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COMMUTATORS ON THE POTENTIAL-THEORETIC ENERGY SPACES

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1. Let $L^1(dx/(1+x^2))$ be the L^1 space of functions on the real line $\boldsymbol{R}$ with respect to the measure $dx/(1+x^2)$. Let C_0^∞ be the totality of infinitely differentiable functions on $\boldsymbol{R}$ with compact support. For $0<\alpha<1$, the energy space E_α with respect to the α-Riesz kernel $|x|^{\alpha-1}$ is the Banach space of functions on $\boldsymbol{R}$ obtained as the completion of C_0^∞ with respect to norm

$$\|f\|_\alpha = \left\{\int_{-\infty}^{\infty} |\xi|^\alpha |\hat{f}(\xi)|^2 d\xi\right\}^{1/2} ,$$

where $\hat{f}$ is the Fourier transform of f [5, p. 352]. The Hilbert transform H is defined by

$$Hf(x) = \frac{1}{\pi}\text{p.v.}\int_{-\infty}^{\infty} \frac{f(y)}{y-x}dy .$$

For $f \in L^1(dx/(1+x^2))$, $[f, H]$ is an operator defined by

$$[f, H]g(x) = H(fg)(x) - f(x)Hg(x)\left(=\frac{1}{\pi}\text{ p.v.}\int_{-\infty}^{\infty} \frac{f(y)-f(x)}{y-x}g(y)dy\right) .$$

In the theory of singular integrals, this operator plays an important role [1]. In this note, we shall characterize the boundedness of $[f, H]$ as an operator from E_α to itself in terms of the BMO space BMO_α with respect to E_α. We say that a non-negative measure $d\mu(x, y)$ in the upper half plane $\boldsymbol{C}_+ = \{(x, y); x \in \boldsymbol{R}, y > 0\}$ is an α-Carleson measure if there exists a constant B such that, for any open set $O \subset \boldsymbol{R}$ with $\text{Cap}_\alpha(O) < \infty$,

$$\iint_{\hat{O}} d\mu(x, y) \leqq B\,\text{Cap}_\alpha(O) ,$$

where $\hat{O} = \cup\{I \times (0, |I|);\ I \text{ component of } O\}$ ($|I|$ is the length of I) and $\text{Cap}_\alpha(\cdot)$ is the capacity with respect to the α-Riesz kernel [5, p. 131]. The minimum of such constants is denoted by $\|d\mu\|_{\text{Car},\alpha}$. Let BMO_α denote the Banach space of functions $f \in L^1(dx/(1+x^2))$, modulo constants, with norm

$$\|f\|_{\text{BMO}_\alpha} = \|\,|\nabla f(x, y)|^2 y^{1-\alpha}\|_{\text{Car},\alpha}^{1/2} ,$$

where $f(x, y)$ is the Poisson extension of f to $\boldsymbol{C}_+$ and $|\nabla f|^2 = |\partial f/\partial x|^2 + |\partial f/\partial y|^2$. We show:

THEOREM. *An operator* $[f, H]$ *is bounded from* E_α *to itself if and only if* $f \in \mathrm{BMO}_\alpha$.

The dual space $E_{-\alpha}$ of E_α is the Banach space of distributions on $\boldsymbol{R}$ obtained as the completion of C_0^∞ with respect to norm

$$\|f\|_{-\alpha} = \left\{\int_{-\infty}^{\infty} |\xi|^{-\alpha} |\hat{f}(\xi)|^2 d\xi\right\}^{1/2} .$$

For $f \in L^1(dx/(1+x^2))$, we put

$$\|f\|_{H_\alpha^1} = \inf \sum_{k=1}^{\infty} \|g_k + i\varepsilon_k H g_k\|_\alpha \|h_k + i\varepsilon_k H h_k\|_{-\alpha} ,$$

where the infimum is taken over all sequences $\{(g_k, h_k, \varepsilon_k)\}_{k=1}^\infty$ of triples such that

$$f = \sum_{k=1}^{\infty} (g_k + i\varepsilon_k H g_k)(h_k + i\varepsilon_k H h_k) \quad (g_k, h_k \in C_0^\infty,\ \varepsilon_k \in \{-1, 1\}) .$$

Let H_α^1 be the Banach space of distributions with respect to norm $\|\cdot\|_{H_\alpha^1}$. Our theorem shows that the dual space of H_α^1 is BMO_α. This corresponds to Fefferman's duality theorem [2, p. 145]. Hence our theorem suggests that BMO_α is useful in studying singular integrals from E_α to itself. The authors express their thanks to Professors Y. Meyer and S. Semmes for some comments about commutators.

2. Throughout this note, we use C for various absolute constants and for various constants depending only on α. For $f \in L^1(dx/(1+x^2))$, we write simply by $f(x, y)$ its Poisson extension to $\boldsymbol{C}_+$. Let $\mathscr{M}f$ denote the non-centered maximal function of f [4, p. 6]. The "if" part is immediately deduced from the following known inequality.

LEMMA 1 ([3]). $\displaystyle\int_0^\infty \mathrm{Cap}_\alpha(x; \mathscr{M}f(x) > \lambda)\lambda d\lambda \leqq C\|f\|_\alpha^2 \quad (f \in E_\alpha)$.

