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UNIFORM DOMAINS

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1. Introduction. Uniform domains in the Euclidean n-space R^n, $n \geqq 2$, were introduced by Martio and Sarvas [MS] who proved injectivity and approximation results for these domains. This notion has turned out to be useful in several questions. For example, Jones proved that BMO and Sobolev functions on uniform domains can be extended to the whole R^n (cf. [Jo_1], [Jo_2], [GO]).

In Section 2 we consider various definitions for uniform domains. All of these are based on the same idea; joining points of the domain by a cigar. In Section 3 we give an essentially different characterization based on compactness. In Section 4 we show that the uniformity of a domain is a local property of its boundary. Section 5 deals with null-sets for uniform domains.

2. Uniformity and cigars.

2.1. Cigars. We shall always assume that the dimension n of the space R^n is at least two. Open balls are written as $B(x, r)$.

Roughly speaking, a domain $D \subset R^n$ is uniform if each pair of points in D can be joined by a cigar which is not too thin or too crooked. One can use several types of cigars. Many of these can be described as follows: Suppose that for every pair of distinct points $a, b \in R^n$ there is given a set $F(a, b)$ of continua containing a and b. Suppose also that for every triple $\gamma = (E, a, b)$ with $E \in F(a, b)$, there is given a continuous function $\lambda: E \to [0, \infty)$ such that $\lambda^{-1}(0) = \{a, b\}$. Then for $r > 0$, we define the *cigar r-neighborhood* of γ of *type* (F, λ) as

$$C = \operatorname{cig}(\gamma, r, F, \lambda) = \cup \{B(x, r\lambda(x)): x \in E\} .$$

Observe that C is an open neighborhood of $E \setminus \{a, b\}$. Suppose that δ is a map which associates to every nondegenerate continuum $E \subset R^n$ a number $\delta(E) > 0$. If $c > 1$, $r = 1/c$ and E satisfies the *turning condition* $\delta(E) \leqq c|a - b|$, we say that C is a *c-cigar* of *type* (F, λ, δ). The continuum E is the *core* of C. In the notation we often replace the triple γ by E if the points a and b are clear from the context.

A domain $D \subset R^n$ is *c-uniform of type* (F, λ, δ) if for each pair of distinct points $a, b \in D$ there is $E \in F(a, b)$ such that $\delta(E) \leqq c|a-b|$ and $\mathrm{cig}(E, 1/c, F, \lambda) \subset D$. Briefly: a and b can be joined by a c-cigar of type (F, λ, δ) in D.

Two definitions for c-uniform domains are *equivalent* if every c-uniform domain $D \subset R^n$ in the sense of one definition is always c_1-uniform in the sense of the other definition with $c_1 = c_1(c, n)$.

Typical examples of non-uniform domains in R^2 are the parallel strip and the complement of a half line. In the first example the cigars are too thin and in the second example too crooked.

2.2. Length cigars. The most common choice for $F(a, b)$ in the literature seems to be the family of all *rectifiable arcs* with end points a and b. Then $\delta(E)$ is the length of E, and $\lambda(x)$ is the length of the shorter one of the two subarcs of E into which x divides E. We call cigars of this type *length cigars* and denote them by $\mathrm{cig}_l(\gamma, r)$.

2.3. Diameter cigars. Another choice is to let $F(a, b)$ be the family of *all arcs* joining a and b. Then $\delta(E)$ is the diameter $d(E)$ of E. If $x \in E \in F(a, b)$, x again divides E into subarcs E_1 and E_2. We set

$$\lambda(x) = \min(d(E_1), d(E_2)) ,$$

call cigars of this type *diameter cigars* and write them as $\mathrm{cig}_d(\gamma, r)$. A slightly different choice for λ is the function

$$\lambda_1(x) = \min_i \max\{|y-x| : y \in E_i\} ,$$

used by Martio in [Mo, 4.5]. Since $\lambda_1 \leqq \lambda \leqq 2\lambda_1$, these choices are equivalent from the point of view of uniform domains. The equivalence of the definitions based on length or diameter cigars was proved by Martio [Mo].

2.4. Distance cigars. In this paper we shall use the triple (F, λ, δ) where $F(a, b)$ is the family of *all continua* containing a and b, $\delta(E) = d(E)$, and

$$\lambda(x) = \min(|x-a|, |x-b|) .$$

We call cigars of this type (F, λ, δ) *distance cigars* or simply *cigars* and write them as $\mathrm{cig}(\gamma, r)$ or $\mathrm{cig}(E, r)$. The function

$$\lambda_2(x) = \frac{|x-a|\,|x-b|}{|a-b|} ,$$

used by Jones [Jo_2] is equivalent to λ provided that E satisfies the turning condition $d(E) \leqq c|a-b|$; then $\lambda/2 \leqq \lambda_2 \leqq c\lambda$.

It is often convenient to allow the core E of a cigar to be an arbitrary continuum instead of an arc, especially if the cigar is constructed as a limit of a sequence of cigars. For a future reference, we give the following classical result on continua (see e.g. [Ku, p. 172]):

2.5. LEMMA. *Let $A \subset R^n$ be a continuum, let $U \subset R^n$ be open, and suppose that $A \setminus U \neq \emptyset$. Then every component of $A \cap \bar{U}$ meets ∂U.*

On the other hand, some constructions are easier to do if the core of the cigar is an arc. The following result enables us to replace arbitrary continua by arcs:

2.6. LEMMA. *Let $\operatorname{cig}(\gamma, 1/c)$ be a c-cigar with $\gamma = (E, a, b)$, and let $c_1 > c$. Then there is an arc E_1 with end points a and b such that for $\gamma_1 = (E_1, a, b)$, $\operatorname{cig}(\gamma_1, 1/c_1)$ is a c_1-cigar contained in $\operatorname{cig}(\gamma, 1/c)$.*

PROOF. Let $\varepsilon > 0$, and consider the cigar $C = \operatorname{cig}(\gamma, \varepsilon)$. It obviously suffices to find an arc E_1 joining a and b in $C \cup \{a, b\}$. For this, it suffices to find an arc in $C \cup \{a\}$ with end point at a. We may assume that $|a - b| = 1$. By 2.5, there is a component A_1 of $\bar{B}(a, 1/2) \cap E$ joining a and $\partial B(a, 1/2)$. Proceeding inductively we find a sequence of continua $A_1 \supset A_2 \supset \cdots$ such that A_j joins a and a point $x_j \in \partial B(a, 2^{-j})$ in $\bar{B}(a, 2^{-j})$. Join x_j and x_{j+1} by a broken line in $B(a, 2^{-j+1}) \cap \operatorname{cig}(\gamma, \varepsilon)$. The desired arc is then contained in the union of these broken lines. $\square$

2.7. Möbius cigars. A somewhat different way to describe a cigar is based on cross ratios. This idea is due to Martio [Mo], see also [Vä$_2$]. It is technically somewhat more complicated than the cigars above, but it is handy when considering Möbius and quasi-Möbius maps in the extended space $\dot{R}^n = R^n \cup \{\infty\}$. If a, b, c, d are distinct points in $\dot{R}^n$, their cross ratio is

$$|a, b, c, d| = \frac{|a - b|\,|c - d|}{|a - c|\,|b - d|}$$

with the obvious modification if one of the points is ∞. The reader should be warned that in the literature, there exist at least four out of the six possible essentially different choices for the order of the points a, b, c, d in the notation of the cross ratio. The choice above is the same as in [Mo], but different (I apologize) from [Vä$_2$]. I think that this order is the easiest to remember.

