COHOMOLOGY GROUPS OF FINITE ABELIAN GROUPS

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Actual determination of cohomology groups is very difficult owing to complicated definition of cohomology. Our aim here is to reduce the condition to conditions on the generators, which corresponds in 2-dimensional case to the work of O. Schreier [4] on group extensions. Our method of proof is not constructive as in Schreier but uses axiomatic cohomology theory recently developed by H. Cartan [1] and S. Eilenberg [2]. As applications we insert a section on galois cohomology and a short proof of R. C. Lyndon's formula [3] for trivial coefficient groups.

1. From axiomatic cohomology theory. We shall summalize here some results due to H. Cartan and S. Eilenberg which are necessary in the sequel.

Let G be a group. By a G-complex we shall mean an exact sequence:

$$0 \longleftarrow Z \overset{\varepsilon}{\longleftarrow} C_0 \overset{d_1}{\longleftarrow} C_1 \overset{d_2}{\longleftarrow} C_2 \longleftarrow \cdots \cdots \overset{d_q}{\longleftarrow} C_q \longleftarrow \cdots$$

of free G-modules C_q, here Z is the additive group of rational integers and to which G operates trivially. For any G-module A we shall consider the module of all G-homomorphisms of C_q into A:

$$\sum_q \mathrm{Hom}_G(C_q, A).$$

This is a group, under addition, with differential operator δ:

$$\delta f(c_q) = f(d_q c_q) \qquad (c_q \in C_q).$$

We shall finally put

$$H^q(G, A) = H^q\Big(\sum_p \mathrm{Hom}_G(C_p, A)\Big)$$

and call the cohomology group of G with coefficients in A defined by the G-complex C.

It may be true that G has many G-complexes; but Cartan-Eilenberg's fundamental result is that any of such G-complexes gives the same cohomology group. Therefore, we can omit the adjective word "defined by the G-complex C" in the definition of cohomology group.

As an existence proof, they gave the usual non-homogeneous G-complex defind as follows. Let $C_q(G)$ be the free G-module with

$$[x_1, \ldots, x_q], \qquad x_1, \ldots, x_q \in G$$

as a G-basis (for $C_0(G)$ the symbol [ ]) and define

$$d_q[x_1, \ldots, x_q] = x_1[x_2, \ldots, x_q] + \sum_{i=1}^{q-1}(-1)^i[x_1, \ldots, x_i x_{i+1}, \ldots, x_q] + (-1)^q[x_1, \ldots, x_{q-1}]$$

as the G-homomorphism $C_q(G) \to C_{q-1}(G)$ (for $\varepsilon : C_0(G) \to Z$ by $\varepsilon[\ ] = 1$).

They also remarked that for special groups one can finds more simple G-complexes. For example, let G be a finite cyclic group with a generator $s_1 : s_1^{n_1} = 1$, Λ the group-ring of G over Z, then

(1) $$\begin{cases} C_q = \Lambda \\ d_{2q} c_{2q} = (1 + s_1 + \ldots + s_1^{n_1 - 1}) c_{2q} \\ d_{2q+1} c_{2q+1} = (1 - s_1) c_{2q+1} \\ \varepsilon 1 = 1 \end{cases}$$

is a G-complex. If G and G' have G-complex C and G'-complex C' then the tensor product complex:

(2) $$\begin{cases} (C \otimes C')_q = \sum_{p=0}^{q} C_p \otimes C'_{q-p} \\ d(c_p \otimes c'_q) = dc_p \otimes c'_q + (-1)^p c_p \otimes dc'_q \\ \varepsilon(c_0 \otimes c'_0) = \varepsilon c_0 \cdot \varepsilon c'_0 \end{cases}$$

is a $G \times G'$-complex.

