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SPECTRUM OF THE SCHRÖDINGER OPERATOR ON A LINE BUNDLE OVER COMPLEX PROJECTIVE SPACES

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1. Introduction. Let $\pi: E \to (M, g)$ be a C^∞ Hermitian line bundle over a C^∞ Riemannian manifold ($n = \dim M$). Let us consider a Schrödinger operator D with a vector potential and a scalar potential, which operates on $C^\infty(E)$, the space of C^∞ sections of E. The operator D is a second order, self-adjoint, elliptic differential operator locally expressed with respect to a local unitary frame of E and local coordinates of M as

$$\begin{aligned} D &= -\sum_{j,k=1}^{n} g^{jk}(\nabla_j + ia_j)(\nabla_k + ia_k) + V \\ &= -\sum_{j,k} g^{jk}\nabla_j\nabla_k - 2i\sum_{j,k} g^{jk}a_j\nabla_k + \sum_{j,k} g^{jk}(a_ja_k - i\nabla_j a_k) + V , \end{aligned}$$

where ∇ is the Levi-Civita connection defined by g, $\alpha = \sum a_j dx^j$ is a locally defined real 1-form, and V is a real C^∞ function on M. The local 1-form $\omega = i\alpha$ is regarded as the connection form of a linear connection $\tilde{d}$ on E which is compatible with the Hermitian structure (cf. [7]). Thus we have a one-to-one correspondence:

$$D \leftrightarrow (M, g; E, \tilde{d}; V) .$$

When $V \equiv 0$, D is called the *Bochner-Laplacian* associated with $\tilde{d}$ (and g), and moreover if $E = M \times \boldsymbol{C}$ and $\tilde{d}$ is a flat connection, then $D = \Delta$ (the Laplace-Beltrami operator). It is interesting to investigate the asymptotic distribution of large eigenvalues of D under the influences of $\tilde{d}$ and V.

Guillemin [4], Weinstein [12], etc. clarified how the scalar potential V exerts an effect upon the spectrum for the case where (M, g) is the sphere or the projective space with the canonical metric. In the previous papers [8] and [9] we made clear for a line bundle over the sphere the relationship between the holonomies of the connection $\tilde{d}$ and the spectrum of the Bochner-Laplacian.

This article studies the effects of the connection $\tilde{d}$ on the spectrum

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of D for a line bundle over the complex projective space CP^n. Let $\mathscr{C}(CP^n)$ be the set of pure-imaginary closed C^∞ 2-forms Ω on CP^n such that $[\Omega/2\pi i]$ is integral. Then, one has the following (cf. Kostant [6]).

PROPOSITION. (1) *The set of equivalence classes of C^∞ complex line bundles over CP^n is in one-to-one correspondence with $Z \cong H^2(CP^n, Z) = \{[\Omega/2\pi i]; \Omega \in \mathscr{C}(CP^n)\}$ ($[\Omega/2\pi i]$ being called the Chern class of the line bundle).*

(2) *For each $\Omega \in \mathscr{C}(CP^n)$,* (i) *there exists a unique (up to the gauge equivalence) linear connection $\tilde{d}$ on the line bundle E with the Chern class $[\Omega/2\pi i]$, whose curvature form is given by Ω, and* (ii) *there exists a Hermitian structure h on E such that the connection $\tilde{d}$ is compatible with h. Conversely, if a 2-form Ω on CP^n satisfies* (i) *and* (ii), *then Ω belongs to $\mathscr{C}(CP^n)$.*

(3) *The Hermitian structure h in the above* (2), (ii) *is given uniquely (up to scalar multiple) on each line bundle E without depending on connections on E.*

Let $\{[E_m]; m \in Z\}$ (m: the Chern number) be the set of equivalence classes of Hermitian line bundles over CP^n. On each line bundle E_m there is a unique *harmonic connection* $\tilde{d}_m$ whose curvature form is a harmonic 2-form (Hodge's theorem). The purposes of this paper are

(1) to compute explicitly the spectrum of the Bochner-Laplacian for the harmonic connection (Proposition 2.3),

(2) to describe for any connection the asymptotic behavior of the spectrum of D in terms of its holonomies along closed geodesics (Theorem 3.1), and

(3) to show that the geometric structure $(E_m, \tilde{d}_m; V = \text{const.})$ is characterized by its spectrum for CP^n ($n \geqq 2$) (Theorem 4.5, which is a generalization of Guillemin's result [4]).

2. Spectra for the harmonic connections. In this section we construct the harmonic connection $\tilde{d}_m$ on each line bundle E_m ($m \in Z$: the Chern number) over CP^n, and compute the spectrum of the Bochner-Laplacian associated with $\tilde{d}_m$.