If $E \subset \dot{R}^n$ is a continuum containing distinct points a and b, we define two *unsymmetric Möbius cigar neighborhoods of* $\gamma = (E, a, b)$:

$$\operatorname{cig}^1_m(\gamma, r) = \{x \in \dot{R}^n : |x, y, a, b| < r \text{ for some } y \in E\}\,,$$

$$\mathrm{cig}_m^2(\gamma, r) = \{x \in \dot{R}^n \colon |x, y, b, a| < r \text{ for some } y \in E\} ,$$

and the *symmetric* Möbius cigar neighborhoods

$$\mathrm{cig}_m(\gamma, r) = \mathrm{cig}_m^1(\gamma, r) \cup \mathrm{cig}_m^2(\gamma, r) ,$$
$$\mathrm{cig}_m^*(\gamma, r) = \mathrm{cig}_m^1(\gamma, r) \cap \mathrm{cig}_m^2(\gamma, r) .$$

A domain $D \subset \dot{R}^n$ is called *Möbius c-uniform* if each pair of distinct points a, b in D can be joined by some $\mathrm{cig}_m(\gamma, 1/c)$ in D. No turning condition for the core E is needed, since it is automatically satisfied. The equivalence of Möbius and ordinary uniformities was proved by Martio [Mo]. The following lemma shows that one can replace the cigars cig_m by cig_m^1, cig_m^2 or cig_m^*:

2.8. LEMMA. *Suppose that E is a continuum in $\dot{R}^n$ containing the distinct points a and b. If $0 < r < 1$, then $\mathrm{cig}_m^1(E, r/(1+r)) \subset \mathrm{cig}_m^2(E, r)$.*

PROOF. Set $r_1 = r/(1+r)$. Performing an auxiliary Möbius map, we may assume that $a = \infty$, $b = 0$. If $x \in \mathrm{cig}_m^1(E, r_1)$, there is $y \in E$ such that

$$\tau = \frac{|x-y|}{|y|} = |x, y, \infty, 0| < r_1 .$$

Since

$$|x| \geqq |y| - |y - x| = |y|(1-\tau) ,$$

we have

$$|x, y, 0, \infty| = \frac{|x-y|}{|x|} \leqq \frac{\tau}{1-\tau} < \frac{r_1}{1-r_1} = r ,$$

and hence $x \in \mathrm{cig}_m^2(E, r)$. □

2.9. DEFINITION. A domain $D \subset R^n$ is *c-uniform*, $c \geqq 1$, if each pair of distinct points $a, b \in E$ can be joined by a (distance) c-cigar in D. The empty set is considered as a c-uniform domain for every $c \geqq 1$. A domain $D \subset \dot{R}^n$ is called c-uniform if $D \cap R^n$ is c-uniform. Remember that we always assume $n \geqq 2$.

2.10. THEOREM. *Definition 2.9 is equivalent to Möbius uniformity and hence to all ordinary definitions of uniformity.*

PROOF. We shall prove the relations

$$\mathrm{cig}(\gamma, r/2) \subset \mathrm{cig}_m(\gamma, r) , \quad \mathrm{cig}_m^*(\gamma, r^2/2) \subset \mathrm{cig}(\gamma, r) ,$$

where $0 < r < 1$, $\gamma = (E, a, b)$, $E \subset R^n$ is a continuum, and in the second condition, E satisfies the turning condition $rd(E) \leqq |a - b|$. The proof is valid in every metric space.

If $x \in \operatorname{cig}(\gamma, r/2)$, there is $y \in E$ with $|x-y| < 2^{-1}r \min(|y-a|, |y-b|)$. Assuming $|x-a| \geqq |x-b|$ we have $|a-b| \leqq 2|x-a|$, and obtain $|x, y, a, b| < r$. Hence $x \in \operatorname{cig}_m(\gamma, r)$.

Next assume that $rd(E) \leqq |a-b|$ and that $x \in \operatorname{cig}_m^*(\gamma, r^2/2)$. Then there is $y \in E$ such that $|x, y, a, b| < r^2/2$. Since $|y-b| \leqq d(E) \leqq |a-b|/r$ and $|y-a| \leqq |a-b|/r$, this implies

$$\frac{|x-y|}{|y-b|} < \frac{r^2}{2} \frac{|x-y| + |y-a|}{|a-b|} \leqq \frac{|x-y|}{2|y-b|} + \frac{r}{2}.$$

Hence $|x-y|/|y-b| < r$. By symmetry, $x \in \operatorname{cig}(\gamma, r)$. By [Vä$_2$, 4.7], the Möbius uniformity of D is equivalent to that of $D \cap R^n$. The theorem follows. □

It is often useful to know that boundary points of a uniform domain can also be joined by a cigar in the domain:

2.11. THEOREM. *Suppose that D is a c-uniform domain in R^n and that $a, b \in \bar{D} \cap R^n$, $a \neq b$. Then there is a c-cigar joining a and b in D.*

PROOF. Choose sequences (a_j) and (b_j) of points in D converging to a and b, respectively. Choose c-cigars $\operatorname{cig}(E_j, 1/c)$ joining a_j and b_j in D. Since $d(E_j) \leqq c|a_j - b_j|$, the continua E_j are contained in a compact subset of R^n. Passing to a subsequence we may therefore assume that the sequence (E_j) converges to a continuum E in the Hausdorff metric. Then E contains a and b, and $d(E) \leqq c|a-b|$. It is easy to see that $\operatorname{cig}(E, 1/c) \subset D$. □

2.12. REMARK. In [Vä$_2$] we also considered uniform *sets* which are not necessarily open. It follows from 2.11 that a set A is c-uniform if and only if $D \subset A \subset \bar{D}$ for some c-uniform domain D.

2.13. Plump sets and c-pairs. We shall give a new characterization for uniform domains, which will be needed in the next section. An open set $U \subset \dot{R}^n$ is *c-plump*, $c \geqq 1$, if for each $x \in U \cap R^n$ and for each $r \in (0, d(U))$, there is $z \in \bar{B}(x, r)$ such that $B(z, r/c) \subset U$. A uniform domain is always plump (see 2.15), but the converse is not true. The plumpness condition is clearly also true for $x \in \bar{U} \cap R^n$. If U is bounded, we easily see that the plumpness condition is also true for $r = d(U)$. The empty set is c-plump for every $c \geqq 1$.

