A NOTE ON PARACOMPACT SPACES

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E. G. Begle [1][1] proved the following theorem:

"Let A be an n-dimensional Hausdorff space which has the star-finite property[2] and B be an n-dimensional compact Hausdorff space, then the product space $A \times B$ has the star-finite property and $\dim(A \times B) \leqq m+n$, where we mean by the dimension of a space the so called Lebesgue's dimension."

In this paper we shall prove that if A is an m-dimensional paracompact[3] normal space and if B is an n-dimensional compact normal space, then the dimension inequality

$$\dim(A \times B) \leqq m+n$$

holds good.

Since a Hausdorff space which satisfies the star-finite property is *a fortiori* paracompact, by Theorem 1 of [2], it is normal. Therefore, for the dimension inequality, our result is a generalization of the Begle's result.

In the sequel, a space is required to be a topological space but not necessarily a T_0-space, and by a normal space we mean a space which satisfies only the normality.

C. H. Dowker defined three kinds of dimension, $\dim_L Y \leqq n$, $\dim_S X \leqq n$ and $\dim_F X \leqq n$[4] for an arbitrary space X, and he proved[5] that if X is a normal space then the above three dimensions are the same, by showing that each of these inequalities holds good if, and only if, for each closed set $X' \subset X$, each continuous mapping f of X' into the n-sphere S^n can be extended to a continuous mapping F of X into S^n. Following Dowker, we define the dimension of a normal space X, $\dim X$, to be the common dimension $\dim_L X = \dim_S X = \dim_F X$.

For the sake of convenience we rewrite the second half of the proof of Theorem 3.5 of [3] as a lemma.

1) Numbers in brackets refer to the references cited at the end of this paper.

2) A space X has the star-finite property if and only if, any open covering of X has a star-finite refinement.

3) A space X is paracompact if, and only if, every open covering X has a locally finite refinement.

4) $\dim_L X \leqq n$ ($\dim_S X \leqq n$ or $\dim_F X \leqq n$) means that for every locally finite (star-finite or finite) covering of X there exists a locally finite (stat-finite or finite) refinement of order $\leqq n+1$. Lebesgue's dimension is nothing but $\dim_F$.

5) See Theorem 3.5 and Corollary 3.6 of [3].

LEMMA 1. *Let X be a normal space such that* dim $X \leqq n$, *and* $\mathfrak{U}$ *be a given locally finite covering* [6] *of X. Then there exists a normal*[7] *refinement* $\mathfrak{V}$ *of* $\mathfrak{U}$ *such the order of* $\mathfrak{V}$ *is not greater than* $n+1$, *and* $\mathfrak{V}$ *is finite if* $\mathfrak{U}$ *is finite.*

Let K be a (not necessary locally finite) simplicial complex. Following J. H. C. Whitehead, we define the topology of K by the conditions:

(1) each closed simplex of K has the topology natural to its affine geometry,

(2) a set of points in K is closed if, and only if, its intersection with each closed simplex is closed.

We denote by $P(K)$ a simplical complex K with such weak topology, and denote by $N(K)$ a simplicial complex K which is topologized by the natural metric[8]. The identical tranformation of $N(K)$ onto $P(K)$ is continuous on every finite subcomplex of $N(K)$. Therefore, by Lemma 1.2 of [3], we have the following lemma.

LEMMA 2. *Let X be a normal space and* $\mathfrak{U}$ *be a locally finite covering of X. Let* ϕ *be a canonical mapping*[9] *of X into the nerve with the natural metric* $N(\mathfrak{U})$ *of* $\mathfrak{U}$. *Then* ϕ *is also a canonical mapping of K into the nerve with the weak topology* $P(\mathfrak{U})$ *of* $\mathfrak{U}$.

Combining Lemma 1 and Lemma 2 we have

LEMMA 3. *Let X be a normal space such that* dim $X \leqq n$ *and let* $\mathfrak{U}$ *be a given locally finite covering of X. Then there exists a locally fintie refinement* $\mathfrak{V}$ *of* $\mathfrak{U}$ *and a continuous mapping* ϕ *of X onto the nerve* $P(\mathfrak{V})$ *of* $\mathfrak{V}$ *such that*

(i) *the order of* $\mathfrak{V}$ *is not greater than* $n+1$,

(ii) *each set V of* $\mathfrak{V}$ *is the inverse image, under* ϕ, *of the open star of the vertex of* $P(\mathfrak{V})$ *corresponding to V.*

LEMMA 4. *A simplicial complex P(K) with the weak topology is a paracompact Hausdorff space, and therefore normal.*

PROOF. It is obvious that the identical transformation of $N(K)$ into $P(K)$ transforms an open set of $N(K)$ onto an open set of $P(K)$. $N(K)$ is a metric space, hence it is a Hausdorff space. Therefore $P(K)$ is a Hausdorff space.

