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INTERSECTION FORMULA FOR STIEFEL-WHITNEY HOMOLOGY CLASSES

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1. Introduction and the statement of results. The purpose of this paper is to study the relationship between the Stiefel-Whitney homology classes of mutually transverse Euler spaces and their intersection in an ambient PL-manifold. Besides manifolds, real analytic spaces are typical examples of mod 2 Euler spaces (cf. Sullivan [11]).

Let (A, B) be a pair of a polyhedron A and a subspace B of A such that rank $H_*(A, B; \boldsymbol{Z}) < \infty$. Denote by $e(A, B)$ the mod 2 Euler number of the pair (A, B). If $B \neq \emptyset$, we write $e(A) = e(A, \emptyset)$.

Let X be a locally compact n-dimensional polyhedron. The polyhedron X is said to be a mod 2 Eulder space (cf. [1], [5]), if the following hold for the subpolyhedron ∂X:

(1) ∂X is $(n-1)$-dimensional or empty.

(2) $e(X, X - x) = \begin{cases} 1 & (x \in X - \partial X) \\ 0 & (x \in \partial X) \end{cases}$

(3) if $\partial X \neq \emptyset$, then $e(\partial X, \partial X - x) = 1$ $(x \in \partial X)$.

Let K be a triangulation of a polyhedron X. Denote by K' the barycentric subdivision of K. If X is an n-dimensional mod 2 Euler space, the sum of all k-simplexes in K' is a mod 2 cycle and defines an element $s_k(X)$ in $H_k(X, \partial X; \boldsymbol{Z}_2)$, which is called the k-th Stiefel-Whitney homology class of X (cf. [1], [5]). We put $s_*(X) = s_0(X) + s_1(X) + \cdots + s_n(X)$. We define the mod 2 fundamental class in $H_n(X, \partial X; \boldsymbol{Z}_2)$ to be $[X] = s_n(X)$. If X is a $\boldsymbol{Z}_2$-homology manifold, then we know that $s_*(X) = [X] \cap w^*(X)$, where $w^*(X)$ is the Stiefel-Whitney cohomology class of X.

Let X be an n-dimensional polyhedron and let K be a triangulation of X. If the union of all n-simplexes are dense in X, the polyhedron is said to be pure n-dimensional. If X is a mod 2 Euler space of pure dimension PL-embedded in a PL-manifold M with $\partial X \subset \partial M$ and $X - \partial X \subset M - \partial M$, then X is called a proper PL-subspace in M. Let a and b be homology classes in $H_*(M, \partial M; \boldsymbol{Z}_2)$. We define the homological intersection by $a \cdot b = [M] \cap (([M] \cap)^{-1} a \cup ([M] \cap)^{-1} b)$.

The main result of this paper is the following:

THEOREM. *Let X and Y be* mod 2 *Euler spaces of pure dimension* *which are proper* PL-*subspaces in a* PL-*manifold* M. *Let* $f: X \to M$, $g: Y \to M$ *and* $h: X \cap Y \to M$ *be the inclusions. If* X *is transverse to* Y, *then* $X \cap Y$ *is a* mod 2 *Euler space and the following holds*:

$$f_* s_*(X) \cdot g_* s_*(Y) = h_* s_*(X \cap Y) \cap w^*(M) .$$

2. **Transversality.** Let X be a polyhedron and let K be a collection of PL-balls in X. We write $|K| = \cup_{\sigma \in K} \sigma$. The collection K is called a ball complex structure on X if the following hold:

(1) X is the disjoint union of the interiors $\operatorname{Int} \sigma$ of all PL-balls σ in K.

(2) If σ is a PL-ball in K, then the boundary $\partial \sigma$ of σ is the union of PL-balls in K.