Consider the complex vector space $C^{n+1} = \{z = (z_0, \cdots, z_n)\}$ with the Hermitian inner product:

$$\langle z, z' \rangle = \sum_{j=0}^{n} z_j \bar{z}'_j \,.$$

C^{n+1} with the real inner product $\langle \cdot, \cdot \rangle_R$ given by

$$\langle z, z' \rangle_R = \sum_{j=0}^{n} (x_j x'_j + y_j y'_j) = \mathrm{Re}\langle z, z' \rangle$$

($z_j = x_j + iy_j$, $z'_j = x'_j + iy'_j$), is identified with $\boldsymbol{R}^{2n+2}$. Let

$$S^{2n+1}_{[2]} = \{z;\ |z|^2 = \langle z, z\rangle = 4\} \subset \boldsymbol{C}^{n+1} \cong \boldsymbol{R}^{2n+2}$$

be the $2n+1$ dimensional sphere with radius 2, and let $\tilde{g}_0$ be the Riemannian metric on it induced from $\langle\cdot, \cdot\rangle_{\boldsymbol{R}}$ in $\boldsymbol{C}^{n+1}$. The circle group $S^1 = \{\varepsilon(t) = e^{it};\ 0 \leqq t \leqq 2\pi\}$ acts freely and isometrically on $S^{2n+1}_{[2]}$ as $z\cdot\varepsilon(t) = e^{it}z$, and we get the Hopf fibre bundle:

$$S^1 \to S^{2n+1}_{[2]} \xrightarrow{\tilde{\pi}} \boldsymbol{C}P^n . \tag{2.1}$$

The tangent bundle of $S^{2n+1}_{[2]}$ is given by

$$TS^{2n+1}_{[2]} = \{(z, u);\ z \in S^{2n+1}_{[2]},\ u \in \boldsymbol{C}^{n+1},\ \langle z, u\rangle_{\boldsymbol{R}} = 0\} .$$

For $z \in S^{2n+1}_{[2]}$, let $V_z = (d\tilde{\pi}_z)^{-1}(0) \subset T_z S^{2n+1}_{[2]}$, and

$$V_z = \{(z, i\lambda z);\ \lambda \in \boldsymbol{R}\} .$$

Let H_z be the orthogonal complement of V_z in $T_z S^{2n+1}_{[2]}$ with respect to the Hermitian product $\langle\cdot, \cdot\rangle$. Thus

$$T_z S^{2n+1}_{[2]} = H_z \oplus V_z \tag{2.2}$$

(H_z being given by $\{(z, u);\ \langle z, u\rangle = 0\}$). Let us define the Riemannian metric g_0 on $\boldsymbol{C}P^n$ so that $d\tilde{\pi}_z\colon H_z \to T_{\tilde{\pi}(z)}\boldsymbol{C}P^n$ is an isometry. Then, g_0 is the Fubini-Study metric of constant holomorphic sectional curvature 1, and all the geodesics of $(\boldsymbol{C}P^n, g_0)$ are closed and have a common length 2π.

The fibration (2.1) is a principal S^1-bundle, and the decomposition (2.2) defines a connection on it (H_z: the horizontal space), whose connection form $\tilde{\omega}$ on $S^{2n+1}_{[2]}$ is given by

$$\tilde{\omega}_z\colon u \mapsto -\frac{i}{4}\langle u, z\rangle \in \boldsymbol{R} \quad (z \in S^{2n+1}_{[2]},\ u \in T_z S^{2n+1}_{[2]}) ,$$

or written as $\tilde{\omega} = \iota^*\theta$ with

$$\theta = -\frac{i}{2|z|^2}\sum_{j=0}^{n}(\bar{z}_j dz_j - z_j d\bar{z}_j) ,$$

$\iota\colon S^{2n+1}_{[2]} \to \boldsymbol{C}^{n+1}$ being the inclusion map. The curvature form $\tilde{\Omega} = d\tilde{\omega}$ is given by $\tilde{\Omega} = \iota^*\Theta$ with

$$\Theta = \frac{i}{|z|^4}\Big(|z|^2\sum_{j=0}^{n} dz_j \wedge d\bar{z}_j - \sum_{j,k=0}^{n}\bar{z}_j z_k dz_j \wedge d\bar{z}_k\Big) .$$

The real 2-form Θ is invariant under the natural action of $\boldsymbol{C}\setminus 0$ on $\boldsymbol{C}^{n+1}\setminus 0$, and is regarded as a 2-form on $\boldsymbol{C}P^n$ represented with respect to homogeneous coordinates.

LEMMA 2.1. *The 2-form Θ on CP^n is harmonic with respect to the metric g_0, and $[\Theta/2\pi]$ is a generator for the second cohomology group $H^2(CP^n, \boldsymbol{Z}) \cong \boldsymbol{Z}$.*

PROOF. (CP^n, g_0) is regarded as a Riemannian symmetric space $U(n+1)/(U(n)\times U(1))$, and Θ is invariant under the action of $U(n+1)$. Hence Θ is harmonic (see [11, p. 26]). Next, consider a 2-dimensional closed submanifold $CP^1 \subset CP^n$. By straightforward calculation using the coordinates $\zeta_j = z_j/z_0$ $(j = 1, \cdots, n)$ we obtain

$$\frac{1}{2\pi}\int_{CP^1}\Theta = \frac{-1}{2\pi i}\int_{C}\frac{d\zeta_1 \wedge d\bar{\zeta}_1}{(1+|\zeta_1|^2)^2} = 1\,. \tag{2.3}$$

Thus the lemma is proved.