Let $f \in \mathrm{BMO}_\alpha$. Without loss of generality, we may assume that f is real-valued. For real-valued functions $u, v \in C_0^\infty$, we have

$$([f, H]u, v) = (H(fu) - fHu, v) = -(f, Hu\cdot v + uHv) ,$$

where $(\cdot, \cdot)$ is the inner product (with respect to dx). Put $U = u - iHu$ and $V = v - iHv$. Then $\|U\|_\alpha \leqq 2\|u\|_\alpha$, $\|V\|_{-\alpha} \leqq 2\|v\|_{-\alpha}$, and $U(x, y)$, $V(x, y)$ are analytic in $\boldsymbol{C}_+$. Since

$$\left\{\iint_{C_+} |V(x, y)|^2 y^{-1+\alpha} dxdy\right\}^{1/2} = C\|V\|_{-\alpha} \leqq C\|v\|_{-\alpha} ,$$

we have

$$
\begin{aligned}
|([f, H]u, v)| &= |\mathrm{Im}(f, \overline{UV})| \leqq |(f, \overline{UV})| \\
&= C\left|\iint_{C_+} \frac{\partial f}{\partial x}(x, y)U(x, y)V(x, y)dxdy\right| \\
&\leqq C\left\{\iint_{C_+} |\nabla f(x, y)|^2 |U(x, y)|^2 y^{1-\alpha}dxdy\right\}^{1/2}\left\{\iint_{C_+} |V(x, y)|^2 y^{-1+\alpha}dxdy\right\}^{1/2} \\
&\leqq C\|v\|_{-\alpha}\left\{\iint_{C_+} |\nabla f(x, y)|^2 |U(x, y)|^2 y^{1-\alpha}dxdy\right\}^{1/2} .
\end{aligned}
$$

Let $O_\lambda = \{x; \mathscr{M}U(x) > \lambda\}$, $O'_\lambda = \{(x, y); |U(x, y)| > \lambda\}$ $(\lambda > 0)$. Then $O'_\lambda \subset \hat{O}_{\eta\lambda}$ $(\lambda > 0)$ for some absolute constant η [4, p. 85]. Lemma 1 gives that

$$
\begin{aligned}
\iint_{C_+} |\nabla f(x, y)|^2 |U(x, y)|^2 y^{1-\alpha}dxdy &= C\int_0^\infty \left\{\iint_{O'_\lambda} |\nabla f(x, y)|^2 y^{1-\alpha}dxdy\right\}\lambda d\lambda \\
&\leqq C\int_0^\infty \left\{\iint_{\hat{O}_{\eta\lambda}} |\nabla f(x, y)|^2 y^{1-\alpha}dxdy\right\}\lambda d\lambda \leqq C\|f\|^2_{\mathrm{BMO}_\alpha}\int_0^\infty \mathrm{Cap}_\alpha(O_{\eta\lambda})\lambda d\lambda \\
&\leqq C\|f\|^2_{\mathrm{BMO}_\alpha}\|U\|^2_\alpha \leqq C\|f\|^2_{\mathrm{BMO}_\alpha}\|u\|^2_\alpha ,
\end{aligned}
$$

which shows that

$$
|([f, H]u, v)| \leqq C\|f\|_{\mathrm{BMO}_\alpha}\|u\|_\alpha\|v\|_{-\alpha} .
$$

Thus $\|[f, H]\|_{\alpha,\alpha} \leqq C\|f\|_{\mathrm{BMO}_\alpha}$, where $\|[f, H]\|_{\alpha,\alpha}$ is the norm of $[f, H]$ from E_α to itself. This completes the proof of the "if" part.

3. The main part of this note is the proof of the "only if" part. We see easily the following lemma.

LEMMA 2. *For* $f \in L^1(dx/(1 + x^2))$, $s > 0$, *we put* $f_s(x) = f(x, s)$ *and* $F_s = f_s - iHf_s$. *Then*

$$
\frac{1}{2}\|[F_s, H]\|_{\alpha,\alpha} \leqq \|[f_s, H]\|_{\alpha,\alpha} \leqq \|[f, H]\|_{\alpha,\alpha} .
$$

Let BMO denote the Banach space of functions f, modulo constants, with norm

$$
\|f\|_{\mathrm{BMO}} = \sup \frac{1}{|I|}\int_I |f(x) - (f)_I|dx ,
$$

where $(f)_I$ is the mean of f over I and the supremum is taken over all intervals I. We show:

LEMMA 3. $\|f\|_{\mathrm{BMO}} \leqq C\|[f, H]\|_{\alpha,\alpha}$.

