge$ (resp. $U_F^\wedge$) with compact support. We fix a regular unramified character $\chi \in X_{\mathrm{reg}}(T)$. Let $\rho(D_\chi)$ be the constituent of $I(\chi)$ with a Whittaker model (Theorem (3.4)) and denote by $\mathscr{W}\mathscr{H}(\chi, \varphi)$ the Whittaker model of $\rho(D_\chi)$ with respect to φ. The following two theorems are main results in this section.

4.2. THEOREM. *For each* $f \in \mathscr{W}\mathscr{H}(\chi, \varphi)$, *there exists a family* $\{\phi_w \in C_0^\infty(U_F^\wedge) \mid w \in W(\chi)\}$ *such that*

$$f(s) = \sum_{w \in W(\chi)} \phi_w(\varphi^s)\delta_B(s)\chi^w(s)$$

for any $s \in S(F)$, *where* $W(\chi)$ *is the subset of* $W_G(S)$ *defined in* (3.1).

4.3. THEOREM. *For every* $w \in W(\chi)$ *and* $\phi \in C_0^\infty(U^\wedge)$, *there exists a*

Whittaker function $f \in \mathscr{W}\mathscr{H}(\chi, \varphi)$ *such that* $f(s) = \phi(\varphi^s)\delta_B(s)\chi^w(s)$ *for any* $s \in S(F)$.

When G is a split group, these theorems were proved by Rodier in [22].

For the proof, we need a few lemmas. If necessary, changing χ to χ^w by $w \in W(\chi)$, one may assume $\rho(D_\chi)$ is the irreducible subrepresentation of $I(\chi)$. Put $\pi = \rho(D_\chi)$ and let V_π be the representation space of π realized in $I(\chi)$.

4.4. LEMMA. *Let θ be a subset of Δ. Then the Jacquet representation $J(P_\theta, G; I(\chi))$ of $I(\chi)$ with respect to (P_θ, M_θ) is $M_\theta(F)$-isomorphic to* $\bigoplus_{w \in W(T,M_\theta)} I(M_\theta, M_\theta \cap B; \chi^w)$.

PROOF. This is proved in four steps.

(Step 1) Applying the Geometrical Lemma to $J(P_\theta, G{:}\ I(\chi))$, one has a numeration $w_1, w_2, \cdots, w_k$ of elements of $W(T, M_\theta)$ and a filtration $0 = J_0 \subset J_1 \subset \cdots \subset J_k = J(P_\theta, G; I(\chi))$ such that J_i/J_{i-1} is $M_\theta(F)$-isomorphic to $I(M_\theta, M_\theta \cap B; \chi^{w_i})$ for $1 \leqq i \leqq k$. Further, the Frobenius reciprocity law implies the isomorphism

$$\mathrm{Hom}_{M_\theta}(J(P_\theta, G; I(\chi)), J_i/J_{i-1}) \simeq \mathrm{Hom}_T\Big(\bigoplus_{w \in W_G(S)} \chi^w, \chi^{w_i}\Big) .$$

Hence, $\mathrm{Hom}_{M_\theta}(J(P_\theta, G; I(\chi)), J_i/J_{i-1})$ is of dimension one. Let A_i be a base of $\mathrm{Hom}_{M_\theta}(J(P_\theta, G; I(\chi)), J_i/J_{i-1})$ for $1 \leqq i \leqq k$.

(Step 2) We show $J_{i-1} \subset \mathrm{Ker}(A_i)$ for $1 \leqq i \leqq k$. For $i = 1$, it is trivial. Assume $i \geqq 2$. Suppose $A_i(J_{i-1})$ is non-trivial. Then, since $A_i(J_{i-1})$ contains the irreducible subrepresentation of $I(M_\theta, M_\theta \cap B; \chi^{w_i})$, the Jacquet module $J(M_\theta \cap B, M_\theta; A_i(J_{i-1}))$ must contain the $T(F)$-irreducible component χ^{w_i}. On the other hand, if we apply the functor $J(M_\theta \cap B, M_\theta; \cdot)$ to the exact sequence $J_{i-1} \to A_i(J_{i-1}) \to 1$, then we obtain the exact sequence

$$\begin{array}{c} J(M_\theta \cap B, M_\theta; J_{i-1}) \to J(M_\theta \cap B; M_\theta; A_i(J_{i-1})) \to 1 , \\ \wr\| \\ \displaystyle\bigoplus_{j=1}^{i-1} \bigoplus_{w \in W_\theta(S)} \chi^{w \cdot w_j} \end{array}$$

where $W_\theta(S)$ denotes the relative Weyl group of M_θ with respect to S. By [1, Lemma 2.11], each coset of $W_\theta(S)\backslash W_G(S)$ contains the one and only one element of $W(T,M_\theta)$. It follows from this fact that $J(M_\theta \cap B, M_\theta; J_{i-1})$ does not contain χ^{w_i}. This is a contradiction.

(Step 3) We show $J_i \not\subset \mathrm{Ker}(A_i)$ for $1 \leqq i \leqq k$. Suppose $\mathrm{Ker}(A_i)$ contains J_i. Then $J(M_\theta \cap B, M_\theta; \mathrm{Ker}(A_i))$ contains the $T(F)$-subrepresentation $\bigoplus_{w \in W_\theta(S)} \chi^{w \cdot w_i}$. However, if we apply the functor $J(M_\theta \cap B, M_\theta; \cdot)$ to the exact sequence

$$1 \to \mathrm{Ker}(A_i) \to J(P_\theta, G; I(\chi)) \to \mathrm{Im}(A_i) \to 1\ ,$$
$$\begin{array}{c} \cap \\ \downarrow \\ I(M_\theta, M_\theta \cap B; \chi^{w_i}) \end{array}$$

then we obtain the exact sequence

$$1 \to J(M_\theta \cap B, M_\theta; \mathrm{Ker}(A_i)) \to \bigoplus_{w \in W_G(S)} \chi^w \to J(M_\theta \cap B, M_\theta; \mathrm{Im}(A_i)) \to 1\ .$$
$$\begin{array}{c} \cap \\ \downarrow \\ \bigoplus_{w \in W_\theta(S)} \chi^{w \cdot w_i} \end{array}$$

This implies a contradiction.

(Step 4) By (Step 2) and (Step 3), the composition of the injection $R_i: J_i/J_{i-1} \hookrightarrow J(P_\theta, G; I(\chi))/J_{i-1}$ and A_i is non-trivial for $1 \leqq i \leqq k$. Thus $A_i \circ R_i$ gives a base of $\mathrm{Hom}_{M_\theta}(I(M_\theta, M_\theta \cap B; \chi^{w_i}), I(M_\theta, M_\theta \cap B; \chi^{w_i}))$ for $1 \leqq i \leqq k$. In particular, A_i is surjective and $\mathrm{Ker}(A_i|_{J_i})$ equals J_{i-1}. Hence, the homomorphism

$$\bigoplus_{i=1}^{k} A_i: J(P_\theta, G; I(\chi)) \to \bigoplus_{i=1}^{k} (J_i/J_{i-1}) \simeq \bigoplus_{w \in W(T, M_\theta)} I(M_\theta, M_\theta \cap B; \chi^w)$$

gives a $M_\theta(F)$-isomorphism. q.e.d.

4.5. COROLLARY. *$J(P_\theta, G; \pi)$ is $M_\theta(F)$-isomorphic to a subrepresentation of $\bigoplus_{w \in W(\chi) \cap W(T, M_\theta)} I(M_\theta, M_\theta \cap B; \chi^w)$.*

This is a result of Lemma (4.4), Theorem (3.2) (3) and the Frobenius reciprocity law.

4.6. We recall the results in [19]. We consider the quotient space

$$V_\pi^\sharp = V_\pi / \mathrm{Span}\{\pi(u')v - v \mid v \in V_\pi,\ u' \in U'(F)\}\ .$$

$U(F)$ acts on $V_\pi^\sharp$ by π. Since the action of $U'(F)$ on $V_\pi^\sharp$ is trivial, $(\pi, V_\pi^\sharp)$ gives rise to a representation of $U(F)/U'(F)$. Then, by [19, Theorem 2 and Proposition 2], there exists the locally free sheaf $\mathscr{F}$ of the complex vector space over $U^\wedge$ satisfying the following:

(4.6.1) Denote by $\Gamma_c(\mathscr{F}, U^\wedge)$ the vector space consisting of sections of $\mathscr{F}$ over $U^\wedge$ with compact support. For $\psi \in U^\wedge$ and $y \in \Gamma_c(\mathscr{F}, U^\wedge)$, let $\mathscr{F}(\psi)$ be the stalk of $\mathscr{F}$ at ψ and $y(\psi)$ the image of y to $\mathscr{F}(\psi)$. Then there exists the isomorphism $\eta^\sharp$ from $V_\pi^\sharp$ onto $\Gamma_c(\mathscr{F}, U^\wedge)$ such that

$$\eta^\sharp(\pi(u)v^\sharp)(\psi) = \psi(u) \cdot \eta^\sharp(v^\sharp)(\psi)$$

for every $u \in U(F)$, $v^\sharp \in V_\pi^\sharp$ and $\psi \in U^\wedge$.

(4.6.2) Let $\eta: V_\pi \to \Gamma_c(\mathscr{F}, U^\wedge)$ be the composition of the natural homomorphism $V_\pi \to V_\pi^\sharp$ and $\eta^\sharp$. Then, for each $\psi \in U^\wedge$, the homomorphism $v \mapsto \eta(v)(\psi)$ from V_π to $\mathscr{F}(\psi)$ is surjective and its kernel is equal to

$V_\pi^\psi(B, G)$ (see (2.3)). Thus $\mathscr{F}(\psi)$ is $U(F)/U'(F)$-isomorphic to the ψ-localization $V_\psi(B, G; \pi)$ of π.
The following assertion is equivalent to the uniqueness of a Whittaker model (c.f. [11], [21]).

(4.6.3) $\mathscr{F}(\psi) \simeq \boldsymbol{C}$ for any non-degenerate character $\psi \in U^\wedge$.
Furthermore, the following is an easy consequence of (4.6.2).

(4.6.4) For $s \in S(F)$, $\psi \in U^\wedge$ and $v \in V_\pi$, $\eta(v)(\psi^s)$ vanishes if and only if $\eta(\pi(s)v)(\psi)$ vanishes.
Using (4.6.3), for each $v \in V_\pi$, we define the $\boldsymbol{C}$-valued function f_v on $G(F)$ by

(4.6.5) $f_v(g) = \eta(\pi(g)v)(\varphi)$, $g \in G(F)$.
Then one has $\mathscr{W}\mathscr{H}(\chi, \varphi) = \{f_v \mid v \in V_\pi\}$.