For each integer m, let ρ_m be the representation of S^1 on $\boldsymbol{C}$ defined by $\rho_m(\varepsilon(t))w = \varepsilon(t)^{-m}w$ $(w \in \boldsymbol{C})$. Let $\pi\colon E_m \to CP^n$ denote the line bundle associated with the principal bundle (2.1) by the representation ρ_m, that is, the quotient manifold of $S^{2n+1}_{[2]} \times \boldsymbol{C}$ with respect to the equivalence relation $(z, w) \sim (z\cdot\varepsilon, \varepsilon^m w)(\varepsilon \in S^1)$. For each $z \in S^{2n+1}_{[2]}$, define $q_z\colon \boldsymbol{C} \to \pi^{-1}(\tilde{\pi}(z))$ by $w \mapsto [(z, w)]$. Let $C^\infty_m(S^{2n+1}_{[2]})$ be the set consisting of every C^∞ function f on $S^{2n+1}_{[2]}$ such that

$$f(z\cdot\varepsilon) = \varepsilon^m f(z) \tag{2.4}$$

for every $z \in S^{2n+1}_{[2]}$ and $\varepsilon \in S^1$, which is called an equivariant function with respect to ρ_m. For $s \in C^\infty(E_m)$, define a C^∞ function $q^\sharp_m s$ on $S^{2n+1}_{[2]}$ by $(q^\sharp_m s)(z) = q_z^{-1}(s(\tilde{\pi}(z))$. Then $q^\sharp_m s$ belongs to $C^\infty_m(S^{2n+1}_{[2]})$ and $q^\sharp_m$ gives a one-to-one correspondence between $C^\infty(E_m)$ and $C^\infty_m(S^{2n+1}_{[2]})$. Let $\tilde{d}_m$ be the linear connection on E_m associated with the connection (2.2) on the principal bundle (2.1), which is defined as the covariant derivative:

$$\tilde{\nabla}^{(m)}_X s = (q^\sharp_m)^{-1} X^\sharp q^\sharp_m s\,,$$

$s \in C^\infty(E_m)$, X being a vector field on CP^n and $X^\sharp$ the horizontal lift of X to $S^{2n+1}_{[2]}$. We calculate the curvature form $\Omega_m(X, Y) = [\tilde{\nabla}^{(m)}_X, \tilde{\nabla}^{(m)}_Y] - \tilde{\nabla}^{(m)}_{[X,Y]}$ of $\tilde{d}_m$ and get $\Omega_m = -im\Theta$. Thus we have the following by Lemma 2.1.

LEMMA 2.2 (1) *The set of equivalence classes of complex line bundles over CP^n is $\{E_m; m \in \boldsymbol{Z}\}$.*

(2) *$\tilde{d}_m$ is a unique harmonic connection on E_m, whose curvature form is $\Omega_m = -im\Theta$.*

REMARK. $\tilde{d}_m$ is the canonical connection (cf. [13, pp. 77–84]) with respect to the holomorphic line bundle structure and the Hermitian structure uniquely given on E_m.

Now we study the Bochner-Laplacian associated with the harmonic connection $\tilde{d}_m$, and compute its spectrum denoted by Spec(L_m). Consider a set of C^∞ vector fields $\{X_1, \cdots, X_{2n}\}$ defined on a neighborhood of $w \in \boldsymbol{C}P^n$ such that $g_0(X_j, X_k)(w) = \delta_{jk}$. Then,

$$(L_m s)(w) = -\sum_{j=1}^{2n} (\tilde{\nabla}^{(m)}_{X_j} \tilde{\nabla}^{(m)}_{X_j} s)(w)$$
$$= -\Big(\sum_{j=1}^{2n} (q_m^\sharp)^{-1}(X_j^\sharp)^2(q_m^\sharp s)\Big)(w) \quad (s \in C^\infty(E_m)) .$$

If we set $L_m^\sharp = (q_m^\sharp)L_m(q_m^\sharp)^{-1}$, $L_m^\sharp$ is a differential operator acting on $C_m^\infty(S_{[2]}^{2n+1})$ and

$$(L_m^\sharp \tilde{s})(z) = -\Big(\sum_{j=1}^{2n} (X_j^\sharp)^2 \tilde{s}\Big)(z) = \Big(\Big(\Delta + \frac{1}{|Z|^2} Z^2\Big)\tilde{s}\Big)(z)$$

holds for $\tilde{s} \in C_m^\infty(S_{[2]}^{2n+1})$, where $w = \tilde{\pi}(z)$, Δ is the Laplace-Beltrami operator on $(S_{[2]}^{2n+1}, \tilde{g}_0)$, and $Z \in V_z$ is the infinitesimal generator of the action of $S^1 = \{\varepsilon(t)\}$. For $\tilde{s} \in C_m^\infty(S_{[2]}^{2n+1})$ we have $Z\tilde{s} = im\tilde{s}$ from (2.4), hence,

$$L_m^\sharp \tilde{s} = \Delta \tilde{s} - \frac{m^2}{4}\tilde{s} . \tag{2.5}$$

Thus, if $L_m s = \lambda s$, then $L_m^\sharp \tilde{s} = \lambda \tilde{s}$ ($\tilde{s} = q_m^\sharp s$) and $\Delta \tilde{s} = (\lambda + m^2/4)\tilde{s}$, that is, $\tilde{s}$ is an eigenfunction of Δ.

