ON A WEAKLY CENTRAL OPERATOR ALGEBRA

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In the previous paper [7], we have defined the weak centrality of an operator algebra modifying I. Kaplansky's definition of the (strong) centrality [6]. Although we have assumed an additional condition in the previous occasion, here we shall study the weak centrality as itself. It will be seen in the below, that a weak central C^*-algebra can be decomposed into C^*-algebras each of which is factorial C^*-algebra. The other purpose of this paper is to prove that any W^*-algebra is weakly central.

1. Definitions and notations. We shall assume in this papar that any algebra which we consider has a unit element. A self-adjoint algebra A of bounded linear operators on a Hilbert space will be called a *C^*-algebra* (*W^*-algebra*), according to I. E. Segal, provided that A is uniformly (weakly) closed in the sense of J. von Neumann [9].

Let Ω be the set of all maximal ideals in the C^*-algebra A. In simple case we shall consider the 0-ideal as its maximal ideal. If S is a nonvacuous subset of Ω, we define M_0 is contained in the closure of S if and only if $M_0 \supset \bigcap_{M \in S} M$. This topological space Ω will be called the *spectrum* of A, according to I. E. Segal [10]. The spectrum Ω becomes, in general, a compact T_1-space. In the commutative case, it is known that the spectrum becomes a T_2-space.

A C^*-algebra A is called *weakly central* provided that two maximal ideals M_1 and M_2 coincide if and only if

$$M_1 \cap Z = M_2 \cap Z$$

where Z means the center of A. It will be seen that A is weakly central if it has at most one maximal ideal. Conversely, if the center of a weakly central algebra A is a field, then A contains at most one maximal ideal. It is not difficult to see that the spectrum of a weakly central C^*-algebra is homeomorphic in its natural mapping to the spectrum of the center, whence it is a compact T_2-space.

In the terminology of N. Jacobson [4], an ideal P of A is called *primitive* if there exists a maximal right ideal M' such that

$$M' : A = \{x \in A \mid ax \in M' \text{ for all } a \in A\} = P.$$

It is known that a C^*-algebra is *semi-simple* in the sense of Jacobson [4], i. e., the intersection of all primitive ideals in a C^*-algebra vanishes. The set of all primitive ideals is called the *structure space* of the algebra with the Stone topogy. A C^*-algebra is *central* if and only if the definitive property of the weak centrality is held for primitive ideals in stead of maximals ideals.

By $\sigma(x)$ we mean the set of all complex numbers $\lambda \neq 0$ such that $x - \lambda 1$ have not inverses and we insert $0 \in \sigma(x)$ unless x has an inverse.

When I is a closed ideal of a C^*-algebra A, we can consider a factor algebra A/I in the usual way and define

$$\| x^\theta \| = \inf_{y \in I} \| x + y \|$$

where x^θ is the class of A/I which contains x. It is known that A/I is a C^*-algebra too.

An algebra will be called *factorial* if its center contains only scalar multiples of the identity.

2. **The decomposition of a weakly central C^*-algebra.** Firstly we shall prove a following lemma.

LEMMA 1. *Let Z be the center of a weakly central C^*-algebra A and let I be a closed ideal A. If the intersection of Z and I is a maximal ideal in Z, then the factor algebra A/I is a factorial C^*-algebra.*

PROOF. Let M_1 and M_2 be two maximal ideals in A which contain I, then by N. Jacobson [4] $M_i \cap Z$ are maximal ideals in Z. Because of $M_i \supset I$, we have $M_i \cap Z \supset I \cap Z$ for $i = 1, 2$. Since $I \cap Z$ is a maximal ideal in Z, we have

$$M_1 \cap Z = M_2 \cap Z = I \cap Z.$$

This implies $M_1 = M_2$ by weak centrality of A. Thus, there is a unique maximal ideal which contains I. In other words A/I has a unique maximal ideal.

