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ON A CERTAIN CLASS OF HOMOGENEOUS PROJECTIVELY FLAT MANIFOLDS

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Among the manifolds with a flat projective structure the homogeneous ones are particularly interesting. So far a general classification of such manifolds does not seem to exist. So the interest mostly has focussed on special cases. Agaoka [1] studied left-invariant projectively flat structures on Lie groups in some detail. Also he obtained a complete classification in the case where the Lie group action is the same as for an irreducible Riemannian symmetric space. Vinberg [12] developed a theory for projectively homogeneous bounded domains in $\boldsymbol{R}^n$. Here we will obtain rather complete results about another class of homogeneous projectively flat manifolds, which we call bihomogeneous.

A differentiable manifold M is said to be bihomogeneous if there exists a pair of Lie groups G_1 and G_2, each acting transitively on M and transformations from different groups commuting with each other. When M has a geometric structure, we say it is bihomogeneous if both G_1 and G_2 preserve the structure in question. In this paper we study bihomogeneous manifolds with flat projective structures. This class of manifolds includes S^3, the real projective space $\boldsymbol{RP}^n$ with two hyperplanes removed, and, more generally, the models which can be described as follows.

Let A be an associative algebra with identity over $\boldsymbol{R}$. The open submanifold M of the projective space $P(A)$ corresponding to the open cone of all units in A is bihomogeneous where G_1 and G_2 are the groups of left and right multiplication, respectively. Our first main result is that the universal covers of these models exhaust all simply-connected bihomogeneous projectively flat manifolds.

We reduce the problem to the study of biinvariant affine connections on Lie groups which are projectively flat. We show, in particular, that the only semisimple Lie groups admitting biinvariant flat projective structures are $SL(n, \boldsymbol{R})$ and $SL(n, \boldsymbol{H})$, where $\boldsymbol{H}$ is the field of quaternions. From biinvariant flat projective structures on $SL(n, \boldsymbol{R})$ and $SL(n, \boldsymbol{H})$ we can easily obtain many examples of compact homogeneous

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projectively flat manifolds as quotient spaces by cocompact discrete subgroups.

Now suppose that M is a manifold with a projective structure P. We say that a Lie group acting freely on M is a *translation group* if all orbits of 1-parameter subgroups are geodesics. We shall determine the projectively flat manifolds which admit transitive translation groups. They turn out to be bihomogeneous and their study is reduced to the study of Lie groups whose (0)-connections are projectively flat. These Lie groups were indeed studied by E. Cartan [3]. We are going to complete his results, provide a simpler proof based on our general method mentioned above, and describe our main result here in the following way:

The following subsets of $\boldsymbol{R}P^n$ *admit a transitive translation group.*

(1) $\boldsymbol{R}P^n - H = \boldsymbol{R}^n$ *where* H *is a hyperplane. The translation group* G *is abelian or* 2-*step nilpotent.*

(2) *Each component of* $\boldsymbol{R}P^n - (H_1 \cup H_2)$, *where* H_1 *and* H_2 *are hyperplanes; the group* G *is solvable but not nilpotent and is characterized by an integer* p, $0 \leqq p \leqq (n-1)/2$.

(3) $\boldsymbol{R}P^{2n+1}$ – *subspace of codimension* 2; G *is uniquely determined.*

(4) $\boldsymbol{R}P^3$; G *is* $SO(3)$.

(5) *Each component of* $\boldsymbol{R}P^3 - Q$ *where* Q *is a quadric of signature* $(-, -, +, +)$; G *is* $PSL(2, \boldsymbol{R})$.

All projectively flat manifolds admitting a transitive translation group are obtained by passing to quotients by discrete central subgroups of G *in case* (1) *or by passing to arbitrary covering spaces in the non-simply connected cases* (3), (4) *and* (5).