4.7. For $\psi \in U^\wedge$, let

$$\Delta(\psi) = \{\sigma(a) \mid a \in \Psi_0 \text{ and } \psi \circ z_a \text{ is non-trivial}\}$$

and $Z_S(\psi)$ the stabilizer of ψ in $S(F)$. Then one has

(4.7.1) $Z_S(\psi) = (\bigcap_{\beta \in \Delta(\psi)} \mathrm{Ker}(\beta))(F)$.
Since $Z_S(\psi)$ acts on $V_\psi(B, G; \pi)$ according to π, the isomorphism $\mathscr{F}(\psi) \simeq V_\psi(B, G; \pi)$ of (4.6.2) induces the action of $Z_S(\psi)$ on $\mathscr{F}(\psi)$. This action is also denoted by π. From the definition (2.3), it follows that the representation $(\pi, \mathscr{F}(\psi))$ of $Z_S(\psi)$ is equivalent to the representation $(\delta_B \otimes J_\psi(B, G; \pi), V_\psi(B, G; \pi))$.

We denote the subset $W(\chi) \cap W(T, M_{\Delta(\psi)})$ of $W_G(S)$ by $W(\chi, \psi)$, that is, $W(\chi, \psi) = \{w \in W(\chi) \mid w^{-1}(\Delta(\psi)) \subset \Phi_+\}$.

4.8. LEMMA. *The representation* $(\pi, \mathscr{F}(\psi))$ *is* $Z_S(\psi)$*-isomorphic to a subrepresentation of* $\bigoplus_{w \in W(\chi, \psi)} \delta_B \cdot \chi^w |_{Z_S(\psi)}$.

PROOF. Put $P = P_{\Delta(\psi)}$, $M = M_{\Delta(\psi)}$. It is easy to see that $\mathscr{F}(\psi)$ is $Z_S(\psi)$-isomorphic to

$$\delta_B|_{Z_S(\psi)} \otimes J_{\psi|_{(U \cap M)}}(M \cap B, M; J(P, G; \pi)) ,$$

where $\psi|_{(U \cap M)}$ is the restriction of ψ to $U(F) \cap M(F)$. Then, it follows from Corollary (4.5) that $\mathscr{F}(\psi)$ is $Z_S(\psi)$-isomorphic to a subrepresentation of

$$\bigoplus_{w \in W(\chi) \cap W(T, M)} \{\delta_B|_{Z_S(\psi)} \otimes J_{\psi|_{(U \cap M)}}(M \cap B, M; I(M, M \cap B; \chi^w))\} .$$

By the definition of $\Delta(\psi)$, $\psi|_{(U \cap M)}$ gives a non-degenerate character of $U(F) \cap M(F)$. Then, it is known by [11, Corollary (1.7)] that $J_{\psi|_{(U \cap M)}}(M \cap B, M; I(M, M \cap B; \chi^w))$ is $Z_S(\psi)$-isomorphic to $\chi^w|_{Z_S(\psi)}$. Here, notice that $Z_S(\psi)$ is central in $M(F)$. q.e.d.

4.9. LEMMA. *Denote by* $\mathrm{Cl}(U_0^\wedge)$ *the closure of* $U_0^\wedge$ *in* $U_F^\wedge$. *Let* ψ *be an element in* $\mathrm{Cl}(U_0^\wedge)$ *and* ξ *an element of the set* $\{\delta_B\cdot\chi^w \mid w\in W(\chi,\psi)\}$. *Let* v *be an element in* V_π *such that* $\eta(v)(\psi)$ *is an eigenvector of* $Z_S(\psi)$ *with the eigen character* $\xi|_{Z_S(\psi)}$. *Then there exists a compact neighbourhood* $\mathfrak{A}$ *of* ψ *in* $U_F^\wedge$ *such that the map* $s\mapsto\xi(s)^{-1}\cdot\eta(\pi(s)v)(\varphi)$ *is constant on* $\{s\in S(F)\mid\varphi^s\in\mathfrak{A}\}$.

PROOF. We verify this lemma in four steps.

(Step 1) Since both π and ξ are smooth, there exists an open compact subgroup $\mathfrak{C}$ of $S(F)$ such that $\pi(s)v=v$ and $\xi(s)=1$ for any $s\in\mathfrak{C}$. Put

$$Z_S(\psi)_1=\{s\in Z_S(\psi)\mid |\alpha(s)|_F\leqq 1\quad\text{for any}\quad\alpha\in\Delta\}\ .$$

Here we can prove that the quotient $Z_S(\psi)_1/(\mathfrak{C}\cap Z_S(\psi)_1)$ is a finitely generated monoid (Appendix). Thus there exists a finite subset $\mathfrak{S}\subset Z_S(\psi)_1$ such that $Z_S(\psi)_1$ is generated by $(\mathfrak{C}\cap Z_S(\psi)_1)\cup\mathfrak{S}$. For each $s\in\mathfrak{S}$, since $\eta(\pi(s)v)(\psi)$ is equal to $\xi(s)\eta(v)(\psi)$, there exists a compact neighbourhood $\mathfrak{A}_s$ of ψ in $U_F^\wedge$ such that $\eta(\pi(s)v)\equiv\xi(s)\eta(v)$ on $\mathfrak{A}_s$. Let $\mathfrak{A}'$ be the intersection of $\mathfrak{A}_s$, $s\in\mathfrak{S}$.

(Step 2) For simplicity, let $\Delta=\{\alpha_1,\alpha_2,\cdots,\alpha_n\}$ and $\Delta(\psi)=\{\alpha_{k+1},\cdots,\alpha_n\}$. By the isomorphism λ_F defined in (4.1), we assume that ψ corresponds to $(t_i)_{1\leqq i\leqq n}\in\bigoplus_{1\leqq i\leqq n}F$. From (4.7.1), it follows

$$\begin{cases} t_i=0 \text{ and } \alpha_i(Z_S(\psi)_1)\subset\mathcal{O}_F \text{ for } 1\leqq i\leqq k\\ t_i\neq 0 \text{ and } \alpha_i(Z_S(\psi)_1)=1 \text{ for } k+1\leqq i\leqq n\ . \end{cases}\tag{4.9.1}$$

For a positive integer p, define the neighbourhood $\mathfrak{B}_p$ of ψ in $U_F^\wedge$ by

$$\mathfrak{B}_p=\left\{\lambda_F((r_i))\,\middle|\,\begin{matrix} r_i\in\mathscr{P}_F^p \text{ for } 1\leqq i\leqq k\\ r_i\in t_i(1+\mathscr{P}_F^p) \text{ for } k+1\leqq i\leqq n\end{matrix}\right\}\ .$$

We take a positive integer p_0 such that $\mathfrak{B}_p\subset\mathfrak{A}'$ if $p\geqq p_0$. Let p be an integer greater than p_0 and $\varphi^{s'}$ be an element in $U_0^\wedge\cap\mathfrak{B}_p$. Now we show $\{\varphi^s\mid s\in s'Z_S(\psi)_1\}\subset\mathfrak{B}_p$. Since $\varphi^s=\lambda_F((\alpha_i(s))_{1\leqq i\leqq n})$ for $s\in S(F)$, φ^s is contained in $\mathfrak{B}_p$ if and only if $\alpha_i(s)$ is contained in $\mathscr{P}_F^p$ or $t_i(1+\mathscr{P}_F^p)$ according to $1\leqq i\leqq k$ or $k+1\leqq i\leqq n$. In particular, one has $\alpha_i(s')\in\mathscr{P}_F^p$ for $1\leqq i\leqq k$ and $\alpha_i(s')\in t_i(1+\mathscr{P}_F^p)$ for $k+1\leqq i\leqq n$. Then, by (4.9.1), $\alpha_i(s'Z_S(\psi)_1)$ is contained in $\mathscr{P}_F^p$ or $t_i(1+\mathscr{P}_F^p)$ according to $1\leqq i\leqq k$ or $k+1\leqq i\leqq n$.

(Step 3) We show there exist integers $p''\geqq p'\geqq p_0$ and an element $\varphi^{s'}\in U_0^\wedge\cap\mathfrak{B}_{p'}$ such that $U_0^\wedge\cap\mathfrak{B}_{p''}$ is contained in $\{\varphi^s\mid s\in s'\cdot\mathfrak{C}\cdot Z_S(\psi)_1\}$. First one can take coweights $\{\beta_1,\beta_2,\cdots,\beta_n\}\subset X_*(S)$ such that

$$\begin{cases}\langle\alpha_i,\beta_j\rangle=0 \text{ if } i\neq j\\ \langle\alpha_i,\beta_i\rangle=m_i>0\ .\end{cases}\tag{4.9.2}$$

Since $\mathfrak{C}$ is open in $S(F)$, there exists a positive integer q such that $\beta_i(1+\mathscr{P}_F^q)\subset\mathfrak{C}$ for $1\leqq i\leqq n$. Then one has

$$\begin{aligned}\{\varphi^s\,|\,s\in\mathfrak{C}\} &= \{\lambda_F((\alpha_i(s))_{1\leqq i\leqq n})\,|\,s\in\mathfrak{C}\}\\ &\supset\left\{\lambda_F\Big(\Big(\alpha_i\Big(\prod_{j=1}^{n}\beta_j(r_j)\Big)\Big)_{1\leqq i\leqq n}\Big)\,\Big|\,r_j\in 1+\mathscr{P}_F^q \text{ for } 1\leqq j\leqq n\right\}\\ &= \{\lambda_F((r_i))\,|\,r_i\in(1+\mathscr{P}_F^q)^{m_i} \text{ for } 1\leqq i\leqq n\}\,.\end{aligned}$$

Thus if we take an integer q' such that $1+\mathscr{P}_F^{q'}$ is contained in $\bigcap_{i=1}^{n}(1+\mathscr{P}_F^q)^{m_i}$, then one has

$$\{\varphi^s\,|\,s\in\mathfrak{C}\}\supset\{\lambda_F((r_i))\,|\,r_i\in 1+\mathscr{P}_F^{q'} \text{ for } 1\leqq i\leqq n\}\,. \tag{4.9.3}$$