Now we recall the definition of transversality according to Buoncristiano, Rourke and Sanderson [3]. Let K be a ball complex structure on a PL-manifold M and let X be a subpolyhedron of M. We say that X is collarable in M, if there exists a collar $c: (\partial M, X \cap \partial M) \times I \to (M, X)$. The polyhedron X is transverse to K if $X \cap \sigma$ is collarable in σ for each PL-ball σ in K. Let X and Y be subpolyhedra in M. The polyhedron X is transverse (or mock-transverse) to Y in M, if there is a ball complex structure K on M with a subcomplex L such that $|L| = Y$ and that X is transverse to K (cf. [3]). By McCrory [9], we know that for collarable polyhedra X and Y in an ambient PL-manifold, the polyhedron X is transverse to Y if and only if Y is transverse to X. Other definitions of transversality were given by Armstrong and Zeeman [2], Stone [12] and McCrory [9]. These definitions are equivalent if subpolyhedra are collarable in an ambient PL-manifold (McCrory [9]).

Let X be a subpolyhedron and N be a PL-submanifold in a PL-manifold. The polyhedron X is block transverse to N if there exists a normal block bundle $\nu = (E, i, N)$ of N such that the restriction $(X \cap E, i|(X \cap N), X \cap N)$ of ν to $X \cap N$ is a block bundle over $X \cap N$ (cf. [10]). Then by [3] we have the following:

PROPOSITION 2.1. *The polyhedron* X *is block transverse to* N *if and only if* N *is transverse to* X.

We need the following to prove our theorem.

LEMMA 2.2. *Let* X *and* Y *be collarable subpolyhedra in a* PL-*manifold* M *and* V *a proper* PL-*submanifold in* M. *Suppose that* X *is transverse to* Y *and* V *is transverse to* $X \cup Y$ *in* M. *Then* $X \cap V$ *is transverse to* $Y \cap V$ *in* V.

LEMMA 2.3. *Let X and Y be collarable subpolyhedra in a* PL-*manifold M and V be a proper* PL-*submanifold in M with a normal block bundle $\nu = (E, i, V)$. Let X be transverse to Y and let $X \cup Y$ be block transverse to ν. Then $X \cap V$ and $Y \cap V$ are transverse to $Y \cap E$ and $X \cap E$ in E, respectively.*

PROOF OF LEMMA 2.2. By assumption, there exists a ball complex structure K which contains a ball complex structure of Y and there exists a subdivision K' of K which contains a ball complex structure of $X \cup Y$ such that X and V are transverse to K and K', respectively. Then for each Δ in K, we see that $V \cap \Delta$ is transverse to $X \cap \Delta$ in Δ. By the symmetry of transversality, we see that $X \cap \Delta$ is transverse to $V \cap \Delta$ in Δ. Then there exists a subdivision L of K such that X is transverse to L and that L contains the ball complex structures of Y and V. Consequently we see that $X \cap V$ is transverse to $L|V$ and contains a ball complex structure of $Y \cap V$. Hence $X \cap V$ is transverse to $Y \cap V$ in V.

q.e.d.

PROOF OF LEMMA 2.3. By Proposition 2.1, the PL-manifold V is transverse to $X \cup Y$ in M. Then, by Lemma 2.2, the intersection $X \cap V$ is transverse to $Y \cap V$ in V. In view of the definition of transversality, there exist a ball complex structure K on V and a subcomplex L such that $|L| = Y \cap V$ and that $X \cap V \cap \sigma$ is collarable in σ, for each PL-ball σ in K. Let $E(\sigma)$ be the block over σ of the block bundle ν. Let $K(E)$ be a ball complex structure on E which consists of blocks $E(\sigma)$ and their faces for σ in V. Define a subcomplex $L(E)$ of $K(E)$ by $L(E) = \{\Delta \in K(E) | \Delta \subset Y\}$. Then $|L(E)| = Y \cap E$ and $X \cap V \cap \Delta$ is collarable in Δ for each PL-ball Δ in $K(E)$. Hence $X \cap V$ is transverse to $Y \cap E$ in E. We see that $Y \cap V$ is transverse to $X \cap E$ in E in the same manner. q.e.d.