In §1, we give several definitions and preliminary results on projective structures, volume elements, equiaffine connections, projective curvature tensor, etc. Some old results in Eisenhart [4] and Schouten [9] are reformulated. In §2 we study bihomogeneous spaces and reduce the problem on bihomogeneous projectively flat manifolds to the study of biinvariant projectively flat affine connections on Lie groups. This is carried out and our first main result is precisely stated in §3. The case of semisimple Lie groups is treated is §4. In §5 we show that there is almost no overlap between the class of manifolds we study here and the class investigated by Vinberg: we prove that the only bihomogeneous bounded domain in $\boldsymbol{R}^n$ is the interior of a simplex. In §6 we reduce the study of projectively flat manifolds admitting transitive translation groups to E. Cartan's problem on Lie groups. We give a complete solution of this problem using a result from §3. In §7 we shall derive the geometric description we already mentioned.

1. Projective structures, equiaffine connections, projective curvature tensors. The notion of flat projective structure can be defined in the following way. A differentiable manifold M has a *flat projective structure* if it has an atlas $\{U_\alpha, \phi_\alpha\}$ where ϕ_α maps the open subset U_α onto an open subset V_α of the real projective space $\boldsymbol{RP}^n$ in such a way that in the non-empty intersection $U_\alpha \cap U_\beta$ the transformation

$$\phi_\beta \phi_\alpha^{-1}\colon \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is the restriction of a projective transformation of $\boldsymbol{RP}^n$ to $\phi_\alpha(U_\alpha \cap U_\beta)$. See Kobayashi [7].

In order to give an equivalent definition which we shall use, we start with affine connections ∇ and $\bar{\nabla}$ with zero torsion tensors defined on a manifold M (or on an open subset of M). We say that ∇ and $\bar{\nabla}$ are *projectively equivalent* if there is a 1-form ρ such that

$$\bar{\nabla}_X Y = \nabla_X Y + \rho(X) Y + \rho(Y) X \tag{1}$$

for all tangent vector fields X and Y. Geometrically, this means that both affine connections have the same family of curves as geodesics (though the affine parameters may not be the same). See Eisenhart [4, p. 87] and Tanaka [11, Appendix].

We say that a differentiable manifold M has a *projective structure* P if M has an open covering $\{U_\alpha\}$, where each U_α has a torsion-free affine connection ∇^α in such a way that in any non-empty intersection $U_\alpha \cap U_\beta$ the affine connections ∇^α and ∇^β are projectively equivalent. We may call such a covering $\{U_\alpha\}$ an atlas for the projective structure P and it is obvious when two atlases define the same projective structure.

PROPOSITION 1. *If M has a projective structure, we can find a torsion-free affine connection ∇ on the whole M such that in a neighborhood of each point, ∇ is projectively equivalent to a local affine connection ∇^α on U_α of the atlas.*

PROOF. Choose a partition of unity $\{f_\alpha\}$ subordinate to $\{U_\alpha\}$ and define a global affine connection $\nabla = \sum f_\alpha \nabla^\alpha$. Note that each point x of M has a compact neighborhod on which all but a finite number of f_α's vanish so that ∇ is locally a finite convex sum of torsion-free affine connections which are projectively equivalent. □

PROPOSITION 2. *Let ∇ be a torsion-free affine connection on a manifold M. Given a volume element* (i.e. *a non-vanishing n-form*) ω *on M, there is a unique torsion-free affine connection $\bar{\nabla}$ on M such that*

(1) $\bar{\nabla}$ *is projectively equivalent to* ∇;

(2) $\bar{\nabla}\omega = 0$ (*that is,* ω *is a parallel relative to* $\bar{\nabla}$).

PROOF. It is easily verified that

$$\rho(X) = \frac{1}{n+1} \frac{(\nabla_X \omega)(X_1, \cdots, X_n)}{\omega(X_1, \cdots, X_n)} \tag{2}$$

is the unique choice for the 1-form in (1), where the right-hand side is independent of the choice of a basis $\{X_1, \cdots, X_n\}$ at each point. □

Proposition 2 can be found in Eisenhart [4, p. 104] and Schouten [9, p. 288].