Let $\mathfrak{U}$ be an arbitrary covering of $P(K)$. By Theorem 3.5 of [7], there exists a simplicial subdivision K_0 of K such that the open star of each vertex of K_0 with respect to K_0 is contained in some element of $\mathfrak{U}$. Let K_0'' be the second barycentric subdivision of K_0. Since as topological spaces $P(K)=P(K_0)=P(K_0'')$, all open stars of vertices of K_0'' with respect to K_0''

6) By a covering we mean open covering

7) See [3, §1, p. 209]

8) See [3, §1]

9) Since $\mathfrak{U}$ is locally finite, by Theorem 11 of [3], there exists a canonical mapping ϕ : $X \to N(\mathfrak{U})$.

form a covering $\mathfrak{V}$ of $P(K)$. It is easily verified that the star of $\mathfrak{V}$ is a refinement of $\mathfrak{U}$. Hence $P(K)$ is fully normal. Therefore, by Theorem 1 of [6], $P(K)$ isparacompact. Also, by Theorem 1 of [2], $P(K)$ is normal.

A simplicial complex $P(K)$ with the weak topology is a special CW-complex in the sense of J.H.C. Whitehead. Let K be a CW-complex. By $\dim^* K$ we shall denote the upper bound of dimensional numbers of all cells on K.

LEMMA 5. $\dim^* K = \dim K$.

PROOF. According to (G) in §5 of [9] K is normal, hence $\dim K$ has the meaning.

Now we assume that $\dim K \leqq m$. Then for every closed subset X of K we have $\dim X \leqq m$. Hence we have $\dim \bar{e}^n \leqq m$, where e^n is an arbitrary cell on K. Let $f: \sigma^n \to \bar{e}^n$ be a characteristic mapping for e^n, where σ^n is a closed euclidean n-simplex. Since f is topological on the interior of σ^n and $\dim \sigma^n = n$, it is easily seen that $n \leqq m$. Therefore we have $\dim^* K \leqq m$.

Next we assume that $\dim^* K \leqq n$. We shall show that for each closed subset $X \subset K$, each mapping $\phi_0: X \to S^n$ can be extended to a mapping $\phi: K \to S^n$. If this is done we have $\dim K \leqq n$ which completes the proof.

Let K^n be the n-skeleton of K, and let $K_n = K^n \cup X$. Since K^0 is discrete ϕ_0 has an extension $\phi^0: K_0 \to S^n$. In order to use the iuduction we suppose that a mapping $\phi^{n-1}: K_{n-1} \to S^n$ $(0 < n \leqq m)$ is defined such that $\phi^{n-1} | X = \phi_0$. Let e^n be an n-cell on K and $f: \sigma^n \to e^n$ be a characteristic mapping for e^n. Let $Y = f^{-1}(K_{n-1})$ and $g_0 = \phi^{n-1} f: Y \to S^n$. Since $\dim \sigma^n = n \leqq m$ and Y is a closed subset of σ^n, g_0 has an extension $g: \sigma^n \to S^m$. Let $\phi^n_{e^n} = gf^{-1}$. Since $Y \supset \partial\sigma^n$, and f is topological on the interior $\sigma^n - \partial\sigma^n$ of σ^n, $\phi^n_{e^n}$ is one-valued. Therefore, by Lemma 3 of [8], $\phi^n_{e^n}$ is continuous. Therefore a transformation $\phi^n: K_n \to S^n$ defined by $\phi^n(P) = \phi^n_{e^n}(p)$ (if $p \in \bar{e}^n$) is continuous and $\phi^n | K^{n-1} = \phi^{n-1}$. Thus by the induction on n, there exists a mapping $\phi: K \to S^m$ such that $\phi | X = \phi_0$.

C.H. Dowker proved[10] that if A is a countably paracompact normal space and if B is a compact metric space then $A \times B$ is normal.

By the same way as his proof we can prove the following result.

LEMMA 6. *Let A be a paracompact normal space and B be a compact normal space. Then the product space $A \times B$ is normal.*

LEMMA 7. *Let A and B be compact normal space. Then we have*

$$\dim (A \times B) \leqq \dim A + \dim B.$$

PROOF. Hemmingen [5] proved the same theorem under the condition that A and B are compact Hausdorff spaces. But it is easily seen, by using Lemma 1 that this condition can be replaced by the weaker condition that A and B are compact normal spaces.

10) [4, Lemma 3].

