ON MAPS OF A $(2n-1)$-DIMENSIONAL SPHERE INTO AN n-DIMENSIONAL SPHERE

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1. Introduction. H. Hopf [3] studied the problem of enumeration of homotopy classes of maps of S^3 on S^2. His useful idea, the socalled Hopf's invariant was extended [4] for mappings of a $(2n-1)$-dimensional sphere into an n-dimensional sphere. L. Pontrjagin [7] succeeded the enumeration of the homotopy classes of maps of a 3-complex K^3 on S^2. There Pontrjagin obtained the result that Hopf's invariant only determines homotopic classes of the maps of S^3 on S^2. H. Whitney [9] reformulated another Hopf's theorem [5] which is essential for the theory of maps, where Whitney introduced two deformation theorems and the definition of a standard map. In this paper we shall attempt the generalization of that Pontrjagin's theorem. We generalize Whitney's deformations and standard maps for this purpose. If two maps of a $(4k-1)$-dimensional sphere into a $2k$-dimensional sphere is given, the necessary and sufficient condition for the homotopy of the two maps is that their Hopf's invariant are equal each other and they are satisfied by some conditions of homotopy. If we set $n=2$, Pontrjagin's theorem follows as a corollary of last theorem by virtue of the Eilenberg's Homotopy Theorem [1].

2. Let $K^{2n-1}, \overset{*}{K}{}^n$ be simplicial subdivisions of S^{2n-1}, S^n respectively and τ_0^n are n-simplex of $\overset{*}{K}{}^n$. We may choose $\overset{*}{K}{}^n$ so that if p_0 is an interior point of τ_0^n and $\bar{p}_0$ is the antipodal point of S^n, each great semicircle from p_0 to $\bar{p}_0$ intersects the boundary $\partial\tau_0^n$ exactly in one point. By pushing along these semicircles, we define a deformation θ_t of the identity $\theta_0(p)\equiv p$ into a map θ_1, where $\theta_1(p)\equiv \bar{p}_1$ for p in $S^n-\tau_0^n$. We now generalize the definition of Whitney's standard map.

DEFINITION 2.1. Let C^k be a complex of K^{2n-1} and f a map of C^k into $\overset{*}{K}{}^n$. We say f is standard if $f(p)\equiv \bar{p}_0 (p\in C^k)$ or $f(p)=\theta_0((\overset{*}{f}(p))\ (p\in C^k)$, where $\overset{*}{f}$ is a suitable simplicial map of C^k onto τ_0^n.

DEFINITION 2.2. Let f be a map of K^{2n-1} into $\overset{*}{K}{}^n$. We say f is standard if K^{2n-1} is a sum of some complexes and f is a standard map on every one of the complexes.

DEFINITION 2.3. τ_0^n which is used in the definition of θ_t, is called the basic simplex for standard maps.

LEMMA 2.1. *Let f is a map of S^{2n-1} into S^n, then there exists a standard map which is homotopic to f.*

PROOF. f maps K^{2n-1} into $\overset{*}{K}{}^{n}$ and there is a simplicial approximation $\overset{*}{f}$ which is homotopic to f.

We set $\overset{**}{f}(p) = \theta_0(\overset{*}{f}(p))$ $(p \in K^{2n-1})$. As θ_t is a deformation from the identity θ_0 into θ_1, $\theta_0(\overset{*}{f}) \simeq \theta_1(\overset{*}{f}) = \overset{**}{f}$ on $\overset{*}{f}{}^{-1}(\tau_0^n)$. On the other hand, $\theta_0(\overset{*}{f}) \simeq \theta_1(\overset{*}{f}) \equiv \bar{p}_0$ on $K^{2n-1} - \overset{*}{f}{}^{-1}(\tau_0^n)$. By Definition 2.2, $\overset{**}{f}$ is a required standard map.