Further, since $\beta_i(\mathscr{O}_F-\{0\})$ is contained in $Z_S(\psi)_1$ for $1\leqq i\leqq k$, one has

$$\{\varphi^s\,|\,s\in Z_S(\psi)_1\}\supset\left\{\lambda_F((r_i))\,\middle|\,\begin{array}{l}r_i\in\mathscr{P}_F^{q''}-\{0\} \text{ for } 1\leqq i\leqq k\\ r_i=1 \text{ for } k+1\leqq i\leqq n\end{array}\right\}, \tag{4.9.4}$$

where let $q''=\max(m_1, m_2, \cdots, m_n)$. Now, we take an integer p' greater than $\max(p_0, q', q'')$ and $\varphi^{s'}\in U_0^\wedge\cap\mathfrak{B}_{p'}$. We also take an integer p'' such that $\mathscr{P}_F^{p''}\subset\alpha_i(s')\cdot\mathscr{P}_F^{p'}$ for $1\leqq i\leqq k$. Then, by (4.9.3) and (4.9.4), one has

$$\begin{aligned}\{\varphi^s\,|\,s\in s'\cdot\mathfrak{C}\cdot Z_S(\psi)_1\} &= \{\lambda_F((\alpha_i(s'\cdot s_1\cdot s_2))_{1\leqq i\leqq n})\,|\,s_1\in\mathfrak{C},\ s_2\in Z_S(\psi)_1\}\\ &\supset\left\{\lambda_F((r_i))\,\middle|\,\begin{array}{l}r_i\in\alpha_i(s')\cdot\mathscr{P}_F^{q''}-\{0\} \text{ for } 1\leqq i\leqq k\\ r_i\in\alpha_i(s')\cdot(1+\mathscr{P}_F^{q'}) \text{ for } k+1\leqq i\leqq n\end{array}\right\}\\ &\supset\left\{\lambda_F((r_i))\,\middle|\,\begin{array}{l}r_i\in\mathscr{P}_F^{p''}-\{0\} \text{ for } 1\leqq i\leqq k\\ r_i\in t_i(1+\mathscr{P}_F^{p''}) \text{ for } k+1\leqq i\leqq n\end{array}\right\}\\ &= U_0^\wedge\cap\mathfrak{B}_{p''}\,.\end{aligned}$$

(Step 4) Let $\mathfrak{B}_{p'}$, $\mathfrak{B}_{p''}$ and $\varphi^{s'}$ be the same as in (Step 3). From $\{\varphi^{s''}\,|\,s''\in s'Z_S(\psi)\}\subset\mathfrak{B}_{p'}\subset\mathfrak{A}'$ (Step 2), it follows

(4.9.5) $\xi(s)^{-1}\cdot\eta(\pi(s)v)-\eta(v)\equiv 0$ on $\{\varphi^{s''}\,|\,s''\in s'Z_S(\psi)\}$

for any $s\in\mathfrak{C}\cup\mathfrak{S}$. By (4.6.4) and the fact that $Z_S(\psi)_1$ is generated by $(\mathfrak{C}\cap Z_S(\psi)_1)\cup\mathfrak{S}$, it is known that (4.9.5) holds for every $s\in\mathfrak{C}\cdot Z_S(\psi)_1$. In particular, one has

$$\xi(s)^{-1}\cdot\eta(\pi(s)v)(\varphi^{s'})-\eta(v)(\varphi^{s'})=0$$

for any $s\in\mathfrak{C}\cdot Z_S(\psi)_1$. Further (4.6.4) implies

$$\eta(\xi(s)^{-1}\pi(s'\cdot s)v-\pi(s')v)(\varphi)=0$$

for any $s\in\mathfrak{C}\cdot Z_S(\psi)_1$, that is,

$$\xi(s)^{-1}\cdot\eta(\pi(s)v)(\varphi)=\xi(s')^{-1}\cdot\eta(\pi(s')v)$$

for any $s\in s'\cdot\mathfrak{C}\cdot Z_S(\psi)_1$. Hence, it follows from (Step 3) that the map

$s \mapsto \xi(s)^{-1} \cdot \eta(\pi(s)v)(\varphi)$ is constant on $\{s \in S(F) | \varphi^s \in \mathfrak{B}_{p''}\}$. q.e.d.

4.10. PROOF OF THEOREM (4.2). Let $f \in \mathscr{W}\mathscr{H}(\chi, \varphi)$ be a Whittaker function. By (4.6.5), f has the form

(4.10.1) $f(g) = f_v(g) = \eta(\pi(g)v)(\varphi),\ g \in G(F)$

for some $v \in V_\pi$. Let $\mathfrak{U}_v$ be the support of $\eta(v)$ in $U^\wedge$. Notice that $\mathfrak{U}_v$ is open-compact. Furthermore, (4.6.4) implies that for $s \in S(F)$,

(4.10.2) $f_v(s) = 0$ if and only if $\varphi^s \notin \mathfrak{U}_v$.

Thus, if $U_0^\wedge \cap \mathfrak{U}_v$ is empty, then f_v vanishes on $S(F)$. Assume $U_0^\wedge \cap \mathfrak{U}_v$ is not empty. We take an element ψ in the closure $\mathrm{Cl}(U_0^\wedge \cap \mathfrak{U}_v)$ of $U_0^\wedge \cap \mathfrak{U}_v$ in $U_F^\wedge$. From Lemma (4.8), it follows that $\eta(v)(\psi)$ has the decomposition of the form $\eta(v)(\psi) = \bigoplus_{w \in W(\chi,\psi)} y(w)$, where $y(w)$ is either zero or an eigenvector in $\mathscr{F}(\psi)$ with the eigencharacter $\delta_B \cdot \chi^w|_{Z_S(\psi)}$. Put $W_v(\chi, \psi) = \{w \in W(\chi, \psi) | y(w) \neq 0\}$. It is known by (4.6.2) that for each $y(w)$ there exists a vector $v(w) \in V_\pi$ such that $y(w) = \eta(v(w))(\psi)$. Thus one has $\eta(v)(\psi) = \bigoplus_{w \in W(\chi,\psi)} \eta(v(w))(\psi)$. Applying Lemma (4.9) to each eigenvector $\eta(v(w))(\psi)$, one can take an open compact neighbourhood $\mathfrak{A}(\psi, w)$ of ψ in $U_F^\wedge$ such that

$$\delta_B \chi^w(s)^{-1} \cdot \eta(\pi(s)v(w))(\varphi) = \text{constant}$$

on $\{s \in S(F) | \varphi^s \in \mathfrak{A}(\psi, w)\}$. We denote this constant by $c(\psi, w)$ for each $w \in W_v(\chi, \psi)$ and put $c(\psi, w) = 0$ for any $w \in W(\chi) - W_v(\chi, \psi)$. Hence, if we put $\mathfrak{A}(\psi) = \bigcap_{w \in W_v(\chi,\psi)} \mathfrak{A}(\psi, w)$, then one has

(4.10.3) $\eta(\pi(s)v)(\varphi) = \sum_{w \in W(\chi)} c(\psi, w) \cdot \delta_B \chi^w(s)$

on $\{s \in S(F) | \varphi^s \in \mathfrak{A}(\psi)\}$. Since $\mathrm{Cl}(U_0^\wedge \cap \mathfrak{U}_v)$ is compact in $U_F^\wedge$, there exist a finite subset $\{\psi_1, \psi_2, \cdots, \psi_k\}$ of $\mathrm{Cl}(U_0^\wedge \cap \mathfrak{U}_v)$ such that

$$\mathrm{Cl}(U_0^\wedge \cap \mathfrak{U}_v) \subset \bigcup_{i=1}^{k} \mathfrak{A}(\psi_i)\ .$$

Let $\mathfrak{C}_i = \mathfrak{A}(\psi_i) - \bigcup_{j=i+1}^{k} \mathfrak{A}(\psi_j)$ for $1 \leqq i \leqq k-1$ and $\mathfrak{C}_k = \mathfrak{A}(\psi_k)$. These subsets are open compact and disjoint each other. For each $w \in W(\chi)$, we define the function $\phi_w \in C_0^\infty(U_F^\wedge)$ by

$$\phi_w(\psi) = \begin{cases} c(\psi_i, w) & \text{if } \psi \in \mathfrak{C}_i \cap \mathfrak{U}_v,\ 1 \leqq i \leqq k \\ 0 & \text{otherwise}\ . \end{cases}$$

Then, by (4.10.1), (4.10.2) and (4.10.3), one has clearly

$$f(s) = \sum_{w \in W(\chi)} \phi_w(\varphi^s) \cdot \delta_B(s) \chi^w(s)$$

for any $s \in S(F)$. q.e.d.

4.11. PROOF OF THEOREM (4.3). For a given $\phi \in C_0^\infty(U^\wedge)$, it is sufficient to prove that there exists a Whittaker function $f \in \mathscr{W}\mathscr{H}(\chi, \varphi)$

such that $f(s)=\phi(\varphi^s)\cdot\delta_B(s)\chi(s)$ for any $s\in S(F)$. Let w' be the element of $W_G(S)$ such that $w'C^+=-C^+$. Then π is $G(F)$-isomorphic to the irreducible quotient representation of $I(\chi^{w'})$, that is, one has a $G(F)$-homomorphism A from $I(\chi^{w'})$ onto V_π. A gives rise to an isomorphism A_φ of $V_\varphi(B, G; I(\chi^{w'}))$ to $\mathscr{F}(\varphi)\simeq V_\varphi(B, G; \pi)$ since both of them have dimension one. Thus, if we denote by η_φ the natural homomorphism from $I(\chi^{w'})$ onto $V_\varphi(B, G; I(\chi^{w'}))$, then one has $\eta(A(y))(\varphi)=A_\varphi(\eta_\varphi(y))$ for any $y\in I(\chi^{w'})$. Hence, if we put $f_y(g)=A_\varphi(\eta_\varphi(I(\chi^{w'})(g)(y)))$ for $y\in I(\chi^{w'})$ and $g\in G(F)$, then one has $\mathscr{W}\mathscr{H}(\chi, \varphi)=\{f_y\,|\,y\in I(\chi^{w'})\}$. Now, we consider the subspace

$$Y=\{y\in I(\chi^{w'})\,|\,\mathrm{supp}(y)\subset B(F)\cdot w'\cdot B(F)\}\ .$$

It is known by [11, Corollary (1.8)] that there exists a non-zero constant C such that

$$A_\varphi(\eta_\varphi(y))=C\cdot\int_{U(F)}y(w'\cdot u)\cdot\varphi(u)^{-1}du$$

for any $y\in Y$. Thus one has

$$\text{(4.11.1)}\quad f_y(s)=C\cdot\delta_B\chi(s)\cdot\int_{U(F)}y(w'\cdot u)\cdot\varphi^s(u)^{-1}du$$

for any $y\in Y$ and $s\in S(F)$. Let du' be the Haar measure of $U'(F)$ obtained by the restriction of du. For $y\in Y$, define

$$y'(u)=\int_{U'(F)}y(w'\cdot u\cdot u')du'\ .$$

Then $y\mapsto y'$ is a linear map from Y onto the space $C_0^\infty(U(F)/U'(F))$ of all locally constant functions on $U(F)/U'(F)$ with compact support. Further, let $y'\mapsto y'^\wedge$ be the Fourier transform of $C_0^\infty(U(F)/U'(F))$ onto $C_0^\infty(U^\wedge)$. For a given $\phi\in C_0^\infty(U^\wedge)$, we can take an element $y\in Y$ such that $\phi=C\cdot y'^\wedge$. Then, by (4.11.1), one has

$$f_y(s)=C\cdot\delta_B\chi(s)\cdot y'^\wedge(\varphi^s)=\delta_B\chi(s)\cdot\phi(\varphi^s)$$

for any $s\in S(F)$. q.e.d.