The Ricci tensor of an affine connection is defined by

$$\mathrm{Ric}(Y, Z) = \text{trace of the linear map } X \mapsto R(X, Y)Z .$$

The Ricci tensor may not be symmetric

PROPOSITION 3. *A torsion-free affine connection* ∇ *has symmetric Ricci tensor if and only if it admits locally a parallel volume element. If the manifold* M *is simply connected, then a torsion-free affine connection with symmetric Ricci tensor admits a global parallel volume element.*

PROOF. From the definition of the Ricci tensor and the first Bianchi identity for the curvature tensor, we get

$$\mathrm{Ric}(Y, Z) - \mathrm{Ric}(Z, Y) = -\mathrm{trace}\, R(Y, Z) .$$

If there is a parallel volume element ω in a neighbourhood U, then the holonomy group at x for U is contained in $SL(T_x(M))$ so that $R(Y, Z)$ has trace 0. Thus Ric is symmetric. The converse can be proved by using a well-known theorem on the Lie algebra of the holonomy group. □

An affine connection with zero torsion is said to be *equiaffine* if it admits a globally defined parallel volume element.

PROPOSITION 4. *Suppose a differentiable manifold* M *has a projective structure* P. *For any given volume element* ω *on* M *we can find an equiaffine connection* (*with zero torsion*) *which induces the given projective structure* P. *Such* ∇ *is unique.*

This follows from Propositions 1 and 2.

An affine connection ∇ with zero torsion and symmetric Ricci tensor is said to be *projectively flat* if in a neighborhood of each point ∇ is projectively equivalent to an affine connection $\bar{\nabla}$ which is flat, that is, the curvature tensor $\bar{R}$ is identically zero. The following is well-known (H. Weyl).

PROPOSITION 5. *A torsion-free equiaffine connection* ∇ *is projectively flat if and only if*

(a) *the projective curvature tensor* W *defined by*

$$W(X, Y)Z = R(X, Y)Z - \frac{1}{(n-1)}\{\mathrm{Ric}(Y, Z)X - \mathrm{Ric}(X, Z)Y\} \tag{3}$$

is identically 0; *and*

(b) *the Ricci tensor satisfies Codazzi equation, that is,*

$$(\nabla_X \mathrm{Ric})(Y, Z) = (\nabla_Y \mathrm{Ric})(X, Z) \tag{4}$$

for all tangent vectors X, Y, Z.

Furthermore, if $\dim M = n \geqq 3$, *then* (b) *follows from* (a). *For* $n = 2$, (a) *is automatically satisfied, so* (b) *is a necessary and sufficient condition for projective flatness.*

REMARK. The projective curvature tensor W defined by (3) is the same for all projectively equivalent equiaffine connections.

For Proposition 5, see Schouten [9, p. 289], Eisenhart [4, p. 95]. It is known that the Levi-Civita connection of a nondegenerate metric g is projectively flat if and only if g has constant sectional curvature.

Finally, a projective structure P defined by an atlas $\{U_\alpha, \nabla^\alpha\}$ is flat if each local affine connection ∇^α is projectively flat. This definition indeed coincides with the definition of flat projective structure given in the first paragraph of this section.

2. Bihomogeneous projectively flat manifolds. By a bihomogeneous manifold M we mean a manifold M with a pair of Lie groups G_1 and G_2 acting effectively and transitively on M in such a way that the transformation by every element a_1 of G_1 commutes with the transformation by every element a_2 of G_2. It is natural to denote the action of G_1 on M on the left:

$$x \in M \mapsto a_1 x \in M\,, \quad a_1 \in G_1$$

and the action of G_2 on M on the right

$$x \in M \mapsto x a_2 \in M\,, \quad a_2 \in G_2$$

and thus we have

$$(a_1 x)a_2 = a_1(x a_2) \quad \text{for all} \quad a_1 \in G_1,\ a_2 \in G_2,\ x \in M\,.$$

PROPOSITION 6. *If* M *is a bihomogeneous manifold with groups* G_1

and G_2, then M is diffeomorphic to a Lie group G in such a way that G_1 corresponds to the group of left translations of G and G_2 to the group of right translations of G. G_1 and G_2 are isomorphic to each other.

PROOF. We show that G_1 acts freely on M. Suppose $a_1 \in G_1$ has a fixed point $x \in M$: $a_1 x = x$. For any y in M there exists $a_2 \in G_2$ such that $y = xa_2$. Then

$$a_1 y = a_1(xa_2) = (a_1 x)a_2 = xa_2 = y ,$$

showing that a_1 acts as the identity transformation on M. Since G_1 is effective, a_1 is the identity element of G_1. Similarly, G_2 acts freely on M.

