ANALYTIC FUNCTIONS STAR-LIKE OF ORDER p IN ONE DIRECTION

TOSHIO UMEZAWA

(Received July 11, 1952).

1. Introduction. Recently A. W. Goodman and M. S. Robertson [1] have studied typically-real functions of order p which were defined as follows:

A function

$$f(z) = \sum_{n=0}^{\infty} b_n z^n \tag{1.1}$$

is said to be a member of the class $T(p)$, if in (1.1) the coefficients b_n are all real and if either (I) $f(z)$ is regular in $|z| \leqq 1$ and $\Im f(e^{i\theta})$ changes sign $2p$ times as $z = e^{\theta}$ traverses the boundary of the unit circle, or (II) $f(z)$ is regular in $|z| < 1$ and if there is a $\rho < 1$ such that for each r in $\rho < r < 1$, $\Im f(re^{\theta})$ changes sign $2p$ times as $z = re^{i\theta}$ traverses the circle $|z| = r$.

Concerning the above class of functions A. W. Goodman [2] has obtained the following result:

Let

$$f(z) = z^q + \sum_{n=q+1}^{\infty} b_n z^n \tag{1.2}$$

be a function of the set $T(p)$. Suppose that in addition to the q-th order zero at $z = 0$, the function $f(z)$ has exactly s zeros, $\beta_1, \beta_2, \cdots, \beta_s$, such that $0 < |\beta_j| < 1$, $j = 1, 2, \ldots, s$. Finally let the non-negative integer t be defined by

$$q + s + t = p \geqq 1 \tag{1.3}$$

and let $m = [(t+1)/2]$. Then

$$|b_n| \leqq B_n, \qquad n = q+1, q+2, \ldots, \tag{1.4}$$

where B_n is defined by

$$F(z) = \frac{z^q}{(1-z)^{2q+2s}} \left(\frac{1+z}{1-z}\right)^{2m} \prod_{j=1}^{s} \left(1 + \frac{z}{|\beta_j|}\right)(1 + z|\beta_j|) \tag{1.5}$$

$$= z^q + \sum_{n=q+1}^{\infty} B_n z^n.$$

When t is odd or when $t = 0$, $F(z) \in T(p)$ and the inequality (1.4) is sharp.

Now in the present paper we shall introduce wider classes of functions, to be defined precisely in § 2, whose coefficients are not necessarily real, proving that inequalities similar to (1.4) can be obtained.

For the special case when $t = 0$ in (1.3) the above work has already been done by the present author. [3]

2. Preliminary considerations.

LEMMA 1. *Let*

$$w = f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{2.1}$$

be regular for $|z| \leqq 1$ *and have* $p(\geqq 0)$ *zeros in* $|z| \leqq 1$, *no zeros on* $|z| = 1$. *Then there exists a point* $\zeta(|\zeta| = 1)$ *for which the following equality holds*

$$\arg f(-\zeta) = \arg f(\zeta) + p\pi. \tag{2.2}$$

This lemma was proved in [3].

DEFINITION 1. The straight line $f(\zeta) \circ f(-\zeta)$ is said to be the diametral line of $f(z)$ when ζ satisfies Lemma 1.

The special case of Lemma 1 and Definition 1 we owe to S. Ozaki [4] and N. G. DeBruijn [5]

LEMMA 1′. *Let* (2.1) *be a function regular for* $|z| \leqq 1$, *and* $f(z) \neq 0$ on $|z| = 1$. *Then there exists at least one diametral line of* $f(z)$ *in the w-plane.*

DEFINITION 2. Let $f(z)$ be regular for $|z| \leqq 1$ and let C be the image curve of $|z| = 1$. If C is cut by a straight line passing through the origin in $2p$, and not more than $2p$ points, then $f(z)$ is said to be star-like of order p in the direction of the straight line. This set of functions is denoted by $S^1(p)$.

Especially when the direction of star-likeness of order p is that of the diametral line of $f(z)$, $f(z)$ is said to belong to the class $D(p)$.

The idea of being star-like in one direction was introduced by M. S. Robertson [6] and also extended to general p by him [7], [8]. And $D(1)$ was studied in [4], [5] and generalized in [3].