4.12. REMARK. The assumption on the characteristic of F was used only in the proof of Lemma (4.9). Thus the other lemmas remain true without this assumption.

5. Parametrization of irreducible representations of ${}^LG$. In order to construct Euler factors, we must study finite dimensional representations of the L-group ${}^LG$ of G. In this section, we give a complete parametrization of equivalence classes of finite dimensional irreducible representations of ${}^LG$.

5.1. Since the minimal splitting field E of G is an unramified extension of F, the Galois group $\Gamma = \mathrm{Gal}(E/F)$ is cyclic. Let σ be a Frobenius element of Γ, hence σ is a generator of Γ. Let $(X^*(T), \Delta_E, X_*(T), \Delta_E^\vee)$ be the based root datum attached to (G, B, T). Here, we define the action of Γ on $X^*(T)$ and $X_*(T)$ by

$$^\gamma\xi(t) = \gamma(\xi(\gamma^{-1}(t))) \,, \quad \langle ^\gamma\xi, \lambda\rangle_T = \langle \xi, {}^\gamma\lambda\rangle_T$$

for $\gamma \in \Gamma$, $\xi \in X^*(T)$, $\lambda \in X_*(T)$, where $\langle\ ,\ \rangle_T$ denotes the natural pairing $X^*(T)\times X_*(T) \to \boldsymbol{Z}$. The dual system $(X_*(T), \Delta_E^\vee, X^*(T), \Delta_E)$ determines uniquely (up to isomorphisms) the connected reductive algebraic group $^LG^0$, the maximal torus $^LT^0$ and the Borel subgroup $^LB^0$ defined over $\boldsymbol{C}$. Since the Galois group Γ acts on $^LG^0$ (see [2]), we can define the semi-direct product $^LG = {}^LG^0 \rtimes \Gamma$. Usually, this is called the "finite Galois form" of the L-group. For $\gamma \in \Gamma$ and $g \in {}^LG^0$, we denote by $^\gamma g$ the transform of g by γ. By the definition, one has

(5.1.1) $\quad {}^\gamma\lambda(t) = \lambda(^\gamma t)$

for every $\lambda \in X^*(^LT^0)$ and $t \in {}^LT^0$.

5.2. By representation of ^LG, we mean a morphism $r\colon {}^LG \to GL_n(\boldsymbol{C})$ of complex algebraic groups. Let $\mathscr{R}(^LG^0)$ (resp. $\mathscr{R}(^LG)$) be the set of all equivalence classes of finite dimensional irreducible representations of $^LG^0$ (resp. ^LG). Let Λ be the set of dominant weights in $X^*(^LT^0)$. Notice that Λ is Γ-invariant. By the classical theory of Cartan and Weyl, $\mathscr{R}(^LG^0)$ is parametrized by Λ, that is, there is a bijection $R^\sim\colon \Lambda \to \mathscr{R}(^LG^0)$. When $R(\lambda)$ is a representative of an equivalence class $R^\sim(\lambda)$ for $\lambda \in \Lambda$, λ is the highest weight of $R(\lambda)$. For $R(\lambda)$ and $\gamma \in \Gamma$, we define the representation $^\gamma R(\lambda)$ of $^LG^0$ by $^\gamma R(\lambda)(g) = R(\lambda)(^\gamma g)$. Then it follows from (5.1.1) that $^\gamma\lambda$ is the highest weight of $^\gamma R(\lambda)$. Let Λ/Γ be the set of Γ-orbits in Λ and $[\lambda] = \{^\gamma\lambda \mid \gamma \in \Gamma\}$ for $\lambda \in \Lambda$. For an orbit $[\lambda] \in \Lambda/\Gamma$, write by $e([\lambda])$ the cardinality of $[\lambda]$. Then we can take representatives $R(\lambda)$ of equivalence classes $R^\sim(\lambda)$ satisfying the following relation:

(5.2.1) $\quad R(^{\sigma^k}\lambda) = {}^{\sigma^k}R(\lambda)$ for any $\lambda \in \Lambda$, $k = 0, 1, \cdots, e([\lambda]) - 1$.

The representation space of $R(^\gamma\lambda)$, $\gamma \in \Gamma$ is denoted by $V_{[\lambda]}$. Hereafter, we fix a set of such representatives $\{(R(\lambda), V_{[\lambda]})\}_{\lambda\in\Lambda}$.

5.3. We fix an orbit $[\lambda] \in \Lambda/\Gamma$. Put $e = e([\lambda])$ and $p = p([\lambda]) = \min\{p' \in \boldsymbol{N} \mid {}^{\sigma^{p'\cdot e}}R(\lambda) \equiv R(\lambda)\}$. Clearly, both integers e and p are divisors of the order $|\Gamma|$ of Γ and depend only on the orbit $[\lambda]$. Now we consider the space $\mathrm{Hom}_{^LG^0}(R(\lambda), {}^{\sigma^e}R(\lambda))$. Since $^{\sigma^e}R(\lambda)$ is $^LG^0$-isomorphic to $R(\lambda)$, it follows from the Schur's lemma that $\mathrm{Hom}_{^LG^0}(R(\lambda), {}^{\sigma^e}R(\lambda))$ is of dimension one. Further, since $^{\sigma^e}R(\lambda)$ has the representation space $V_{[\lambda]}$, $\mathrm{Hom}_{^LG^0}(R(\lambda),$

${}^{\sigma^e}R(\lambda))$ may be considered as a subspace of $\mathrm{End}(V_{[\lambda]})$. Thus, one can take the power of the elements of $\mathrm{Hom}_{{}^LG^0}(R(\lambda), {}^{\sigma^e}R(\lambda))$. Then, it is easily seen that each element Q of $\mathrm{Hom}_{{}^LG^0}(R(\lambda), {}^{\sigma^e}R(\lambda))$ satisfies the following:

(5.3.1) Q^p is a scalar operator on $V_{[\lambda]}$.

(5.3.2) Q^k is not a scalar operator for any $1 \leqq k < p$.

(5.3.3) If the order of Q is finite, then p divides it.

Let $V_{[\lambda]}^{\lambda}$ be the common highest weight space of $R(\lambda)$ and ${}^{\sigma^e}R(\lambda)$. As a result of above properties, there exists a unique element $Q_0 \in \mathrm{Hom}_{{}^LG^0}(R(\lambda), {}^{\sigma^e}R(\lambda))$ such that

(5.3.4) $Q_0|_{V_{[\lambda]}^{\lambda}} =$ the identity map of $V_{[\lambda]}^{\lambda}$.

Then the order of Q_0 is exactly p. Q_0 is called "the primitive element" of $[\lambda]$. Define the subset $A_{[\lambda]}$ of $\mathrm{Hom}_{{}^LG^0}(R(\lambda), {}^{\sigma^e}R(\lambda))$ by

$$A_{[\lambda]} = \{\zeta_{|\Gamma|/e}^k \cdot Q_0 \mid k = 1, 2, \cdots, |\Gamma|/e\} ,$$

where $\zeta_{|\Gamma|/e} = \exp(2\pi\sqrt{-1}e/|\Gamma|)$. Since one has

$$\mathrm{Hom}_{{}^LG^0}(R({}^\gamma\lambda), {}^{\sigma^e}R({}^\gamma\lambda)) = \mathrm{Hom}_{{}^LG^0}(R(\lambda), {}^{\sigma^e}R(\lambda)) = \boldsymbol{C}Q_0$$

as a subspace of $\mathrm{End}(V_{[\lambda]})$ for every $\gamma \in \Gamma$, $A_{[\lambda]}$ depends only on the orbit $[\lambda]$.

5.4. We construct irreducible representations of ${}^LG$. First, for $[\lambda] \in \Lambda/\Gamma$ and $Q \in A_{[\lambda]}$, we define the representation $(R(\lambda, Q), V_{[\lambda]})$ of ${}^LG^0 \rtimes \langle\sigma^{e([\lambda])}\rangle$ by

$$R(\lambda, Q)(g \rtimes \sigma^{k\cdot e([\lambda])}) = R(\lambda)(g)\cdot Q^k ,$$

where $\langle\sigma^{e([\lambda])}\rangle$ denotes the cyclic group generated by $\sigma^{e([\lambda])}$. It is easy to verify that this is well-defined and irreducible. For $Q_1, Q_2 \in A_{[\lambda]}$, $R(\lambda, Q_1)$ is ${}^LG^0 \rtimes \langle\sigma^{e([\lambda])}\rangle$-isomorphic to $R(\lambda, Q_2)$ if and only if $Q_1 = Q_2$. Next we consider the representation $r(\lambda, Q)$ of ${}^LG$ induced by $R(\lambda, Q)$. By standard arguments of the representation theory, we obtain the following:

(5.4.1) The restriction of $r(\lambda, Q)$ to ${}^LG^0 \rtimes \langle\sigma^{e([\lambda])}\rangle$ is ${}^LG^0 \rtimes \langle\sigma^{e([\lambda])}\rangle$-isomorphic to $\bigoplus_{k=0}^{e([\lambda])-1} R({}^{\sigma^k}\lambda, Q)$.

(5.4.2) $r(\lambda, Q)$ is irreducible.

In particular, (5.4.1) implies that $r(\lambda, Q)$ is ${}^LG$-isomorphic to $r({}^\gamma\lambda, Q)$ for any $\gamma \in \Gamma$. Hence $r(\lambda, Q)$ depends only on the orbit $[\lambda]$ and $Q \in A_{[\lambda]}$. Thus we denote it by $r([\lambda], Q)$. Furthermore, one has

(5.4.3) $r([\lambda_1], Q_1) \simeq r([\lambda_2], Q_2)$ if and only if $[\lambda_1] = [\lambda_2]$ and $Q_1 = Q_2$.

Let $r^\sim([\lambda], Q)$ be the equivalence class containing $r([\lambda], Q)$. Then, by (5.4.2) and (5.4.3), we obtain an injection

$$r^\sim\colon \coprod_{[\lambda]\in\Lambda/\Gamma} A_{[\lambda]} \to \mathscr{R}({}^LG),\ ([\lambda], Q) \mapsto r^\sim([\lambda], Q) .$$

We prove the following:

5.5. PROPOSITION. *The map $r^{\sim}$ is bijective.*

PROOF. It is enough to prove that for each irreducible representation r of ${}^LG$ there exist the orbit $[\lambda] \in \Lambda/\Gamma$ and $Q \in A_{[\lambda]}$ such that r is ${}^LG$-isomorphic to $r([\lambda], Q)$.