TRANSVERSALITY THEOREM 2.4 ([3], [9]). *Let X and Y be collarable subpolyhedra of a* PL-*manifold M and let $X \cap \partial M$ be transverse to $Y \cap \partial M$ in ∂M. Then there exists an arbitrarily small ambient isotopy h_t of M such that $h_t|\partial M$ is the identity for all t and that $h_1(X)$ is transverse to Y in M.*

The first half of our theorem is the following proposition:

PROPOSITION 2.5. *Let X and Y be* mod 2 *Euler spaces which are proper* PL-*subspaces in a* PL-*manifold M. If X is transverse to Y, then $X \cap Y$ is a* mod 2 *Euler space with the boundary $\partial X \cap \partial Y$.*

To prove this proposition, we rewrite the definition of mod 2 Euler spaces in the following form:

LEMMA 2.6. *Let X be a polyhedron and let ∂X be a subpolyhedron of X. Let K be a ball complex structure on X and let L be a subcomplex of K such that $|L| = \partial X$. The polyhedron X is a* mod 2 *Euler space with the boundary ∂X if and only if the following holds*:

(1) $^{\#}\{\tau \in K | \tau \gneqq \sigma\}$ *is even for σ in $K - L$.*

(2) $^{\#}\{\tau \in K | \tau \gneqq \sigma\}$ *is odd for σ in L.*

(3) $^{\#}\{\tau \in K | \tau \gneqq \sigma\}$ *is even for σ in L.*

PROOF OF PROPOSITION 2.5. Let K be a ball complex structure on M and let L be a subcomplex such that $|L| = Y$ and that $X \cap \sigma$ is collarable in σ for each PL-ball σ in K. By induction on the codimension of σ, we easily see that $X \cap \sigma$ is a mod 2 Euler space with the boundary $X \cap \partial\sigma$. This means that $X \cap \sigma = X \cap Y \cap \sigma$ is a mod 2 Euler space for each PL-ball σ in L. By Lemma 2.6, we see that $X \cap Y$ is a mod 2 Euler space with the boundary $X \cap \partial Y = \partial X \cap \partial Y$ if Y is a mod 2 Euler space. q.e.d.

3. Characterization of Stiefel-Whitney homology classes. Let $\xi = (E, i, A)$ be a block bundle over a polyhedron A. Denote by $\bar{E}$ the total space of the sphere bundle associated with ξ. Let $\mathfrak{B}_*(E, \bar{E})$ be the bordism group of compact mod 2 Euler spaces. We can define a homomorphism $e_\xi: \mathfrak{B}_*(E, \bar{E}) \to \boldsymbol{Z}_2$ by using the transversality theorem. (See [6] for details.) Let U_ξ be the Thom class of ξ and let $\bar{w}(\xi)$ be the dual Stiefel-Whitney cohomology class of ξ.

We have the following proposition ([6; Lemmas 3.2 and 3.3]):

PROPOSITION 3.1. *For every map $\varphi: X \to E$ in $\mathfrak{B}_*(E, \bar{E})$, we have $\langle U_\xi \cup i^{*-1}\bar{w}(\xi), \varphi_* s_*(X)\rangle = e_\xi(\varphi, X)$. Furthermore, the dual Stiefel-Whitney cohomology class $\bar{w}(\xi)$ is completely characterized by this identity.*

Let M be a PL-manifold and let $\bar{M}$ and $\tilde{M}$ be codimension zero submanifolds of ∂M such that $\partial M = \bar{M} \cup \tilde{M}$ and $\bar{M} \cap \tilde{M} = \partial\bar{M} = \partial\tilde{M}$. Let X be a mod 2 Euler space PL-embedded in M such that $\partial X \subset \tilde{M}$ and $X - \partial X \subset M - \partial M$. We denote by $f: (X, \partial X) \to (M, \tilde{M})$ the inclusion. Let $\mathfrak{N}_*(M, \bar{M})$ be the differentiable unoriented bordism group and let $\mathfrak{B}_*(M, \bar{M})$ be the bordism group of compact mod 2 Euler spaces. We have a natural homomorphism $b: \mathfrak{N}_*(M, \bar{M}) \to \mathfrak{B}_*(M, \bar{M})$. Now we define homomorphisms $\bar{e}_f: \mathfrak{B}_*(M, \bar{M}) \to \boldsymbol{Z}_2$ and $e_f = \bar{e}_f \circ b$. Let $\varphi: V \to M$ be a map in $\mathfrak{B}_*(M, \bar{M})$. Then there exists a PL-embedding $\psi: (V, \partial V) \to (M \times D^k, \bar{M} \times D^k)$ for k sufficiently large, such that $\psi \simeq \varphi \times \{0\}$. By using the transversality theorem, we may assume that $\psi(V)$ is transverse to $X \times D^k$ in $M \times D^k$. Define $\bar{e}_f$ by $\bar{e}_f(\varphi, V) = e(\psi(V) \cap X \times D^k)$, where e takes the mod 2 Euler number. The homomorphism $\bar{e}_f$ is well-defined by the transversality theorem and Proposition 2.5.