Let $\mathscr{P}_{p,q}$ be the space of homogeneous polynomials of degree p in $z \in \boldsymbol{C}^{n+1}$ and of degree q in $\bar{z}$ ($p, q = 0, 1, 2, \cdots$), and let $\mathscr{H}_{p,q}$ be the subspace of $\mathscr{P}_{p,q}$ consisting of harmonic ones, i.e., $P \in \mathscr{P}_{p,q}$ such that

$$\Delta_0 P = -\frac{1}{4}\sum_{j=0}^{n} \frac{\partial^2 P}{\partial z_j \partial \bar{z}_j} = 0 .$$

We denote the space of restrictions of elements in $\mathscr{P}_{p,q}$ (resp. $\mathscr{H}_{p,q}$) to $S_{[2]}^{2n+1} (\subset \boldsymbol{C}^{n+1})$ by $\tilde{\mathscr{P}}_{p,q}$ (resp. $\tilde{\mathscr{H}}_{p,q}$). It is easy to see (cf. [1, pp. 159–160]) that each element of $\tilde{\mathscr{H}}_{p,q}$ is an eigenfunction of Δ on $(S_{[2]}^{2n+1}, \tilde{g}_0)$ with the eigenvalue $(p + q)(p + q + 2n)/4$. Since we have $P(e^{it}z, \overline{e^{it}z}) = e^{i(p-q)t}P(z, \bar{z})$ for $P(z, \bar{z}) \in \mathscr{P}_{p,q}$, $\tilde{\mathscr{P}}_{p,q}$ belongs to $C_m^\infty(S_{[2]}^{2n+1})$ if and only if $p - q = m$. Therefore, for

$$P \in \begin{cases} \tilde{\mathscr{H}}_{k+|m|,k} & (m \geqq 0) \\ \tilde{\mathscr{H}}_{k,k+|m|} & (m < 0) \end{cases} \tag{2.6}$$

we have from (2.5)

$$L_m^\sharp P = \Big\{\Big(k + \frac{|m|}{2}\Big)\Big(k + \frac{|m|}{2} + n\Big) - \frac{m^2}{4}\Big\}P$$

($k = 0, 1, 2, \cdots$). Noting that

$$\tilde{\mathscr{P}}_{p,q} = \tilde{\mathscr{H}}_{p,q} \oplus \tilde{\mathscr{P}}_{p-1,q-1} = \bigoplus_{l=0}^{r} \tilde{\mathscr{H}}_{p-l,q-l} \quad (r = \min(p, q))$$

(cf. [1, p. 160]), we see that $\bigoplus_{p-q=m} \tilde{\mathscr{H}}_{p,q}$ is L^2-dense in $C_m^\infty(S_{[2]}^{2n+1})$, and

$$\begin{aligned}\dim \tilde{\mathscr{H}}_{p,q} &= \dim \tilde{\mathscr{P}}_{p,q} - \dim \tilde{\mathscr{P}}_{p-1,q-1} \\ &= \binom{n+p}{p}\binom{n+q}{q} - \binom{n+p-1}{p-1}\binom{n+q-1}{q-1}.\end{aligned}$$

Noticing that $q_m^\#: C^\infty(E_m) \to C_m^\infty(S_{[2]}^{2n+1})$ is L^2 bi-continuous, we get the following.

PROPOSITION 2.3. *Let $E_m \to (CP^n, g_0)$ ($m \in \boldsymbol{Z}$) be a complex line bundle with the harmonic connection $\tilde{d}_m$ whose curvature form is $\Omega_m = -im\Theta$, and let L_m be the Bochner-Laplacian associated with $\tilde{d}_m$. Then,* Spec(L_m) *consists of eigenvalues*

$$\lambda_k^{(m)} = \left(k + \frac{|m|}{2}\right)\left(k + \frac{|m|}{2} + n\right) - \frac{m^2}{4}, \quad k = 0, 1, 2, \cdots,$$

where the multiplicity of $\lambda_k^{(m)}$ is equal to

$$\binom{k+|m|+n}{k+|m|}\binom{k+n}{k} - \binom{k+|m|+n-1}{k+|m|-1}\binom{k+n-1}{k-1},$$

and the space of eigensections associated with $\lambda_k^{(m)}$ is $\{s = (q_m^\#)^{-1}P$; P being given by (2.6)$\}$.

REMARK. This result is a generalization of Theorem 5.1 in [7] where we considered the case of line bundles over $CP^1 = S^2$

3. Holonomies and spectrum. Let $E \to (CP^n, g_0)$ be a Hermitian line bundle with a linear connection $\tilde{d}$ compatible with the Hermitian structure, and let V be a real C^∞ function on CP^n. We will study the asymptotic distribution of large eigenvalues of the operator D associated with $(E, \tilde{d}; V)$, and derive a result similar to that in [9].