Let Z^θ be the center of A/I and let M^θ be the unique maximal ideal of A/I. We shall assume that A/I is not factorial, then any maximal ideal of Z is not the 0-ideal. Thus $M^\theta \cap Z^\theta$ is a nontrivial maximal ideal of Z^θ. Let P^θ be any primitive ideal of A/I, then $P^\theta \cap Z^\theta$ is a maximal ideal of Z^θ [4]. On the other hand $P^\theta \subset M^\theta$ and hence we have

$$P^\theta \cap Z = M^\theta \cap Z$$

for any primitive ideal P^θ of A/I. Since A/I is a C^*-algebra, the intersection of all primitive ideals of A/I vanishes. This contradicts to non-triviality of $M^\theta \cap Z$. That is, A/I is a factorial C^*-algebra.

LEMMA 2. *Let Ω be the spectrum of the center Z of a weakly central C^*-algebra A. For any $\zeta \in \Omega$, we define I_ζ as the intersection of all closed ideals in A which contain ζ. Then*

$$\bigcap_{\zeta \in \Omega} I_\zeta = 0.$$

PROOF. Let P be any primitive ideal of A, then $P \cap Z$ is a maximal ideal of Z. We shall denote this maximal ideal ζ_1. By definition of I_{ζ_1}, we have $P \supset I_{\zeta_1}$. This shows that $\bigcap_{\zeta \in \Omega} I_\zeta$ is contained in the intersection of all primitive ideals of A. This proves the lemma.

THEOREM 1. *Let A be a weakly central C^*-algebra. Then there exist a compact Hausdorff space Ω and a factorial C^*-algebra A_ζ for each $\zeta \in \Omega$ such that for all $x \in A$ there exists a function $x(\zeta)$ which is defined on Ω and $x(\zeta) \in A_\zeta$ for all $\zeta \in \Omega$ (we denote this correspondence by $x \sim x(\zeta)$) and satisfy following conditions*:

(1) $x = 0$ *if and only if* $x(\zeta) = 0$ *for all* $\zeta \in \Omega$,

(2) $\alpha x + \beta y \sim (\alpha x + \beta y)(\zeta) = \alpha x(\zeta) + \beta y(\zeta)$ *for any complex numbers* α *and* β,

(3) $x^* \sim x(\zeta)^*$,

(4) $xy \sim x(\zeta)y(\zeta)$,

(5) $\|x\| = \sup_{\zeta \in \Omega} \|x(\zeta)\|_\zeta$ *where* $\|\cdot\|_\zeta$ *is the norm on* A_ζ,

(6) $\|x(\zeta)\|_\zeta$ *is a continuous function on* Ω.

Furthermore, let $\mathfrak{A}$ *be a normed algebra of functions* $\theta(\zeta)$ *which are defined on* Ω *and* $\theta(\zeta) \in A_\zeta$ *for all* $\zeta \in \Omega$ *and satisfies* (1) – (6). *If* $\mathfrak{A}$ *contains all* $x \sim x(\zeta)$ *where* $x \in A$, *then* $\mathfrak{A}$ *coincides with* A.

PROOF. Let Ω be the spectrum of A, then Ω is a compact Hausdorff space. For each $\zeta \in \Omega$ we define a closed ideal I_ζ as Lemma 2. If we define

$$A_\zeta = A/I_\zeta\,,$$

then A_ζ is a factorial C*-algebra by Lemma 2.

Let $x(\zeta)$ be the image of the natural mapping of $x \in A$ in $A_\zeta = A/I_\zeta$. We shall denote this correspondence by $x \sim x(\zeta)$. Then each function $x(\zeta)$ is defined on Ω and

$$x(\zeta) \in A_\zeta \qquad \text{for all } \zeta \in \Omega.$$

It is obvious that $x \sim x(\zeta)$ satisfies (2), (3) and (4). For any $\zeta \in \Omega$, $x(\zeta) = 0$ is equivalent to $x \in I_\zeta$. It follows that $x(\zeta) = 0$ for all $\zeta \in \Omega$ implies

$$x \in \bigcap_{\zeta \in \Omega} I_\zeta,$$

that is, $x = 0$. Conversely, it is clear that $x = 0$ implies that $x(\zeta) = 0$ for all $\zeta \in \Omega$. This proves (1).