If the plumpness condition is satisfied for every $x \in \partial U \cap R^n$ and $r \in (0, d(\partial U))$, U is $2c$-plump. To show this, assume that $x \in U \cap R^n$ and that $r \in (0, d(U))$. If $B(x, r/2) \subset U$, we can choose $z = x$. If $B(x, r/2) \not\subset U$, there is a point $a \in B(x, r/2) \cap \partial U$. If $r/2 < d(\partial U)$, there is $z \in \bar{B}(a, r/2) \subset \bar{B}(x, r)$ such that $B(z, r/2c) \subset U$. If $r/2 \geqq d(\partial U)$, then $R^n \setminus U \subset \bar{B}(a, r/2)$, and we can choose z to be a point such that $|z - x| = r$ and such that x lies on the line segment with end points z and a.

Let D be a domain in R^n and let $c \geqq 1$. A pair (B_1, B_2) of balls $B_j = B(x_j, r_j) \subset D$ is a *c-pair* in D if

(1) $r_1/r_2 \in [1/2, 2]$,

(2) $|x_1 - x_2| \leqq 4c \max(r_1, r_2)$.

2.14. LEMMA. *Let $D \subset R^n$ be a c-plump domain, and let $a, b \in \bar{D} \cap R^n$ with $|a - b| = r > 0$. Then there are sequences (a_j) and (b_j) such that setting $B_j = B(a_j, 2^{-j}r/c)$, $B'_j = B(b_j, 2^{-j}r/c)$ we have:*

(1) *The pairs (B_1, B'_1), (B_j, B_{j+1}) and (B'_j, B'_{j+1}) are c-pairs in D.*

(2) $|a_j - a| \leqq 2^{-j}r$, $|b_j - b| \leqq 2^{-j}r$.

PROOF. Since $r = |a - b| \leqq d(D)$, we can find $a_j \in \bar{B}(a, 2^{-j}r)$ such that $B(a_j, 2^{-j}r/c) \subset D$. Similarly, there are $b_j \in \bar{B}(b, 2^{-j}r)$ such that $B(b_j, 2^{-j}r/c) \subset D$. The condition (2) is clearly true, and (1) is easy to check. □

2.15. THEOREM. *Suppose that $D \subset R^n$ is a domain, that $c \geqq 1$, $c_0 \geqq 1$, and that*

(1) *D is c-plump,*

(2) *the centers of the balls of every c-pair in D can be joined by a c_0-cigar in D.*

Then D is c_1-uniform for some $c_1 = c_1(c, c_0)$. Conversely, every c_0-uniform domain D satisfies (1) *and* (2) *with $c = 4c_0$.*

PROOF. Let a and b be distinct points in D. Set $|a - b| = r$ and choose sequences (a_j) and (b_j) as in Lemma 2.14. For each $j \geqq 1$ join a_j to a_{j+1} by a c_0-cigar $\operatorname{cig}(E_j, 1/c_0)$ in D, and similarly b_j to b_{j+1} by $\operatorname{cig}(F_j, 1/c_0)$. Observe that

$$d(E_j) \leqq c_0 |a_{j+1} - a_j| \leqq 2^{-j+1} c_0 r \ ,$$

and similarly for $d(F_j)$. Finally join a_1 and b_1 by a c_0-cigar $\operatorname{cig}(E^*, 1/c_0)$ in D. Then the union of all continua E^*, E_j, F_j together with a and b is a continuum E. We show that E is the core of the desired c_1-cigar joining a and b in D.

We shall first find an upper bound for $d(E)$. Let $z \in E$. Assume that $z \in E_j$. Then

$$|z - a| \leqq |z - a_{j+1}| + |a_{j+1} - a| \leqq 2^{-j+1} c_0 r + 2^{-j-1} r \leqq (c_0 + 1) r \ .$$

Similarly, $z \in F_j$ implies $|z - b| \leqq (c_0 + 1) r$, and thus $|z - a| \leqq (c_0 + 2) r$.

If $z \in E^*$, we have

$$|z - a| \leqq |z - a_1| + |a_1 - a| \leqq c_0|a_1 - b_1| + r/2 \leqq (2c_0 + 1)r .$$

Hence $d(E) \leqq 6c_0 r$.

It remains to find an upper bound c_1 for the function

$$u(z) = \frac{\min(|z - a|, |z - b|)}{d(z, \partial D)}$$

over $x \in E\setminus\{a, b\}$. Assume first that $z \in E_j$. As above, we obtain $|z - a| \leqq 2^{-j}(2c_0 + 1)r$. We consider three cases:

Case 1. $|z - a_j| \leqq 2^{-j-1}r/c$. Since $B_j \subset D$, we have $d(z, \partial D) \geqq 2^{-j-1}r/c$. Hence $u(z) \leqq 2c(2c_0 + 1)$.

Case 2. $|z - a_{j+1}| \leqq 2^{-j-2}r/c$. A similar argument gives $u(z) \leqq 4c(2c_0 + 1)$.

Case 3. The cases 1 and 2 do not occur. Since $\operatorname{cig}(E_j, 1/c_0) \subset D$, we have $d(z, \partial D) \geqq 2^{-j-2}r/cc_0$, and hence $u(z) \leqq 4cc_0(2c_0 + 1)$.

The case $z \in F_j$ is similar and the case $z \in E^*$ almost similar.

Conversely, assume that D is c_0-uniform. The condition (2) is trivially true. Let $x \in D$ and let $r \in (0, d(D))$. Choose $y \in D$ with $|y - x| = r/2$. There is a c_0-cigar $\operatorname{cig}(E, 1/c_0)$ joining x and y in D. Choose $z \in E$ with $|z - x| = r/4$. Then $B(z, r/4c_0) \subset D$. Thus D is $4c_0$-plump. $\square$

2.16. Remark. Martin [Mn] has given the following interesting characterization for the uniformity of a domain $D \subset R^n$: There is $c \geqq 1$ such that each pair a, b of points in D is contained in a c-bilipschitz image $C \subset D$ of a ball. Gehring and Hag [GH] have recently characterized uniformity in terms of a min-max property of curves.

2.17. John domains. For completeness, we include a discussion of John domains, but this concept is not used in the later sections. However, these domains are closely related to uniform domains. Indeed, the original definition for uniform domains in [MS] was based on John domains. The definition of a John domain is usually given in terms of *carrots* rather than cigars (see, however, 2.20). As in 2.1, suppose that for each pair a, b of distinct points in R^n there is given a set $F(a, b)$ of continua containing a and b. We also assume that for every triple $\gamma = (E, a, b)$ with $E \in F(a, b)$ there is given a continuous function $\lambda\colon E \to [0, \infty)$, but instead of $\lambda^{-1}(0) = \{a, b\}$ we assume that $\lambda^{-1}(0) = \{a\}$. If $r > 0$, the *carrot r-neighborhood* of γ of *type* (F, λ) is the set

$$\operatorname{car}(\gamma, r, F, \lambda) = \cup\{B(x, r\lambda(x))\colon x \in E\} ,$$

also written as $\operatorname{car}(E, r, F, \lambda)$. Thus $\operatorname{car}(\gamma, r, F, \lambda)$ is an open neighborhood of $E \setminus \{a\}$. If $c \geqq 1$, we say that $\mathrm{car}(\gamma, 1/c, F, \lambda)$ is a *c-carrot* of type (F, λ) joining a and b. No turning condition is given on E.