Let $Q_{\tilde{d}}(c)$ denote the *holonomy* of $\tilde{d}$ along a closed curve c in CP^n. Each element (x, ξ) of the unit cosphere bundle $S^*CP^n = \{(x, \xi) \in T^*CP^n;$ $|\xi| = (\sum g_0^{jk}(x)\xi_j\xi_k)^{1/2} = 1\}$ corresponds to a closed geodesic γ of (CP^n, g_0). Hence we have a C^∞ map

$$\bar{Q}_{\tilde{d}}: S^*CP^n \to S^1 = \{e^{2\pi i\theta}; 0 \leqq \theta < 1\}$$

by $\bar{Q}_{\tilde{d}}(x, \xi) = Q_{\tilde{d}}(\gamma)$. On the manifold S^*CP^n there exists the volume form $d\,\mathrm{vol}$ induced from the symplectic volume form $dx_1 \wedge \cdots \wedge dx_{2n} \wedge d\xi_1 \wedge \cdots \wedge d\xi_{2n}$ on T^*CP^n.

For $0 \leqq a < b < 1$, we set

$$\tilde{J}[a, b] = \{e^{2\pi i\theta}; a \leqq \theta \leqq b\} \subset S^1 ,$$

and

$$J_k[a, b] = [\lambda_k + aC_k, \lambda_k + bC_k] , \quad k = 0, 1, 2, \cdots ,$$

where

$$\lambda_k = \left(k + \frac{n}{2}\right)^2 ,$$

and $C_k = \lambda_{k+1} - \lambda_k = 2k + n + 1$. Let

$$\mathrm{Spec}(D) = \{(0 \leqq) \mu_0 \leqq \mu_1 \leqq \cdots \leqq \mu_j \leqq \cdots\} .$$

Then we have the following.

THEOREM 3.1. *Suppose* $\mathrm{vol}\{\bar{Q}_{\tilde{d}}^{-1}(e^{2\pi i a})\} = \mathrm{vol}\{\bar{Q}_{\tilde{d}}^{-1}(e^{2\pi i b})\} = 0$. *Then*

$$\#\{\mu_j \in J_k[a, b]\} = (2\pi)^{-2n} \mathrm{vol}\{\bar{Q}_{\tilde{d}}^{-1}(\tilde{J}[a, b])\} k^{2n-1} + o(k^{2n-1})$$

as $k \to \infty$, *where* $\#$ *denotes the cardinality.*

REMARK. The highest order term in the above expansion does not depend on the scalar potential V.

We prove this theorem as an application of the theorem by Colin de Verdière [2].

First we note the following.

LEMMA 3.2. *For the harmonic connection* $\tilde{d}_m$ *on* E_m,

$$\bar{Q}_{\tilde{d}_m}(x, \xi) = (-1)^m$$

holds for every $(x, \xi) \in S^*CP^n$.

PROOF. The group $U(n+1)$ acts transitively on the space of all closed geodesics in (CP^n, g_0), and the curvature form $\Omega_m = -im\Theta$ of $\tilde{d}_m$ is invariant under the action of $U(n+1)$ on CP^n. Therefore we see that

$$Q_{\tilde{d}_m}(\gamma) = \exp\left(-\int_\Sigma \Omega_m\right)$$

(Σ: a surface with $\partial\Sigma = \gamma$) takes constant value for every closed geodesic γ of (CP^n, g_0). From (2.3) we get $Q_{\tilde{d}_m}(\gamma) = e^{\pi i m} = (-1)^m$.

On the other hand, the spectrum of the operator associated with $(E_m, \tilde{d}_m; 0)$ consists of

$$\lambda_k^{(m)} = \begin{cases} \lambda_{k+p} - \left(p^2 + \dfrac{n^2}{4}\right) & (|m| = 2p) \\ \lambda'_{k+p} - \left(p^2 + p + \dfrac{n^2}{4} + \dfrac{1}{2}\right) & (|m| = 2p+1) \end{cases} \tag{3.1}$$

with $k=0, 1, 2, \cdots$, where $\lambda_k' = \lambda_k + (C_k/2)$.

We set

$$P_m = \left(L_m + \frac{m^2+n^2}{4}\right)^{1/2} .$$

Then P_m is a self-adjoint elliptic pseudo-differential operator of order 1 operating on $C^\infty(E_m)$ with the principal symbol

$$\sigma(P_m)(x, \xi) = (\sum g_0^{jk}(x)\xi_j\xi_k)^{1/2} = |\xi| ,$$

and eigenvalues of P_m are $k + \{(|m|+n)/2\}$ $(k=0, 1, 2, \cdots)$. Suppose the operator D is defined on E_m, and we put