LEMMA 2. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a member of the class* $S^1(p)$ *or* $D(p)$. *Further let* $f(z)$ *have* s *zeros* $\beta_1, \beta_2, \dots, \beta_s$ *such that* $0 < |\beta_j| < 1$, $j = 1, 2, \dots, s$. *Then the function* $F(z)$ *defined by*

$$F(z) = f(z)g(z), \quad g(z) = z^s \Big/ \prod_{j=1}^{s} (z - \beta_j)(1 - \bar{\beta}_j z)$$

is also a member of the class $S^1(p)$ *or* $D(p)$, *respectively.*

PROOF. Regularity of $F(z)$ in $|z| \leqq 1$ is evident. Now we easily see that

$$g(e^{i\theta}) = 1 \Big/ \prod_{j=1}^{s} |e^{i\theta} - \beta_j|^2.$$

Hence $\arg F(e^{i\theta}) = \arg f(e^{i\theta})$ for every θ. Consequently if $f(z) \in S^1(p)$ or $D(p)$, then $F(z) \in S^1(p)$ or $D(p)$, respectively.

DEFINITION 3. The harmonic function $V(r, \theta)$ is said to have a change of sign at $\theta=\theta_j$ if there exists an $\varepsilon>0$ such that for $0<\delta<\varepsilon$

$$V(r,\theta_j+\delta)V(r,\theta_j-\delta)<0. \tag{2.3}$$

Note that in (2.3) r is constant.

DEFINITION 4. $\Im f(z)=V(r,\theta)$ is said to change sign q times on $|z|=r$ if there are q values of $\theta,\theta_1,\theta_2,\cdots,\theta_q$ such that

(a) inequality (2.3) holds for each θ_j, $j=1,2,\ldots,q$,

(b) $\theta_j\not\equiv\theta_k \pmod{2\pi}$ if $j\neq k$,

(c) if θ_s is any value of θ for which $V(r,\theta)$ has a change of sign then for one of the θ_j, $j=1,\ldots,q$ $\theta_s\equiv\theta_j \pmod{2\pi}$.

LEMMA 3. *If $V_1(r,\theta)$ and $V_2(r,\theta)$ both have a change of sign at θ_j, then the product $V_1(r,\theta)V_2(r,\theta)$ does not have a change of sign at θ_j. If $V_1(r,\theta)$ has a change of sign at θ_j and if $V_2(r,\theta)$ does not have a change of sign at θ_j then the product has a change of sign at θ_j.*

The above two definitions and Lemma 3 were used in [1].

LEMMA 4. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$, $|a_0|=1$, be regular in $|z|\leqq 1$ and $\Im f(re^{i\theta})$ change sign $2p$ times on $|z|=r$ for r near 1. Further let $f(z)$ have no zeros in $|z|<1$. Then*

$$f(z)\ll\left(\frac{1+z}{1-z}\right)^{p+1}. \tag{2.4}$$

PROOF. Since $f(z)$ is free of zeros in $|z|<1$, there is a function $h(z)=(f(z))^{1/p+1}$ such that $h(z)=\sum_{n=0}^{\infty}\alpha_nz^n$, $|\alpha_0|=1$, is regular and has no zeros in $|z|<1$. We shall see that for $|z|<1$, $\Re re^{i\alpha}h(z)\geqq 0$ with a proper choice of α.

Since $h(z)$ is free of zeros on $|z|\leqq 1$ by the above assumption, if we take a branch of $\arg h(re^{i\theta})$, $\arg h(re^{i\theta})$ is a one-valued continuous function in the interval $0\leqq\theta\leqq 2\pi$. Hence for our purpose it is sufficient to show that $\max_{0\leqq\theta\leqq 2\pi}\arg h(re^{i\theta})-\min_{0\leqq\theta\leqq 2\pi}\arg h(re^{i\theta})\leqq\pi$ which means $h(re^{i\theta})$ lies on a half plane.