LEMMA 8. *Let $P = P(K)$ be an m-dimensional simplicial complex with the weak topology and B be an n-dimensional compact normal space. Then*

$$\dim (P \times B) \leqq m + n.$$

PROOF. According to Lemma 4 and 6, $P \times B$ is normal, hence dim $(P \times B)$ has the meaning.

Let X be a given closed set of $P \times B$ and ϕ_0 be a given mapping of X into S^{m+n}. Then it is sufficient to show that ϕ_0 can be extended to a mapping ϕ of $P \times B$ into S^{m+n}.

Since P has the weak topology and B is compact, a function f defined on a closed subset $F \subset P \times B$ is continuous if, and only if, for each closed simplex $\sigma \subset P$, f is continuous on $(\sigma \times B) \cap F$.

According to this fact and Lemma 7, ϕ_0 can be stepwise extended to a mapping ϕ: $P \times B \rightarrow S^{m+n}$.

REMARK. By Lemma 4, a simplicial complex with weak topology is paracompact. I do not know if all CW-complexes are paracompact or not. In vertue of (H) in §5 of [9] and Theorem 4 of [4], all CW-complexes are countably paracompact. But if K is an CW-complex and B is a compact normal space then it can be proved by using the special characters of CW-complexe that $K \times B$ is normal and dim $(K \times B) \leqq$ dim $K +$ dim B holds good. For our purpose this is not necessary, and so we shall omit the proof.

Now we prove the following theorem which is our purpose.

THEOREM. *Let A be an m-dimensional paracompact normal space and let B be an n-dimensional compact normal space. Then $A \times B$ is paracompact normal and*

$$\dim (A \times B) \leqq m + n.$$

PROOF. By Lemma 6, $A \times B$ is normal hence $\dim(A \times B)$ has the meaning. By Theorem 5 of [2], $A \times B$ is paracompact.

Now let $\mathfrak{W}_0$ de an arbitrary locally finite covering of $A \times B$. Let a be any point of A. Each point of $a \times B$ is contained in an open set of the form $U \times V$, U open in A, V open in B, such that $U \times V$ is containd in an open set of $\mathfrak{W}_0$. For a fixed point $a \in A$, the set of all such U's is a covering of B and hence a finite number of them, say $V_{a,1}, \dots . V_{a,k(a)}$, form a covering $\mathfrak{V}_a$ of B. Let U_a be the intersection of the corresponding U's.

The collection $\{U_a\}$ of all such sets U_a is a covering of A. Since A is paracompact normal and dim $A \leqq m$, by Lemma 3, there exists a locally finite refinement $\mathfrak{U}$ of $\{U_a\}$ and a mapping ϕ of A onto the nerve with the weak topology $P(\mathfrak{U})$ of $\mathfrak{U}$ such that

(i) the order of $\mathfrak{U}$ is not greater than $n+1$, i.e. dim* $P(\mathfrak{U}) \leqq n$,

(ii) each open set U of $\mathfrak{U}$ is the inverse image, under ϕ, of the open star of the vertex of $P(\mathfrak{U})$ correspondig to U.

We construct a covering $\mathfrak{W}$ of $A \times B$ as follows: each set U of $\mathfrak{U}$ is contained in some U_a and with each U_a is associated a covering $\mathfrak{V}_a$ of B. Form the product of U with each set of $\mathfrak{V}_a$. The totality of these products forms a covering $\mathfrak{W}$ of $A\times B$, and by construction, $\mathfrak{W}$ is a refinement of $\mathfrak{W}_0$.

Let θ be a mapping of $A\times B$ onto $P(\mathfrak{U})\times B$ defined by $\theta(a\times b) = \phi(a)\times b$ $(a\in A,\ b\in B)$, where ϕ is the above mapping of A onto $P(\mathfrak{U})$. Each element of $\mathfrak{W}$ is thus mapped by θ onto an open set of $P(\mathfrak{U})\times B$, so $\mathfrak{X} = \theta(\mathfrak{W})$ is a covering of $P(\mathfrak{U})\times B$.

Now, according to Lenma 5 and 8, we have dim $(P)(\mathfrak{U})\times B)\leqq m+n$. By Lemma 4, 6 and Theorem 5 of [2], $P(\mathfrak{U})\times B$ is paracompact normal, therefore dim $(P(\mathfrak{U})\times B) = \dim_L(P(\mathfrak{U})\times B)\leqq m+n$ means that there exists a locally finite refinement $\mathfrak{Y}$ of $\mathfrak{X}$, of the order $\leqq m+n+1$. Then the covering $\theta^{-1}(\mathfrak{Y})$ is locally finite covering of $A\times B$, of order not greater than $m+n+1$, and by (ii), $\theta^{-1}(\mathfrak{Y})$ ia a refinement of $\mathfrak{U}$, which proves the theorem.

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