PROOF. For an interval I, χ denotes its characteristic function and $\lambda(x) = (x - x_0)\chi(x)$, where x_0 is the midpoint of I. We have

$$
\begin{aligned}
|I|\int_I |f(x)-(f)_I|\,dx &= \int_I\left|\int_I (f(x)-f(y))dy\right|dx \\
&= \int_I\left|\int_I \frac{f(y)-f(x)}{y-x}\{(y-x_0)-(x-x_0)\}dy\right|dx \\
&= \pi\int_{-\infty}^{\infty} |\chi(x)[f,H]\lambda(x)-\lambda(x)[f,H]\chi(x)|\,dx\,.
\end{aligned}
$$

Note that $\|\chi\|_\alpha = C|I|^{(1-\alpha)/2}$ and $\|\lambda\|_\alpha = C|I|^{(3-\alpha)/2}$. Let $g=|[f,H]\chi|$. Then Parseval's equality shows that

$$
\begin{aligned}
\|g\|_\alpha &= C\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|g(x)-g(y)|^2}{|x-y|^{1+\alpha}}dxdy\right\}^{1/2} \\
&\leqq C\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|[f,H]\chi(x)-[f,H]\chi(y)|^2}{|x-y|^{1+\alpha}}dxdy\right\}^{1/2} \\
&= C\|[f,H]\chi\|_\alpha \leqq C\|[f,H]\|_{\alpha,\alpha}\|\chi\|_\alpha \leqq C\|[f,H]\|_{\alpha,\alpha}|I|^{(1-\alpha)/2}\,,
\end{aligned}
$$

and hence

$$
\begin{aligned}
\int_{-\infty}^{\infty}\lambda(x)g(x)dx &\leqq |I|\int_{-\infty}^{\infty}\chi(x)\overline{g(x)}dx = C|I|\int_{-\infty}^{\infty}\hat{\chi}(\xi)\overline{\hat{g}(\xi)}d\xi \\
&\leqq C|I|\left\{\int_{-\infty}^{\infty}|\xi|^{-\alpha}|\hat{\chi}(\xi)|^2d\xi\right\}^{1/2}\left\{\int_{-\infty}^{\infty}|\xi|^{\alpha}|\hat{g}(\xi)|^2d\xi\right\}^{1/2} \\
&= C|I|^{1+(1+\alpha)/2}\|g\|_\alpha \leqq C\|[f,H]\|_{\alpha,\alpha}|I|^2\,.
\end{aligned}
$$

Let $h=|[f,H]\lambda|$. Then, in the same manner as above,

$$
\|h\|_\alpha \leqq C\|[f,H]\|_{\alpha,\alpha}\|\lambda\|_\alpha \leqq C\|[f,H]\|_{\alpha,\alpha}|I|^{(3-\alpha)/2}\,,
$$

and hence

$$
\int_{-\infty}^{\infty}\chi(x)h(x)dx \leqq C|I|^{(1+\alpha)/2}\|h\|_\alpha \leqq C\|[f,H]\|_{\alpha,\alpha}|I|^2\,.
$$

Consequently we have

$$
\frac{1}{|I|}\int_I |f(x)-(f)_I|\,dx \leqq C\|[f,H]\|_{\alpha,\alpha}\,,
$$

which shows that $\|f\|_{\mathrm{BMO}} \leqq C\|[f,H]\|_{\alpha,\alpha}$. q.e.d.

Let $L^2(1-\alpha, \boldsymbol{C})$ denote the L^2 space on the complex plane $\boldsymbol{C}$ with respect to the measure $|y|^{1-\alpha}dxdy$. The norm is denoted by $\|\cdot\|_{L^2(1-\alpha,\boldsymbol{C})}$. Let T^* be an operator defined by

$$
T^*u(x,y) = \sup_{\varepsilon>0}\left|\iint_{|\zeta-z|>\varepsilon}\frac{u(s,t)}{(\zeta-z)^2}dsdt\right| \quad (\zeta = s+it,\ z = x+iy)\,.
$$

The 2-dimensional (centered) maximal operator $\tilde{\mathscr{M}}$ is defined by

$$\tilde{\mathscr{M}}u(x, y) = \sup_{\varepsilon>0} \frac{1}{\pi\varepsilon^2} \iint_{D((x,y),\varepsilon)} |u(s, t)| dsdt ,$$

where $D((x, y), \varepsilon)$ is the open disk of center (x, y) and of radius ε. It is well-known that T^*, $\tilde{\mathscr{M}}$ are bounded operators from $L^2(1-\alpha, \boldsymbol{C})$ to itself [4, pp. 21 and 56]. For $\beta \in \boldsymbol{R}$ and a function $u(x, y)$ in $\boldsymbol{C}_+$, we write

$$|||u|||_\beta = \left\{\iint_{\boldsymbol{C}_+} |u(x, y)|^2 y^\beta dxdy\right\}^{1/2} .$$

We show:

LEMMA 4. *Let $g(x, y)$ be a function in $\boldsymbol{C}_+$ such that $|||g|||_{1+\alpha} < \infty$. Then there exists a function $h(x, y)$ in $\boldsymbol{C}_+$ such that*

$$\bar{\partial} h(x, y) = g(x, y) \quad ((x, y) \in \boldsymbol{C}_+) , \quad |||h|||_{-1+\alpha} \leqq C|||g|||_{1+\alpha} ,$$

where $\bar{\partial} = (\partial/\partial x + i\partial/\partial y)/2$.