By choosing an arbitrary point x_0 in M as the origin we can define a diffeomorphism f_1 of G_1 onto M by $f_1(a_1) = a_1 x_0$. Since $f_1(b_1 a_1) = (b_1 a_1)x_0 = b_1 f_1(x_0)$, we see that the action of G_1 on M corresponds to the action of G_1 on itself by left translations. This way we can identify M with G_1, and similarly for M and G_2.

For each a_1 in G_1 we can find a unique $a_2 \in G_2$ such that $a_1 x_0 = x_0 a_2$. It is easy to verify that $\phi: a_1 \in G_1 \to a_2 \in G_2$ determined in this way is an isomorphism of G_1 onto G_2. □

Now let M and $\bar{M}$ be differentiable manifolds with projective structures P and $\bar{P}$, respectively. A diffeomorphism f of M onto $\bar{M}$ is called a projective transformation if for each local connection $\bar{\nabla}^\alpha$ of the atlas for $\bar{M}$ the pull-back $f^*\bar{\nabla}^\alpha$ is projectively equivalent to every local affine connection of the atlas for M. For a manifold M with a projective structure P, a projective transformation of M onto itself is also called an automorphism of (M, P). The group of all automorphisms is denoted by $\mathrm{Aut}(M, P)$.

A manifold M with a projective structure P is bihomogeneous if the action of each group G_i, $i = 1, 2$, preserves P. In this case, M may be thought of as the space of a Lie group G provided with a biinvariant projective structure, namely, a projective structure which is preserved by left and right translations.

We want to reduce the study of this structure to that of a biinvariant affine connection which is projectively flat. For this purpose, we prove the following general result.

THEOREM 1. *Let G/H be a homogeneous space of a connected Lie group G. Assume*

(1) *G/H admits a projective structure invariant by G;*

(2) *G/H admits a volume element ω invariant by each $g \in G$ up to*

a scalar multiple, that is, for each $g \in G$ there is a positive constant $c(g)$ such that $g^*\omega = c(g)\omega$.
Then there is on G/H a unique invariant torsion-free affine connection ∇ which is equiaffine relative to ω and which induces the given projective structure.

PROOF. By Proposition 4 we can choose an equiaffine connection ∇ relative to ω which induces the given projective structure. For each $g \in G$, the transform $g^*\nabla$ of the connection ∇ is equiaffine relative to the pull-back $g^*\omega = c(g)\omega$ and hence relative to ω. By the uniqueness part of Proposition 2 we conclude that $g^*\nabla = \nabla$. Thus ∇ is invariant by every $g \in G$. The uniqueness of ∇ is clear. □

REMARK. A typical situation of a volume element as in condition (2) is the usual volume element on the affine space $\boldsymbol{R}^n$ regarded as a homogeneous space $A(n, \boldsymbol{R})/GL(n, \boldsymbol{R})$ of the affine group $A(n, \boldsymbol{R})$. An orientable homogeneous space G/H with compact H, in particular a Lie group G itself with $H = \{e\}$, has an invariant volume element.

We obtain

COROLLARY. *Let G be a connected Lie group with a biinvariant projective structure. Then there is a unique torsion-free biinvariant affine connection which is equiaffine relative to a left invariant volume element and which induces the given projective structure.*

PROOF. Consider G as a homogeneous space $\tilde{G}/D$, where $\tilde{G} = G \times G$ and D is the diagonal subgroup of $\tilde{G}$. A left invariant volume element on G satisfies condition (2) in Theorem 1 for the action of $(a, b) \in \tilde{G}$, which is $x \in G \mapsto axb^{-1} \in G$. In fact, $R^*_{b^{-1}}\omega$ is left invariant and hence equal to a constant multiple of ω. □

3. Biinvariant projectively flat affine connections on Lie groups. Our problem is now to study biinvariant projectively flat affine connections on Lie groups. Let $\mathfrak{g}$ be the Lie algebra of a Lie group G. Recall that for any left invariant geometric structure on G (such as a Riemannian metric, an affine connection, etc.) there is an algebraic structure on $\mathfrak{g}$. If ∇ is a left invariant affine connection on G, then it can be expressed as a bilinear mapping

$$(5) \qquad (X, Y) \in \mathfrak{g} \times \mathfrak{g} \mapsto \nabla_X Y \in \mathfrak{g} .$$