$$D = L_m + Q .$$

Let $\{e_\kappa\}$ be a family, associated with an open covering of CP^n, of local sections of E_m such that $|e_\kappa| = 1$. Let $\{\omega_\kappa = i\alpha_\kappa\}$ (resp. $\{\omega_\kappa^{(m)} = i\alpha_\kappa^{(m)}\}$) be the system of connection forms of $\tilde{d}$ (resp. $\tilde{d}_m$) with respect to $\{e_\kappa\}$. Then $\beta_\kappa = \alpha_\kappa - \alpha_\kappa^{(m)}$ does not depend on κ and defines a global real 1-form β on CP^n, and Q is represented locally as

$$Q = -2i \sum_{j,k=1}^{2n} g_0^{jk} b_j \nabla_k + V + \sum_{j,k=1}^{2n} g_0^{jk}(b_j b_k - i\nabla_j b_k + 2a_j^{(m)} b_k) ,$$

where $\beta = \sum b_j dx^j$ and $\alpha_\kappa^{(m)} = \sum a_j^{(m)} dx^j$. Consider the averaged operator of Q:

$$Q_{\mathrm{av}} = \frac{1}{2\pi}\int_0^{2\pi} \exp(-itP_m) Q \exp(itP_m) dt ,$$

which is a self-adjoint pseudo-differential operator of order 1 and the following lemma is obtained.

LEMMA 3.3. (1) *The principal symbol $\sigma(Q_{\mathrm{av}})$ of Q_{av} is homogeneous of degree* 1 *in ξ, and satisfies*

$$(3.2) \qquad (-1)^m \exp\{-\pi i \sigma(Q_{\mathrm{av}})(x, \xi)\} = \bar{Q}_{\tilde{d}}(x, \xi)$$

for $(x, \xi) \in S^*CP^n$.

(2) $[P_m, Q_{\mathrm{av}}] = 0$.

(3) *Let* $\mathrm{Spec}(L_m + Q_{\mathrm{av}}) = \{\mu_j'\}_{j=0}^\infty$. *Then there exists a constant C not depending on j such that*

$$(3.3) \qquad |\mu_j' - \mu_j| \leqq C .$$

PROOF. (1) Let $(x(t), \xi(t))(0 \leqq t \leqq 2\pi)$ be a closed orbit of the Hamiltonian flow associated with $\sigma(P_m)$, which is just the geodesic flow, on T^*CP^n through (x, ξ). Let $\omega = i\alpha = i(\alpha^{(m)} + \beta)$ be the connection 1-

form of $\tilde{d}$ with respect to a unitary frame of E_m over a neighborhood of the closed geodesic $\gamma = \{x(t); 0 \leqq t \leqq 2\pi\}$. By Egorov's theorem we have

$$\begin{aligned}\sigma(Q_{\mathrm{av}})(x, \xi) &= \frac{1}{2\pi}\int_0^{2\pi}\sigma(Q)(x(t), \xi(t))dt \\ &= \frac{|\xi|}{\pi}\int_0^{2\pi}\sum_j b^j(x(t))\frac{\xi_j}{|\xi|}(t)dt = \frac{|\xi|}{\pi}\int_\gamma \beta\,,\end{aligned}$$

$(b^j = \sum g_0^{jk} b_k)$. On the other hand, by virtue of Lemma 3.2, we have for $(x, \xi) \in S^*CP^n$,

$$\bar{Q}_{\tilde{d}}(x, \xi) = \exp\Big(-\int_\gamma \omega\Big) = \exp\Big(-i\int_\gamma (\alpha^{(m)} + \beta)\Big) = (-1)^m \exp\Big(-i\int_\gamma \beta\Big)\,.$$

Hence we get (3.2).

(2) and (3) are the same as [12, Lemma 1.1] and [9, Lemma 3.2].

We carry out the proof of Theorem 3.1 along the same line as in [9] by applying the theorem of Colin de Verdière [2] to the commuting operators (P_m, Q_{av}). Let $\Lambda = \{(\bar{\lambda}_{k,j}^{(m)}, \kappa_{k,j})\}$ be the set of eigenvalues of (P_m, Q_{av}), where

$$\bar{\lambda}_{k,1}^{(m)} = \cdots = \bar{\lambda}_{k,N_k}^{(m)} = \bar{\lambda}_k^{(m)} = k + \frac{|m| + n}{2}$$

(N_k being the multiplicity of $\bar{\lambda}_k^{(m)}$). Then, for $\mathrm{Spec}(L_m + Q_{\mathrm{av}}) = \{\mu'_{k,j}\}$, we have

$$\mu'_{k,j} = \lambda_{k,j}^{(m)} + \kappa_{k,j} \quad (\text{where } \lambda_{k,j}^{(m)} = \lambda_k^{(m)})\,,$$

$\kappa_{k,j}$ being the difference between $\mu'_{k,j}$ and $\lambda_k^{(m)}$.

LEMMA 3.4 (see [9, Lemma 3.3]). *Let*

$$M = \underset{(x,\xi) \in S^*CP^n}{\mathrm{Max}} \sigma(Q_{\mathrm{av}})(x, \xi) \quad (\geqq 0)\,.$$

Then we have

$$|\kappa_{k,j}| \leqq Mk + M'\,,$$

M' being some positive constant.