Let f be a simplicial map of K^{2n-1} into $\overset{*}{K}{}^{n}$. Following Hopf's notation [3, 4] we denote by $\varphi_{S^{2n-1}}(p_0)$ the inverse image of p_0 for f. It is well known that $\varphi_{S^{2n-1}}(p_0)$ is the sum of some triangulated manifolds M_i $(i = 1, 2, \cdots k)$. This triangulation depends on the triangulation K^{2n-1} of S^{2n-1}. When M_i is mapped on p_0 by f we denote the complex which is mapped on τ_0^n by R_i (p_0). If we assume that the subdivision K^{2n-1} is sufficiently fine, it follows that $R_i \cap R_j = 0$ $(i \neq j)$ by virtue of the relation $M_i \cap M_j = 0$ $(i \neq j)$. Every $(2n-1)$-simplex $T_\alpha{}^{2n-1}$ of K^{2n-1} which is mapped on τ_0^n necessarily intersects $\varphi_{S^{2n-1}}(p_0)$. As we easily deduce, on account of the fact $R_i \cap R_j = 0$, that only one of M_i's intersects a arbitrary preasigned simplex T_α^{2n-1}. $\varphi_{S^{2n-1}}(p_0) \cap T_\alpha^{2n-1} = M_i \cap T_\alpha^{2n-1}$ is an $(n-1)$-simplex $(a_0^\alpha, a_1^\alpha, \cdots, a_{n-1}^\alpha)$. Any two of its n-simplexes mapped on τ_0^n have a common 1-simplex. These 1-simplexes determine an $(n-1)$-simplex which is a face of T_α^{2n-1}. We denote it by $(e_0^\alpha, e_1^\alpha, \cdots, e_{n-1}^\alpha)$. Then $\{a_0^\alpha, a_1^\alpha, \cdots, a_n^{\alpha-1};\ e_0^\alpha, e_1^\alpha, \cdots, e_{n-1}^\alpha\}$ is a $(2n-1)$-simplex which we denote by $\overset{*}{T}{}_\alpha^{2n-1}$. Such $(2n-1)$-simplex is obtained from every simplex of M_i. $\sum_\alpha (e_0^\alpha, e_1^\alpha, \cdots, e_{n-1}^\alpha)$ constitutes a triangulated manifold $\bar{M}_i$. We consider the following complex from n-faces of $\overset{*}{T}{}_\alpha^{2n-1}$:

$$(a_0^\alpha, a_1^\alpha, \cdots, a_{n-1}^\alpha, e_0^\alpha) + (-1)^{n-1}(a_1^\alpha, \cdots, a_{n-1}^\alpha, e_0^\alpha, e_1^\alpha)$$
$$+ (-1)^{2(n-1)}(a_2^\alpha, \cdots, a_{n-1}^\alpha, e_0^\alpha, e_1^\alpha, e_2^\alpha) + \ldots + (-1)^{(n-1)^2}(a_{n-1}^\alpha, e_0^\alpha, e_1^\alpha, \cdots, e_{n-1}^\alpha).$$

The sum of these complexes for every α such that $M_i \cap T_\alpha^{2n-1} \neq 0$, is given to constitute a n-manifold bounded by M_i and $\bar{M}_i$. This manifold is denoted by $\bar{K}_{i0}$. We replace $(e_0^\alpha, e_1^\alpha, \cdots, e_{n-1}^\alpha)$ by $(e_p^\alpha, e_{p+1}^\alpha, \cdots, e_{n-1}^\alpha, e_0^\alpha, \cdots, e_{p-1}^\alpha)$. Then

$$\sum_\alpha [(a_0^\alpha, a_1^\alpha, \cdots, a_{n-1}^\alpha, e_p^\alpha) + (-1)^{n-1}(a_1^\alpha, a_2^\alpha, \cdots, a_{n-1}^\alpha, e_\alpha^p, e_{p+1}^\alpha)$$
$$+ (-1)^{2(n-1)}(a_2^\alpha, a_3^\alpha, \cdots, a_{n-1}^\alpha, e_p^\alpha, e_{p+1}^\alpha, e_{p+2}^\alpha) + \cdots$$
$$+ (-1)^{(n-1)^2}(a_{n-1}^\alpha, e_p^\alpha, e_{p+1}^\alpha, \cdots, e_{p-1}^\alpha, e_0^\alpha, e_1^\alpha \cdots, e_{p-1}^\alpha)]$$

constitutes a similar n-manifold bounded by M_i and $\bar{M}_i$. We shall denote it by $\bar{K}_{ip}$. As $\bar{M}_i \sim 0$ in S^{2n-1}, there exists an n-complex $\bar{\bar{K}}_i$ bounded by $\bar{M}_i$. Therefore $K_{ip} = \bar{K}_{ip} + \bar{\bar{K}}_i$ is n-complex bounded by M_i. As p_0 is an interior point of τ_0^n, p_0 and vertices of each $(n-1)$-face of τ_0^n make n-simplexes $\tau_{00}^n, \tau_{01}^n, \cdots, \tau_{0n}^n$. If we replace τ_0^n by $\tau_{00}^n, \cdots, \tau_{0n}^n$ in $\overset{*}{K}{}^{n}$, then the complex thus obtained is more fine simplicial subdivision than $\overset{*}{K}{}^{n}$. We shall denote the resulted complex by the same notation $\overset{*}{K}{}^{n}$ for berevity. This notation will not throw us into confusion. Similarly we can consider a fine simplicial triangulation of K^{2n-1} by inverse image of $\overset{*}{K}{}^{n}$ by f and a suitable additional subdivision. We shall also denote it by K^{2n-1}.

DEFINITIOIN 2.4, We paste some simplexes of a manifold in Euclidean space and we say that the resulted complex is a manifold having some singular simplexes.