PROPOSITION 3.2. *Let* $f: X \to M$ *be as above. For every map* $\varphi: V \to M$ *in* $\mathfrak{N}_*(M, \bar{M})$, *we have* $\langle([M]\cap)^{-1}(f_*s_*(X)\cap \bar{w}(M)), \varphi_*s_*(V)\rangle = e_f(\varphi, V)$. *Furthermore the homology class* $f_*s_*(X)$ *is completely characterized by this identity.*

PROPOSITION 3.3. *In the same situation as in Proposition 3.2, for every map* $\varphi: V \to M$ *in* $\mathfrak{B}_*(M, \bar{M})$, *we have* $\langle([M]\cap)^{-1}(f_*s_*(X)\cap \bar{w}(M)), \varphi_*s_*(V)\rangle = \bar{e}_f(\varphi, V)$.

PROOF OF PROPOSITION 3.2. Let $\psi: \to M\times D^k$ be a map such that $\psi \simeq \varphi\times\{0\}$. Then $\langle([M]\cap)^{-1}(f_*s_*(X)\cap \bar{w}(M)), \psi_*s_*(V)\rangle = \langle([M\times D^k]\cap)^{-1}((f\times \mathrm{id})_*s_*(X\times D^k)\cap \bar{w}(M\times D^k), \psi_*s_*(V)\rangle$. Therefore we have only to give the proof for the case where $\varphi: V\to M$ is a PL-embedding and $\varphi(V)$ is transverse to X in M.

Let $\nu = (E, \varphi_E, V)$ be a normal block bundle of $\varphi: V \to M$ and let U_ν be the Thom class of ν. Then $[E]\cap U_\nu = \varphi_{E*}[V]$. Since $s_*(V) = [V]\cap w^*(V)$, we have

$$\begin{aligned}&\langle([M]\cap)^{-1}(f_*s_*(X)\cap \bar{w}(M)), \varphi_*s_*(V)\rangle\\ &\quad = \langle([M]\cap)^{-1}(f_*s_*(X)\cap \bar{w}(M)), \varphi_*([V]\cap w^*(V))\rangle\\ &\quad = \langle U_\nu \cup \varphi_E^{*-1}\varphi^*\bar{w}(M)\cup \varphi_E^{*-1}w(V), [E]\cap \varphi_E^{*-1}\varphi^*([M]\cap)^{-1}f_*s_*(X)\rangle .\end{aligned}$$

If we define $f_E: X\cap E\to E$ by $f_E(x)=f(x)$, then $[E]\cap\varphi_E^{*-1}\varphi^*([M]\cap)^{-1}f_*s_*(X) = f_{E*}s_*(X\cap E)$. On the other hand, we know that $\varphi^*\bar{w}(M)\cap w^*(V) = \bar{w}(\nu)$. Hence $\langle([M]\cap)^{-1}(f_*s_*(X)\cap \bar{w}(M)), \varphi_*s_*(V)\rangle = \langle U_\nu\cup\varphi_E^{*-1}\bar{w}(\nu), f_{E*}s_*(X\cap E)\rangle$, which is equal to $e(X\cap\varphi(V))$ by Proposition 3.1. In view of the definition of e_f, we have $e_f(\varphi, V) = e(X\cap\varphi(V))$. Thus we obtain the formula. The uniqueness of $f_*s_*(V)$ is clear (cf. [6, Lemma 5.3]). q.e.d.