(Step 1) Let (r, X) be an irreducible representation of ${}^LG$ and (r', X) the restriction of r to ${}^LG^0$. Since each representation of ${}^LG^0$ is completely reducible, r' decomposes to the direct sum of irreducible representations of ${}^LG^0$, that is,

$$\begin{aligned} r' &= m_1 r_1 \oplus m_2 r_2 \oplus \cdots \oplus m_e r_e \\ X &= X_1^{\oplus m_1} \oplus X_2^{\oplus m_2} \oplus \cdots \oplus X_e^{\oplus m_e} \end{aligned}, \tag{5.5.1}$$

where (r_i, X_i) $1 \leqq i \leqq e$ are irreducible represetations of ${}^LG^0$ which are not equivalent each other and m_i is the multiplicity of r_i for $1 \leqq i \leqq e$. It follows from the irreducibility of r that e is less than or equal to $|\Gamma|$. One may assume $1 \leqq m_1 \leqq m_2 \leqq \cdots \leqq m_e$.

(Step 2) We show that (r', X) has the decomposition of the form

$$(r', X) = (r', X_1^{\oplus m}) \oplus (r', r(\sigma^{-1})X_1^{\oplus m}) \oplus \cdots \oplus (r', r(\sigma^{-e+1})X_1^{\oplus m}) \tag{5.5.2}$$

where $m = m_1$. First, we show that $X_1, r(\sigma^{-1})X_1, \cdots, r(\sigma^{-e+1})X_1$ are ${}^LG^0$-modules which are not isomorphic each other. Suppose that $r(\sigma^{-k})X_1$ is ${}^LG^0$-isomorphic to $r(\sigma^{-k'})X_1$ for some $0 \leqq k < k' \leqq e-1$. Then X_1 is ${}^LG^0$-isomorphic to $r(\sigma^{k-k'})X_1$ and hence $X_1^{\oplus m}$ coinsides with $r(\sigma^{k-k'})X_1^{\oplus m}$. The subspace $X_1^{\oplus m} \oplus r(\sigma^{-1})X_1^{\oplus m} \oplus \cdots \oplus r(\sigma^{k-k'+1})X_1^{\oplus m}$ is a proper ${}^LG$-invariant subspace of X. This contradicts to the irreducibility of r. Next, since (r', X) contains exactly e inequivalent irreducible representations of ${}^LG^0$, these are completely exhausted by $\{(r', r(\sigma^{-i})X_1) \mid 0 \leqq i \leqq e-1\}$. Thus $r(\sigma^{-e})X_1$ is ${}^LG^0$-isomorphic to X_1 and $r(\sigma^{-e})X_1^{\oplus m}$ equals $X_1^{\oplus m}$. Then, the subspace $X_1^{\oplus m} \oplus r(\sigma^{-1})X_1^{\oplus m} \oplus \cdots \oplus r(\sigma^{-e+1})X_1^{\oplus m}$ is ${}^LG$-invariant, hence this equals X.

(Step 3) Compairing (5.5.1) with (5.5.2), one has $m = m_1 = m_2 = \cdots = m_e$ and $r(\sigma^{-i})X_1 = X_{i+1}$ for $0 \leqq i \leqq e-1$ after a change of numeration. By $r(g)r(\sigma^{-i}) = r(\sigma^{-i})^{\sigma^i}r_1(g)$ on X_1 for any $g \in {}^LG^0$, (r_{i+1}, X_{i+1}) is ${}^LG^0$-isomorphic to $({}^{\sigma^i}r_1, X_1)$. In the result, r' has the irreducible decomposition of the form

$$(r', X) = (r_1, X_1)^{\oplus m} \oplus ({}^{\sigma}r_1, X_1)^{\oplus m} \oplus \cdots \oplus ({}^{\sigma^{e-1}}r_1, X_1)^{\oplus m}. \tag{5.5.3}$$

Further, by $r(\sigma^{-e})X_1^{\oplus m} = X_1^{\oplus m}$, $(r, X_1^{\oplus m})$ gives rise to a representation of ${}^LG^0 \rtimes \langle\sigma^e\rangle$. We denote it by $(\mathbb{R}, Y)$. From the Frobenius reciprocity law and (5.5.3), it follows that r is ${}^LG$-isomorphic to the representation $\mathrm{Ind}({}^LG, {}^LG^0 \rtimes \langle\sigma^e\rangle; \mathbb{R})$ induced from $(\mathbb{R}, Y)$. The ${}^LG$-irreducibility of r implies the ${}^LG^0 \rtimes \langle\sigma^e\rangle$-irreducibility of $\mathbb{R}$.

(Step 4) We show that if $(\mathbb{R}, Y) = (r, X_1^{\oplus m})$ is ${}^LG^0 \rtimes \langle\sigma^e\rangle$-irreducible, then one has $m = 1$. We must investigate the $\langle\sigma^e\rangle$-action on Y. It is known by the irreducibility of $\mathbb{R}$ that Y has the decomposition of the form

$$Y = X_1 \oplus \mathbb{R}(\sigma^{-e})X_1 \oplus \cdots \oplus \mathbb{R}(\sigma^{-(m-1)e})X_1 .$$

Since ${}^{\sigma^e}r_1$ is ${}^LG^0$-isomorphic to r_1, there exists a ${}^LG^0$-isomorphism $A: ({}^{\sigma^e}r_1, X_1) \to (r_1, X_1)$. Then A^j gives a ${}^LG^0$-isomorphism from $({}^{\sigma^{je}}r_1, X_1)$ to (r_1, X_1) for any $j \in \boldsymbol{Z}$. On the other hand, $\mathbb{R}(\sigma^{-je})$ gives a ${}^LG^0$-isomorphism from $({}^{\sigma^{je}}r_1, X_1)$ to $(\mathbb{R}|_{{}^LG^0}, \mathbb{R}(\sigma^{-je})X_1)$ for $0 \leqq j \leqq m-1$. Thus we obtain the ${}^LG^0$-isomorphism

$$\bigoplus_{j=0}^{m-1} A^j \circ \mathbb{R}(\sigma^{je}) : (\mathbb{R}|_{{}^LG^0}, \bigoplus_{j=0}^{m-1} \mathbb{R}(\sigma^{-je})X_1) \to (m\cdot r_1, X_1^{\oplus m}) .$$

For $v = (x_0, x_1, \cdots, x_{m-1}) \in X_1^{\oplus m}$, $k \in \boldsymbol{Z}$ and $g \in {}^LG^0$, we define the action of $m\cdot r_1(g \rtimes \sigma^{-ke})$ on v by

$$m\cdot r_1(g \rtimes \sigma^{-ke})(v) = (r_1(g)A^k(x_{m-k}), r_1(g)A^k(x_{m-k+1}), \cdots, r_1(g)A^k(x_{m-k-1})) ,$$

where indices of x are taken by modulo m. By this action, $(m\cdot r_1, X_1^{\oplus m})$ is considered as a representation of ${}^LG^0 \rtimes \langle\sigma^e\rangle$ and $\bigoplus_{j=0}^{m-1} A^j \circ \mathbb{R}(\sigma^{je})$ gives rise to a ${}^LG^0 \rtimes \langle\sigma^e\rangle$-isomorphism of $(\mathbb{R}, Y)$ to $(m\cdot r_1 \cdot X_1^{\oplus m})$. If m is greather than one, then the subspace $\{(x, x, \cdots, x) \mid x \in X_1\}$ of $X_1^{\oplus m}$ is ${}^LG^0 \rtimes \langle\sigma^e\rangle$-invariant. Thus, if $(\mathbb{R}, Y)$ is irreducible, then m is necessarilly equal to one.

(Step 5) One has $Y = X_1$ by (Step 4). Let λ be the highest weight of (r_1, X_1). Then there exists a ${}^LG^0$-isomorphism A' of (r_1, X_1) to $(R(\lambda), V_{[\lambda]})$. If we put $Q = A' \circ \mathbb{R}(\sigma^e) \circ A'^{-1}$, then it is easy to show that Q is contained in $A_{[\lambda]}$ and $(\mathbb{R}, Y)$ is ${}^LG^0 \rtimes \langle\sigma^e\rangle$-isomorphic to $R(\lambda, Q)$. Therefore, by (Step 3), one has $r \simeq r([\lambda], Q)$. q.e.d.

5.6. Finally, we define the notation. For $r = r^{\sim}([\lambda], \zeta^k_{|\Gamma|/e([\lambda])} \cdot Q_0) \in \mathscr{R}({}^LG)$, define $e(r) = e([\lambda])$, $c(r) = 2\pi k(|\Gamma| \cdot \log(q_F))^{-1}\sqrt{-1}$ and $\xi_r = \sum_{\lambda' \in [\lambda]} \lambda'$, where Q_0 is the primitive element of $[\lambda]$. ξ_r is an element in the set $X^*({}^LT^0)^\Gamma$ consisting of Γ-invariant elements in $X^*({}^LT^0)$. By the definition, $X^*({}^LT^0)^\Gamma$ equals $X_*(S)$. Thus ξ_r is contained in $X_*(S)$. Further, we put

$$\mathscr{R}_0({}^LG) = \{r \in \mathscr{R}({}^LG) \mid \langle\alpha, \xi_r\rangle = 0 \text{ for any } \alpha \in \Delta\}$$

and $\mathscr{R}_+({}^LG) = \mathscr{R}({}^LG) - \mathscr{R}_0({}^LG)$.

6. Construction of Euler factors. In this section, we define local zeta integrals and construct Euler factors. When G is a split classical type group and r is the standard representation of ${}^LG$, our definition of the zeta integral coincides with that given by Rodier [22].

6.1. We use the same notation as in Sections 4 and 5. Let $C_0^\infty(F)$ be the set of all locally constant functions on F with compact support. Let φ be the non-degenerate character of $U(F)$ defined in (4.1). For $\chi \in X_{\mathrm{reg}}(T)$, let $\mathscr{W}\mathscr{H}(\chi, \varphi)$ be the Whittaker model of the constituent $\rho(D_\chi)$ of $I(\chi)$ with respect to φ. For $r \in \mathscr{R}({}^LG)$, $f \in \mathscr{W}\mathscr{H}(\chi, \varphi)$ and $s \in \boldsymbol{C}$, we define the zeta integral by

(6.1.1) $\quad Z(s, r, f) = \int_{F^*} f(\xi_r(t))|t|_F^s \cdot \delta_B^{-1}(\xi_r(t))dt,$

where dt is the Haar measure on F^* such that $\int_{\mathscr{O}_F^*} dt = 1$. First, we prove the convergence of this integral.

6.2. PROPOSITION. *Let f be in $\mathscr{W}\mathscr{H}(\chi, \varphi)$ and r in $\mathscr{R}_+({}^LG)$. Then the integral of* (6.1.1) *is absolutely convergent for* $\mathrm{Re}(s) \gg 0$.