Now we shall prove (5). Clearly $\|x(\zeta)\|_\zeta \leqq \|x\|$ for all $\zeta \in \Omega$ and we have

$$\|x\| \geqq \sup_{\zeta \in \Omega} \|x(\zeta)\|_\zeta.$$

We shall show that supremum attains for some $\zeta \in \Omega$. Because of the identity $\|xx^*\| = \|x\|^2$, one needs to prove this only in case $x \geqq 0$, and we may assume $\|x\| = 1$. To say that $\|x(\zeta)\|_\zeta$ is less than 1 is equivalent to say that $\sigma(x(\zeta))$ does not contain 1, that is, $1 - x(\zeta)$ has an inverse. This is equivalent to $1 - x$ has an inverse modulo I_ζ. If $\|x(\zeta)\|_\zeta$ is less than 1 for all $\zeta \in \Omega$, then $1 - x$ has an inverse modulo every I_ζ. As we showed above, any primitive ideal contains some I_ζ, this implies $1 - x$ has an inverse modulo every primitive ideal. It is known that this means that $1 - x$ has an inverse in A (cf. [5]). This contradicts to $\|x\| = 1$.

Next we shall prove (6). In the first place, we shall show that $\|x(\zeta)\|_\zeta$

is continuous at 0, that is, if $x(\zeta_0)=0$ then for any $\varepsilon>0$ there exists a neighborhood V of ζ_0 such that

$$\|x(\zeta)\|_\zeta<\varepsilon \qquad \text{for all } \zeta\in V.$$

Let I be the set of all $y\in A$ such that the corresponding function $y(\zeta)$ vanishes in a neighborhood of ζ_0. Clearly I is an ideal of A. Let $\bar{I}$ be the closure of I. Obviously $\bar{I}\subset I_{\zeta_0}$. Since Z is represented by all continuous numerical valued function on Ω, $\bar{I}\cap Z$ is maximal in Z. Therefore,

$$\bar{I}\cap Z=I_{\zeta_0}\cap Z,$$

since $I_{\zeta_0}\cap Z$ is a maximal in Z. By the definition of I_{ζ_0}, we have $I_{\zeta_0}\subset\bar{I}$. Thus we have $I_{\zeta_0}=\bar{I}$. Obviously $x\in I_{\zeta_0}$, then for any $\varepsilon>0$ there exists $y\in I$ such that

$$\|x-y\|<\varepsilon.$$

This implies that

$$\|x(\zeta)-y(\zeta)\|_\zeta<\varepsilon \qquad \text{for all } \zeta\in\Omega.$$

By the definition of I, there exists a neighborhood V of ζ_0 such that

$$y(\zeta)=0 \qquad \text{for all } \zeta\in V.$$

By the above, we have

$$\|x(\zeta)\|_\zeta<\varepsilon \qquad \text{for all } \zeta\in V.$$

We pass the general proof of continuity. We assume that

$$\|x(\zeta_0)_{\zeta_0}\|=r\neq 0.$$

One needs to show that for any $\varepsilon>0$ there exists a neighborhood V of ζ_0 such that

$$r-\varepsilon<\|x(\zeta)\|<r+\varepsilon \qquad \text{for all } \zeta\in V.$$

We may assume the $x\geqq 0$ as previous. Let S be the set of all $\zeta\in\Omega$ such that $\|x(\zeta)\|_\zeta\leqq r-\varepsilon$ and let ζ_1 be an element of the closure of S.