A domain $D \neq R^n$ is called a *c-John domain* of *type* (F, λ), $c \geqq 1$, if there is $x_0 \in D$, called the *John center* of D, such that every point in D can be joined to x_0 by a c-carrot of type (F, λ) in D. The equivalence of two types are defined as for uniform domains (2.1). A c-uniform domain could be defined as a domain such that each pair a, b of distinct points in D is contained in a c-John domain $G \subset D$ with $d(G) \leqq c|a - b|$.

As with cigars, we consider three types of carrots, which are mutually equivalent:

(1) *Length carrots.* We let $F(a, b)$ be the family of all rectifiable arcs E with end points a and b. Now $\lambda(x)$ is the length of the subarc E_x of E with end points a and x. This is the most usual choice in the literature. One often also gives an upper bound for the lengths of the arcs E, but the number $cd(x_0, \partial D)$ is always such a bound.

(2) *Diameter carrots.* Now $F(a, b)$ is the family of all arcs joining a and b, and $\lambda(x) = d(E_x)$. We denote these carrots by $\mathrm{car}_d(\gamma, r)$.

(3) *Distance carrots.* Here we let $F(a, b)$ be the set of all continua containing a and b, and $\lambda(x) = |x - a|$. We write carrots of this type as $\mathrm{car}(\gamma, r)$ or $\mathrm{car}(E, r)$ and call them simply *carrots*.

The equivalence of the types (1) and (2) was proved in [MS, 2.7]. In fact, [MS] used a variation of (2) with paths instead of arcs, but a path can always be replaced by an arc joining the same points in the induced order [Wh, p. 39]. We prove the equivalence of the types (2) and (3):

2.18. THEOREM. *Suppose that $D \subset R^n$ is a domain.*

(a) *If D is a c-John domain in the diameter sense* (2), *it is a c-John domain in the distance sense* (3).

(b) *If D is a c-John domain in the distance sense* (3), *it is a c_1-John domain in the diameter sense* (2) *with $c_1 = c_1(c)$.*

PROOF. The part (a) is trivial. Suppose that D is a c-John domain in the distance sense. Let x_0 be the corresponding John center of D. Replacing c by a slightly larger number we may assume that the cores of all carrots are arcs (cf. 2.6). Set $\alpha = d(x_0, \partial D)$. We first observe that $D \subset B(x_0, c\alpha)$. Indeed, if $x \in D$, choose a c-carrot $\mathrm{car}(E, 1/c)$ joining x to x_0 in D. Then $B(x_0, |x - x_0|/c) \subset D$, which implies $|x - x_0| \leqq c\alpha$.

Suppose that $x_1 \in D$. We want to find a diameter c-carrot $\mathrm{car}_d(E, 1/c_1)$ joining x_1 to x_0 in D. If $|x_1 - x_0| < \alpha$, E can be chosen to be a straight line segment. We may thus assume that $|x_1 - x_0| \geqq \alpha$. Set $r = d(x_1, \partial D)/2$. Then $|x_1 - x_0| \geqq r$. Let k be the unique positive integer for which

$$2^{k-1}r \leqq |x_1 - x_0| < 2^k r \text{ .}$$

We define inductively points $y_j \in \partial B(x_1, 2^{j-1}r) \cap D$, $1 \leqq j \leqq k$, as follows: Let $y_1 \in \partial B(x_1, r)$ be arbitrary. Assume that the points $y_1, \cdots, y_j$ have been chosen. Join y_j to x_0 by a (distance) c-carrot $\operatorname{car}(E_j, 1/c)$ in D, where E_j is an arc from y_j to x_0. Let y_{j+1} be the first point of E_j meeting $\partial B(x_1, 2^j r)$.

For $1 \leqq j \leqq k-1$ we let A_j be the subarc of E_j with end points y_j and y_{j+1}. We also set $A_k = E_k$ and let A_0 denote the line segment from x_1 to y_1. Then the union

$$E = A_0 \cup \cdots \cup A_k$$

has a natural structure of the image of a path from x_1 to x_0. Set $d_j = d(A_0 \cup \cdots \cup A_j)$. It suffices to show that if $x \in A_j$, then the function $u(x) = d_j / d(x, \partial D)$ has an upper bound $c_1 = c_1(c)$. Observe that $d_j \leqq 2^{j+1}r$ for $j \leqq k-1$.

If $j = 0$, we have $u(x) \leqq 1$. Suppose that $1 \leqq j \leqq k-1$. We consider two cases:

Case 1. $|x - y_j| \geqq 2^{j-3}r/c$. Since $\operatorname{car}(E_j, 1/c) \subset D$, we have $d(x, \partial D) \geqq 2^{j-3}r/c^2$, and hence $u(x) \leqq 16c^2$.

Case 2. $|x - y_j| \leqq 2^{j-3}r/c$. If $j \geqq 2$, then $y_j \in E_{j-1}$. Since $\operatorname{car}(E_{j-1}, 1/c) \subset D$, we have $d(y_j, \partial D) \geqq |y_j - y_{j-1}|/c \geqq 2^{j-2}r/c$. This is clearly also true if $j = 1$. Hence $d(x, \partial D) \geqq 2^{j-3}r/c$, which implies $u(x) \leqq 16c$.

Finally assume that $j = k$. Now

$$d_k \leqq d(D) \leqq 2c\alpha \leqq 2c|x_1 - x_0| < 2^{k+1}cr \text{ .}$$

Considering two cases as above we obtain $u(x) \leqq 16c^3$. □

2.19. Remark. The relation $D \subset B(x_0, c\alpha)$ holds for every c-John domain of type (F, λ) provided that $\lambda(x) \geqq |x - a|$, which is true for each of the three types considered above. In particular, every John domain of such a type is bounded. Remember that we excluded the case $D = R^n$.