PROOF. Put $\xi = \xi_r$. By Theorem (4.2), f has the form

$$f(g) = \sum_{w \in W(\chi)} \phi_w(\varphi^g) \cdot \delta_B \chi^w(g)$$

on $S(F)$, where ϕ_w are elements in $C_0^\infty(U_F^\wedge)$. Thus the integral is equal to

$$\int_{F^*} \{ \sum_{w \in W(\chi)} \phi_w(\varphi^{\xi(t)}) \chi^w(\xi(t)) \} |t|_F^s dt \, .$$

Here, using the isomorphism $\lambda_F \colon \bigoplus_{a \in \Psi_0} F \to U_F^\wedge$, one has

$$\varphi^{\xi(t)} = \lambda_F((t^{\langle \sigma(a), \xi \rangle})_{a \in \Psi_0}) \, ,$$

hence

(6.2.1) $\quad \phi_w(\varphi^{\xi(t)}) = \phi_w \circ \lambda_F((t^{\langle \sigma(a), \xi \rangle})_{a \in \Psi_0}).$

Let h_w be the function on F defined by the right hand side of (6.2.1). Notice that $\langle \sigma(a), \xi \rangle$ is non-negative integer for each $a \in \Psi_0$. By the assumption on r, $\langle \sigma(a), \xi \rangle$ is positive for at least one root ray $a \in \Psi_0$. Thus h_w is contained in $C_0^\infty(F)$ for every $w \in W(\chi)$. Therefore, the integral

$$\int_{F^*} \{ \sum_{w \in W(\chi)} h_w(t) \chi^w(\xi(t)) \} \cdot |t|_F^s dt$$

is absolutely convergent for $\mathrm{Re}(s) \gg 0$. q.e.d.

6.3. REMARK. If r is in $\mathscr{R}_0({}^LG)$, then $\xi_r(t)$ is contained in $Z_S(\varphi)$ for any $t \in F^*$. Thus the integral has the form

$$\int_{F^*} \{ \sum_{w \in W(\chi)} \phi_w(\varphi) \chi^w(\xi_r(t)) \} \cdot |t|_F^s dt \, .$$

Generally, this integral is not convergent for any $s \in \boldsymbol{C}$.

6.4. Next we construct an Euler factor $L(s, r, \chi)$ as the "greatest common divisor" of $\{Z(s, r, f) \mid f \in \mathscr{W}\mathscr{H}(\chi, \varphi)\}$. Usually, a function $L(s)$ on $\boldsymbol{C}$ is called an Euler factor if $L(s)$ has the form $L(s) = P(q_F^{-s})^{-1}$, where $P(X)$ is a polynomial in $\boldsymbol{C}[X]$ with constant term 1. For $(r, \chi) \in \mathscr{R}_+({}^LG) \times X_{\mathrm{reg}}(T)$, let $P(r, \chi)$ be the set of polynomials $P(X) \in \boldsymbol{C}[X]$ such that $P(q_F^{-s})Z(s, r, f)$ is an entire function of s for every $f \in \mathscr{W}\mathscr{H}(\chi, \varphi)$. Clearly, $P(r, \chi)$ is an ideal of $\boldsymbol{C}[X]$. Further we obtain the following:

6.5. THEOREM. *For any* $(r, \chi) \in \mathscr{R}_+({}^LG) \times X_{\mathrm{reg}}(T)$, $P(r, \chi)$ *is a non-trivial principal ideal of* $\boldsymbol{C}[X]$ *and has the generator* $P_{r,\chi}(X) \in \boldsymbol{C}[X]$ *with* $P_{r,\chi}(0) = 1$.

PROOF. We fix $(r, \chi) \in \mathscr{R}_+({}^LG) \times X_{\mathrm{reg}}(T)$ and put $\xi = \xi_r$. Let $k = \min\{\langle \sigma(a), \xi \rangle > 0; a \in \Psi_0\}$ be the positive integer. For $h \in C_0^\infty(F)$, we define the function $h^k \in C_0^\infty(F)$ by $h^k(t) = h(t^k)$. For $h \in C_0^\infty(F)$ and $w \in W(\chi)$, put

$$Z_w(s, r, h) = \int_{F^*} h(t)\chi^w(\xi(t)) \cdot |t|_F^s dt .$$

Then, by Theorem (4.2), one has

$$\text{(6.5.1)} \quad \{Z(s, r, f) \mid f \in \mathscr{W}\mathscr{H}(\chi, \varphi)\} \subset \{ \sum_{w \in W(\chi)} c_w Z_w(s, r, h_w) \mid c_w \in \boldsymbol{C},\ h_w \in C_0^\infty(F)\} .$$

On the other hand, by Theorem (4.3), one has

$$\text{(6.5.2)} \quad \{ \sum_{w \in W(\chi)} c_w Z_w(s, r, h_w^k) \mid c_w \in \boldsymbol{C},\ h_w \in C_0^\infty(F)\} \subset \{Z(s, r, f) \mid f \in \mathscr{W}\mathscr{H}(\chi, \varphi)\} .$$

For an unramified character μ of F^*, define $P^\mu(X) = 1 - \mu(\varpi_F)X$. Then it is well known that each $P^{\chi^w \circ \xi}(X)$ has the following properties:

(6.5.3) $P^{\chi^w \circ \xi}(q_F^{-s})Z_w(s, r, h)$ is an entire function of s for all $h \in C_0^\infty(F)$.

(6.5.4) When h_0 is the characteristic function of $\mathscr{O}_F$, then $P^{\chi^w \circ \xi}(q_F^{-s})Z_w(s, r, h_0) \equiv 1$.

Now we define an equivalence relation $\sim_r$ of $W(\chi)$ by $w \sim_r w'$ if and only if $\chi^w \circ \xi \equiv \chi^{w'} \circ \xi$. Let $W(\chi)/\sim_r$ be the set of equivalence classes with respect to this relation and $[W(\chi)]_r$ a set of representatives of $W(\chi)/\sim_r$. Let

(6.5.5) $P_{r,\chi}(X) = \prod_{w \in [W(\chi)]_r} P^{\chi^w \circ \xi}(X)$.

It follows from (6.5.1) and (6.5.3) that $P_{r,\chi}(X)$ is contained in $P(r, \chi)$. Thus $P(r, \chi)$ is non-trivial. We show $P_{r,\chi}$ generates $P(r, \chi)$. If $P(X)$ is in $P(r, \chi)$, then $P(q_F^{-s})Z_w(s, r, h_0)$ is an entire function of s for every $w \in W(\chi)$ by (6.5.2). Hence (6.5.4) implies that $P(X)$ is divided by $P^{\chi^w \circ \xi}(X)$ for every $w \in W(\chi)$, that is, $P(X)$ is contained in $P_{r,\chi}(X) \cdot \boldsymbol{C}[X]$. q.e.d.

6.6. By Theorem (6.5), we define the Euler factor $L(s, r, \chi)$ by $L(s, r, \chi) = P_{r,\chi}(q_F^{-s})^{-1}$. This is independent of the choice of the non-degenerate character φ of $U(F)$.

7. **Comparison of $L(s, r, Sp(\chi))$ and $L(s, r, \chi)$.** In this section, we give a relation between our Euler factors $L(s, r, \chi)$ and Langlands' Euler factors $L(s, r, Sp(\chi))$ for $(r, \chi) \in \mathscr{R}_+({}^LG) \times X_{\mathrm{reg}}(T)$.

7.1. We recall the Langlands' Euler factor $L(s, r, Sp(\chi))$. Let K be a hyperspecial maximal compact subgroup of $G(F)$. Let $({}^LG^0 \rtimes \sigma)_{\mathrm{s.s.}}/\mathrm{Int}({}^LG^0)$ be the set of semisimple conjugacy classes of ${}^LG$. An admissible representation (π, V_π) of $G(F)$ is called K-spherical if V_π has a non-zero K-invariant vector. Let $\mathscr{S}(G, K)$ be the set of all equivalence classes of irreducible K-spherical representations of $G(F)$. It is well known that for every unramified character $\chi \in \mathrm{Hom}(T(F)/T_0, \boldsymbol{C}^*)$, $I(\chi)$ contains a unique K-spherical constituent $Sp(\chi)$. Further, for $\chi, \chi' \in \mathrm{Hom}(T(F)/T_0, \boldsymbol{C}^*)$, $Sp(\chi)$ is $G(F)$-isomorphic to $Sp(\chi')$ if and only if χ' equals χ^w for some $w \in W_G(S)$. Thus one has an injection

(7.1.1) $\mathrm{Hom}(T(F)/T_0, \boldsymbol{C}^*)/W_G(S) \to \mathscr{S}(G, K): [\chi] \mapsto Sp^\sim(\chi)$,

where $[\chi]$ denotes the $W_G(S)$-orbit of χ and $Sp^\sim(\chi)$ denotes the equivalence class containing $Sp(\chi)$. The Satake isomorphism implies that this map is bijective. On the other hand, Langlands constructed a bijection

(7.1.2) ν: $\mathrm{Hom}(T(F)/T_0, C^*)/W_G(S) \to ({}^LG^0 \rtimes \sigma)_{\mathrm{s.s.}}/\mathrm{Int}({}^LG^0)$

(see Borel [2]). Using these bijection, the Euler factor $L(s, r, Sp(\chi))$ attached to $(r, Sp^\sim(\chi)) \in \mathscr{R}({}^LG) \times \mathscr{S}(G, K)$ is defined to be

$$L(s, r, Sp(\chi)) = \det(1 - r(g_\chi \rtimes \sigma)q_F^{-s})^{-1},$$

where $g_\chi \rtimes \sigma$ is an element in the conjugacy class $\nu([\chi])$.

7.2. THEOREM. *For any $(r, \chi) \in \mathscr{R}_+({}^LG) \times X_{\mathrm{reg}}(T)$, $L(e(r)(s - c(r)), r, \chi)^{-1}$ is a factor of $L(s, r, Sp(\chi))^{-1}$ as a polynomial of q_F^{-s}, where $e(r)$ and $c(r)$ are numbers defined in* (5.6).