For $0 \leqq a < b < 1$ we set

$$a'(\text{resp. } b') = \begin{cases} a & (\text{resp. } b) \qquad \text{if } m \text{ is even}\,, \\ a - \frac{1}{2}\Big(\text{resp. } b - \frac{1}{2}\Big) & \text{if } m \text{ is odd}\,. \end{cases}$$

For sufficiently small $\varepsilon > 0$ we consider the following conic subsets of

$R^2 = \{(x_1, x_2)\}$:

$$C_N^{\pm\varepsilon} = \{(x_1, x_2); 2(a' \mp \varepsilon + N)x_1 \leqq x_2 \leqq 2(b' \pm \varepsilon + N)x_1, x_1 > 0\} ,$$

for $N = 0, \pm 1, \pm 2, \cdots$. By virtue of (3.1) and (3.3) we have

$$\#\left\{\bigcup_{N=-q}^{q}(\Lambda \cap C_N^{-\varepsilon} \cap \{x_1 = \bar{\lambda}_{k-N}^{(m)}\})\right\} \leqq \#\{\mu_j \in J_k[a, b]\} \leqq \#\left\{\bigcup_{N=-q}^{q}(\Lambda \cap C_N^{+\varepsilon} \cap \{x_1 = \bar{\lambda}_{k-N}^{(m)}\})\right\}$$

for sufficiently large k, where q is an integer satisfying $2q > M$ for the constant M in Lemma 3.4. Let

$$p = (\sigma(P_m), \sigma(Q_{\mathrm{av}})): T^*CP^n \setminus 0 \to R^2 ,$$

and let $\widetilde{\mathrm{vol}}$ denote the volume on the hypersurface $\sigma(P_m)(x, \xi) = |\xi| = \text{const.}$ induced from that on T^*CP^n. If $a' \pm \varepsilon + N$ and $b' \pm \varepsilon + N$ $(-q \leqq N \leqq q)$ are regular values of $\sigma(Q_{\mathrm{av}})|_{S^*CP^n}$, then by the theorem of Colin de Verdière [2, Theorem 0.8] we have

$$\begin{aligned}
&\#\left\{\bigcup_{N=-q}^{q}(\Lambda \cap C_N^{\pm\varepsilon} \cap \{x_1 = \bar{\lambda}_{k-N}^{(m)}\})\right\} \\
&\quad = \sum_N (2\pi)^{-2n}\, \widetilde{\mathrm{vol}}\{p^{-1}(C_N^{\pm\varepsilon} \cap \{x_1 = \bar{\lambda}_{k-N}^{(m)}\})\} + O(k^{2n-2}) \\
&\quad = (2\pi)^{-2n} \sum_N \mathrm{vol}\{(x, \xi) \in S^*CP^n; 2(a' \mp \varepsilon + N) \leqq \sigma(Q_{\mathrm{av}})(x, \xi) \leqq 2(b' \pm \varepsilon + N)\} \\
&\qquad \times \left(k - N + \frac{|m| + n}{2}\right)^{2n-1} + O(k^{2n-2}) \\
&\quad = (2\pi)^{-2n}\, \mathrm{vol}\{\bar{Q}_{\tilde{a}}^{-1}(\tilde{J}^{\pm\varepsilon})\} k^{2n-1} + O(k^{2n-2}) ,
\end{aligned}$$

where $\tilde{J}^{\pm\varepsilon} = \exp(2\pi i[a \mp \varepsilon, b \pm \varepsilon]) \subset S^1$, and the last equality is derived from (3.2). Now, instead of ε we choose a sequence $\{\varepsilon_\nu(>0); \nu = 1, 2, \cdots\}$ such that $a' \pm \varepsilon_\nu + N$ and $b' \pm \varepsilon_\nu + N$ are regular values of $\sigma(Q_{\mathrm{av}})|_{S^*CP^n}$, and $\varepsilon_\nu \downarrow 0$ as $\nu \to \infty$ (Sard's theorem). Note that $\mathrm{vol}\{\bar{Q}_{\tilde{a}}^{-1}(e^{2\pi i a})\} = \mathrm{vol}\{\bar{Q}_{\tilde{a}}^{-1}(e^{2\pi i b})\} = 0$ by assumption, and we obtain Theorem 3.1 by $\nu \to \infty$.

REMARK. In general, for a vector bundle over a $C_{2\pi}$-manifold we have in [10] a formula similar to that in Theorem 3.1 about the asymptotic distribution of the spectrum.

4. Cluster theorem. We give the following definition for the spectrum $\mathrm{Spec}(D) = \{\mu_j\}$ of the operator D.

DEFINITION. The spectrum of D is said to *make clusters of type* $\{a\}$ $(0 \leqq a < 1)$ if there is a constant M such that

$$\mathrm{Spec}(D) \subset \bigcup_{k=0}^{\infty}[\lambda_k + aC_k - M, \lambda_k + aC_k + M] .$$

Noticing (3.1) and that V is regarded as a bounded operator, we see that the spectrum of $D = L_m + V$ makes clusters of type $\{0\}$ if m is even, or of type $\{1/2\}$ if m is odd. Moreover, we have the following.