2.20. John domains and cigars. We next show that John domains can also be characterized in terms of cigars simply by dropping the turning condition from the defintion of a c-uniform domain. Thus we say that a domain $D \subset R^n$ is a *c-John domain in the cigar sense* if for each pair a, b of distinct points in D there is a continuum E containing a and b such that $\operatorname{cig}(E, 1/c) \subset D$. Compared with the definition in 2.17 there is one essential difference: There are unbounded domains satisfying the new condition, for example, a half space. For bounded domains, however, the two definitions are equivalent:

2.21. THEOREM. *Suppose that D is a domain in R^n.*

(a) *If D is a c-John domain in the (distance) carrot sense, it is a c-John domain in the cigar sense.*

(b) *If D is bounded and a c-John domain in the cigar sense, it is a c_1-John domain in the carrot sense with $c_1 = c_1(c)$.*

PROOF. (a) Let x_0 be the John center of D, and suppose that $a, b \in D$. Join a and b to x_0 by c-carrots $\mathrm{car}(E, 1/c)$ and $\mathrm{car}(F, 1/c)$ in D. Then $\mathrm{cig}(E \cup F, 1/c)$ joins a and b in D.

(b) Choose points $a, b \in D$ with $|a - b| = \alpha \geqq d(D)/2$, and join them by a continuum E with $\mathrm{cig}(E, 1/c) \subset D$. Choose a point $x_0 \in E$ with $|x_0 - a| = |x_0 - b|$. Then $B(x_0, \alpha/2c) \subset D$. Suppose that $x_1 \in D$, and join x_1 to x_0 by a continuum F with $\mathrm{cig}(F, 1/c) \subset D$. If $x \in F$ and $|x - x_0| \geqq \alpha/4c$, then $|x - x_1| \leqq d(D) \leqq 8c|x - x_0|$, and thus $d(x, \partial D) \leqq |x - x_1|/8c^2$. If $x \in F$ and $|x - x_0| \leqq \alpha/4c$, then $d(x, \partial D) \geqq \alpha/4c \geqq d(D)/8c \geqq |x - x_1|/8c$. Hence $\mathrm{car}(F, 1/8c^2) \subset D$, and thus D is a $8c^2$-John domain in the carrot sense. □

2.22. REMARK. O. Martio (unpublished) has given the following characterization for a John domain $D \subset R^n$. There are $x_0 \in D$ and $c \geqq 1$ such that for each point $x \in D$ there are $r > 0$ and a c-bi-Lipschitz mapping $f: B(0, r) \to D$ with $f(0) = x_0$ and $x \in fB(0, r)$ (cf. 2.16).

2.23. REMARK. The definition 2.9 of uniform domains makes sense in every metric space. Many results and proofs of this paper are valid in this general case. A notable exception is Section 3 where we make effective use of the similarity maps and the local compactness of R^n.

In Lemma 2.8 we made use of the inversion in R^n. A general metric space can be isometrically embedded into a normed space V. The inversion $u(x) = x/|x|^2$ in V changes the cross ratios at most by the factor 3^4 [Vä$_2$, 1.6]. Hence the analogue of 2.8 is true in every metric space with $r/(r + 1)$ replaced by $r/3^8(r + 1)$. The proof of 2.10 shows that uniformity and Möbius uniformity are equivalent in every metric space. However, Möbius uniformity makes also sense in an extended space $X \cup \{\infty\}$. On the other hand, the Möbius uniformity of a domain D in the extended space is often equivalent to the uniformity of $D \setminus \{\infty\}$ [Vä$_2$, 4.7].

3. Uniformity and compactness. In this section we give a characterization for uniform domains in $\dot{R}^n$ in terms of compactness.

3.1. Terminology. If X is a compact metric space, the set

$$K(X) = \{A: \varnothing \neq A \subset X, A \text{ compact}\}$$

with the Hausdorff metric is a compact metric space, and the subfamily of all continua is compact [Ku, pp. 45, 47, 139]. We shall only consider the case where X is the extended n-space $\dot{R}^n = R^n \cup \{\infty\}$ with the spherical metric. We write

$$K^n = K(\dot{R}^n) .$$

Closed unbounded sets $A \subset R^n$ will also be considered as elements of K^n, identifying A with $A \cup \{\infty\}$.

If $H \subset K^n$, we let $\operatorname{sim} H$ denote the family of all images of the members of H under similarity maps of $\dot{R}^n$. Furthermore, we write

$$H^2 = \{A \in H\colon \{0, e_1\} \subset \partial A\}$$

where $e_1 = (1, 0, \cdots, 0)$.

We say that a family $H \subset K^n$ is *stable* if (1) $\operatorname{sim} H = H$ and (2) H^2 is compact. The family $H = \{\dot{R}^n\}$ is trivially stable, since $H^2 = \varnothing$.

For any $H \subset K^n$ we let $\sigma(H)$ denote the union of all stable subfamilies of H.

We shall show that the set of the complements of all uniform domains is precisely $\sigma(H)$ where H is the family of the complements of all domains in $\dot{R}^n$. For that we need a similar characterization for the plump open sets.

In another paper [Vä$_4$] I prove that if $H \subset K^n$ is any family which is invariant under quasisymmetric maps, then $\sigma(H)$ has the same property. It follows that the uniformity of a domain is invariant under quasisymmetric maps of the complement of the domain, and the same is true for the plumpness of an open set.

We first give an alternative characterization of $\sigma(H)$. An auxiliary result is needed:

3.2. LEMMA. *If $H \subset K^n$ and* $\operatorname{sim} H = H$, *then* $(\operatorname{cl} H^2)^2 = (\operatorname{sim} \operatorname{cl} H^2)^2 = (\operatorname{cl} H)^2$.

PROOF. Clearly $\operatorname{sim} H = H$ implies $\operatorname{sim} \operatorname{cl} H = \operatorname{cl} H$. Hence $\operatorname{cl} H^2 \subset \operatorname{sim} \operatorname{cl} H^2 \subset \operatorname{cl} H$. Thus it suffices to show that $(\operatorname{cl} H)^2 \subset (\operatorname{cl} H^2)^2$. Suppose that $A \in (\operatorname{cl} H)^2$. Then $\{0, e_1\} \subset \partial A$, and there is a sequence $A_j \in H$ converging to A. Choose points $a_j, b_j \in \partial A_j$ with $a_j \to 0$, $b_j \to e_1$. Let α_j be a similarity which maps (a_j, b_j) to $(0, e_1)$. We may assume that the sequence (α_j) converges uniformly to a similarity α with $\alpha | \{0, e_1\} = \mathrm{id}$. Then $B_j = \alpha^{-1}\alpha_j A_j \in H^2$ and $B_j \to A$. Hence $A \in (\operatorname{cl} H^2)^2$. □

3.3. THEOREM. *Let $H \subset K^n$ with* $\operatorname{sim} H = H$, *and suppose that $A \in K^n$ has at least two finite boundary points. Then $A \in \sigma(H)$ if and only if* $\operatorname{cl}((\operatorname{sim}\{A\})^2) \subset H^2$.

PROOF. Set $M = (\mathrm{sim}\{A\})^2$. If $A \in \sigma(H)$, A belongs to a stable family $L \subset H$. Then $M \subset L^2$, and hence $\mathrm{cl}\, M \subset L^2 \subset H^2$.