PROOF. We put

$$h(x, y) = \frac{2i}{\pi} \iint_{C_+} \frac{g(s, t)t}{(\zeta - z)(\bar{\zeta} - z)} dsdt \quad (\zeta = s + it,\ z = x + iy) .$$

(This form was communicated by Professor S. Semmes; in the earlier draft, the authors did not use this form.) Then $\bar{\partial} h(x, y) = g(x, y)$ $((x, y) \in \boldsymbol{C}_+)$. Suppose that the support of g is compact and contained in $\boldsymbol{C}_+$. Then we have easily $|||h|||_{-1+\alpha} < \infty$. For $u \in L^2(1-\alpha, \boldsymbol{C})$, we have

$$\left|\iint_{C_+} h(x, y)u(x, y)dxdy\right| = \left|\iint_{C_+} g(s, t)Su(s, t)tdsdt\right| \leqq |||g|||_{1+\alpha} |||Su|||_{1-\alpha} ,$$

where

$$Su(s, t) = \frac{2}{\pi} \iint_{C_+} \frac{u(x, y)}{(\zeta - z)(\bar{\zeta} - z)} dxdy \quad (\zeta = s + it,\ z = x + iy) .$$

We have, for $(s, t) \in \boldsymbol{C}_+$,

$$Su(s, t) = \frac{2}{\pi} \iint_{D((s,t),t)} + \frac{2}{\pi} \iint_{C_+ - D((s,t),t)} = L_1 + L_2 ,$$

$$|L_1| \leqq \frac{C}{t} \iint_{D((s,t),t)} \frac{|u(x, y)|}{|\zeta - z|} dxdy \leqq C\tilde{\mathscr{M}}u(s, t)$$

and

$$|L_2| \leqq \frac{2}{\pi} \left|\iint_{C_+ - D((s,t),t)} \left\{\frac{1}{(\zeta - z)(\bar{\zeta} - z)} - \frac{1}{(\zeta - z)^2}\right\} u(x, y)dxdy\right| + \frac{2}{\pi} T^*u(s, t) \leqq C\tilde{\mathscr{M}}u(s, t) + \frac{2}{\pi} T^*u(s, t) .$$

Hence

$$|Su(s,t)| \leqq C\tilde{\mathscr{M}}u(s,t) + \frac{2}{\pi}T^*u(s,t) \quad ((s,t) \in \boldsymbol{C}_+) .$$

Consequently

$$(1) \qquad \left|\iint_{C_+} h(x,y)u(x,y)dxdy\right| \leqq C|||g|||_{1+\alpha}\{\|\tilde{\mathscr{M}}u\|_{L^2(1-\alpha,\boldsymbol{C})} + \|T^*u\|_{L^2(1-\alpha,\boldsymbol{C})}\} \leqq C|||g|||_{1+\alpha}\|u\|_{L^2(1-\alpha,\boldsymbol{C})} .$$

We now choose

$$u(x,y) = \begin{cases}\overline{h(x,y)}y^{-1+\alpha} & ((x,y)\in \boldsymbol{C}_+)\\ 0 & ((x,y)\in \boldsymbol{C}-\boldsymbol{C}_+) .\end{cases}$$

Since $\|u\|_{L^2(1-\alpha,C)} = |||h|||_{-1+\alpha} < \infty$, (1) yields that $|||h|||_{-1+\alpha} \leqq C|||g|||_{1+\alpha}$.

In the general case, we restrict g to $\{(x,y); |x|\leqq n, 1/n \leqq y \leqq n\}$; say g_n. Let h_n be the function corresponding to g_n. Then

$$|||h_n|||_{-1+\alpha} \leqq C|||g_n|||_{1+\alpha} \leqq C|||g|||_{1+\alpha} .$$

Letting n tend to infinity, we obtain $|||h|||_{-1+\alpha} \leqq C|||g|||_{1+\alpha}$. q.e.d.

LEMMA 5. *Let f be a differentiable function on $\boldsymbol{R}$ satisfying $\sup\{|f(x)| + |f'(x)|; x\in\boldsymbol{R}\} < \infty$. We put*

$$D^\alpha f(x) = -i\int_0^\infty \frac{f(x+t)-f(x-t)}{t^{1+\alpha}}dt .$$

Then, for any $u \in C_0^\infty$,

$$\left\{\iint_{C_+} |(D^\alpha f)(x,y)|^2|\nabla U(x,y)|^2 y^{1+\alpha}dxdy\right\}^{1/2} \leqq C\|f\|_{\mathrm{BMO}}\|u\|_\alpha ,$$

where $U = u - iHu$.