PROOF. We use the same notation as in section 5. We fix $(r, \chi) \in \mathscr{R}_+({}^LG) \times X_{\mathrm{reg}}(X)$ and put $\xi = \xi_r$, $e = e(r)$, $c = c(r)$, $\zeta = \exp(2\pi\sqrt{-1}e/|\Gamma|)$. By Proposition (5.5), r is represented by $r^\sim([\lambda], Q)$ for some $[\lambda] \in \Lambda/\Gamma$ and $Q \in A_{[\lambda]}$. Further Q has the form $\zeta^k \cdot Q_0$, where Q_0 is the primitive element of $[\lambda]$. We also denote by r the irreducible representation $r([\lambda], Q)$ which was constructed in (5.4). Let n be the degree of the irreducible representation $(R(\lambda), V_{[\lambda]})$ of ${}^LG^0$ and $h_1 = \lambda$, $h_2, \cdots, h_n$ the collection of all weights of $R(\lambda)$. Let v_i be an eigenvector corresponding to the weight h_i for $1 \leqq i \leqq n$. We recall that the representation space of r consists of all functions ϕ: ${}^LG \to V_{[\lambda]}$ such that

$$\phi((g' \rtimes \sigma^{je})g) = R(\lambda)(g')Q^j(\phi(g))$$

for any $g' \rtimes \sigma^{je} \in {}^LG^0 \rtimes \langle\sigma^e\rangle$ and $g \in {}^LG$. Thus the representation space of

r has a basis $\{v^{i,j} \mid 0 \leqq j \leqq e-1,\ 1 \leqq i \leqq n\}$ such that

$$v^{i,j}(\sigma^m) = \begin{cases} v_i & \text{if} \quad j = m \\ 0 & \text{if} \quad j \neq m \end{cases}.$$

Under this basis, $r(\sigma)$ is represented by the matrix of the form

$$r(\sigma) = \begin{pmatrix} 0 & 1_n & 0 & 0 \cdots 0 \\ 0 & 0 & 1_n & 0 \cdots 0 \\ \vdots & \vdots & & \ddots \\ 0 & 0 & & 1_n \\ Q & 0 & & \cdots\cdots 0 \end{pmatrix}. \tag{7.2.1}$$

Moreover, by [2, Lemma (6.5)], the semisimple conjugacy class $\nu([\chi])$ has a representative $g_\chi \rtimes \sigma$ in ${}^LT^0 \rtimes \sigma$. Thus $r(g_\chi)$ is represented by a diagonal matrix of the form

(7.2.2) $r(g_\chi) = \operatorname{diag}(A, {}^\sigma A, \cdots, {}^{\sigma^{e-1}}A)$,

where let ${}^{\sigma^j}A = \operatorname{diag}({}^{\sigma^j}h_1(g_\chi), \cdots, {}^{\sigma^j}h_n(g_\chi))$ for $0 \leqq j \leqq e-1$. From (7.2.1) and (7.2.2), it follows that $L(s, r, Sp(\chi))$ is equal to

(7.2.3) $\det(1 - A \cdot {}^\sigma A \cdots {}^{\sigma^{e-1}}A \cdot Q_0 \cdot \zeta^k \cdot q_F^{-es})^{-1}$.

Next let $W_G(S)\lambda$ be the $W_G(S)$-orbit of λ. One may assume $W_G(S)\lambda = \{h_1, h_2, \cdots, h_d\}$. Let $V^i_{[\lambda]}$ be the eigenspace of h_i for $1 \leqq i \leqq d$. Clearly, $V_{[\lambda]}$ has a direct sum decomposition of the form

(7.2.4) $V_{[\lambda]} = \bigoplus_{i=1}^d V^i_{[\lambda]} \bigoplus V^0_{[\lambda]}$.

Now, it is known by [2, Lemma (6.2)] that each element $w \in W_G(S)$ has a representative $^*w \in N_{{}^LG^0}({}^LT^0)$ which is fixed under σ, where $N_{{}^LG^0}({}^LT^0)$ is the normalizer of ${}^LT^0$ in ${}^LG^0$. From this fact, it follows that each $h_i \in W_G(S)\lambda$ is σ^e-invariant. In particular, $V^i_{[\lambda]}$ is Q_0-invariant for $1 \leqq i \leqq d$. Hence the restriction of Q_0 to $\bigoplus_{i=1}^d V^i_{[\lambda]}$ is the diagonal matrix $\operatorname{diag}(b_1, b_2, \cdots, b_d)$. For $1 \leqq i \leqq d$, we take $w_i \in W_G(S)$ such that $h_i = \lambda^{w_i}$. Then one has

(7.2.5) $R(\lambda)(^*w_i)Q_0 = Q_0R(\lambda)(^*w_i)$, $R(\lambda)(^*w_i)V^1_{[\lambda]} = V^i_{[\lambda]}$

for $1 \leqq i \leqq d$. By the definition of Q_0 and (7.2.5), one has $b_1 = b_2 = \cdots = b_d = 1$. Thus Q_0 has the form

$$Q_0 = \begin{pmatrix} 1_d & 0 \\ 0 & Q_0' \end{pmatrix}, \quad Q_0' = Q_0|_{V^0_{[\lambda]}}. \tag{7.2.6}$$

Combining (7.2.3) and (7.2.6), one has

$$\begin{aligned} L(s, r, Sp(\chi)) &= \prod_{i=1}^d \{1 - (h_i + {}^\sigma h_i + \cdots + {}^{\sigma^{e-1}}h_i)(g_\chi) \cdot \zeta^k \cdot q_F^{-es}\}^{-1} \times P(s) \\ &= \prod_{i=1}^d \{1 - (\lambda + {}^\sigma\lambda + \cdots + {}^{\sigma^{e-1}}\lambda)^{w_i}(g_\chi) \cdot q_F^{-e(s-c)}\}^{-1} \times P(s) \end{aligned}$$

$$= \prod_{i=1}^{d} \{1 - \xi^{w_i}(g_\chi)\cdot q_F^{-e(s-c)})^{-1} \times P(s) ,$$

where $P(s) = \det(1 - B\cdot Q_0'\cdot\zeta^k\cdot q_F^{-es})^{-1}$ and B is a diagonal matrix of the form

$$\mathrm{diag}\left(\left(\sum_{j=0}^{e-1} {}^{\sigma^j}h_{d+1}\right)(g_\chi), \cdots, \left(\sum_{j=0}^{e-1} {}^{\sigma^j}h_n\right)(g_\chi)\right) .$$

On the other hand, by the proof of Theorem (6.5), $L(e(s-c), r, \chi)$ is equal to

$$\prod_{w\in[W(\chi)]_r} (1 - \chi\circ\xi^w(\varpi_F)q_F^{-e(s-c)})^{-1} .$$

Therefore, in order to finish the proof, we must show the following:

(7.2.7) A map $[W(\chi)]_r \to W_G(S)\lambda$: $w \mapsto \lambda^w$ is injective,

(7.2.8) $\chi\circ\eta(\varpi_F) = \eta(g_\chi)$ for any $\eta\in X_*(S) = X^*({}^LT^0)^\Gamma$.

(7.2.7) is obvious from the definition of the equivalence relation $\sim_r$ and (7.2.8) is easily shown from the construction of the bijections of (7.1.1) and (7.1.2) (c.f. [2, Chapter III]). q.e.d.

7.3. COROLLARY. *Let $r = r^\sim([\lambda], Q)$ be in $\mathscr{R}_+({}^LG)$ and χ in $X_{\mathrm{reg}}(T)$. Assume that the pair (r, χ) satisfies the following:*

(7.3.1) *$W_G(S)\lambda$ coincides with the collection of all weights of $R(\lambda)$.*

(7.3.2) $[W(\chi)]_r = W_G(S)$.

Then $L(e(r)(s - c(r)), r, \chi)$ equals $L(s, r, Sp(\chi))$.

7.4. REMARK. For the condition (7.3.1), we refer the reader to Bourbaki [6]. There are few irreducible representations satisfying this condition. For examples of the pair (r, χ) satisfying the conditions (7.3.1) and (7.3.2), see Rodier [22]. Finally, we note that the constituent $\rho(D_\chi)$ does not necessarily coincide with $Sp(\chi)$.

8. Examples.

Let F be a non-archimedean local field of characteristic zero.

8.1. Let E be the unramified extension of F with degree four. First we consider the case of $G = R_{E/F}(SL_2)$. We take a Borel subgroup B, a maximal torus T and a maximal F-split torus S as follows:

$$B(F) = \left\{\begin{pmatrix} t & b \\ 0 & t^{-1}\end{pmatrix} \middle| t\in E^*, b\in E\right\} ,$$

$$T(F) = \left\{\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix} \middle| t\in E^*\right\} ,$$

$$S(F) = \left\{\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix} \middle| t\in F^*\right\} .$$

Let $W_G(S) = \{1, w\}$ and $\Psi^\vee = \{\pm a^\vee\}$, where $a^\vee(t) = \mathrm{diag}(t, t^{-1})$ for $t \in F^*$. For $z \in \boldsymbol{C}$, an unramified character μ_z (resp. χ_z) of F^* (resp. $S(F)$) is defined by $\mu_z(t) = |t|_F^z$ (resp. $\chi_z(a^\vee(t)) = |t|_F^z$) for $t \in F^*$. Then $z \mapsto \chi_z$ gives rise to an isomorphism from $\boldsymbol{C}/2\pi(\log(q_F))^{-1}\sqrt{-1}\boldsymbol{Z}$ onto $\mathrm{Hom}(S(F)/S_0, \boldsymbol{C}^*)$. Since $\chi_z^w = \chi_{-z}$, it is enough to consider the set $\{\chi_z \mid z \in \Sigma_F\}$, where Σ_F is the region $\{z \in \boldsymbol{C} \mid \mathrm{Re}(z) \geqq 0,\ 0 \leqq \mathrm{Im}(z) < 2\pi(\log(q_F))^{-1}\}$. Put $z_0 = \pi(\log(q_F))^{-1}\sqrt{-1}$. Then, one has $X_{\mathrm{reg}}(T)/W_G(S) = \{\chi_z \mid z \in \Sigma_F - \{0, z_0\}\}$ and

$$\text{(8.1.1)} \qquad \begin{cases} H(\chi_z) = \varnothing,\ W(\chi_z) = W_G(S) & \text{if } z \in \Sigma_F - \{4\} \\ H(\chi_4) = \{a^\vee\},\ W(\chi_4) = \{1\}\,. \end{cases}$$