Conversely, assume that $\mathrm{cl}\, M \subset H^2$. Then $L = \mathrm{sim}\,\mathrm{cl}\, M \subset H$. Since $\partial A \cap R^n$ contains at least two points, $A \in L$. Hence it suffices to show that L is stable. Clearly $\mathrm{sim}\, L = L$. From 3.2 it follows that $L^2 = (\mathrm{cl}\, M)^2$. Since $\mathrm{cl}\, M \subset H^2$, $L^2 = \mathrm{cl}\, M$, and thus L^2 is compact. □

3.4. THEOREM. *For $c \geqq 1$ let L_c be the family of all $A \in K^n$ such that $\dot{R}^n \setminus A$ is c-plump. Then L_c is compact and stable. Conversely, for every stable family $L \subset K^n$ there is $c \geqq 1$ such that $L \subset L_c$.*

PROOF. Assume that $A_j \in L_c$ and that $A_j \to A \in K^n$. To prove that L_c is compact we must show that $U = \dot{R}^n \setminus A$ is c-plump. Suppose that $x \in U \cap R^n$ and that $0 < r < d(U)$. We may assume that the sets $U_j = \dot{R}^n \setminus A_j$ have diameters greater than r and that $x \in U_j$ for all j. Since U_j is c-plump, we can choose $z_j \in \bar{B}(x, r)$ with $B(z_j, r/c) \subset U_j$. We may assume that $z_j \to z \in \bar{B}(x, r)$. Then $B(z, r/c) \subset U$. Hence U is c-plump.

Clearly $\mathrm{sim}\, L_c = L_c$. Assume that $B_j \in L_c^2$ and that $B_j \to B \in K^n$. To prove that L_c is stable we must show that $B \in L_c^2$. By the first part of the proof, B belongs to L_c. Since $\{0, e_1\} \subset \partial B_j$, $d(\dot{R}^n \setminus B_j) \geqq 1$. For every given $r \in (0, 1)$ we can choose $y_j \in \bar{B}(0, r)$ with $B(y_j, r/c) \subset U_j$. We may assume that $y_j \to y \in \bar{B}(0, r)$. Then $B(y, r/c) \subset \dot{R}^n \setminus B$, and thus $0 \in \partial U$. Similarly $e_1 \in \partial U$, and thus $B \in L_c^2$. We have proved that L_c is stable.

Assume that the last part of the theorem is false. Then there is a stable family $L \subset K^n$ and sequences $A_j \in L$, $a_j \in \partial A_j \cap R^n$ and $r_j \in (0, d(\partial A_j))$ such that $d(z, A_j) \leqq r_j/j$ for all $z \in \bar{B}(a_j, r_j)$; see 2.13. We may assume that $\partial A_j \cap \partial B(a_j, r_j) \neq \emptyset$. For every j choose a similarity α_j such that

$$\alpha_j(a_j) = 0\ , \quad \alpha_j B(a_j, r_j) = B(0, 1) = B^n\ , \quad e_1 \in \partial \alpha_j A_j\ .$$

Then $C_j = \alpha_j A_j \in L^2$. Since L^2 is compact, we may assume that $C_j \to C \in L^2$. Since $d(z, C_j) \leqq 1/j$ for all $z \in \bar{B}^n$, we have $\bar{B}^n \subset C$. Hence $0 \notin \partial C$, which gives a contradiction $C \notin L^2$. □

3.5. COROLLARY. *An open set $U \subset \dot{R}^n$ is plump if and only if $\dot{R}^n \setminus U \in \sigma(K^n)$.*

3.6. THEOREM. *For $c \geqq 1$ let M_c be the family of all $A \in K^n$ such that $\dot{R}^n \setminus A$ is a c-uniform domain. Then M_c is compact and stable. Conversely, if $M \subset K^n$ is a stable family such that $\dot{R}^n \setminus A$ is connected for every $A \in M$, then $M \subset M_c$ for some $c \geqq 1$.*

PROOF. Suppose that $A_j \in M_c$ and that $A_j \to A \in K^n$. To show that $D = \dot{R}^n \setminus A$ is a c-uniform domain let $a, b \in D \cap R^n$. Then $a, b \in D_j = R^n \setminus A_j$

for large j. Join a and b by a c-cigar $\operatorname{cig}(E_j, 1/c)$ in D_j. We may assume that (E_j) converges to a continuum E with $d(E) \leqq c|a - b|$. It is easy to see that $\operatorname{cig}(E, 1/c) \subset D$. Hence $A \in M_c$, and M_c is compact.

Clearly $\operatorname{sim} M_c = M_c$. By 2.15, $M_c \subset L_{4c}$ where L_c is as in 3.4. Since M_c is compact, 3.4 implies that $\operatorname{cl} M_c^2 \subset L_{4c}^2 \cap M_c = M_c^2$. Hence M_c is stable.

Suppose that the last part of the theorem is false. By 3.4 there is $c \geqq 1$ such that all members of M are c-plump. By 2.15 we can find sequences of sets $A_j \in M$ and c-pairs $(B(x_j, r_j), B(y_j, s_j))$ of balls in $D_j = R^n \setminus A_j$ such that the centers x_j, y_j cannot be joined by a j-cigar in D_j. We may assume that $r_j \leqq s_j$. Then

$$r_j \leqq |x_j - y_j| \leqq 8cr_j$$

for all j.

Set $Q_j = \partial A_j \cap \bar{B}(x_j, 10cr_j)$. If $d(Q_j) \leqq r_j$, it is easy to join x_j and y_j in D_j by a j-cigar for large j. We may thus assume that $d(Q_j) \geqq r_j$ for all j. Choose points $a_j, b_j \in Q_j$ with $|a_j - b_j| \geqq r_j$. For every j choose a similarity α_j such that $\alpha_j(a_j) = 0$, $\alpha_j(b_j) = e_1$. Since M is stable, the sets $C_j = \alpha_j A_j$ are in M^2, and we may assume that $C_j \to C \in M$. Setting $L_j = |a_j - b_j|^{-1} = \operatorname{lip} \alpha_j$ we have $1/20c \leqq L_j r_j \leqq 1$. Passing again to a subsequence we may assume that

$$\alpha_j(x_j) \to x' , \quad \alpha_j(y_j) \to y' , \quad L_j r_j \to r'$$

with

$$x', y' \in \bar{B}(0, 20c) , \quad 1/20c \leqq r' \leqq 1 , \quad |x' - y'| \geqq r' .$$

Then the balls $B(x', r')$ and $B(y', r')$ are contained in $D = R^n \setminus C$. Since $C \in M$, D is connected, and we can join x' and y' by a continuum $E \subset D$. Choose $b \geqq 1$ such that $d(E) \leqq b|x' - y'|$ and $\operatorname{cig}(E, 1/b) \subset D$. Let E_j be the union of E and the two line segments joining x' to $\alpha_j(x_j)$ and y' to $\alpha_j(y_j)$. Then for large j, $\operatorname{cig}(E_j, 1/2b)$ is a j-cigar joining $\alpha_j(x_j)$ and $\alpha_j(y_j)$ in $R^n \setminus C_j$. Applying the map α_j^{-1} we obtain a j-cigar joining x_j and y_j in D_j. This contradiction completes the proof. □