PROOF. Since

$$|||\nabla U|||_{1-\alpha} = C\|U\|_\alpha \leqq C\|u\|_\alpha ,$$

it is sufficient to show that

$$(2) \qquad |(D^\alpha f)(x,y)| \leqq C\|f\|_{\mathrm{BMO}}/y^\alpha \quad ((x,y)\in\boldsymbol{C}_+) .$$

Without loss of generality, we may assume that $x = 0$. We have, with $I = (-y, y)$,

$$\begin{aligned}|(D^\alpha f)(0,y)| &= \frac{1}{\pi}\left|\int_0^\infty \frac{1}{t^{1+\alpha}}\left\{\int_{-\infty}^\infty\left(\frac{y}{(s-t)^2+y^2} - \frac{y}{(s+t)^2+y^2}\right)(f(s)-(f)_I)ds\right\}dt\right| \\ &\leqq \frac{1}{\pi}\left|\int_0^{2y}\frac{1}{t^{1+\alpha}}\left\{\int_{-\infty}^\infty\right\}dt\right| + \frac{1}{\pi}\left|\int_{2y}^\infty \frac{1}{t^{1+\alpha}}\left\{\int_I\right\}dt\right| + \frac{1}{\pi}\left|\int_{2y}^\infty\frac{1}{t^{1+\alpha}}\left\{\int_{I^c}\right\}dt\right| \\ &= L_1 + L_2 + L_3 .\end{aligned}$$

Since

$$\int_{-\infty}^{\infty}\frac{y}{s^2+y^2}|f(s)-(f)_I|\,ds \leqq C\|f\|_{\mathrm{BMO}} \quad ([2,\ \text{p. } 142])\ ,$$

$$\begin{aligned}|L_1| &\leqq C\int_0^{2y}\frac{1}{t^{1+\alpha}}\left\{\frac{t}{y}\int_{-\infty}^{\infty}\frac{y}{s^2+y^2}|f(s)-(f)_I|\,ds\right\}dt\\ &\leqq C\int_0^{2y}\frac{1}{t^{\alpha}}dt\,\|f\|_{\mathrm{BMO}}/y \leqq C\|f\|_{\mathrm{BMO}}/y^{\alpha}\ .\end{aligned}$$

We have

$$|L_2| \leqq C\int_{2y}^{\infty}\frac{1}{t^{1+\alpha}}\left\{\frac{1}{y}\int_I|f(s)-(f)_I|\,ds\right\}dt \leqq C\|f\|_{\mathrm{BMO}}/y^{\alpha}\ .$$

It remains to estimate L_3. We have

$$\begin{aligned}&\int_{2y}^{\infty}\frac{1}{t^{1+\alpha}}\left\{\int_{I^c}\frac{y}{(s-t)^2+y^2}(f(s)-(f)_I)\,ds\right\}dt\\ &\quad=\int_{2y}^{\infty}\frac{1}{t^{1+\alpha}}\left\{\int_{-\infty}^{-y}\right\}dt+\sum_{n=1}^{\infty}\int_{2^ny}^{2^{n+1}y}\frac{1}{t^{1+\alpha}}\left\{\int_y^{2^{n-1}y}\right\}dt\\ &\qquad+\sum_{n=1}^{\infty}\int_{2^ny}^{2^{n+1}y}\frac{1}{t^{1+\alpha}}\left\{\int_{2^{n-1}y}^{2^{n+2}y}\right\}dt+\sum_{n=1}^{\infty}\int_{2^ny}^{2^{n+1}y}\frac{1}{t^{1+\alpha}}\left\{\int_{2^{n+2}y}^{\infty}\right\}dt\\ &\quad=L_{31}+L_{32}+L_{33}+L_{34}\ .\end{aligned}$$

We have easily $|L_{31}| \leqq C\|f\|_{\mathrm{BMO}}/y^{\alpha}$. Since

$$|(f)_I-(f)_{(y,2^{n-1}y)}| \leqq Cn\|f\|_{\mathrm{BMO}} \quad (n \geqq 2)\ ([2,\ \text{p. } 142])\ ,$$

$$\begin{aligned}|L_{32}| &\leqq C\sum_{n=2}^{\infty}\int_{2^ny}^{2^{n+1}y}\frac{1}{t^{1+\alpha}}\left\{\frac{1}{2^ny}\int_y^{2^{n-1}y}|f(s)-(f)_I|\,ds\right\}dt\\ &\leqq C\sum_{n=2}^{\infty}\int_{2^ny}^{2^{n+1}y}\frac{1}{t^{1+\alpha}}\left\{\frac{1}{2^ny}\int_y^{2^{n-1}y}|f(s)-(f)_{(y,2^{n-1}y)}|\,ds+n\|f\|_{\mathrm{BMO}}\right\}dt\\ &\leqq C\|f\|_{\mathrm{BMO}}\sum_{n=2}^{\infty}n\int_{2^ny}^{2^{n+1}y}\frac{1}{t^{1+\alpha}}dt \leqq C\|f\|_{\mathrm{BMO}}/y^{\alpha}\ .\end{aligned}$$

We have

$$\begin{aligned}|L_{34}| &\leqq C\sum_{n=1}^{\infty}\int_{2^ny}^{2^{n+1}y}\frac{1}{t^{1+\alpha}}\left\{\int_{2^{n+2}y}^{\infty}\frac{y}{s^2+y^2}|f(s)-(f)_I|\,ds\right\}dt\\ &\leqq C\int_{2y}^{\infty}\frac{1}{t^{1+\alpha}}dt\int_{-\infty}^{\infty}\frac{y}{s^2+y^2}|f(s)-(f)_I|\,ds \leqq C\|f\|_{\mathrm{BMO}}/y^{\alpha}\ .\end{aligned}$$