THEOREM 4.1. *Let D be the Schrödinger operator associated with $(E, \tilde{d}; V)$ over CP^n. A necessary and sufficient condition for the spectrum of D to make clusters is that $\tilde{d}$ is a harmonic connection if $n \geqq 2$, and is that the curvature form Ω of $\tilde{d}$ is an odd 2-form, i.e., $\tau^*\Omega = -\Omega$ for the antipodal map $\tau: CP^1 \to CP^1$ if $n = 1$.*

PROOF. By virtue of Theorem 3.1 if the spectrum of D makes clusters of type $\{a\}$, then $\bar{Q}_{\tilde{d}}(x, \xi) = e^{2\pi i a}$ for every $(x, \xi) \in S^*CP^n$. Note that $\bar{Q}_{\tilde{d}}(x, -\xi) = (\bar{Q}_{\tilde{d}}(x, \xi))^{-1}$, and a must be equal to 0 or 1/2. Moreover, a is equal to 0 (resp. 1/2) if D is defined on E_m with even m (resp. odd m). Indeed, consider a one parameter family $\tilde{d}(s)$ $(0 \leqq s \leqq 1)$ of linear connections with $\tilde{d}(0) = \tilde{d}_m$ and $\tilde{d}(1) = \tilde{d}$, which are defined by the connection forms $\{\omega_\kappa^{(m)} + is\beta\}$, $\omega_\kappa^{(m)}$ being the connection form of $\tilde{d}_m$ and β a real 1-form (cf. § 3). It follows from the continuity of the holonomies $\bar{Q}_{\tilde{d}(s)}(\cdot)$ with respect to s that the types of clusters for $\tilde{d}_m$ and $\tilde{d}$ coincide. Thus, if the spectrum makes clusters, then

$$\int_\gamma \beta = 0 \tag{4.1}$$

holds for every closed geodesic γ of (CP^n, g_0). The proof of the theorem for the case of $n = 1$ has been carried out in [8, Proposition 4.4 and Theorem 4.5]. In the case $n \geqq 2$ the theorem is derived from the following lemma proved by Gasqui and Goldschmidt [3].

LEMMA 4.2. *If a 1-form β on CP^n with $n \geqq 2$ satisfies* (4.1) *for every closed geodesic γ of (CP^n, g_0), then β is exact.*

Next, the distribution of the eigenvalues in the k-th interval $I_k = [\lambda_k + aC_k - M, \lambda_k + aC_k + M]$ is studied in the same way as that by Colin de Verdière. For a real C^∞ function V we define a C^∞ function on S^*CP^n by

$$\hat{V}(x, \xi) = \frac{1}{2\pi}\int_0^{2\pi} V(x(t))dt ,$$

where $x(t)(0 \leqq t \leqq 2\pi)$ is a closed geodesic of (CP^n, g_0) with the initial condition $x(0) = x$, $\dot{x}(0)^* = \xi$ ($*$: $TCP^n \to T^*CP^n$ being the bundle isomorphism defined by g_0). For $a < b$ we set

$$I_k^{(m)}[a, b] = [\lambda_k^{(m)} + a, \lambda_k^{(m)} + b] .$$

PROPOSITION 4.3 ([2]). *Let* $\mathrm{Spec}(L_m + V) = \{\mu_j\}_{j=0}^\infty$. *Suppose* $\mathrm{vol}\{\hat{V}^{-1}(a)\} =$

$\mathrm{vol}\{\hat{V}^{-1}(b)\} = 0$. *Then we have*

$$\#\{\mu_j \in I_k^{(m)}[a, b]\} = (2\pi)^{-2n}\,\mathrm{vol}\{\hat{V}^{-1}([a, b])\}k^{2n-1} + o(k^{2n-1})$$

as $k \to \infty$.

Concerning the function $\hat{V}$, the following is known [5, p. 128].

LEMMA 4.4. *Let* V *be a* C^∞ *function on* $CP^n (n \geqq 2)$ *and* c *be a constant. Then,* $\hat{V} = c$ *if and only if* $V = c$.

As a consequence of the above results we have:

THEOREM 4.5. *Let* D *be the Schrödinger operator associated with* $(E, \tilde{d}; V)$ *over* $CP^n (n \geqq 2)$. *Then,* $\mathrm{Spec}(D) = \{\mu_j\}_{j=0}^{\infty}$ *makes clusters if and only if* $\tilde{d}$ *is a harmonic connection. Moreover when* $\tilde{d}$ *is harmonic,*

$$\operatorname*{Max}_{\mu_i, \mu_j \in I_k} |\mu_i - \mu_j|$$

($I_k = [\lambda_k + aC_k - M, \lambda_k + aC_k + M]$: *the* k*-th cluster*) *tends to zero as* $k \to \infty$ *if and only if* V *is a constant function.*

COROLLARY 4.6. *Let* c *be a constant.* $(E_m, \tilde{d}_m; V \equiv c)$ *over* $CP^n (n \geqq 2)$ *is characterized by the spectrum* $\{\lambda_k^{(m)} + c; k = 0, 1, 2, \cdots\}$ *of the associated operator* D.

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