3.7. COROLLARY. *Let H be the family of all $A \in K^n$ such that $R^n \setminus A$ is connected. Then a domain $D \subset R^n$ is uniform if and only if $\dot{R}^n \setminus D \in \sigma(H)$.*

3.8. COROLLARY. *A domain $D \subset \dot{R}^n$ is not uniform if and ony if there is a sequence (α_j) of similarities such that*

(1) $\{0, e_1\} \subset \alpha_j \partial D$,

(2) $\dot{R}^n \setminus \alpha_j D \to A \in K^n$, *and either* $0 \in \operatorname{int} A$ *or* $R^n \setminus A$ *is not connected.*

3.9. REMARK. The case $0 \in \operatorname{int} A$ in 3.8 occurs if the cigars are too thin and the other case if they are too crooked. For example, let D be the strip $0 < x_2 < 1$ in R^2. Setting $\alpha_j(x) = x/j$ we have $\dot{R}^2 \setminus \alpha_j D \to \dot{R}^2$. Next let D be the complement of the half line $\{re_1 : r \geqq 0\}$ in R^2. For $\alpha_j(x) = j(x - e_1)$ the sequence $\dot{R}^2 \setminus \alpha_j D$ converges to the x_1-axis.

Of course, the situation with arbitrary domains may be much more complicated.

4. **Local uniformity.** In this section we show that the uniformity of a domain D is a local property of ∂D. This probably belongs to the folklore.

4.1. THEOREM. *Suppose that $D \subset R^n$ is a bounded domain and that $c \geqq 1$, $0 < r < d(D)$. Suppose also that if $z \in \partial D$, then every pair of points in $D \cap B(z, r)$ can be joined by a c-cigar in D. Then D is c_1-uniform with $c_1 = 40c^3 d(D)/r$.*

PROOF. We may assume that $d(D) = 1$. Observe first that if $a, b \in D$ with $|a - b| \leqq r/2$, then a and b can be joined by a c-cigar in D. Indeed, if $d(a, \partial D) < r/2$, then a and b belong to $B(z, r)$ for some $z \in \partial D$. If $d(a, \partial D) \geqq r/2$, then $B(a, r/2) \subset D$, and there is even a 1-cigar joining a and b in D.

Let a and b be arbitrary points in D. We want to join them by a c_1-cigar in D. By the remark above, we may assume that $|a - b| > r/2$. Set $q = 1/20c$. Since D is connected, there is a finite sequence $a = x_0, x_1, \cdots, x_s = b$ of points in D such that

$$2qr \leqq |x_j - x_{j-1}| \leqq 5qr$$

for all $j = 1, \cdots, s$. Since $q \leqq 1/20$, $|x_j - x_{j-1}| \leqq r/4$ for all j. Hence we can join x_{j-1} and x_j by a c-cigar $\operatorname{cig}(E_j, 1/c)$ in D. By Lemma 2.5, there is a continuum $A \subset \{x \in E_1 : |x - a| \leqq |x - x_1|\}$ containing a and a point u_1 with $|u_1 - a| = |u_1 - x_1|$. Similarly choose a continuum $B \subset \{x \in E_s : |x - b| \leqq |x - x_{s-1}|\}$ containing b and a point u_s with $|u_s - b| = |u_s - x_{s-1}|$. For $1 \leqq j \leqq s - 1$ we let u_j be an arbitrary point in E_j satisfying $|u_j - x_{j-1}| = |u_j - x_j|$. Since

$$d(E_j) \leqq c|x_j - x_{j-1}| \leqq 5qrc = r/4$$

for all j, $|u_{j+1} - u_j| \leqq r/2$ for $1 \leqq j \leqq s - 1$. Hence there is a c-cigar $\operatorname{cig}(F_j, 1/c)$ joining u_j and u_{j+1} in D. Then $F = A \cup F_1 \cup \cdots \cup F_{s-1} \cup B$ is a continuum joining a and b. We claim that it is the core of a c_1-cigar in D.

Since $d(F) \leqq d(D) = 1$ and since $|a - b| \geqq r/2$, F satisfies the turning condition $d(F) \leqq (2/r)|a-b| \leqq c_1|a-b|$. It remains to show that $u(z) \leqq c_1$ for $z \in F$ where

$$u(z) = \frac{\min(|z-a|, |z-b|)}{d(z, \partial D)} .$$

Case 1. $z \in A$ or $z \in B$. Now $u(z) \leqq c \leqq c_1$.

Case 2. $z \in F_j$ for some j. Since $d(D) = 1$, if suffices to show that $d(z, \partial D) \geqq 1/c_1$. Since $\operatorname{cig}(E_j, 1/c) \subset D$ and since $|u_j - x_j| \geqq |x_j - x_{j-1}|/2 \geqq qr$, we have $d(u_j, \partial D) \geqq qr/c$. Hence $d(z, \partial D) \geqq qr/2c \geqq 1/c_1$ if $|z - u_j| \leqq qr/2c$, and the same is true if $|z - u_{j+1}| \leqq qr/2c$. If these distances are at least $qr/2c$, the condition $\operatorname{cig}(F_j, 1/c) \subset D$ implies that $d(z, \partial D) \geqq qr/2c^2 = 1/c_1$. □

4.2. REMARKS. 1. It follows from 4.1 that the (ε, δ)-domains of Jones [Jo_2] are uniform if they are bounded. For unbounded domains this is not true.

2. We obtain a version of 4.1 for unbounded domains by adding the condition that each pair of points in $D \setminus B(0, 1/r)$ can be joined by a c-cigar in D. Then D is c_1-uniform with some $c_1 = c_1(c, r)$.

Alternatively, one can use the spherical metric. In fact, the spherical metric gives the same class of uniform domains in R^n as the Euclidean metric. This follows from the fact that the identity map of R^n is a Möbius transformation (preserves cross ratios) with respect to these metrics and from the Möbius invariance of uniform domains.

3. Theorem 4.1 and its proof are valid in every metric space.

5. Null-sets for uniform domains.

5.1. We say that a closed set $A \subset R^n$ is a *null-set for uniform domains* or an NUD set if $\operatorname{int} A = \varnothing$ and if $R^n \setminus A = D$ is a uniform domain. If D is c-uniform, we say that A is c-NUD. The main result of this section is Theorem 5.4 in which we show that removing an NUD set from a uniform domain yields a uniform domain. We first give easy estimates for the dimension of an NUD set. Let $\dim A$ and $\dim_H A$ denote the topological and the Hausdorff dimension of A, respectively.

5.2. THEOREM. *If* $A \subset R^n$ *is* c-NUD, *then* $\dim A \leqq n-2$ *and* $\dim_H A \leqq \alpha = \alpha(c, n) < n$.