2. Incidence matrices for a finite groups. Let G be a finite group and $\{C_q\}$ be a G-complex such that each C_q is a free G-module with finite basis. We now fix one of its basis as

$$C_q = \Lambda e_q^1 + \ldots + \Lambda e_q^Q$$

where Λ is the group-ring of G over Z. Then the G-homomorphism d_{q+1} is represented by a matrix with elements in Λ:

$$\begin{matrix} d_{q+1} e_{q+1}^1 = \eta_{11} e_q^1 + \ldots + \eta_{1Q} e_q^Q \\ \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ d_{q+1} e_{q+1}^R = \eta_{R1} e_q^1 + \ldots + \eta_{RQ} e_q^Q \end{matrix} \qquad (\eta_{ij} \in \Lambda).$$

We shall call this matrix $\eta(q+1)$ an incidence matrix of G. Then Cartan-Eilenberg's results can be translated into the

THEOREM 1. *Let G be a finite group, $\eta(q)$, $q = 1, 2, \ldots,$ one of its incidence matrices and A any G-module, then*

$$H^q(G, A) \cong \mathfrak{a}/\eta(q)\mathfrak{b}$$

where ${}^t\mathfrak{a} = (a_1, \ldots, a_R)$ are vectors with elements in A such

$$\eta(q+1)\mathfrak{a} = 0$$

while ${}^t\mathfrak{b} = (b_1, \ldots, b_Q)$ are arbitrary vectors in A.

The proof is immediate and merely put

$$a_i = f(e_{q+1}^i) \qquad f \in \mathrm{Hom}_G(C_{q+1}, A) \qquad i = 1, \ldots, R,$$

$$b_j = g(e_q^j) \qquad g \in \mathrm{Hom}_G(C_q, A) \qquad j = 1, \cdots, Q.$$

Let G' be another finite group with incidence matrices $\eta'(q)$, then a system of incidence matrices of $G \times G'$ is given by

$$(3)\ \eta''(q+1) = \left(\begin{array}{c|c|c|c|c}
\begin{matrix}\eta'(q+1) & & \\ & \ddots & \\ & & \eta'(q+1)\end{matrix} & 0 & 0 & & \\ \hline
\eta(1)\otimes 1'_q & \begin{matrix}-\eta'(q) & & \\ & \ddots & \\ & & -\eta'(q)\end{matrix} & 0 & & \\ \hline
0 & \eta(2)\otimes 1'_{q-1} & \begin{matrix}\eta'(q-1) & & \\ & \ddots & \\ & & \eta'(q-1)\end{matrix} & & \\ \hline
 & & \ddots & \ddots & \\ \hline
 & & & \eta(q)\otimes 1'_1 & \begin{matrix}(-1)^q\eta'(1) & & \\ & \ddots & \\ & & (-1)^q\eta'(1)\end{matrix} \\ \hline
 & & & 0 & \eta(q+1)\otimes 1'_0
\end{array}\right)$$

where $1'_p$ is the unit matrix of degree equal to the rank of C'_p.

This is immediate from the definition (2) of tensor product complex, if we arrange for columns

$$C_0 \otimes C'_q, \quad C_1 \otimes C'_{q-1}, \cdots, C_q \otimes C'_0$$

and rows

$$d(C_0 \otimes C'_{q+1}), \ d(C_1 \otimes C'_q), \cdots, \ d(C_{q+1} \otimes C'_0)$$

conveniently.

3. Incidence matrices of finite abelian groups. Actual computation of incidence matrices for a finite group is in general very tedious; but for abelian groups this is very systematically done by the formula (3).

Let G be a finite abelian group with m-generators $s_1, \cdots, s_m$, $s_i^{n_i} = 1$ $(i = 1, \cdots, m)$ and put for simplicity's sake

$$\begin{aligned} \Delta_i &= 1 - s_i \\ N_i &= 1 + s_i + \cdots + s_i^{n_i - 1} \end{aligned} \qquad i = 1, \cdots, m.$$

We now define a system of incidence matrices, common to all abelian groups with same number of generators, which we shall write in the sequel as

$$\eta(q\,;m) \qquad\qquad q=1,2,\ldots\ldots$$

For $m=1$ we can take by (1)

$$(4)\qquad \begin{cases} \eta(2n-1\,;1)=(\Delta_1) \\ \eta(2n\,;1)=(N_1) \end{cases} \qquad\qquad n=1,2,\ldots\ldots$$