We assume that there exists $r_1\in\sigma(x(\zeta_1))$ such that $r_1>r-\varepsilon$. Now we shall define a continuous function $f(t)$ which is defined on $(0,\infty)$ such that

$$f(t)\begin{cases} =0 & \text{if } 0\leqq t\leqq r-\varepsilon,\\ =1 & \text{if } t=r_1.\end{cases}$$

We consider the commutative C^*-algebra A_x which is generated by x. A_x can be represented as a ring of all continuous numerical functions on $\sigma(x)$ (vanishing at origin if $0\in\sigma(x)$) by a theorem due to I. Gelfand. If we restrict $f(t)$ on $\sigma(x)$, we have a continuous function on $\sigma(x)$ and so we have a corresponding element of A_x. We shall denote this element by $f(x)$. Analogously, we can define $f(x(\zeta))$ and we have an identity:

$$f(x(\zeta))=f(x)(\zeta)$$

which was proved in [6]. If $\zeta\in S$ then

$$r-\varepsilon\geqq\lambda \text{ for all } \lambda\in\sigma(x)$$

by the definition of S. Therefore

$$f(x)(\zeta) = f(x(\zeta)) = 0 \quad \text{for } \zeta \in S.$$

By the continuity of $\| x(\zeta) \|_\zeta$ at 0, this implies that $f(x(\zeta_1)) = 0$. On the other hand $f(x(\zeta_1)) \neq 0$ since $f(t)$ does not vanish at $r_1 \in \sigma(x(\zeta_1))$. This is a contradiction, that is

$$r - \varepsilon \geqq \lambda \quad \text{for all} \quad \lambda \in \sigma(x(\zeta_1)).$$

This implies that $\| x(\zeta_1) \|_{\zeta_1} \leqq r - \varepsilon$, that is, $\zeta_1 \in S$. Thus, the closedness of S was proved. If we denote the complement of S by V_1, V_1 is a neighborhood of ζ_0 such that

$$\| x(\zeta) \|_\zeta > r - \varepsilon \quad \text{for all} \quad \zeta \in V_1.$$

It was proved in [6, Lemma 3.3] that there exists a neighborhood V_2 of ζ_0 such that

$$\| x(\zeta) \|_\zeta < r + \varepsilon \quad \text{for all} \quad \zeta \in V_2.$$

It is clear that $V = V_1 \cap V_2$ has our property. This proves (6).

Finally we shall prove the remainder part of the theorem. Let $\mathfrak{A}$ be a normed algebra of functions $\theta(\zeta)$ which are defined on Ω and $\theta(\zeta) \in A_\zeta$ for all $\zeta \in \Omega$ and satisfies (1)–(6). Furthermore we assume that $\mathfrak{A}$ containes A. Take any fixed element $\theta(\zeta)$ of $\mathfrak{A}$, then by definition of A_ζ there exists an element $x(\zeta) \in A_\zeta$ such that $\theta(\zeta) = x(\zeta)$ for a fixed $\zeta \in \Omega$. From (6), this implies that for any $\varepsilon > 0$ there exists a neighborhood $U(\zeta)$ of ζ such that

$$(*) \qquad \| \theta(\zeta') - x(\zeta') \|_{\zeta'} < \varepsilon \quad \text{for all} \quad \zeta' \in U(\zeta).$$

If we correspond such neighborhood $U(\zeta)$ for each $\zeta \in \Omega$, then the family $\{U(\zeta) \mid \zeta \in \Omega\}$ is an open covering of Ω. From the compactness of Ω, there exists a finite covering

$$U(\zeta_1), \cdots, U(\zeta_n).$$

It is known that there exist non negative continuous functions $f_1(\zeta), f_2(\zeta), \cdots, f_n(\zeta)$ such that

$$\sum_{i=1}^{n} f_i(\zeta) = 1$$

and $f_i(\zeta)$ vanishes outside of $U(\zeta_i)$ for each i (cf. [1]). For each i, let x_i be the element which is determined by (*). Since each $f_i(\zeta)$ determines an element of Z, $x(\zeta)$ determines an element of A if we define

$$x(\zeta) = \sum_{i=1}^{n} f_i(\zeta) x_i(\zeta).$$

From the definition of $U(\zeta_i)$

$$\| \theta(\zeta) - x_i(\zeta) \|_\zeta < \varepsilon \qquad \text{for } \zeta \in U(\zeta_i).$$