Let us suppose that the above inequality does not hold. Then we have

$$\max_{0\leqq\theta\leqq 2\pi}\arg f(re^{i\theta})-\min_{0\leqq\theta\leqq 2\pi}\arg f(re^{i\theta})>(p+1)\pi$$

since $\arg f(z)=(p+1)\arg h(z)$. Hence $f(z)$ changes sign at least $p+1$ times on the arc of the circle $z=re^{i\theta}$, $\theta_1\leqq\theta\leqq\theta_2$, where $\arg f(re^{i\theta_1})=\min_{0\leqq\theta\leqq 2\pi}\arg f(re^{i\theta})$ and $\arg f(re^{i\theta_2})=\max_{0\leqq\theta\leqq 2\pi}\arg f(re^{i\theta})$. But $f(z)$ has no zeros, so that on the full circle $z=re^{i\theta}$, $-\pi+\varepsilon<\theta\leqq\pi+\varepsilon$, $\Delta\arg f(z)=0$. Therefore on the full circle $f(z)$ must change sign at least $2(p+1)$ times. This contradicts to our assumption.

Hence $Re^{i\alpha}h(z)\geqq 0$. By Carathéodory's Theorem, $h(z)\ll(1+z)/(1-z)$ and hence (2.4) is proved.

3. On the class $S(p)$.

THEOREM 1. *Let*

$$f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n \tag{3.1}$$

be a function of the set $S^1(p)$. Suppose that in addition to the q-th order zero at $z=0$ the function $f(z)$ has exactly s zeros, $\beta_1, \beta_2, \ldots, \beta_s$ such that $0<|\beta_j|<1$, $j=1,2,\ldots,s$. Finally let the non-negative integer t be defined by

$$q+s+t=p \geqq 1. \tag{3.2}$$

Then

$$|a_n| \leqq A_n, \qquad n=q+1,\ q+2,\ldots, \tag{3.3}$$

where A_n is defined by

$$\begin{aligned} F(z) &= \frac{z^q}{(1-z)^{2q+2s}}\left(\frac{1+z}{1-z}\right)^{2t+1}\prod_{j=1}^{s}\left(1+\frac{z}{|\beta_j|}\right)(1+z|\beta_j|) \\ &= z^l + \sum_{n=q+1}^{\infty} A_n z^n. \end{aligned} \tag{3.4}$$

PROOF. Let us put

$$E(z) = f(z)\cdot z^s \Big/ \prod_{j=1}^{s}(z-\beta_j)(1-\bar{\beta}_j z) \tag{3.5}$$

then by Lemma 2 $E(z) \in S^1(p)$, since $f(z) \in S^1(p)$ and

$$(-1)^s \prod_{i=1}^{s} \beta_i E(z) = z^{q+s} + \alpha_{q+s+1} z^{q+s+1} + \cdots = \psi(z) \in S^1(p). \tag{3.6}$$

We wish now to show that

$$\psi(z) \ll \frac{z^{q+s}}{(1-z)^{2q+2s}}\left(\frac{1+z}{1-z}\right)^{2t+1}. \tag{3.7}$$

For the purpose it will be sufficient to assume that the direction of star-likeness of order p is that of the real axis since otherwise we may consider $e^{i\alpha}f(z)$ with a proper choice of α. Then by our hypothesis $\Im\,\psi(z)$ changes sign $2p$ times on $|z|=1$, i.e. at θ_j $(j=1,2,\ldots,2p)$.

Let

$$\varphi(z) = (-1)^{q+s+1}\exp\left(-\frac{i}{2}\sum_{k=1}^{2(q+s)}\theta_{s_k}\right)\cdot\prod_{k=1}^{2(q+s)}(e^{i\theta_{s_k}}-z)/z^{q+s} \tag{3.8}$$

where θ_{s_k} $(k=1,2,\ldots,2(q+s))$ are chosen arbitrarily from θ_s $(s=1,2,\ldots,2p)$ then

$$\varphi(e^{i\theta}) = -2^{2(q+s)}\prod_{k=1}^{2(q+s)}\sin\frac{\theta_{s_k}-\theta}{2}. \tag{3.9}$$

Hence $\varphi(e^{i\theta})$ changes sign $2(q+s)$ times on $|z|=1$, i.e. at θ_{s_k} $(k=1,2,\ldots,2(q+s))$.