This transition comes simply by observing that if X and Y are left invariant vector fields on G (i.e. elements of $\mathfrak{g}$), then so is $\nabla_X Y$. See Nomizu [8]. Since the torsion tensor is 0 we have

(6) $$\nabla_X Y - \nabla_Y X = [X, Y] \quad \text{for} \quad X, Y \in \mathfrak{g} .$$

Biinvariance of the affine connection of G leads to the invariance of the bilinear mapping (5) by the adjoint mapping in $\mathfrak{g}$:

(7) $$\nabla_{[Z,X]} Y + \nabla_X [Z, Y] = [Z, \nabla_X Y] .$$

The connection being equiaffine relative to a left invariant volume element on G we have

(8) $$\operatorname{trace} \nabla_X = 0 \quad \text{for} \quad X \in \mathfrak{g} .$$

Somewhat more generally, even if the affine connection ∇ is not equiaffine relative to a left invariant volume element on G, assume it has symmetric Ricci tensor (that is, it admits locally a parallel volume element—see Proposition 3). Let γ be the normalized Ricci tensor $\gamma(Y, Z) = \mathrm{Ric}(Y, Z)/(n-1)$, where $n = \dim G$. The projective curvature tensor W is given by

$$W(X, Y)Z = R(X, Y)Z - \{\gamma(Y, Z)X - \gamma(X, Z)Y\} ,$$

which is an algebraic expression in $\mathfrak{g}$ like (5), (6) and others. Thus the projective flatness is given by

(9) $$R(X, Y)Z = \gamma(Y, Z)X - \gamma(X, Z)Y .$$

We now prove

LEMMA. *Let ∇ be a biinvariant torsion-free affine connection on a Lie group and consider its Lie algebra expression on the Lie algebra $\mathfrak{g}$. If ∇ is flat (that is, $R = 0$), then $\mathfrak{g}$ becomes an associative algebra over $\boldsymbol{R}$ with respect to multiplication $X \cdot Y = \nabla_X Y$.*

PROOF. $R = 0$ gives

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = 0$$

and hence

(10) $$X \cdot (Y \cdot Z) - Y \cdot (X \cdot Z) - (X \cdot Y - Y \cdot X) \cdot Z = 0 .$$

On the other hand, biinvariance of ∇ leads to

$$\nabla_{[X,Y]} Z + \nabla_Y [X, Z] = [X, \nabla_Y Z]$$

that is,

$$(X \cdot Y - Y \cdot X) \cdot Z + Y \cdot (X \cdot Z - Z \cdot X) = X \cdot (Y \cdot Z) - (Y \cdot Z) \cdot X .$$

Combining this with (10) we get $Y \cdot (Z \cdot X) = (Y \cdot Z) \cdot X$, thus proving associativity of the multiplication.

Conversely, let A be any associative algebra over $\boldsymbol{R}$. Then A can be made into a Lie algebra $\mathfrak{g}$ by setting $[X, Y] = X \cdot Y - Y \cdot X$. Also the bilinear mapping $(X, Y) \mapsto \nabla_X Y = X \cdot Y$ is biinvariant and flat. Hence the simply connected Lie group G with $\mathfrak{g}$ as its Lie algebra will have a biinvariant, torsion-free, flat affine connection.

For the study of left-invariant *complete* torsion-free flat affine connections on Lie groups, see Auslander [2].