Thus we have a following inequality:

$$\| \theta(\zeta) - x(\zeta) \|_\zeta = \| \sum_{i=1}^{n} f_i(\zeta)\theta(\zeta) - \sum_{i=1}^{n} f_i(\zeta) x_i(\zeta) \|_\zeta$$

$$\leq \sum_{i=1}^{n} f_i(\zeta) \| \theta(\zeta) - \mathfrak{X}_i(\zeta) \|_\zeta < \sum_{i=1}^{n} f_i(\zeta)\varepsilon = \varepsilon.$$

From (5), this proves that every element of $\mathfrak{A}$ is a limit of elements of A, that is, $\mathfrak{A} \subset A$ since A is complete.

Thus Theorem 1 was proved completely.

Theorem 1 and Theorem 3.1 of [6] imply the following theorem.

THEOREM 2. *Under the notation of Theorem* 1, *any closed* (*right, left*) *ideal* L *in* A *has the following from*: *for each* $\zeta \in \Omega$ *a closed* (*right, left*) *ideal* L_ζ *in* A_ζ *is given, and* L *consists of all* $x \in A$ *with* $x(\zeta) \in L_\zeta$ *for all* $\zeta \in \Omega$.

3. The weak centrality of W^***-algebras.** In this section we shall prove the following theorem:

THEOREM 3. *A* W^**-algebra is a weakly central* C^**-algebra.*

To prove this theorem we shall use the following:

LEMMA 3 (J. Dixmier [2]). *Let* x *be an element of a* W^**-algebra* A. *Consider the linear combination*

$$\sum_{i=1}^{n} \lambda_i u_i x u_i^{-1}$$

where u_i *are unitary elements of* A *and* $\lambda_i \geqq 0$ *and* $\sum_{i=1}^{n} \lambda_i = 1$. *Let* K_x *be the set of their uniform limits. Then the intersection of* K_x *and the center* Z *of* A *is not empty. Moreover, for any positive* ε *and a pair* x, y *of* A *there exist* $x' \in K_x \cap Z$, $y' \in K_y \cap Z$ *and unitary elements* $u_1, \ldots, u_n$ *of* A *and* $\lambda_i \geqq 0$ $(\sum_{i=1} \lambda_i = 1)$ *such that*

$$\| \sum_{i=1}^{n} \lambda_i u_i x u_i^{-1} - x' \| < \varepsilon, \qquad \| \sum_{i=1}^{n} \lambda_i u_i y u_i^{-1} - y' \| < \varepsilon.$$

PROOF OF THEOREM 2. Let M_1, M_2 be two distinct maximal ideals in A. One needs only to prove $M_1 \cap Z \neq M_2 \cap Z$. From the maximality of M_1, M_2, we have

$$A = M_1 + M_2.$$

So, there exist $x_i \in M_i (i = 1, 2)$ such that

$$1 = x_1 + x_2.$$

From Lemma 3 for any $\varepsilon > 0$ there exist $x_1' \in K_{x_1} \cap Z$, $x_2' \in K_{x_2} \cap Z$, unitary elements $u_i \in A$ and $\lambda_i \geqq 0$ $(\sum_{i=1}^{n} \lambda_i = 1)$ such that

$$\left\|\sum_{i=1}^{n}\lambda_i u_i x_1 u_i^{-1} - x_1'\right\| < \varepsilon, \quad \left\|\sum_{i=1}^{n}\lambda_i u_i x_2 u_i^{-1} - x_2'\right\| < \varepsilon.$$

Hence we have

$$\left\|\sum_{i=1}^{n}\lambda_i u_i (x_1 + x_2)u_i^{-1} - (x_1' + x_2')\right\|$$

$$\leqq \left\|\sum_{i=1}^{n}\lambda_i u_i x_1 u_i^{-1} - x_1'\right\| + \left\|\sum_{i=1}^{n}\lambda_i u_i x_2 u_i^{-1} - x_2'\right\| < 2\varepsilon.$$