For $m=2$, using the formula (3), we define

$$(5)\qquad \begin{cases} \eta(2n-1\,;2)=\underbrace{\left(\begin{array}{ccccccc} \Delta_2 & & & & & & \\ \Delta_1 & -N_2 & & & & & \\ & N_1 & \Delta_2 & & & & \\ & & \Delta_1 & \ddots & & & \\ & & & \ddots & -N_2 & & \\ & & & & N_1 & \Delta_2 & \\ & & & & & \Delta_1 & \end{array}\right)}_{2n-1}\Bigg\}\,2n \\[2ex] \eta(2n\,;2)=\underbrace{\left(\begin{array}{cccccc} N_2 & & & & & \\ \Delta_1 & -\Delta_2 & & & & \\ & N_1 & N_2 & & & \\ & & \Delta_1 & \ddots & & \\ & & & \ddots & N_2 & \\ & & & & \Delta_1 & -\Delta_2 \\ & & & & & \Delta_1 \end{array}\right)}_{2n}\Bigg\}\,2n+1 \end{cases} \qquad n=1,2,\ldots\ldots$$

And inductively, if we know $\eta(q\,;m-1)$, $q=1,2,\ldots\ldots$, we put

$$(6)\qquad \eta(2n-1;m)=\left(\begin{array}{c|c|c|c|c} \Delta_m & 0 & 0 & & \\ \hline \begin{matrix}\Delta_{m-1}\\ \vdots \\ \Delta_1\end{matrix} & \begin{matrix}-N_m & & \\ & \ddots & \\ & & -N_m\end{matrix} & 0 & & \\ \hline 0 & \eta(2;m-1) & \begin{matrix}\Delta_m & & \\ & \ddots & \\ & & \Delta_m\end{matrix} & & \\ \hline & & \ddots & \ddots & \\ \hline & & & \eta(2n-2;m-1) & \begin{matrix}\Delta_m & & \\ & \ddots & \\ & & \Delta_m\end{matrix} \\ \hline & & & 0 & \eta(2n-1;m-1) \end{array}\right)$$

$$n=1,2,\ldots\ldots$$

$$\eta(2n\,;m)=\left(\begin{array}{c|c|c|c|c}
N_m & 0 & 0 & & \\
\hline
\begin{matrix}\Delta_{m-1}\\ \vdots \\ \Delta_1\end{matrix} & \begin{matrix}-\Delta_m & & \\ & \ddots & \\ & & -\Delta_m\end{matrix} & 0 & 0 & \\
\hline
\eta & \eta(2;m-1) & \begin{matrix}N_m & & \\ & \ddots & \\ & & N_m\end{matrix} & & \\
\hline
 & & \ddots & \ddots & \\
\hline
 & & & \begin{matrix}\eta(2n-1; \\ m-1)\end{matrix} & \begin{matrix}-\Delta_m & & \\ & \ddots & \\ & & -\Delta_m\end{matrix} \\
\hline
 & & & & \eta(2n\,;m-1)
\end{array}\right)$$

For the computation of q-cohomology group it is necessary to consider vectors in A with length equal to the column number $\#\eta(q+1\,;m)$ of $\eta(q+1\,;m)$. Now, for any $q=1,2,\cdots\cdot$

$$\#\,\eta(q+1\,;1)=1$$

and $\#\,\eta(q+1\,;m)$ is defined inductively, in virtue of formula (6), by

$$\#\,\eta(q+1\,;m)=\sum_{p=1}^{q+1}\#\,\eta(p;m-1),$$

hence

$$\#\,\eta(q+1\,;m)=(-1)^q\binom{-m}{q}\qquad q=0,1,2,\cdots\cdot. \tag{7}$$

4. A generalization of Schreier's condition. We shall now want to write incidence matrices of the preceding section explicitly at least for low dimensional cases.