LEMMA 2.2. *K_{ip} may be chosen as a manifold which has some singular simplexes of at most one-dimension.*

PROOF. Let us consider that a fixed point of S^{2n-1} is a point at infinity, then S^{2n-1} may be regarded as the sum of the point at infinity and a $(2n-1)$-dimensional Euclidean space. Of course, we don't take the point at infinity on $\varphi_{S^{2n-1}}(p_0)$. Let $T^{n-1}_{i\alpha}$ be any $(n-1)$-simplex of $\bar{M}_i$ and we project every $T^{n-1}_{i\alpha}$ from a suitable point and denote by $P^n_{i\alpha}$ the resulted sets. We shall prove that $\bar{K}_{ip}+\Sigma_\alpha P^n_{i\alpha}$ is an n-manifold which has boundary M_i and some singular simplexes of at most one-dimension. At first it is clear that $\bar{K}_{ip}+\Sigma_\alpha P^n_{i\alpha}$ has these properties except for singularity of at most one-dimension. Let $T^{n-1}_{i\alpha 1}$ and $T^{n-1}_{i\alpha 2}$ be a pair of adjacent simplexes of $\bar{M}_i$ and $T^{n-1}_{i\beta 1}$, $T^{n-1}_{i\beta 2}$ be another adjacent pair. We may assume $(T^{n-1}_{i\alpha 1}\cup T^{n-1}_{i\alpha 2})\cap(T^{n-1}_{i\beta 1}\cup T^{n-1}_{i\beta 2})=0$. If $P^n_{i\alpha 1}\cup P^n_{i\alpha 2}$ and $P^n_{i\beta 1}\cup P^n_{i\beta 2}\cup\bar{K}_{ip}$ have a common n-simplex $(a_0, a_1, \dots, a_n)$, we replace the one in either of the two complexes as follows: We consider an $(n+1)$-simplex $(a_0, a_1, \dots, a_{n+1})$ which has n-face $(a_0a_1\dots a_n)$ and take an interior point b of the former. We shall replace $(a_0a_1\dots a_n)$ of $P^n_{i\alpha 1}\cup P^n_{i\alpha 2}$ by $[(ba_1\dots a_n)-(ba_0a_2\dots a_n)+\dots+(-1)^n(ba_0a_1\dots a_{n-1})]$. If K^{2n-1}, $\overset{*}{K}{}^{n}$ are divided finely by b, $f(b)$ respectively and K^{2n-1} is divided suitably moreover, f may be considered as a simplicial mapping. The resulted complexes are also denoted by K^{2n-1}, $\overset{*}{K}{}^{n}$. However this notation will not confuse us. Suppose that such process is done for every common n-simplex of $P^n_{i\alpha 1}\cup P^n_{i\alpha 2}$ and $P^n_{i\beta 1}\cup P^n_{i\beta 2}\cup\bar{K}_{ip}$, then they have simplexes at most of dimension $n-1$. If $n-1=1$, the proof is complete. Hence we assume $n-1>1$. We denote one of the common $(n-1)$-simplexes by $(a_0a_1\cdots a_{n-1})$ and replace it as follows: We consider n-simplexes $(a_0a_1\dots a_n)$ and $(a_0a_1\dots a'_n)$ of $P^n_{i\alpha 1}\cup P^n_{i\alpha 2}$ which have the $(n-1)$-face $(a_0a_1\dots a_{n-1})$ and take its interior point b and b', respectively. At first, we replace $(a_0a_1\cdots a_{n-1}a_n)$ and $(a_0a_1\dots a_{n-1}a'_n)$ of $P^n_{i\alpha 1}\cup P^n_{i\alpha 2}$ by

$$C^n_1\equiv[(ba_1\dots a_n)-(ba_0a_2\cdots a_n)+\dots+(-1)^{n-1}(ba_0a_1\cdots a_{n-1}a_n)]$$

and

$$C^n_2\equiv[(b'a_1\cdots a'_n)-(b'_na_0a_2\cdots a'_n)+\dots+(-1)^{n-1}(ba_0a_1\cdots a_{n-1}\,a'_n)]$$

respectively. Secondly we can consider an n-dimensional regularly connected complex C^n bounded by

$[(ba_1\cdots a_{n-1})-(ba_0a_2\cdots a_{n-1})+\cdots+(-1)^{n-1}(ba_0a_1\cdots a_{n-1})]$
and
$[(b'a_1\cdots a_{n-1})-(b'a_0a_2\cdots a_{n-1})+\cdots+(-1)^{n-1}(b'a_0a_1\cdots a_{n-1})]$
so that does not intersect $P^n_{r\beta 1}\cup P^n_{i\beta 2}\cup K^n_{ip}$.