Next we prove

THEOREM 2. *Let ∇ be a biinvariant torsion-free affine connection on a Lie group G. Assume ∇ has symmetric Ricci tensor and is projectively flat. Then the Lie algebra $\mathfrak{g}$ can be enlarged to a Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \boldsymbol{R}e$ which admits a biinvariant torsion-free flat affine connection $\tilde{\nabla}$ (so that Theorem 2 applies and $\tilde{\mathfrak{g}}$ becomes an associative algebra with identity e). If furthermore ∇ is assumed to be equiaffine relative to a left-invariant volume element, then $\mathfrak{g}$ can be recovered in $\tilde{\mathfrak{g}}$ as*

$$\mathfrak{g} = \{u \in \tilde{\mathfrak{g}};\ \mathrm{trace}\{v \in \tilde{\mathfrak{g}} \to u \cdot v \in \tilde{\mathfrak{g}}\} = 0\}\,. \tag{11}$$

PROOF. In the direct sum $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \boldsymbol{R}e$, as a Lie algebra, we define a bilinear mapping $\nabla: \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ by

$$\tilde{\nabla}_X Y = \nabla_X Y - \gamma(X, Y)e \quad \text{for} \quad X, Y \in \mathfrak{g} \tag{12}$$

and

$$\tilde{\nabla}_X e = \tilde{\nabla}_e X = X \quad \text{for} \quad X \in \mathfrak{g} \quad \text{and} \quad \tilde{\nabla}_e e = e\,. \tag{13}$$

We now wish to verify that $\tilde{\nabla}$ corresponds to a biinvariant torsion-free flat affine connection on a Lie group with Lie algebra $\tilde{\mathfrak{g}}$.

First, the torsion tensor is 0, namely $\tilde{\nabla}_u v - \tilde{\nabla}_v u = [u, v]$ for all $u, v \in \tilde{\mathfrak{g}}$. The only non-trivial case is for $X, Y \in \mathfrak{g}$

$$\begin{aligned}\tilde{\nabla}_X Y - \tilde{\nabla}_Y X &= \nabla_X Y - \gamma(X, Y)e - (\nabla_Y X - \gamma(Y, X)e) \\ &= \nabla_X Y - \nabla_Y X = [X, Y]\,,\end{aligned}$$

because of the symmetry of γ.

The curvature tensor $\tilde{R}(X, Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z$ can be computed as

$$\begin{aligned}\tilde{R}(X, Y)Z = R(X, Y)Z &- \gamma(Y, Z)X + \gamma(X, Z)Y \\ &- \gamma(X, \nabla_Y Z)e + \gamma(Y, \nabla_X Z)e + \gamma(\nabla_X Y, Z)e - \gamma(\nabla_Y X, Z)e\,.\end{aligned}$$

The first three terms on the right hand side give 0 because of (9). We have

$$\begin{aligned}\gamma(\nabla_X Y, Z) + \gamma(Y, \nabla_X Z) &= -(\nabla_X \gamma)(Y, Z) \\ \gamma(X, \nabla_Y Z) + \gamma(\nabla_Y X, Z) &= -(\nabla_Y \gamma)(X, Z)\,.\end{aligned}$$

Since projective flatness implies Codazzi equation for the Ricci tensor (4), we have $(\nabla_X\gamma)(Y, Z) = (\nabla_Y\gamma)(X, Z)$. Thus we find $\tilde{R}(X, Y)Z = 0$.

The remaining verification of $\tilde{R}(X, Y)e = 0$, $\tilde{R}(X, e)Y = 0$ and $\tilde{R}(X, e)e = 0$, where $X, Y \in \mathfrak{g}$, is straightforward. We have thus $\tilde{R} = 0$.

Biinvariance of $\tilde{\nabla}$ can be checked as follows. For $X, Y, Z \in \mathfrak{g}$, we have

$$\begin{aligned}\tilde{\nabla}_{[X,Y]}Z + \tilde{\nabla}_Y[X, Z] = \nabla_{[X,Y]}Z - \gamma([X, Y], Z)e + \nabla_Y[X, Z] \\ - \gamma(Y, [X, Z])e = \nabla_{[X,Y]}Z + \nabla_Y[X, Z] ,\end{aligned}$$

since

$$\gamma([X, Y], Z) + \gamma(Y, [X, Z]) = 0$$

because γ is biinvariant, like ∇ and R. On the other hand

$$[X, \tilde{\nabla}_Y Z] = [X, \nabla_Y Z - \gamma(Y, Z)e] = [X, \nabla_Y Z]$$

Thus biinvariance of ∇ on $\mathfrak{g}$ leads to

$$\tilde{\nabla}_{[X,Y]}Z + \tilde{\nabla}_Y[X, Z] = [X, \tilde{\nabla}_Y Z] .$$

It is now easy to complete the verification of the biinvariance of $\tilde{\nabla}$.