For this purpose we shall use for q-dimensional vector a the following arrangement of indices:

$$\mathrm{a}=(a_{i_q\cdots\cdot i_1}),\qquad a_{i_q\cdots\cdot i_1}\in A$$

where $i_q\geqq\cdots\cdot\geqq i_1$ are taken from $1,2,\cdots\cdot,m$. The total number of elements is in fact

$$(-1)^q\binom{-m}{q}$$

i.e., by (7), the column number of $\eta(q+1\,;m)$.

It is convenient to write

$$\delta\,\mathrm{a}=\eta(q+1\,;m)\mathrm{a}.$$

Then, from table (6), we have the following recurrence formula:

$$(8)\qquad \begin{cases} \delta a_{i_q \cdots i_1} = \varepsilon(q,0) a_{i_q - 1 \cdots i_1} \\ \qquad\qquad \text{if } i_q = \cdots = i_1, \\ \delta a_{i_q \cdots i_1} = \iota_{i_q \cdots i_{r+1}} \delta a_{i_r \cdots i_1} + \varepsilon(q,r) a_{i_q - 1 \cdots i_1} \\ \qquad\qquad \text{if } i_q = \cdots = i_{r+1} > i_r \geqq \cdots \geqq i_1. \end{cases}$$

Here we use the notations

$$\begin{aligned} \varepsilon(q,r) &= (-1)^r \Delta_{i_q} && \text{if } q-r \text{ is odd} \\ &= (-1)^r N_{i_q} && \text{if } q-r \text{ is even,} \end{aligned}$$

and $\iota_{i_q \cdots i_{r+1}}$ is an operator on $a_{j_s \cdots j_1}$ with $i_{r+1} \geqq j_s \geqq \cdots \geqq j_1$, commutative with Δ_i, N_j, such that

$$\iota_{i_q \cdots i_{r+1}} a_{j_s \cdots j_1} = a_{i_q \cdots i_{r+1} j_s \cdots j_1}.$$

For example

$q = 1:$

$$\delta a_i = \Delta_i a,$$

$q = 2:$

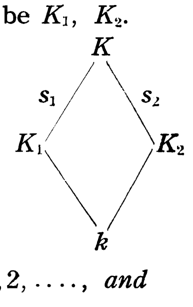
$$\begin{aligned} \delta a_{ii} &= N_i a_i \\ \delta a_{ij} &= \Delta_j a_i - \Delta_i a_j && (i > j), \end{aligned}$$

$q = 3:$

$$\begin{aligned} \delta a_{iii} &= \Delta_i a_{ii} \\ \delta a_{iij} &= \Delta_j a_{ii} - N_i a_{ij} && (i > j) \\ \delta a_{ijj} &= N_j a_{ij} + \Delta_i a_{jj} && (i > j) \\ \delta a_{ijk} &= \Delta_k a_{ij} - \Delta_j a_{ik} + \Delta_i a_{jk} && (i > j > k). \end{aligned}$$

The equation $\delta a = 0$ for the case of $q = 3$ is precisely the Schreier's condition ([4]; Satz III), under which the module A can be extended to a group B with $B/A \cong G$. Therefore, above formulas give a generalization of Schreier's condition.

5. **Application to galois cohomology.** Let K/k be an abelian extension with galois group G which has two generators s_1, s_2. The invariant subfields of s_1, s_2 be K_1, K_2.



We want to determine the cohomology groups of G with coefficients in K^*, the multiplicative group of non-zero elements of K.

But this seems very difficult and we have only the following

THEOREM 2. *Odd-dimensional cohomology group* $H^{2n+1}(G, K^*)$ *contains* $H^3(G, K^*)$ *for any* $n = 1, 2, \ldots,$ *and*

$$(9)\qquad H^3(G, K^*) \cong (N_1 K_2^* \cap N_2 K_1^*)/N_1 N_2 K^*.$$

REMARK. The structure of H^3 was also obtained by Prof. T. Tannaka.