We replace $(a_0a_1\cdots\cdots a_{n-1}a_n)+(a_0a_1\cdots a_{n-1}a'_n)$ by $C^n_1+C^n+C^n_2$. Suppose that such process is done for every common $(n-1)$-simplex of $P^n_{i\alpha 1}\cup P^n_{i\alpha 2}$ and $P^n_{i\beta 1}\cup P^n_{i\beta 2}\cup \overline{K}_{ip}$, then they have common simplexes at most of $(n-2)$-dimensions. If we take care of the fact that every common $(n-2)$-simplex is a common face of some n-simplexes (its number need not be for the common $(n-2)$-simplex necessary two), we can perform similar process. By a repetition of similar processes, we can lower the dimension of common simplexes of $P^n_{i\alpha 1}\cup P^n_{i\alpha 2}$ and $P^n_{i\beta 1}\cup P^n_{\beta 2}\cup \overline{K}^n_i$ till at most 1. Because the dimension of the last common simplex is calculated from the dimensions of S^{2i-1}, $P^n_{i\alpha 1}\cup P^n_{i\alpha 2}$ and $P^n_{i\beta 1}\cup P^n_{i\beta 2}\cup \overline{K}^n_{ip}$.

3. Whitney [8] gave us special deformations for a map of an n-sphere into another n-sphere. We generalize these deformations for our purpose. Let ξ_0 be an interior point of τ^n_0, $\bar{\xi}_0$ be the antipodal point of S^n and K^n_{ip} be the complex constructed in Lemma 2.2.

We assume that $\sigma=(a_0a_1\cdots a_n)$, $\sigma'=-(a'_0a_1\cdots a_n)$ are oriented n-simplexes of K^n_{ip} with the common $(n-1)$-face $\tau=(a_1a_2\cdots a_n)$.

LEMMA 3.1. *Let f be a standard map of K^n_{ip} into S_n and $f(\sigma)=\pm S_n$, $=\xi_0$, then there is a standard map g which is homotopic to f and $g(\sigma)=\bar{\xi}_0$, $g(\sigma')=\pm S^n$ leaving the degree of $K^n_{ip}-(\sigma+\sigma')$ fixed.*

LEMMA 3.2. *Let f be a standard map of K^n_{ip} into S^n and $f(\sigma)=+S^n$, $f(\sigma')=-S^n$, then there is a standard map g which is homotopic to f and $g(\sigma)=\bar{\xi}_0$, $g(\sigma')=\bar{\xi}_0$ leaving the degree of $K^n_{ip}-(\sigma+\sigma')$ fixed.*

PROOF OF LEMMA 3.1. We consider the case where τ has no singular 1-simplex at first. Its proof is equal to Whitney's, but I recall it for convenience of the case where τ has singular simplex. Set $\tau=(a_1,a_2,\ldots,a_n)$, $\sigma_1=(a_0,a_2,\ldots,a_n)$, $\sigma'_1=(a'_0,a_2,\cdots,a_n)$.

Let ε_1 and ε_2 be the affine maps of σ_1 into τ and σ'_1 sending a_0 into a_1 and a'_0 respectively. For each p in σ_1, let $\alpha(p,u)$ be points which run linearly along the segments $\overline{p\ \varepsilon_1(p)}$ and $\overline{\varepsilon_1(p)\ \varepsilon_2(p)}$ as u run from 0 to 1 and from 1 to 2.

Set
$$\phi_t[\alpha(p,u)]=\begin{cases} f[\alpha(p,u-t)] & (t\leqq u)\\ f[\alpha(p,0)] & (t>u)\end{cases}$$
and $\phi_t(p)=f(p)$ in $K^n_{ip}-(\sigma+\sigma')$. As $f(p)\equiv\bar{\xi}_0$ in $\partial\sigma+\partial\sigma'$, ϕ_t is clearly a deformation of $\phi_0=f$ into a map ϕ_1. The map ϕ_1 in σ' is obtained from the following relation:

$$\phi_1[\alpha(p, u)] = f[\alpha(p, 1-u)] \qquad (1 \leqq u).$$

Hence $\phi_1(\sigma') = f(\sigma) = \pm S^n$. Also $\phi_1(\sigma) = \phi_1[\alpha(p, u)] = f[\alpha(p, 0)] = \xi_0$ $(1 > u)$. Then ϕ_1 is a required map g.