Since $|(f)_I-(f)_{(t-y,t+y)}| \leqq Cn\|f\|_{\mathrm{BMO}}$ $(t\in(2^ny,\ 2^{n+1}y))$,

$$\begin{aligned}|L_{33}| &\leqq \sum_{n=1}^{\infty}\int_{2^n y}^{2^{n+1}y}\frac{1}{t^{1+\alpha}}\left\{\int_{2^{n-1}y}^{2^{n+2}y}\frac{y}{(s-t)^2+y^2}|f(s)-(f)_{(t-y,t+y)}|ds + Cn\|f\|_{\mathrm{BMO}}\right\}dt \\ &\leqq \sum_{n=1}^{\infty}\int_{2^n y}^{2^{n+1}y}\frac{1}{t^{1+\alpha}}\left\{\int_{-\infty}^{\infty}\frac{y}{(s-t)^2+y^2}|f(s)-(f)_{(t-y,t+y)}|ds\right\}dt + C\|f\|_{\mathrm{BMO}}/y^{\alpha} \\ &\leqq C\|f\|_{\mathrm{BMO}}\left\{\sum_{n=1}^{\infty}\int_{2^n y}^{2^{n+1}y}\frac{1}{t^{1+\alpha}}dt + 1/y^{\alpha}\right\} \leqq C\|f\|_{\mathrm{BMO}}/y^{\alpha}\,.\end{aligned}$$

Consequently,

$$\left|\int_{2y}^{\infty}\frac{1}{t^{1+\alpha}}\left\{\int_{I^c}\frac{y}{(s-t)^2+y^2}(f(s)-(f)_I)ds\right\}dt\right| \leqq C\|f\|_{\mathrm{BMO}}/y^{\alpha}\,. \tag{3}$$

In the same manner, we have (3) with $(s-t)^2+y^2$ replaced by $(s+t)^2+y^2$. Thus $|L_3| \leqq C\|f\|_{\mathrm{BMO}}/y^{\alpha}$. This completes the proof of (2). q.e.d.

We now show the main lemma in this note.

LEMMA 6. *Let F_s be the function in Lemma 2. Then, for any $u \in C_0^{\infty}$,*

$$\left\{\iint_{C_+}|(D^{\alpha}F_s)(x, y)|^2|U(x, y)|^2 y^{1-\alpha}dxdy\right\}^{1/2} \leqq C\|[f, H]\|_{\alpha,\alpha}\|u\|_{\alpha}\,,$$

where $U = u - iHu$.

PROOF. Let V be a function in $L^1(dx/(1+x^2))$ such that $V(x, y)$ is analytic in C_+. Parseval's equality and Lemma 2 show that

$$\begin{aligned}\left|\iint_{C_+}\overline{(D^{\alpha}F_s)(x, y)}U(x, y)V(x, y)y^{-1+\alpha}dxdy\right| &= C|(F_s, UV)| \\ &= C|([F_s, H]\bar{U}, V)| \leqq C\|[F_s, H]\|_{\alpha,\alpha}\|\bar{U}\|_{\alpha}\|V\|_{-\alpha} \\ &\leqq C\|[f, H]\|_{\alpha,\alpha}\|u\|_{\alpha}\|V\|_{-\alpha}\,.\end{aligned} \tag{4}$$

Let $g(x, y) = (D^{\alpha}F_s)(x, y)\partial\overline{U(x, y)}$ $(\partial = (\partial/\partial x - i\partial/\partial y)/2)$. Then Lemmas 2, 3 and 5 show that

$$|||g|||_{1+\alpha} \leqq C\|F_s\|_{\mathrm{BMO}}\|U\|_{\alpha} \leqq C\|[F_s, H]\|_{\alpha,\alpha}\|u\|_{\alpha} \leqq C\|[f, H]\|_{\alpha,\alpha}\|u\|_{\alpha}\,.$$

Let $h(x, y)$ be the function associated with $g(x, y)$ in Lemma 4. We put

$$V_0(x, y) = (D^{\alpha}F_s)(x, y)\overline{U(x, y)} - h(x, y)\,. \tag{5}$$

Then $V_0(x, y)$ is analytic in C_+ ($V_0(x, y)$ is an analytic extension of a function in $L^1(dx/(1+x^2))$, say $V_0(x)$) and

$$\begin{aligned}\|V_0\|_{-\alpha} = C|||V_0|||_{-1+\alpha} &\leqq C|||(D^{\alpha}F_s)\bar{U}|||_{-1+\alpha} + C|||h|||_{-1+\alpha} \\ &\leqq C|||(D^{\alpha}F_s)U|||_{-1+\alpha} + C\|[f, H]\|_{\alpha,\alpha}\|u\|_{\alpha}\,.\end{aligned}$$