Let $\Gamma = \langle \sigma \rangle$ be the Galois group of E over F. ${}^LG^0$ is isomorphic to $PSL_2(\boldsymbol{C}) \times PSL_2(\boldsymbol{C}) \times PSL_2(\boldsymbol{C}) \times PSL_2(\boldsymbol{C})$ and the action of Γ on ${}^LG^0$ is given by ${}^\sigma(g_1, g_2, g_3, g_4) = (g_4, g_1, g_2, g_3)$. Let $\mathfrak{T} = \left\{\begin{pmatrix} b & \\ & b^{-1}\end{pmatrix} \middle| b \in \boldsymbol{C}^*\right\}\Big/\{\pm 1_2\}$ be a maximal torus of $PSL_2(\boldsymbol{C})$. Then $X^*({}^LT^0)$ is identified with $X^*(\mathfrak{T})^{\oplus 4}$. Let λ be the dominant weight in $X^*(\mathfrak{T})$ defined by $\lambda\left(\begin{pmatrix} b & \\ & b^{-1}\end{pmatrix}\right) = b^2$. Since $X^*(\mathfrak{T}) = \boldsymbol{Z}\lambda$, the set of dominant weights of $X^*({}^LT^0)$ is identified with $(\boldsymbol{Z}_+\lambda)^{\oplus 4}$, where $\boldsymbol{Z}_+$ is the set of non-negative integers. We consider three domiant weights $\lambda_1 = (\lambda, 0, 0, 0)$, $\lambda_2 = (\lambda, \lambda, \lambda, 0)$ and $\lambda_3 = (\lambda, 0, \lambda, 0)$. Let $r_i = r([\lambda_i], Q_0)$ be the irreducible representations of ${}^LG$ constructed in (5.4), where Q_0 is the primitive element of $[\lambda_i]$ for $1 \leqq i \leqq 3$. Notice that one has $A_{[\lambda_i]} = \{1\}$ for $i = 1, 2$ and $A_{[\lambda_3]} = \{\pm Q_0\}$ For an unramified character μ_z of F^*, define $L_F(s, \mu_z) = (1 - \mu_z(\varpi_F)q_F^{-s})^{-1}$, Then, by simple calculation, we obtain the following:

$$\begin{aligned} L(s, r_1, Sp(\chi_z)) &= L_F(4s, \mu_0)L_F(4s, \mu_z)L_F(4s, \mu_z^{-1})\,, \\ L(s, r_2, Sp(\chi_z)) &= \{L_F(4s, \mu_0)\}^7\{L_F(4s, \mu_z)L_F(4s, \mu_z^{-1})\}^6 \\ &\quad \times \{L_F(4s, \mu_z^2)L_F(4s, \mu_z^{-2})\}^3\{L_F(4s, \mu_z^3)L_F(4s, \mu_z^{-3})\} \\ L(s, r_3, Sp(\chi_z)) &= L_F(2s, \mu_0)L_F(2s, \mu_z)L_F(2s, \mu_z^{-1}) \\ &\quad \times L_F(4s, \mu_0)L_F(4s, \mu_z)L_F(4s, \mu_z^{-1}) \end{aligned}$$

for any $z \in \Sigma_F$. On the other hand, by (8.1.1), we obtain

$$L(e(r_1)(s - c(r_1)), r_1, \chi_z) = \begin{cases} L_F(4s, \mu_z)L_F(4s, \mu_z^{-1}) & \text{if} \quad z \in \Sigma_F - \{0, 4, z_0\}\,, \\ L_F(4s, \mu_z) & \text{if} \quad z = 4 \end{cases}$$

$$\begin{aligned} &L(e(r_2)(s - c(r_2)), r_2, \chi_z) \\ &\quad = \begin{cases} L_F(4s, \mu_z^3)L_F(4s, \mu_z^{-3}) & \text{if} \quad z \in \Sigma_F - \{4, jz_0/3; 0 \leqq j \leqq 5\}\,, \\ L_F(4s, \mu_z^3) & \text{if} \quad z \in \{4, jz_0/3; j = 1, 2, 4, 5\} \end{cases} \end{aligned}$$

$$L(e(r_3)(s - c(r_3)), r_3, \chi_z) = \begin{cases} L_F(2s, \mu_z)L_F(2s, \mu_z^{-1}) & \text{if} \quad z \in \Sigma_F - \{0, 4, z_0\}\,. \\ L_F(2s, \mu_z) & \text{if} \quad z = 4 \end{cases}$$

8.2. Let E be the unramified quadratic extension of F and $\Gamma = \{1, \sigma\}$ the Galois group of E over F. Next, we treat the unitary group with order three and Witt index one, that is,

$$G(F) = \left\{ g \in GL_3(E) \,\middle|\, g\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}\sigma({}^t g) = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \right\}.$$

We take a Borel subgroup B, a maximal torus T and a maximal F-split torus S as follows:

$$B(F) = \left\{ \begin{pmatrix} t & x & y \\ 0 & \delta & -\sigma(x) \\ 0 & 0 & \sigma(t)^{-1} \end{pmatrix} \,\middle|\, \begin{matrix} t, \delta \in E^*, \ \delta\cdot\sigma(\delta) = 1 \\ x, y \in E, \ -x\cdot\sigma(x) = y + \sigma(y) \end{matrix} \right\},$$

$$T(F) = \left\{ \begin{pmatrix} t & & 0 \\ & \delta & \\ 0 & & \sigma(t)^{-1} \end{pmatrix} \,\middle|\, t, \delta \in E^*, \ \delta\cdot\sigma(\delta) = 1 \right\},$$

$$S(F) = \left\{ \begin{pmatrix} t & & \\ & 1 & \\ & & t^{-1} \end{pmatrix} \,\middle|\, t \in F^* \right\}.$$

Let $W_G(S) = \{1, w\}$ and $\Psi^\vee = \{\pm a^\vee\}$, where $a^\vee(t) = \mathrm{diag}(t, 1, t^{-1})$ for $t \in F^*$. For $z \in \boldsymbol{C}$, let χ_z be the unramified character of $S(F)$ defined by $\chi_z(a^\vee(t)) = |t|_F^z$ for $t \in F^*$. Then, by the same reason as in (8.1), $X_{\mathrm{reg}}(T)/W_G(S)$ is equal to $\{\chi_z \mid z \in \Sigma_F - \{0, z_0\}\}$. Further one has

$$(8.2.1)\quad \begin{cases} H(\chi_z) = \varnothing, \ W(\chi_z) = W_G(S) & \text{if } \ z \in \Sigma_F - \{2, 1 + z_0\} \\ H(\chi_z) = \{a^\vee\}, \ W(\chi_z) = \{1\} & \text{if } \ z \in \{2, 1 + z_0\} \end{cases}.$$

Now σ acts on ${}^L G^0 = GL_3(\boldsymbol{C})$ by

$${}^\sigma g = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} {}^t g^{-1} \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}$$

for $g \in {}^L G^0$. Let λ, λ' be dominant weights of ${}^L T^0$ defined by $\lambda(\mathrm{diag}(b_1, b_2, b_3)) = b_1$ and $\lambda'(\mathrm{diag}(b_1, b_2, b_3)) = b_1 b_2$. We consider two dominant weights $\lambda_1 = \lambda$ and $\lambda_2 = \lambda + \lambda'$. Let $r_i = r([\lambda_i], Q_0)$ be the irreducible representations of ${}^L G$ for $i = 1, 2$. Then one has

$$L(s, r_1, Sp(\chi_z)) = L_F(2s, \mu_0)L_F(2s, \mu_z)L_F(2s, \mu_z^{-1}),$$
$$L(s, r_2, Sp(\chi_z)) = L_F(2s, \mu_0)L_F(s, \mu_z)L_F(s, \mu_z^{-1})L_F(2s, \mu_z)L_F(2s, \mu_z^{-1})$$

for any $z \in \Sigma_F$. On the other hand, by (8.2.1), one has

$$L(e(r_1)(s-c(r_1)), r_1, \chi_z) = \begin{cases} L_F(2s, \mu_z)L_F(2s, \mu_z^{-1}) & \text{if} \quad z \in \Sigma_F - \{0, 2, z_0, 1+z_0\}\,, \\ L_F(s, \mu_z) & \text{if} \quad z \in \{2, 1+z_0\} \end{cases}$$

$$L(e(r_2)(s-c(r_2)), r_2, \chi_z) = \begin{cases} L_F(s, \mu_z)L_F(s, \mu_z^{-1}) & \text{if} \quad z \in \Sigma_F - \{0, 2, z_0, 1+z_0\}\,. \\ L_F(s, \mu_z) & \text{if} \quad z \in \{2, 1+z_0\} \end{cases}$$

Appendix. For a subset $\theta \subset \Delta$, let

$$S_\theta = \{s \in S(F) \mid \alpha(s) = 1 \quad \text{for any} \quad \alpha \in \theta\}$$

and

$$S_{\theta,1} = \{s \in S_\theta \mid |\alpha(s)|_F \leqq 1 \quad \text{for any} \quad \alpha \in \Delta\}\,.$$

We show if $\mathfrak{C}$ is an open compact subgroup of $S(F)$ then $S_{\theta,1}/(\mathfrak{C} \cap S_{\theta,1})$ is a finitely generated monoid. Let S_0 be the maximal compact subgroup of $S(F)$. Since $\mathfrak{C} \cap S_0$ has the finite index in S_0, it is enough to prove the claim for $\mathfrak{C} = S_0$. Let $\Delta - \theta = \{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ and $v\colon S_\theta \to \boldsymbol{Z}^{\oplus k}$ the homomorphism defined by $v(s) = (-\log_{q_F}(|\alpha_i(s)|_F))_{1\leqq i\leqq k}$ for $s \in S(F)$. Then $\mathrm{Ker}(v)$ contains $S_0 \cap S_\theta$ and $\mathrm{Ker}(v)/(S_0 \cap S_\theta)$ is finitely generated as monoid. Let $\{\beta_1, \beta_2, \cdots, \beta_k\}$ be coweights such that

$$\begin{cases} \langle \alpha, \beta_j \rangle = 0 & \text{for} \quad \alpha \in \theta \\ \langle \alpha_i, \beta_j \rangle = 0 & \text{if} \quad i \neq j \\ \langle \alpha_i, \beta_i \rangle = m_i > 0 & \end{cases}$$

for any $1 \leqq i, j \leqq k$. Since $\beta_i(\mathcal{O}_F) \subset S_{\theta,1}$, we know

$$\bigoplus_{i=1}^k m_i \boldsymbol{Z}_+ \subset v(S_{\theta,1}) \subset \bigoplus_{i=1}^k \boldsymbol{Z}_+\,,$$

where $\boldsymbol{Z}_+$ is the set of non-negative integers. Hence there is an exact sequence of monoids as follows:

$$\begin{array}{c} 1 \to \bigoplus_{i=1}^k m_i \boldsymbol{Z}_+ \to v(S_{\theta,1}) \to v(S_{\theta,1})/\bigoplus_{i=1}^k m_i \boldsymbol{Z}_+ \to 1\,. \\ \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad\qquad\qquad \bigoplus_{i=1}^k (\boldsymbol{Z}/m_i\boldsymbol{Z}) \end{array}$$

From this exact sequence, it follows that $v(S_{\theta,1})$ is finitely generated. Furthermore, by the exact sequence

$$1 \to \mathrm{Ker}(v)/(S_0 \cap S_\theta) \to S_{\theta,1}/(S_0 \cap S_\theta) \to v(S_{\theta,1}) \to 1\,,$$

$S_{\theta,1}/(S_0 \cap S_\theta)$ is also finitely generated.

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