Seconly, let τ have some singular 1-simplexes. We assume that $\sigma \cap \sigma' = \tau$, $\overset{*}{\sigma} \cap \overset{*}{\sigma'} = \overset{*}{\tau}$ have common singular 1-simplexes. All possible cases are classified essentially in the following three:

$$\text{(i)} \qquad \begin{cases} f(\sigma) = \pm S^n, & f(\sigma') = \bar{\xi}_0, \\ f(\overset{*}{\sigma}) = \pm S^n, & f(\overset{*}{\sigma'}) = \bar{\xi}_0, \end{cases}$$

$$\text{(ii)} \qquad \begin{cases} f(\sigma) = \pm S^n, & f(\sigma') = \bar{\xi}_0, \\ f(\overset{*}{\sigma}) = \bar{\xi}_0, & f(\overset{*}{\sigma'}) = \bar{\xi}_0, \end{cases}$$

$$\text{(iii)} \qquad \begin{cases} f(\sigma) = \pm S^n, & f(\sigma') = \bar{\xi}_0 \\ f(\overset{*}{\sigma}) = \pm S^n, & f(\overset{*}{\sigma'}) = \pm S^n. \end{cases}$$

In (i), we may take the common one as the deformation for $\sigma + \sigma'$ and $\overset{*}{\sigma} + \overset{*}{\sigma'}$. In (ii), by the deformation for $\sigma + \sigma'$, $\sigma \cap \sigma' = \tau$ is removed from a face of τ^n_{00} to another face and this doesn't contradict to $f(\overset{*}{\sigma} + \overset{*}{\sigma'}) = \bar{\xi}_0$. We shall consider the case (iii) in the proof of Lemma 3.2. If another face of $\sigma + \sigma'$ has also a singular 1-simplex, the above deformation may be done at will.

Proof of Lemma 3.2. Whitney's deformation must be modified a little for the case that K^n_{ip} has singular simplexes. At first, let τ have no singulsr 1-simpexes. We denote by λ, λ' the affine maps of σ and σ' into τ^n_{00} respectively such that

$$\begin{aligned} f(p) &= \theta_1(\lambda(p)) && (p \in \sigma) \\ &= \theta_1(\lambda'(p)) && (p \in \sigma'). \end{aligned}$$

Set $\tau^n_{00} = (b_0 b_1 \cdots b_n)$, $\lambda(a_i) = b_{\kappa_i}$, and $\lambda'(a'_0) = b_{h_0}$, $\lambda'(a_i) = b_{h_i}$ $(i > 0)$. As $f(\sigma) = -f(\sigma')$ and $d_f(\sigma) = d_f(a_0 a_1 \cdots a_n) = -d_f(\sigma') = d_f(-\sigma') = d_f(a'_0 a_1 \cdots a_n)$, $b_{k_0} b_{k_1} \cdots b_{k_n}$ is an even permutation of $b_{h_0} b_{h_1} \cdots b_{h_n}$. Applying Whitney's Lemma [8], we find a deformation λ_n of σ' in τ^n_{00} such that $\lambda_0 \equiv \lambda'$, λ' is affine, and

$$\lambda_1(a'_0) = \lambda(a_0), \ \lambda_1(a_i) = \lambda(a_i) \qquad (i > 0) \tag{1}$$

If we put

$$\phi_t(p) = \begin{cases} \theta_1(\lambda_t(p)) & (n \in \sigma') \\ f(p) & (p \in K^n_{ip} - \sigma'), \end{cases} \tag{2}$$

then ϕ_t is a deformation of $\phi_0 = f$ into ϕ_1.

Set $\tau = (a_1, a_2, \ldots, a_n)$, $\sigma_1 = (a_0, a_2, \ldots, a_n)$ and $\sigma'_1 = (a'_0, a_2, \cdots, a_n)$. Let δ_1 and δ_2 be the affine maps of τ into σ_1 and σ'_1 sending a_1 into a_0 and a'_0 respectively. For each p in τ, let $\beta(p, u)$ and $\beta'(p, u)$ be points which run linearly along the segments $\overline{\delta_1(p)\, p}$ and $\overline{\delta_2(p)\, p}$ as u run from 0 to 1 respectively. Then (1) and (2) give us

$$\phi_1[\beta(p, u)] = \phi_1[\beta'(p, u)].$$

We set $\phi_1 = \phi^*$ and consider the following deformation:

$$\overset{*}{\phi}_t[\beta(p, u)] = \overset{*}{\phi}_t[\beta'(p, u)] = \overset{*}{\phi}[\beta(p, (1-t)u)]$$
$$\overset{*}{\phi}_t(p) = \overline{\xi}_0 \qquad (p \in K^n_{ip} - (\sigma + \sigma')).$$

As $\overset{*}{\phi}_1(p) = \overline{\xi}_0 \, (p \in \sigma + \sigma'), \overset{*}{\phi}_1$ is a required map g of Lemma 3.2. Secondly, let $\sigma \cap \sigma' = \tau$ have some singular 1-simplexes. We assume that $\sigma \cap \sigma' = \tau$ and $\overset{*}{\sigma} \cap \overset{*}{\sigma}{}' = \overset{*}{\tau}$ have some common singular 1-simplexes. As we may prove analogously the similar cases of (i), (ii) of Lemma 3.1, we shall investigate only the following case:

$$f(\sigma) = \pm S^n, \qquad f(\sigma') = \overline{\xi}_0,$$
$$f(\overset{*}{\sigma}) = \pm S^n, \qquad f(\overset{*}{\sigma}{}') = \pm S^n.$$