In other words

$$\| 1 - (x_1' + x_2') \| < 2\varepsilon.$$

On the other hand, $K_{x_i} \subset M_i$ $(i = 1, 2)$ by the definition of K_{x_i} and the closedness of M_i, since they are maximal. If

$$M_1 \cap Z = M_2 \cap Z,$$

then

$$x_1' + x_2' \in M_1 \cap Z = M_2 \cap Z.$$

The closedness of $M_1 \cap Z$ implies that

$$1 \in M_1 \cap Z = M_2 \cap Z.$$

This is a contradiction. Therefore $M_1 \cap Z \neq M_2 \cap Z$. This proves the theorem.

It may be somewhat interesting to observe that Theorem 3 implies a result due to J. W. Calkin [3], since the full operator algebra on a Hilbert space is weakly central by the above theorem, that is, we have the next.

COROLLARY (J. W. Calkin). *A full operator algebra on a Hilbert space has a unique maximal ideal.*

REMARK 1. Above Corollary is valid for any factor in the sense of J. von Neumann.

REMARK 2. If the structure space of a W^*-algebra is T_1-space, then an ideal is primitive if and only if it is maximal. Therefore this structure space is necessary Hausdorff space.

4. Some applications. In this section we shall consider only W^*-algebra. A W^*-algebra is weakly central, thus it can be decomposed in the sense of Theorem 1. The purpose of this section is to give some remarks on the components. We shall use notations in Theorem 1 in the below.

PROPOSITION 1. *Let $e(\zeta_0) \in A_{\zeta_0}$ be a projection, then there exist a projection $e \in A$ such that the value of e at ζ_0 is $e(\zeta_0)$.*

PROOF. Clearly we can take $x_1 \in A$ such that $x_1(\zeta_0) = e(\zeta_0)$. If we put $x_2 = x_1 x_1^*$, then

$$x_2(\zeta_0) = x_1(\zeta_0)x_1(\zeta_0)^* = e(\zeta_0)e(\zeta_0)^* = e(\zeta_0).$$

That is, there exists a self-adjoint and non-negative $x_2 \in A$ such that $x_2(\zeta_0) = e(\zeta_0)$.

Let S be the set of all $\zeta \in \Omega$ for which $\| x_2(\zeta) \|_\zeta \geqq 1$. Define a function $f(\zeta)$ on Ω as following:

$$f(\zeta) \begin{cases} = 1 & \text{if } \zeta \bar{\in} S, \\ = \dfrac{1}{\| x_2(\zeta) \|_\zeta} & \text{if } \zeta \in S. \end{cases}$$

Then it is clear that $f(\zeta)$ is continuous on Ω and hence $f(\zeta)$ determines an element f of Z. If we define $x = fx_2$ then

$$x \sim x(\zeta) = f(\zeta) x_2(\zeta),$$

that is, x is self adjoint and $\| x \| \leqq 1$ and moreover $x(\zeta_0) = e(\zeta_0)$.

From [6, Lemma 3.3] there exists a neighborhood U of ζ_0 such that $\sigma(x(\zeta))$ does not contain λ such that $1/3 \leqq \lambda \leqq 2/3$ for all $\zeta \in U$. It is known that Ω is totally disconnected and there exists a closed $V \subset U$ of ζ_0. Now let $f(\zeta)$ be the characteristic function of V and the corresponding central element be f. If we put $x' = fx$, then $x'(\zeta) = x(\zeta)$ for all $\zeta \in V$. In particular

$$x'(\zeta_0) = e(\zeta_0).$$

Obviously $\lambda \bar{\in} \sigma(x'(\zeta))$ for all $\zeta \in \Omega$ and for all λ such that $1/3 \leqq \lambda \leqq 2/3$. Hence we can prove $\lambda \bar{\in} \sigma(x')$ by an analogous way to the proof of Theorem 1.