PROOF OF PROPOSITION 3.3. We can inductively construct a cohomology class $\Phi(f) = \Phi^0(f) + \Phi^1(f) + \cdots + \Phi^n(f)$ in $H^*(M, \bar{M}; \boldsymbol{Z}_2)$ satisfying $\langle\Phi(f), \varphi_*s_*(V)\rangle = \bar{e}_f(\varphi, V)$ for each (φ, V) in $\mathfrak{B}_*(M, \bar{M})$. We define $\Phi^0(f)$ in $H^0(M, \bar{M}; \boldsymbol{Z}_2)$ by $\Phi^0(f)(\varphi_*s_0(V)) = \bar{e}_f(\varphi, V)$ for (φ, V)in $\mathfrak{B}_0(M, \bar{M})$ and $\Phi^k(f)$, $k\geqq 1$, in the same way as in [6]. The uniqueness of such a cohomology class is also obtained and we have $\Phi(f) = ([M]\cap)^{-1}(f_*s_*(X)\cap\bar{w}(M))$ by Proposition 3.2. q.e.d.

4. Proof of the theorem. In order to prove the theorem, we need the following Halperin type formula ([4], [7]), whose proof can be found in [8].

PROPOSITION 4.1. *Let* $\xi = (E, i, X)$ *be a block bundle over a* mod 2

Euler space X. Then $i_*s_*(X) = (s_*(E) \cap U_\xi) \cap i^{*^{-1}}\bar{w}(\xi)$.

PROOF OF THE THEOREM. The case where X and Y are collarable implies the general case. Thus we may suppose that X and Y are collarable in M. Let $p(f, g) = \langle ([M]\cap)^{-1}\{(f_*s_*(X) \cdot g_*s_*(Y) \cap \bar{w}(M)) \cap \bar{w}(M)\}, \varphi_*s_*(V)\rangle$. We will prove that $p(f, g) = e_h(\varphi, V)$ for each (φ, V) in $\mathfrak{N}_*(M, \bar{M})$. This implies our theorem by Proposition 3.2.

Let $\varphi: V \to M$ be a map in $\mathfrak{N}_*(M, \bar{M})$. We can choose a PL-embedding $\psi: V \to M \times D^\alpha$ for α sufficiently large that ψ is homotopic to $\varphi \times \{0\}$: $V \to M \times D^\alpha$ and $\psi(V)$ is transverse to $(X \cup Y) \times D^\alpha$ in $M \times D^\alpha$. Hence we give the proof only when $\varphi: V \to M$ is a PL-embedding such that $\varphi(V)$ is transverse to $X \cup Y$ in M. We thus assume that $\varphi: V \to M$ is a PL-embedding with a normal bundle $\nu = (E, \varphi_E, V)$. We have the following commutative diagram:

$$\begin{array}{ccccc}
X & \xleftarrow{\ j_X\ } & X \cap E & \xleftarrow{\ \varphi_X\ } & X \cap \varphi(V) \\
{\scriptstyle f}\downarrow & & {\scriptstyle f_E}\downarrow & & \\
M & \xleftarrow{\ j\ } & E & \xleftarrow{\ \varphi_E\ } & V \\
{\scriptstyle g}\uparrow & & {\scriptstyle g_E}\uparrow & \nwarrow {\scriptstyle g_V} & \\
Y & \xleftarrow{\ j_Y\ } & Y \cap E & \xleftarrow{\ \varphi_Y\ } & Y \cap \varphi(V)
\end{array}$$

Here all maps except $\varphi_E: V \to E$ are inclusions and $\nu(\varphi_X) = (X \cap E, \varphi_X, X \cap \varphi(V))$ and $\nu(\varphi_Y) = (Y \cap E, \varphi_Y, Y \cap \varphi(V))$ are block bundles. Let U_E be the Thom class of the normal block bundle $\nu = (E, \varphi_E, V)$, that is, $[E] \cap U_E = \varphi_{E*}[V]$. Let $\bar{w}(\nu)$ be the dual Stiefel-Whitney cohomology class of the normal block bundle ν. Note that $\bar{w}(\nu) = \varphi^*\bar{w}(M) \cup w^*(V)$ and $s_*(V) = [V] \cap w^*(V)$. Then we have