Let

$$G(z)=\psi(z)\varphi(z)=e^{i\theta}+\sum_{n=1}^{\infty}\gamma_n z^n, \tag{3.10}$$

then $G(z)$ is regular for $|z|\leqq 1$ and $G(z)$ has no zeros in $|z|<1$. Moreover by Lemma 3 $\Im G(e^{i\theta})$ changes sign $2p-2(q+s)=2t$ times on $|z|=1$. Furthermore $G(e^{i\theta})$ touches the real axis at the origin $2(q+s)$ times as z moves along $|z|=1$. It should be noticed that $G(e^{i\theta})$ does not touch the real axis at the other points. Hence $\Im G(re^{i\theta})$ for r near 1 changes sign more than $2t$ times in general. It does not, however, change sign more than $4t$ times. This fact can be seen as follows. The image region of $|z|\leqq 1$ under $G(z)$ contains a part of the real axis $(0+x,\ 0-x)$ for properly small x, at most t times since $\Im G(e^{i\theta})$ changes sign $2t$ times. If we consider $\Im G(re^{i\theta})$, the points of contact stated above are removed and new changes of sign appear at most $2t$ in the neighbourhood of the origin. Hence $\Im G(re^{i\theta})$ changes sign at most $4t$ times on $|z|=r$ for r near 1. Consequently by Lemma 4,

$$G(z)\ll\left(\frac{1+z}{1-z}\right)^{2t+1}. \tag{3.11}$$

On the other hand from (3.10) we have

$$\psi(z)=G(z)/\varphi(z)=z^{q+s}G(z)\Big/(-1)^{q+s+1}\exp\left(-\frac{i}{2}\sum_{k=1}^{2(q+s)}\theta_{s_k}\right)\prod_{k=1}^{2(q+s)}(e^{i\theta_{s_k}}-z)$$

which is dominated by

$$\frac{z^{q+s}}{(1-z)^{2(q+s)}}\left(\frac{1+z}{1-z}\right)^{2t+1} \tag{3.12}$$

since we have (3.11).

From (3.4) and (3.5) we have

$$f(z)=\psi(z)\prod_{i=1}^{s}(z-\beta_i)(1-\bar\beta_i z)\Big/\prod_{i=1}^{s}\beta_i z^s$$

which is dominated by

$$\frac{z^q}{(1-z)^{2(q+s)}}\left(\frac{1+z}{1-z}\right)^{2t+1}\prod_{i=1}^{s}\left(1+\frac{z}{|\beta_i|}\right)(1+|\beta_i|z)=F(z)$$

since we have (3.12). q. e. d.

COROLLARY 1. *Let $f(z)$ satisfy the condition of Theorem* 1, *and let $F(z)$ be given by* (3.4), *then for* $0\leqq r<1$

$$|f^{(j)}(re^{i\theta})|\leqq F^{(j)}(r), \qquad j=0,1,2,\ldots.$$

This is a trivial consequence of Theorem 1, since all the coefficients in $F(z)$ are positive.

4. On the class $D(p)$.

THEOREM 2. *Let*

$$(4.1) \qquad f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n$$

be a function of the set $D(p)$. Suppose that in addition to the q-th order zero at $z=0$, the function $f(z)$ has exactly s zeros, $\beta_1, \beta_2, \ldots, \beta_s$ such that $0 < |\beta_j| < 1$, $j = 1, 2, \ldots, s$. Finally let t be defined by

$$(4.2) \qquad q + s + t = p \geqq 1.$$

Then

$$(4.3) \qquad |a_n| \leqq C_n, \qquad n = q+1,\ q+2, \ldots,$$

where C_n is defined by

$$(4.4) \qquad F(z) = \frac{z^q}{(1-z)^{2q+2s}}\left(\frac{1+z}{1-z}\right)^{2t} \prod_{j=1}^{s}\left(1 + \frac{z}{|\beta_j|}\right)(1 + z|\beta_j|)$$

$$= z^q + \sum_{n=q+1}^{\infty} C_n z^n.$$

PROOF. Although we can complete this proof by a slight modification of the method used in §3, some notes must be added.