When $\sigma \cap \sigma' = \tau$ receive the deformation of Lemma 3.1 and $\overset{*}{\sigma} \cap \overset{*}{\sigma}{}' = \overset{*}{\tau}$ receive the deformation of Lemma 3.2, τ and $\overset{*}{\tau}$ are removed in the same state by above deformations. The common 1-simplexes of τ and $\overset{*}{\tau}$ don't contradict to two deformations. When another faces of $\sigma + \sigma'$ have singular simplexes, the deformation of Lemma 3.2 may be done at will.

4. We introduce following several definitions. Definition 4.1 is used by Hurewicz [6] already. Let f_1, f_2 be continuous mappings of S^{2n-1} into S^n.

DEFINITION 4.1. If f_1 and f_2 are homotopic on the m-dimensional skeleton of K^{2n-1}, then we say that f_1 is homotopic to f_2 in m-dimension and we denote $f_1 \underset{m}{\simeq} f_2$.

DEFINITION 4.2. If $f_1 \underset{m+1}{\simeq} f_2$ follows from $f_1 \underset{m}{\simeq} f_2$, we say that (f_1, f_2) are raised by one dimension from m-dimensional homotopy.

DEFINITION 4.3. If $f \underset{m+1}{\simeq} 0$, follows from $f \underset{m}{\simeq} 0$, we say that f is raised by one dimension from m-dimensional 0-homotopy.

We shall consider a map f of S^{2n-1} into S^n. In order to characterize it by Hopf's invariant, we assume that n is even according to Hopf's remark [8].

THEOREM 4.1. *Let f be a map of S^{4k-1} into S^{2k}. If $f|R^{2n-1}(p_0)$ can be raised by $(2k-1)$-dimensions from $(2k)$-dimensional 0-homotopy leaving $f^{-1}(p_0)$ fixed and its Hopf's invariant equal to 0, then f is homotopic to 0.*

PROOF. By Freudenthal's Lemma [2], $\phi_{S^{4k-1}}(p_0)$ may be considered as only one triangulated manifold M^{2k-1}. There is a manifold K^{2k}_p bounded by M^{2k-1} which has at most some singular 1-simplexes by Lemma 2.2.

The map f can be considered as standard map without any loss of generality by Lemma 2.1, when we use Lemma 3.1 and Lemma 3.2.

As Hopf's invariant is equal to zero, there are a set of $2k$-simplexes $\sigma_1, \sigma_2, \cdots, \sigma_s; \sigma'_1, \sigma'_2, \dots, \sigma'_s$, on K^{2k}_p, where σ_i and σ'_i are mapped on S^{2k}

positively and negatively, respectively. For σ_i and σ'_i $(i = 1, 2, \ldots, s)$, there are regularly connected chains $\sigma_i + \sigma_{i_1} + \sigma_{i_2} + \ldots + \sigma_{i_k} + \sigma'_i$ on K_p^{2k}. It may be supposed that $d_f(\sigma_i) = +1$, $d_f(\sigma_{i_1}) = d_f(\sigma_{i_2}) = d_f(\sigma_{i_k}) = \ldots 0$, $d_f(\sigma'_i) = -1$. Using Lemma 3.1, we deform f in $\sigma_i + \sigma_{i_1}$, then in $\sigma_{i_1} + \sigma_{i_2}$, etc; then using Lemma 3.2, we deform the map in $\sigma_{i_k} + \sigma'_i$. The new map f' has as its degree $d_{f'}(\sigma_i) = d_{f'}(\sigma_{i_1}) \cdots = d_{f'}(\sigma'_i) = 0$. We continue in this manner till no simplexes are mapped positively and none are mapped negatively over S^{2k}. Then

$$f \mid K_p^{2k} \simeq 0. \qquad \text{rel. } f^{-1}(p_0).$$

By Definition, K^{2k} and $\bar{K}_p^{2k}$ have no common $2k$-simplex for any p. When f is a simplicial mapping, we consider the state where K_p^{2k} are mapped on τ_{00}^{2k}. As we have defined in §2, the part of K_p^{2k} which are faces of $\overset{*}{T}{}_\alpha^{4k-1}$, is the following complex:

$$\begin{aligned}(a_0^\alpha, a_1^\alpha, \ldots, a_{2k-1}^\alpha, e_p^\alpha) &+ (-1)^{2k-1}(a_1^\alpha, a_2^\alpha, \ldots, a_{2k-1}^\alpha, e_p^\alpha, e_{p+1}^\alpha) \\ &+ (-1)^{2(2k-1)}(a_2^\alpha, a_3^\alpha, \ldots, a_{2k-1}^\alpha, e_p^\alpha, e_{p+1}^\alpha, e_{p+2}^\alpha) + \ldots \\ &+ (-1)^{(2k-1)^2}(a_{2k-1}^\alpha, e_p^\alpha, \cdots, e_{2k-1}^\alpha, e_0^\alpha, e_1^\alpha, \ldots, e_{p-1}^\alpha).\end{aligned}$$