By the lemma, $\tilde{\mathfrak{g}}$ becomes an associative algebra with multiplication $u \cdot v = \tilde{\nabla}_u v$. We get $X \cdot e = X = e \cdot X$ and $e \cdot e = e$, so that e is the identity for the algebra.

Assume condition (8) for ∇ in $\mathfrak{g}$. For $X \in \mathfrak{g}$ we have

$$X \cdot Y = \nabla_X Y - \gamma(X, Y)e \quad \text{and} \quad X \cdot e = X ,$$

so

$$\mathrm{trace}\{u \in \tilde{\mathfrak{g}} \mapsto X \cdot u \in \tilde{\mathfrak{g}}\} = \mathrm{trace}\{Y \in \mathfrak{g} \mapsto \nabla_X Y \in \mathfrak{g}\} = 0$$

by assumption. Obviously, trace $\{u \in \tilde{\mathfrak{g}} \mapsto e \cdot u \in \tilde{\mathfrak{g}}\} = \dim \tilde{\mathfrak{g}}$. Hence we have (11). □

Conversely, we may start with an associative algebra A with identity over $\boldsymbol{R}$. Let $\dim_{\boldsymbol{R}} A = n + 1$. As we already know, we can make A into a Lie algebra $\mathfrak{g}$ by defining $[u, v] = uv - vu$ and obtain a biinvariant torsion-free flat affine connection by $\tilde{\nabla}_u v = uv$. Let

$$\tau(u) = \mathrm{trace}\{v \in \tilde{\mathfrak{g}} \mapsto uv \in \tilde{\mathfrak{g}}\} .$$

We have $\tau(e) = n + 1$. Let $\mathfrak{g} = \{u \in \tilde{\mathfrak{g}}; \tau(u) = 0\}$. Then $\mathfrak{g}$ is a Lie subalgebra such that $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \boldsymbol{R}e$ and $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] \subset \mathfrak{g}$. For X, Y in $\mathfrak{g}$, define $\nabla_X Y$ to be the $\mathfrak{g}$-component of $\tilde{\nabla}_X Y$. We can verify that this defines a biinvariant torsion-free affine connection in $\mathfrak{g}$ which is projectively flat.

Another way of viewing this situation is to consider the group U of all units (i.e. invertible elements) in the associative algebra A. The quotient group $\hat{G} = U/\boldsymbol{R}^*e$, where $\boldsymbol{R}^*$ is the multiplicative group of all nonzero reals, can be considered as an open subset of the real projective space $A/\boldsymbol{R}^*e$. Thus $\hat{G}$ admits a natural flat projective structure in such a way that the group $\hat{G}$ acts on the left as well as on the right as projective transformations. We may summarize this as follows;

THEOREM 3. *Every bihomogeneous projectively flat manifold is equivariantly projectively diffeomorphic to a Lie group with a biinvariant projective structure. Every such Lie group G can be obtained as follows: Take the group of units U in the corresponding associative algebra with identity and form $U/\boldsymbol{R}^*e$. Take the universal covering group $\hat{G}$ of $U/\boldsymbol{R}^*e$ and take the quotient $G = \tilde{G}/\Gamma$, where Γ is a central discrete subgroup of G.*

REMARK. The connection between associative algebras with identity and commuting pairs of transitive projective group actions on domains in $\boldsymbol{R}P^n$ was already known to E. Study [10] in 1890.

COROLLARY. *There is a natural one-one correspondence between*

(a) *simply connected bihomogeneous projectively flat manifolds up to equivariant projective diffeomorphism;*

(b) *associative algebras with identity over $\boldsymbol{R}$ up to algebraic isomorphism.*

Study [10] classified all associative algebras over $\boldsymbol{R}$ of dimensions 2 and 3. For higher dimensions, see Happel [5]. Vinberg [12] studied projectively homogeneous bounded domains in $\boldsymbol{R}^n$ using a certain kind of non-associative algebras.

REMARK. In order to emphasize the fact that bihomogeneity is not only the manifold but also a choice of a pair of groups G_1 and G_2 acting on it, we mention the following example.