Employing the notations in §3, we have by Lemma 2, $E(z) \in D(p)$ and $\psi(z) \in D(p)$.

Now we wish to show that

$$\psi(z) \ll \frac{z^{q+s}}{(1-z)^{2q+2s}}\left(\frac{1+z}{1-z}\right)^{2t}.$$

For the purpose it will be sufficient to assume that the diametral line in which direction $\psi(z)$ is star-like of order p is $\psi(1) \circ \psi(-1)$, since otherwise we may consider $\psi(\zeta z) = g(z)$ for which $g(1) \circ g(-1)$ is the diametral line.

Let $\psi(1) = w = |w| e^{-i\alpha}$ then by our hypothesis $\Im e^{i\alpha}\psi(e^{i\theta})$ changes sign at θ_s $(s = 1, 2, \ldots, 2p)$ where, in particular, $\theta_1 = 0$, $\theta_j = \pi (1 < j \leqq 2p)$.

Defining $\varphi(z)$ as in §3, in particular, with $\theta_{s_1} = \theta_1 = 0$, $\theta_{s_l} = \theta_j = \pi$, we have

$$\psi(z) = e^{-i\alpha} G(z)/\varphi(z)$$

$$= -e^{-i\alpha} G(z)/(1-z^2)(-1)^{q+s} \exp\left(-\frac{i}{2}\sum_{k \neq 1, l}^{2(q+s)} \theta_{s_k}\right) \prod_{k \neq 1, l}^{2(q+s)} (e^{i\theta_{s_k}} - z) z^{-(q+s)}$$

which is dominated by

$$\left(\frac{1+z}{1-z}\right)^{2t+1} \cdot \frac{1}{1-z^2} \cdot \frac{z^{q+s}}{(1-z)^{2(q+s-1)}}$$

since we again have $G(z) \ll \left(\frac{1+z}{1-z}\right)^{t+1}$. Hence we have (4.3) by the same way as in Theorem 1. q.e.d.

Obviously we can obtain a corollary similar to the one in §3. But we refrain from describing it.

COROLLARY 2. *Let $f(z)$ in the form* (3.1) *be regular for* $|z| \leqq 1$ *and assigned with the same zeros as in Theorem* 2. *Suppose that $f(z)$ satisfies one of the following conditions.*

i) $f(1) =$ *real and* $f(-1) =$ *real and* $\Im f(e^{i\theta})$ *changes sign* $2p$ *times on* $|z| = 1$.

ii) *All the coefficients are real and* $\Im f(e^{i\theta})$ *changes sign* $2p$ *times on* $|z| = 1$.

Then (4.3) *holds.*

PROOF. i) In this case the diametral line of $f(z)$ is evidently the real axis and star-like of order p in this direction by our hypothesis which proves the corollary by using Theorem 2.

ii) This is a direct consequence of the preceding i).

5. Classes of functions related to $S^1(p)$ or $D(p)$.

DEFINITION 5. Let $f(z)$ be regular for $|z| \leqq 1$ and C be the image curve of $|z| = 1$. Let, further, P be the orthogonal projection of $f(e^{i\theta})$ onto a straight line. Then P will move on the straight line both positively or nagatively when θ varies from 0 to 2π. If P changes its direction of movement $2p$ times varies from 0 to 2π, then $f(z)$ is said to be convex of order p in the direction when θ which is perpendicular to the straight line. This set of functions is denoted by $K^1(p)$ and has been studied by M. S. Robertson [9] recently. Especially if, when we represent $f(z)$, $zf'(z)$ in the same plane, the straight line is parallel to a diametral line of $zf'(z)$, then $f(z)$ is said to be a member of $F(p)$.

LEMMA 6. *$f(z)$ is a member of the class $K^1(p)$ or $F(p)$ if and only if $zf'(z)$ belongs to the class $S^1(p)$ or $D(p)$, respectively.*

This is a generalization of M. S. Robertson's Lemma, and for $F(p)$ and $D(p)$ was proved in [3]. Analogous proof can be made for $K^1(p)$ and $S^1(p)$.