Hence, by (4),

$$|||(D^\alpha F_s)U|||_{-1+\alpha}^2 = \left|\iint_{C_+} \overline{(D^\alpha F_s)(x, y)}U(x, y)\{V_0(x, y) + h(x, y)\}y^{-1+\alpha}dxdy\right|$$
$$\leqq C\|[f, H]\|_{\alpha,\alpha}\|u\|_\alpha\|V_0\|_{-\alpha} + |||(D^\alpha F_s)U|||_{-1+\alpha}|||h|||_{-1+\alpha}$$
$$\leqq C\{|||(D^\alpha F_s)U|||_{-1+\alpha}\|[f, H]\|_{\alpha,\alpha}\|u\|_\alpha + \|[f, H]\|_{\alpha,\alpha}^2\|u\|_\alpha^2\} ,$$

which shows that

$$|||(D^\alpha F_s)U|||_{-1+\alpha} \leqq C\|[f, H]\|_{\alpha,\alpha}\|u\|_\alpha . \qquad \text{q.e.d.}$$

LEMMA 7. *Let F_s, u and U be the same as in Lemma 6. We put*

$$U_{F_s}(x, y) = \int_0^\infty (U(x, y + t) - U(x, y))(D^\alpha F_s)(x, y + t)t^{-2+\alpha}dt .$$

Then

$$|||U_{F_s}|||_{1-\alpha} \leqq C\|[f, H]\|_{\alpha,\alpha}\|u\|_\alpha .$$

PROOF. We may assume that $\|[f, H]\|_{\alpha,\alpha} = 1$. Inequality (2) combined with Lemmas 2 and 3 shows that

$$|U_{F_s}(x, y)|^2 \leqq C\left\{\int_0^\infty \frac{|U(x, y + t) - U(x, y)|}{t^{2-\alpha}(y + t)^\alpha}dt\right\}^2$$
$$= C\left\{\int_0^y + \int_y^\infty\right\}^2 \leqq C\left\{y^{-\alpha}\int_0^y |U(x, y + t) - U(x, y)|^2 t^{-3+\alpha}dt\right.$$
$$\left. + y^{-1}\int_y^\infty |U(x, y + t) - U(x, y)|^2 t^{-2}dt\right\} ,$$

and hence

$$|||U_{F_s}|||_{1-\alpha}^2 \leqq C\iint_{C_+}\left\{\int_0^\infty |U(x, y + t) - U(x, y)|^2 t^{-3+\alpha}dt\right\}y^{1-2\alpha}dxdy$$
$$+ C\iint_{C_+}\left\{\int_0^\infty |U(x, y + t) - U(x, y)|^2 t^{-2}dt\right\}y^{-\alpha}dxdy$$
$$= C\int_0^\infty |\hat{U}(\xi)|^2\left(\int_0^\infty e^{-2\xi y}y^{1-2\alpha}dy\right)\left(\int_0^\infty |e^{-\xi t} - 1|^2 t^{-3+\alpha}dt\right)d\xi$$
$$+ C\int_0^\infty |\hat{U}(\xi)|^2\left(\int_0^\infty e^{-2\xi y}y^{-\alpha}dy\right)\left(\int_0^\infty |e^{-\xi t} - 1|^2 t^{-2}dt\right)d\xi$$
$$= C\int_0^\infty \xi^\alpha|\hat{U}(\xi)|^2 d\xi = C\|U\|_\alpha^2 \leqq C\|u\|_\alpha^2 . \qquad \text{q.e.d.}$$

4. We now show the "only if" part. Let $f \in L^1(dx/(1 + x^2))$ satisfy $\|[f, H]\|_{\alpha,\alpha} < \infty$. We may assume that f is real-valued. Let F_s, u, U, V_0 be the functions in Lemma 6 and (5). Then

$$L = \left|\iint_{C_+} \overline{\left(\frac{\partial}{\partial x}F_s\right)(x, y)}U(x, y)V_0(x, y)dxdy\right|$$

$$
\begin{aligned}
&= C|(F_s, UV_0)| = C|([F_s, H]\bar{U}, V_0)| \\
&\leqq C\|[f, H]\|_{\alpha,\alpha}\|u\|_\alpha\|V_0\|_{-\alpha} \\
&\leqq C\|[f, H]\|_{\alpha,\alpha}\|u\|_\alpha\{|||(D^\alpha F_s)U|||_{-1+\alpha} + \|[f, H]\|_{\alpha,\alpha}\|u\|_\alpha\} \\
&\leqq C\|[f, H]\|^2_{\alpha,\alpha}\|u\|^2_\alpha
\end{aligned}
$$

and

$$
\begin{aligned}
L &\geqq \left|\iint_{C_+}\overline{\Big(\frac{\partial}{\partial x}F_s\Big)(x, y)}U(x, y)(D^\alpha F_s)(x, y)\overline{U(x, y)}dxdy\right| \\
&\quad - \left|\left|\left|\Big(\frac{\partial}{\partial x}F_s\Big)U\right|\right|\right|_{1-\alpha}|||h|||_{-1+\alpha} \\
&\geqq \left|\iint_{C_+}\overline{\Big(\frac{\partial}{\partial x}F_s\Big)(x, y)U(x, y)}U(x, y)(D^\alpha F_s)(x, y)dxdy\right| \\
&\quad - C\left|\left|\left|\Big(\frac{\partial}{\partial x}F_s\Big)U\right|\right|\right|_{1-\alpha}\|[f, H]\|_{\alpha,\alpha}\|u\|_\alpha .
\end{aligned}
$$