PROOF. If $\dim A = n-1$, $R^n \setminus A$ is not locally connected on the boundary. The theorem follows from [Mo, 6.5 and 6.7]. □

5.3. REMARKS. 1. If A is closed in R^{n-1}, it is easy to see that A is NUD in R^n if and only if A is porous in R^{n-1}, that is, there is $c \geqq 1$ such that every ball $\bar{B}^{n-1}(x, r)$ contains a point z with $B^{n-1}(z, r/c) \cap A = \varnothing$.

2. Let H be the set of all closed sets $A \subset R^n$ with $\dim A \leqq n-2$. It is easy to see that with the notation of 3.1, $\sigma(H)$ is precisely the family of all NUD sets in R^n.

3. There exist countable sets which are not NUD. For example, the set of integers is not NUD in R^2.

4. From [GM, 2.4 and 2.18] it follows that a NUD set is always NED (nullset for extremal distances).

5.4. THEOREM. *Suppose that $A \subset R^n$ is c-NUD and that $D \subset R^n$ is a c-uniform domain. Then $D \setminus A$ is a c_1-uniform domain with $c_1 = 54c^5$.*

PROOF. Let $a, b \in D \setminus A$. We show that there is a c_1-cigar in D joining a and b. Set $c' = 4c/3$. Applying Lemma 2.6 we find an arc E from a to b such that $\mathrm{cig}(E, 1/c')$ is a c'-cigar in D. Let $x_0 \in E$ be a point for which $|x_0 - a| = |x_0 - b|$. Set

$$q = 1/8c^3\ , \quad \lambda(x) = \min(|x-a|, |x-b|)\ ,$$

and define the sequence $x_1, x_2, \cdots$ of points in E inductively as follows: Orient E from a to b. Then x_{j+1} is the last point of $E \cap \bar{B}(x_j, q\lambda(x_j))$. Similarly define $x_{-1}, x_{-2}, \cdots$ by letting x_{-j-1} be the first point in $E \cap \bar{B}(x_{-j}, q\lambda(x_{-j}))$. The sequence $x_1, x_2, \cdots$ converges to a point $b' \in E$. For $k > j$ we have $|x_k - x_j| \geqq q\lambda(x_j)$, which implies $\lambda(b') = 0$, and thus $b' = b$. Consequently, $x_j \to b$ as $j \to \infty$, and similarly $x_j \to a$ as $j \to -\infty$.

Since $R^n \setminus A = G$ is a c-uniform domain, it follows from 2.11 that we can join x_{j-1} and x_j by a c-cigar $\mathrm{cig}(E_j, 1/c)$ in G for every integer j. For each j choose $y_j \in E_j$ with $|y_j - x_{j-1}| = |y_j - x_j|$, and join y_j and y_{j+1} by a c-cigar $\mathrm{cig}(F_j, 1/c)$ in G. We claim that the union F of all F_j, $j \in Z$, and the pair $\{a, b\}$ is the core of the desired c_1-cigar from a to b in $D \setminus A$.

We first estimate the diameters of the sets F_j. If $j > 0$, we obtain

$$\begin{aligned} d(F_j) &\leqq c|y_{j+1} - y_j| \leqq c(d(E_j) + d(E_{j+1})) \\ &\leqq c^2(|x_j - x_{j-1}| + |x_{j+1} - x_j|) \leqq c^2 q(\lambda(x_{j-1}) + \lambda(x_j))\ . \end{aligned} \tag{5.5}$$

A similar argument gives

$$d(F_j) \leqq c^2 q(\lambda(x_j) + \lambda(x_{j+1}))$$

for $j < 0$, and

$$d(F_0) \leqq 2c^2 q\lambda(x_0)\ .$$

It follows that $d(F_j) \to 0$ as $j \to \infty$ or $j \to -\infty$. Hence F is a continuum.

We next estimate the function $\lambda(z)$ for $z \in F$. Suppose that $z \in F_j$ for some $j > 0$. Then (5.5) implies

$$|z - x_j| \leqq d(F_j) + d(E_j) \leqq c^2 q(2\lambda(x_{j-1}) + \lambda(x_j))\ .$$

Here $\lambda(x_{j-1}) \leqq \lambda(x_j) + q\lambda(x_{j-1})$. Since $q \leqq 1/8$, we obtain $\lambda(x_{j-1}) \leqq (8/7)\lambda(x_j)$. Consequently,

$$|z - x_j| \leqq \lambda(x_j)/2c \ . \tag{5.6}$$

Thus

$$\lambda(z) \leqq \lambda(x_j) + |z - x_j| \leqq 3\lambda(x_j)/2 \ . \tag{5.7}$$

Similar arguments show that (5.6) and (5.7) are true also for $z \in F_j$ with $j \leqq 0$.

Since $\mathrm{cig}(E, 1/c') \subset D$, (5.6) implies for $z \in F_j$:

$$d(z, \partial D) \geqq d(x_j, \partial D) - |z - x_j| \geqq \lambda(x_j)/c' - \lambda(x_j)/2c = \lambda(x_j)/4c \ . \tag{5.8}$$

We next estimate $d(z, A)$ for $z \in F$. We only consider the case $z \in F_j$, $j > 0$. Since $\mathrm{cig}(E_j, 1/c) \subset G$, the ball $B(y_j, r_j)$ is in G, where

$$r_j = |y_j - x_j|/c \geqq |x_{j-1} - x_j|/2c = \lambda(x_{j-1})/16c^4 \ .$$

Since $\lambda(x_{j-1}) \geqq (8/9)\lambda(x_j)$, we have $r_j \geqq \lambda(x_j)/18c^4$.

Set $\lambda_j(z) = \min(|z - y_{j+1}|, |z - y_j|)$. If $\lambda_j(z) \leqq \lambda(x_j)/36c^4$, we have

$$d(z, A) \geqq \lambda(x_j)/36c^4 \ .$$

If $\lambda_j(z) \geqq \lambda(x_j)/36c^4$, then $\mathrm{cig}(F_j, 1/c) \subset G$ implies

$$d(z, A) \geqq \lambda_j(z)/c \geqq \lambda(x_j)/36c^5 \ .$$

Together with (5.7) and (5.8) these estimates show that $\mathrm{cig}(F, 1/c_1) \subset D \setminus A$.

We finally verify the turning condition $d(F) \leqq c_1|a - b|$. If $z \in F_j$, (5.6) gives

$$|z - a| \leqq |z - x_j| + |x_j - a| \leqq 3d(E)/2 \ .$$

Since $d(E) \leqq c'|a - b|$, we obtain $d(F) \leqq 4c|a - b|$. □

5.9. Corollary. *If A and B are c-NUD in R^n, then $A \cup B$ is c_1-NUD with $c_1 = 54c^5$.*

Added in proof. The recent paper [Ge] of Gehring contains several characterizations for uniform domains.

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