THEOREM 3. *Let*

$$f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n \tag{5.1}$$

be a function of the set $F(p)$. Suppose that in addition to the q-th order critical points at $z = 0$, the function $f(z)$ has exactly s critical points at $\alpha_1, \alpha_2, \ldots, \alpha_s$, such that $0 < |\alpha_j| < 1$, $j = 1, 2, \ldots, s$. Finally let t be defined by $q + s + t = p \geqq 1$. Then

$$|a_n| \leqq q\, C_n/n, \qquad n = q+1,\ q+2, \ldots, \tag{5.2}$$

$$|f(re^{i\theta})| \leqq q \int_0^r \frac{F(r)}{r} dr \qquad \text{for } r < 1, \tag{5.3}$$

$$|f'(re^{i\theta})| \leqq q\, F(r)/r \qquad \text{for } r < 1, \tag{5.4}$$

where C_n, $F(r)$ are defined by

$$F(z) = \frac{z^q}{(1-z)^{2q+2s}}\left(\frac{1+z}{1-z}\right)^{2t}\prod_{j=1}^{s}\left(1+\frac{z}{|\alpha_j|}\right)(1+z|\alpha_j|) \tag{5.5}$$

$$= z^q + \sum_{n=q+1}^{\infty} C_n z^n.$$

PROOF. Since $f(z) \in F(p)$

$$\frac{1}{q} zf'(z) = z^q + \frac{1}{q}\sum_{n=q+1}^{\infty} n\, a_n z^n \in D(p)$$

by Lemma 6. By using Theorem 2 we have (5.2) and (5.4). By integrating along a radius we have

$$|f(re^{i\theta})| = \left|\int_0^z f'(z)dz\right| \leqq \int_0^r |f'(re^{i\theta})|dr \leqq q\int_0^r \frac{F(r)}{r}\,dr \quad \text{for } r<1,$$

which completes the proof.

REMARK. A theorem for $K^1(p)$ analogous to the above one can be derived by using Theorem 1 and Lemma 6. But we do not arrange it here.

COROLLARY 3. *Let $f(z)$ in the form* (5.1) *be regular for $|z| \leqq 1$ and assigned with the same critical points as in Theorem* 3. *Suppose that $f(z)$ satisfies one of the following conditions.*

(i) *$f'(1) = real$, $f'(-1) = real$, and $f(z)$ is convex of order p in the direction of the imaginary axis.*

(ii) *In* (5.1) *the coefficients are all real and $f(z)$ is convex of order p in the direction of the imaginary axis.*

Then (5.2), (5.3) *and* (5.4) *hold.*

REFERENCES

[1] A. W. GOODMAN AND M. S. ROBERTSON, A class of multivalent functions, Trans. Amer. Math. Soc., 70(1951), 127-136.

[2] A. W. GOODMAN, Typically-real functions with assigned zeros, Proc. Amer. Math. Soc., 2(1951), 349-357.

[3] T. UMEZAWA, A class of Multivalent Function with Assigned Zeros, Proc. Amer. Math. Soc., 3(1952)813-20.

[4] S. OZAKI, On the theory of multivalent functions II, Sci. Rep. Tokyo Bunrika Daigaku, 4(1941), 45-78.

[5] N. G. DEBRUIJN, Ein Satz über schlichte Funktionen, Nederl. Akad. Wetensch., 44(1941), 47-49.

[6] M. S. ROBERTSON, Analytic functions star-like in one direction, Amer. Journ. of Math., 58(1936), 465-472.

[7] M. S. ROBERTSON, A representation of all analytic functions in terms of functions with positive real parts, Ann. of Math., 38(1937), 770-783.

[8] M. S. ROBERTSON, The variation of the sign of V for an analytic function $U+iV$, Duke Math. Journ., 5(1939), 512-519.

[9] M. S. ROBERTSON, A coefficient problem for functions regular in an anulus, Canadian Journ. of Math. 4(1952)407-423.

GUMMA UNIVERSITY, MAEBASHI.