We put $G(x, y) = U(x, y)(D^\alpha F_s)(x, y)$ and

$$
\tilde{D}^{1-\alpha}G(x, y) = -\int_0^\infty (G(x, y+t) - G(x, y))t^{-2+\alpha}dt .
$$

Since

$$
\begin{aligned}
\tilde{D}^{1-\alpha}G(x, y) &= -U(x, y)\int_0^\infty((D^\alpha F_s)(x, y+t) - (D^\alpha F_s)(x, y))t^{-2+\alpha}dt \\
&\quad - U_{F_s}(x, y) = CU(x, y)\Big(\frac{\partial}{\partial x}F_s\Big)(x, y) - U_{F_s}(x, y) ,
\end{aligned}
$$

we have, by Lemma 7,

$$
\begin{aligned}
&\left|\iint_{C_+}\overline{\Big(\frac{\partial}{\partial x}F_s\Big)(x, y)U(x, y)}U(x, y)(D^\alpha F_s)(x, y)dxdy\right| \\
&\quad = \left|\iint_{C_+}\overline{\Big(\frac{\partial}{\partial x}F_s\Big)(x, y)U(x, y)}\tilde{D}^{1-\alpha}G(x, y)y^{1-\alpha}dxdy\right| \\
&\quad = \left|C\iint_{C_+}\left|\Big(\frac{\partial}{\partial x}F_s\Big)(x, y)\right|^2|U(x, y)|^2y^{1-\alpha}dxdy\right. \\
&\qquad \left. - \iint_{C_+}\overline{\Big(\frac{\partial}{\partial x}F_s\Big)(x, y)U(x, y)}U_{F_s}(x, y)y^{1-\alpha}dxdy\right| \\
&\quad \geqq C\left|\left|\left|\Big(\frac{\partial}{\partial x}F_s\Big)U\right|\right|\right|^2_{1-\alpha} - \left|\left|\left|\Big(\frac{\partial}{\partial x}F_s\Big)U\right|\right|\right|_{1-\alpha}|||U_{F_s}|||_{1-\alpha} \\
&\quad \geqq C|||\,|\nabla F_s|U|||^2_{1-\alpha} - C|||\,|\nabla F_s|U|||_{1-\alpha}\|[f, H]\|_{\alpha,\alpha}\|u\|_\alpha .
\end{aligned}
$$

Thus

$$\| \| |\nabla F_s| U \| \|_{1-\alpha}^2 \leqq C \| \| |\nabla F_s| U \| \|_{1-\alpha} \|[f, H]\|_{\alpha,\alpha} \|u\|_\alpha + C\|[f, H]\|_{\alpha,\alpha}^2 \|u\|_\alpha^2 ,$$

which gives that

$$(6) \qquad \iint_{C_+} |\nabla F_s(x, y)|^2 |U(x, y)|^2 y^{1-\alpha} dxdy \leqq C\|[f, H]\|_{\alpha,\alpha}^2 \|u\|_\alpha^2 \quad (u \in C_0^\infty) .$$

The standard argument shows that (6) holds for any $u \in E_\alpha$.

Let O be an open set in $\boldsymbol{R}$ with $\mathrm{Cap}_\alpha(O) < \infty$. Then there exists a non-negative function $u_O \in E_\alpha$ such that $\|u_O\|_\alpha^2 = \mathrm{Cap}_\alpha(O)$ and $u_O(x) \geqq C$ on O [5, p. 138]. Let $U_O = u_O - iHu_O$. Then $|U_O(x, y)| \geqq C$ on $\hat{O}$. Hence (6) shows that

$$\iint_{\hat{O}} |\nabla F_s(x, y)|^2 y^{1-\alpha} dxdy \leqq C \| \| |\nabla F_s| U_O \| \|_{1-\alpha}^2$$
$$\leqq C\|[f, H]\|_{\alpha,\alpha}^2 \|u_O\|_\alpha^2 = C\|[f, H]\|_{\alpha,\alpha}^2 \mathrm{Cap}_\alpha(O) .$$

Thus

$$\iint_{\hat{O}} |\nabla f(x, y+s)|^2 y^{1-\alpha} dxdy = \iint_{\hat{O}} |\nabla f_s(x, y)|^2 y^{1-\alpha} dxdy$$
$$\leqq \iint_{\hat{O}} |\nabla F_s(x, y)|^2 y^{1-\alpha} dxdy \leqq C\|[f, H]\|_{\alpha,\alpha}^2 \mathrm{Cap}_\alpha(O) .$$

Letting s tend to 0,

$$\iint_{\hat{O}} |\nabla f(x, y)|^2 y^{1-\alpha} dxdy \leqq C\|[f, H]\|_{\alpha,\alpha}^2 \mathrm{Cap}_\alpha(O) .$$

Since O is arbitrary, we have $\|f\|_{\mathrm{BMO}_\alpha} \leqq C\|[f, H]\|_{\alpha,\alpha}$. This completes the proof of the "only if" part.

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