PROOF. We shall write additively. Let ${}^t\mathfrak{a} = (a_1, a_2, \ldots.)$ be a vector in K^* with length $\# \eta(2n+2;2)$ such that $a_1 = a_4 = a_5 = \ldots. = 0$. In order that $\mathfrak{a}$ be a cocycle:

$$\eta(2n+2;2)\mathfrak{a} = 0$$

it is necessary and sufficient that

$$a = N_2 a_2 = -N_1 a_3 \in N_1 K_2^* \cap N_2 K_1^*.$$

If it is cohomologous to 0: $\mathfrak{a} \sim 0$, then

$$a = N_2 a^2 = -N_1 a_3 \in N_1 N_2 K^*.$$

Conversely, if $a = N_1 N_2 b_2$ with $b_2 \in K^*$ we put

$$a_2^1 = N_1 b_2, \qquad a_3^1 = -N_3 b_2.$$

Then $N_2 a_2 = N_2 a_2^1$, $N_1 a_3 = N_1 a_3^1$, therefore by Hilbert's lemma, there exist b_1, $b_3 \in K^*$ such that

$$a_2 = N_1 b_2 + \Delta_2 b_1, \qquad \Delta_1 b_1 = 0,$$
$$a_3 = -N_2 b_2 + \Delta_1 b_3, \qquad \Delta_2 b_3 = 0$$

i. e., $\mathfrak{a} \sim 0$.

We have thus proved, that $H^{2n+1}(G, K^*)$ contains a subgroup consists of cocycles

(10) $$^t\mathfrak{a} = (0, a_2, a_3, 0, 0, \ldots.)$$

isomorphic to $(N_1 K_2^* \cap N_2 K_1^*)/N_1 N_2 K^*$. But if $n=1$, by Hilbert's lemma, each cohomology class contains an element of the form (10). Therefore

$$H^3(G, K^*) \cong (N_1 K_2^* \cap N_2 K_1^*)/N_1 N_2 K^*.$$

THEOREM 3. *For any $n = 1, 2, \ldots.$, we have*

(11) $$H^{4n-1}(G, K^*) \supseteqq \underbrace{H^3(G, K^*) + \ldots. + H^3(G, K^*)}_{n},$$

(12) $$H^{4n+1}(G, K^*) \supseteqq H^5(G, K^*) + \underbrace{H^3(G, K^*) + \ldots. + H^3(G, K^*)}_{n-1}.$$

The proof of (11) is based upon (9). For the proof of (12), it is necessary to write $H^5(G, K^*)$ in similar but somewhat complicated form. These verifications are however easy, therefore we omit the proof.

REMARKS. If K is a $\mathfrak{p}$-adic field, then the right hand side of (9) is 1. On the other hand, Mr. H. Kuniyoshi has remarked that for algebraic number fields this is identical with

total norm-residues/norms.

Therefore, it is always a finite group and not necessarily 1.

Combined with (11), (12) it follows that, for algebraic number fields, odd-dimensional cohomology groups of dimension $\geqq 7$ are not necessarily isomorphic to 3-dimensional one.

6. Application to Lyndon's formula. Let G be a finite abelian group with m-generators each of which has order n_i such that

$$n_{i+1} | n_i \qquad i = 1, \ldots, m-1.$$

We now compute cohomology groups $H^i(G, Z)$ of G with coefficients in the additive group of rational integers Z considered as a trivial G-module.