Now we consider the commutative C^*-subalgebra $A_{x'}$ which is generated by x'. Then $A_{x'}$ is represented as a ring of all continuous functions on $\sigma(x')$ (vanishing at 0 when x' has not an inverse). Let $f(t)$ be a function on $(0, \infty)$ such that

$$g(t) \begin{cases} = 1 & \text{if } 2/3 \leqq t \leqq 1, \\ = 0 & \text{otherwise,} \end{cases}$$

then $g(t)$ is continuous on $\sigma(x')$. Thus $g(t)$ is determines an element $e = g(x') \in A_{x'}$. Obviously e is projection. By the identity

$$g(x')(\zeta_0) = g(x'(\zeta_0)) = g(e(\zeta_0)),$$

the value of e at ζ_0 is $e(\zeta_0)$. This proves the Proposition.

PROPOSITION 2. *Let $e(\zeta)$, $f(\zeta) \in A_\zeta$ be two projections for which there exists an element $x(\zeta) \in A_\zeta$ such that*

$$e(\zeta) = x(\zeta) x(\zeta)^* \quad \textit{and} \quad f(\zeta) = x(\zeta)^* x(\zeta).$$

Then there exist two prjections e, f and a partially isometric element x such that

$$e = xx^* \quad \textit{and} \quad f = x^* x.$$

PROOF. Let $x_1 \in A$ be any element of A taking $x(\zeta)$ as its value at ζ and $x_1 = xy$ be the canonical decomposition in usual sense. By a Lemma of F. J. Murray and J. von Neumann [10], x any y belong to A. We shall show that $x(\zeta)$ as its value at ζ. By the definition of canonical decomposition, $y^2 = x_1 x_1^*$, then

$$y(\zeta)^2 = x_1(\zeta)x_1(\zeta)^* = e(\zeta).$$

Since $y(\zeta)$ is positive, we have $y(\zeta) = e(\zeta)$. The initial set of partially isometric operator x is the range of y. That is, $x = xh$, where h is the projection on the range of y. Obviously $h \in A$. From above we have

$$h(\zeta) = e(\zeta) = y(\zeta).$$

Then we have

$$x(\zeta) = x(\zeta)h(\zeta) = x(\zeta)y(\zeta).$$

On the other hand

$$x_1(\zeta) = x(\zeta)y(\zeta).$$

Hence $x_1(\zeta) = x(\zeta)$.

If we define $e = xx^*$ and $f = x^*x$, then it is obviously that e and f satisfy our properties.

REFERENCES

[1] N. BOURBAKI, Elements de mathematique, vol. 3. Topologie generale, Actualites Sci. et Indust., Paris (1948).

[2] J. DIXMIER, Les anneaux d'operateure de class fini, Ann. Ec. Norm., 66(1949), 209–261.

[3] J. W. CALKIN, Two-sided ideals and congruences in the ring of bounded operators in Hilbert space, Ann. Math., 42(1941), 839–873.

[4] N. JACOBSON, The radical and semi-simplicity for arbitrary rings, Amer. Journ. Math., 67(1945), 300–320.

[5] I. KAPLANSKY, Normed algebras, Duke Math. Journ., 16(1949), 339–418.

[6] I. KAPLANSKY, The structure of certain opertor algebras, Trans. Amer. Math. Soc., 70(1951), 219–255.

[7] Y. MISONOU AND M. NAKAMURA, Centering of an operator algebra, Tohoku Math. Journ., 3(1951). 243–248.

[8] F. J. MURRAY AND J. VON NEUMANN, On rings of operators, Ann. Math., 37(1936), 116–229.

[9] J. VON NEUMANN, Zur Algebra der Funktionaloperatoren, Math. Ann., 102(1929), 370–427.

[10] I. E. SEGAL, The group algebra of a locally compact group, Trans. Amer. Math. Soc., 61(1947), 69–105.

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