If $(a_{2k-1}^\alpha, e_p^\alpha, e_{p+1}^\alpha, \ldots, e_{2k-1}^\alpha, e_0^\alpha, e_1^\alpha, \ldots, e_{p-1}^\alpha)$ is mapped on τ_{00}^{2k}, other simplexes are mapped on faces of τ_{00}^{2k} and their dimensions depend on numbers of e_i^α. $(a_{2k-1}^\alpha, e_p^\alpha, e_{p+1}^\alpha, \cdots e_{2k-1}^\alpha, e_0^\alpha, e_1^\alpha, \ldots, e_{p-1}^\alpha)$ is mapped on τ_{00}^{2k} in the same manner for each p except for orientation.

We may neglect this orientation when we take care of this similar property for all α. If $(a^\alpha_{2k-1}, e_p^\alpha, e_{p+1}^\alpha, \cdots, e_{2k-1}^\alpha, e_0^\alpha, e_1^\alpha, \ldots, e_{p-1}^\alpha)$ is not mapped on τ_{00}^{2k}, we may neglect T_α^{4k-1}. In other words, we may consider that $(a_0^\alpha, a_1^\alpha, \ldots, a_{2k-1}^\alpha)$ contract to a point and T_α^{4k-1} is empty. By Lemma 2.2, $\bar{\bar{K}}^{2k}$ is not mapped on τ_{00}^{2k}, then we may consider only K_p for the degree based on τ_{00}^n. On the other hand, deformations of Lemma 3.1 can be introduced leaving the degree of $\bar{\bar{K}}_p^{2k}$ fixed. If we deform f to the standard map by Lemma 2.1, f maps $\Sigma_p K_p^{2k}$ on $\bar{\xi}_0$ and f maps $\bar{K}^{2k} - \Sigma_p K_p^{2k}$ on $\bar{\xi}_0$ by the above remark. Hence $f \mid \widetilde{K}^{2k} \simeq 0$. rel. $f^{-1}(p_0)$, where $\widetilde{K}^{2k}$ is $2k$-dimensional skeleton of $R^{4k-1}(p_0)$.

By the assumption of this theorem, $f \mid \widetilde{K}^{4k-1} \simeq 0$ rel. $f^{-1}(p_0)$, where $\widetilde{K}^{4k-1}$ is a $(4k-1)$-dimensional skeleton of $R^{4k-1}(p_0)$. Therefore $f \simeq 0$.

THOREM 4.2. *Let f_1 and f_2 be continuous mappings of S^{4k-1} into S^{2k} and $f_1 \mid R^{4k-1}(p_0)$, $f_2 \mid R^{4k-1}(p_0)$ be raised by $(2k-1)$-dimensions from $2k$-dimensional homotopy leaving $f_1^{-1}(p_0)$ and $f_2^{-1}(p_0)$ fixed respectively and Hopf's invariant $\gamma(f_1)$ be equal to $\gamma(f_2)$, then $f_1 \simeq f_2$.*

PROOF. We consider Cartesian $(n+1)$-space $\mathfrak{C}^{n+1}$ and its subsets

$$S^n = \{x \in \mathfrak{C}^{n+1} : \quad \sum_{i=1}^{n+1} x_i^2 = 1\},$$

$$E_+^n = \{x \in S^n : \quad x_{n+1} \geqq 0\},$$
$$E_-^n = \{x \in S^n : \quad x_{n+1} \leqq 0\},$$
$$S_0^{n-1} = \{x \in S^n : \quad x_{n+1} = 0\}.$$

We define ϕ_1 as follows:

ϕ_1 maps E_+^{4k-1} onto S^{4k-1}

ϕ_1 is a homeomorphism on $E_+^{4k-1} - S_0^{4k-2}$.

$\phi_1(S_0^{4k-2}) = P$, where P is a fixed point on S^{4k-1}.

$d(\phi_1) = 1$.

We also define ϕ_2 as follows:

ϕ_2 maps E_-^{4k-1} onto S^{4k-1}.

ϕ_2 is a homeomorphism on $E_-^{4k-1} - S_0^{4k-2}$.

$\phi_2(S_0^{4k-2}) = P$.

$d(\phi_2) = -1$.