$$\begin{aligned}
p(f, g) &= \langle ([M]\cap)^{-1}\{(f_*s_*(X) \cdot g_*s_*(Y) \cap \bar{w}(M)) \cap \bar{w}(M)\}, \varphi_*s_*(V)\rangle \\
&= \langle ([M]\cap)^{-1}f_*s_*(X) \cup ([M]\cap)^{-1}g_*s_*(Y) \cup \bar{w}(M) \\
&\quad \cup \bar{w}(M), \varphi_*([V] \cap w^*(V))\rangle \\
&= \langle U_E \cup \varphi_E^{*^{-1}}\bar{w}(\nu) \cup \varphi_E^{*^{-1}}\varphi^*([M]\cap)^{-1}g_*s_*(Y) \\
&\quad \cup \varphi_E^{*^{-1}}\varphi^*\bar{w}(M), [E] \cap \varphi_E^{*^{-1}}\varphi^*([M]\cap)^{-1}f_*s_*(X)\rangle .
\end{aligned}$$

By the naturality of the Stiefel-Whitney homology classes and simple calculation, we have

$$\varphi_E^{*^{-1}}\varphi^*([M]\cap)^{-1}g_*s_*(Y) = ([E]\cap)^{-1}g_{E*}s_*(Y \cap E)$$

and

$$[E] \cap \varphi_E^{*^{-1}}\varphi^*([M]\cap)^{-1}f_*s_*(X) = f_{E*}s_*(X \cap E) .$$

Let U_Y be the Thom class of the block bundle $\nu(\varphi_Y) = (Y \cap E, \varphi_Y, Y \cap \varphi(V))$, that is, $[Y \cap E] \cap U_Y = \varphi_{Y*}[Y \cap \varphi(V)]$. Then

$$\begin{aligned} & U_E \cup \varphi_E^{*^{-1}} \bar{w}(\nu) \cup \varphi_E^{*^{-1}} \varphi^*([M] \cap)^{-1} g_* s_*(Y) \\ &= U_E \cup \varphi_E^{*^{-1}} \bar{w}(\nu) \cup ([E] \cap)^{-1} g_{E*} s_*(Y \cap E) \\ &= ([E] \cap)^{-1} (g_{E*} s_*(Y \cap E) \cap \{U_E \cup \varphi_E^{*^{-1}} \bar{w}(\nu)\}) \\ &= ([E] \cap)^{-1} g_{E*}(\{s_*(Y \cap E) \cap U_Y\} \cap \varphi_Y^{*^{-1}} \bar{w}(\nu(\varphi_Y))\,. \end{aligned}$$

By Proposition 4.1, we have $(s_*(Y \cap E) \cap U_Y) \cap \varphi_Y^{*^{-1}} \bar{w}(\nu(\varphi_Y)) = \varphi_{Y*} s_*(Y \cap \varphi(V))$. Noting that $\varphi_E^{*^{-1}} \varphi^* \bar{w}(M) = j^* \bar{w}(M) = \bar{w}(E)$, we have

$$\begin{aligned} p(f, g) &= \langle ([E] \cap)^{-1} g_{E*} \varphi_{Y*} s_*(Y \cap \varphi(V)) \cup \bar{w}(E)), f_{E*} s_*(X \cap E) \rangle \\ &= \langle ([E] \cap)^{-1} g_{V*}(s_*(Y \cap \varphi(V)) \cap \bar{w}(E), f_{E*} s_*(X \cap E) \rangle\,. \end{aligned}$$

Since $Y \cap \varphi(V)$ is transverse to $X \cap E$ in E by Lemma 2.3, we have $\langle ([E] \cap)^{-1} g_{V*}(s_*(Y \cap \varphi(V)) \cap \bar{w}(E), f_{E*} s_*(X \cap E)) \rangle = \bar{e}_{g_V}(f_E, X \cap E)$ by Proposition 3.3. In view of the definitions of $\bar{e}_{g_V}$ and e_h, we have $\bar{e}_{g_V}(f_E, X \cap E) = e(X \cap Y \cap \varphi(V)) = e_h(\varphi, V)$. Hence $p(f, g) = e_h(\varphi, V)$ for each (φ, V) in $\mathfrak{N}_*(M, \bar{M})$. By Proposition 3.2, we have $(f_* s_*(X) \cdot g_* s_*(Y)) \cap \bar{w}(M) = h_* s_*(X \cap Y)$. q.e.d.

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