We treat only the even-dimensional case: $q = 2n$; odd-dimensional case can be treated similarly. Let $\mathbf{a}$ be a vector of length $\# \eta(2n+1; m)$ and write it as

$$\mathbf{a} = \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{2n+1} \end{bmatrix}$$

where each $\mathbf{a}_i$ is a vector of length $\#\eta\ (i;\ m-1)$. Then the condition

$$\eta(2n+1; m)\mathbf{a} = 0$$

decomposes into

$$(13) \qquad \begin{cases} \eta(2i-1; m-1)\mathbf{a}_{2i-1} - n_m \mathbf{a}_{2i} = 0 \\ \eta(2i; m-1)\mathbf{a}_{2i} = 0 \end{cases} \qquad (1 \leqq i \leqq n),$$

$$(14) \qquad \eta(2n+1; m-1)\mathbf{a}_{2n+1} = 0.$$

The conditions (13) are equivalent to

$$\mathbf{a}_{2i-1} \quad \text{arbitrary} \qquad (1 \leqq i \leqq n)$$

$$\mathbf{a}_{2i} = \frac{1}{n_m}\eta(2i-1; m-1)\mathbf{a}_{2i-1}.$$

While condition (14) is that $\mathbf{a}_{2n+1}$ be a cocycle for the subgroup G_1 generated in G by first $m-1$ generators.

Similarly, if we write the general vector $\mathbf{b}$ of length $\# \eta(2n : m)$ as

$$\mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_{2n} \end{bmatrix},$$

each $\mathbf{b}_i$ is of length $\# \eta(i, m-1)$, the condition $\mathbf{a} = \eta(2n; m)\mathbf{b}$ can be written as

$$(15) \qquad \begin{cases} \mathbf{a}_{2i-1} = \eta(2i-2; m-1)\mathbf{b}_{2i-2} + n_m \mathbf{b}_{2i-1} \\ \mathbf{a}_{2i} \quad = \eta(2i-1; m-1)\mathbf{b}_{2i-1} \end{cases} \qquad (1 \leqq i \leqq n),$$

$$(16) \qquad \mathbf{a}_{2n+1} = \eta(2n; m-1)\mathbf{b}_{2n}.$$

From $n_m | n_i$ $(i = 1, \ldots, m-1)$ it follows that

$$\eta(2i-2; m-1)\mathbf{b}_{2i-2} \subseteqq n_m \mathbf{b}_{2i-1}.$$

Therefore, the factor groups of (13) by (15) are

$$\begin{bmatrix} \mathbf{a}_{2i-1} \\ \\ \eta(2i-1; m-1)\mathbf{a}_{2i-1} \end{bmatrix} \Bigg/ \begin{bmatrix} n_m \mathbf{b}_{2i-1} \\ \\ \eta(2i-1; m-1)\mathbf{b}_{2i-1} \end{bmatrix} \cong \# \eta(2i-1; m-1) \cdot Z/(n_m)$$

$$i = 1, \ldots, n,$$

where $Z/(n_m)$ denotes the cyclic group of order n_m, and multiplication by natural number $\# \eta(2i-1; m-1)$ means repeated direct sum.

Since the factor group of (14) by (16) is

$$H^{2n}(G_1, Z)$$

we have

$$H^{2n}(G, Z) \cong H^{2n}(G_1, Z) + \left(\sum_{i=1}^{n} \#\, \eta(2i-1; m-1)\right) \cdot Z/(n_m).$$

For odd-dimensional cases we can show

$$H^{2n+1}(G, Z) \cong H^{2n+1}(G_1, Z) + \left(\sum_{i=1}^{n} \#\, \eta(2i; m-1)\right) \cdot Z/(n_m).$$

If we insert the value of

$$\#\, \eta(i\,; m-1) = (-1)^{i-1} \binom{-m+1}{i-1}$$

into these equations, we get immediately the following formula of Lyndon ([3]; Theorem 6)

$$H^q(G, Z) \cong \sum_{j=1}^{m} \left(\sum_{i=0}^{q-2} (-1)^q \binom{-j}{i}\right) . Z/(n_j) \qquad (q \geqq 2). \tag{17}$$

In paticular:

$m = 1$

$$H^{2n}(G, Z) \cong Z/(n_1), \qquad H^{2n+1}(G, Z) \cong 0,$$

$m = 2$

$$H^{2n}(G, Z) \cong Z/(n_1) + n \cdot Z/(n_2), \quad H^{2n+1}(G, Z) \cong n \cdot Z/(n_2).$$

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