We may assume $f_1(P) = f_2(P) = Q$ without any loss of generality.

Let R, S $(\neq Q)$ be different points on S^{2k}. From $\gamma(f_1) = \gamma(f_2)$, it follows that $v(f_1^{-1}(R), f_1^{-1}(S)) = v(f_2^{-1}(R), f_2^{-1}(S))$, where v means the looping coefficient. Secondly we construct a map F of S^{4k-1} into S^{2k} as follows:

$$F = \begin{cases} f_1\,\phi_1 & \text{on } E_+^{4k-1} \\ f_2\,\phi_2 & \text{on } E_-^{4k-1}. \end{cases}$$

Then $\gamma(F) = v(F^{-1}(R), F^{-1}(S)) = v(f_1^{-1}(R), f_1^{-1}(S)) - v(f_2^{-1}(R), f_2^{-1}(S)) = 0$. If we denote by $[F], [f_1], [f_2]$ the homotopy classes respectively, $[F] = [f_1] - [f_2]$.

By the assumption of this theorem, $F \mid R^{4k-1}(p_0)$ is raised by $(2k-1)$-dimensions from $2k$-dimensional 0-homotopy. Therefore, by Theorem 4.1,

$$F \simeq 0, \text{ then } f_1 \simeq f_2.$$

Pontrjagin's theorem [7] may be obtained from Theorem 4.2 as its special case we shall prove it in the following line:

THEOREM 4.3 (*Pontrjagin's theorem*). *If f_1 and f_2 are maps of S^3 on S^2 and $\gamma(f_1)$ is equal to $\gamma(f_2)$, then f_1 is homotopic to f_2.*

PROOF. We use Eilenberg's homotopy theorem [1].

Let K be a locally finite complex, K' be a closed subcomplex of K, and Y be n-simple. If the cohomology group $H^n_{\pi_n(Y)}(K - K') = 0$ and $f_1, f_2 \in Y^K$, then

$$f_1 | K' + K^{n+1} \simeq f_2 | K' + K^{n-1} \qquad \text{rel. } A$$

implies

$$f_1 | K' + K^n \simeq f_2 | K' + K^n \qquad \text{rel. } A$$

for any subset A of K'. In our case, we replace $K, K' (= A)$ by $R_{f_1}(p_0) \cup R_{f_2}(p_0), f_1^{-1}(p_0) \cup f_2^{-1}(p_0)$ respectively. We construct a map F of S^3 into S^2 similarly as the proof of Theorem 4.2. Then $R_{f_1}(p_0) \cap R_{f_2}(p_0) = 0$ and $R_{f_i}(p_0)$ $(i = 1, 2)$ is homeomorphic to $S^1 \times E^2$, where $R_{f_1}(p_0)$ and $R_{f_2}(p_0)$ are the inverse images of τ_0 for f_1 and f_2 respectively. As $\gamma(f_1)$ is equal to $\gamma(\gamma(f_2), f_1|K^2 \simeq f_2|K^2$ rel. $f_1^{-1}(p_0) \cup f_2^{-1}(p_0)$. On the other hand we can calculate that $H^3_{\pi_3(S^2)}[R_{f_1}(p_0) \cup R_{f_2}(p_0) - f_1^{-1}(p_0) \cup f_2^{-1}(p_0)]$ is equal to zero. Then $f_1|K^3 \simeq f_2|K^3$ rel. $f_1^{-1}(p_0) \cup f_2^{-1}(p_0)$, therefore $f_1 \simeq f_2$.

BIBLIOGRAPHY

[1] S. EILENBERG, Cohomology and continuous mappings, Ann. of Math., 41(1940), 231–251.
[2] H. FREUDENTHAL, Über die Klassen der Sphärenabbildungen I, Compositio Math., 5(1937), 299–314.
[3] H. HOPF, Über die Abbildungen der dreidimensionalen Sphären auf die Kugelfläche, Math. Ann., 104(1931), 637–665.
[4] H. HOPF, Über die Abbildungen von Sphären auf Sphären niedrigere Dimension, Fund. Math., 25(1935), 427–440.
[5] H. HOPF, Die Klassen der Abbildungen der n-dimensionalen Polyeder auf der n-dimensionale Sphäre, Comment. Math. Helv., 5(1933), 39–54.
[6] W. HUREWICZ, Algebraic and topological classification of mappings, Bull. Amer. Math. Soc., 52(1946), 830.
[7] L. PONTRJAGIN, A classification of mappings of the three dimensional complex into the twodimensional sphere, Rec. Math. N. S., 9(1941), 331–363.
[8] H. WHITNEY, On the maps of an n-sphere into another n-sphere, Duke Math. Jour., 3(1937), 46–50.
[9] H. WHITNEY, The maps of an n-complex into an n-sphere, Duke Math. Jour., 3(1937), 51–55.

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