The ordinary affine space $\boldsymbol{R}^3$ may be considered as a (projectively, in fact, affinely) flat space with G_1 and G_2 as usual translations: $\mathfrak{x} \in \boldsymbol{R} \to \mathfrak{a} + \mathfrak{x} = \mathfrak{x} + \mathfrak{a}$, $\mathfrak{a} \in G_1 = G_2$. Or we may identify the points $(x, y, z) \in \boldsymbol{R}^3$ with the elements

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

of the 3-dimensional Heisenberg group H and regard G_1 as H acting on itself by left multiplication and G_2 by right multiplication. In this case, G_1 and G_2 are non-abelian. The Lie algebra $\mathfrak{h}$ consisting of all matrices

$$\begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix}$$

is enlarged to the associative algebra $A = \tilde{\mathfrak{h}}$ consisting of all matrices

$$\begin{bmatrix} a & x & z \\ 0 & a & y \\ 0 & 0 & a \end{bmatrix} \quad a \in \boldsymbol{R} .$$

We may also consider two other projectively flat manifolds $PSL(2, \boldsymbol{R})$, which is bihomogeneous with a unique choice of the pair (G_1, G_2), and $\boldsymbol{R}P^3$, which is bihomogeneous where the choice of (G_1, G_2) is unique up to conjugacy within the automorphism group.

4. Semisimple Lie groups. In this section, we prove

THEOREM 4. *Let G be a semisimple Lie group that admits a bi-invariant torsion-free equiaffine relative to a left-invariant volume element connection which is projectively flat. Then its Lie algebra is isomorphic to $\mathfrak{sl}(n, \boldsymbol{R})$ or to $\mathfrak{sl}(n, \boldsymbol{H})$, where $\boldsymbol{H}$ is the quaternion field.*

PROOF. By Theorem 3 we obtain an associative algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \boldsymbol{R}e$. By the lemma below, we see that this associative algebra $\tilde{\mathfrak{g}}$ is semisimple if the Lie algebra $\mathfrak{g}$ is semisimple. The structure theorem of Wedderburn implies that $\tilde{\mathfrak{g}}$ splits as $\tilde{\mathfrak{g}}_1 + \tilde{\mathfrak{g}}_2 + \cdots + \tilde{\mathfrak{g}}_k$, where each $\tilde{\mathfrak{g}}_i$ is isomorphic to $\mathfrak{gl}(n, \boldsymbol{R})$, $\mathfrak{gl}(n, \boldsymbol{C})$ or $\mathfrak{gl}(n, \boldsymbol{H})$. (See van der Waerden [13], p. 54). If there were more than one summand, then $z = m_2 e_1 - m_1 e_2$, where e_i is the identity of $\tilde{\mathfrak{g}}_i$ and $m_i = \dim \tilde{\mathfrak{g}}_i$ for $i = 1, 2$, would be in the center of $\mathfrak{g}$. In fact, for each $x \in \mathfrak{g}$, written $x = x_1 + x_2 + \cdots + x_k$ with $x_i \in \tilde{\mathfrak{g}}_i$, we have

$$\begin{aligned}[x, z] &= (x_1 + x_2 + \cdots + x_k)(m_2 e_1 - m_1 e_2) - (m_2 e_1 - m_1 e_2)(x_1 + x_2 + \cdots + x_k) \\ &= m_2 x_1 - m_1 x_2 - (m_2 x_1 - m_1 x_2) = 0 .\end{aligned}$$

We still have to show that $z \in \mathfrak{g}$. This can be done by checking that

$$\operatorname{trace}\{u \in \tilde{\mathfrak{g}} \mapsto zu \in \tilde{\mathfrak{g}}\} = 0 .$$

Therefore we conclude that $\tilde{\mathfrak{g}}$ is either $\mathfrak{gl}(n, \boldsymbol{R})$ or $\mathfrak{gl}(n, \boldsymbol{C})$ or $\mathfrak{gl}(n, \boldsymbol{H})$. The case $\mathfrak{gl}(n, \boldsymbol{C})$ must be excluded because the element ie is in the center of $\mathfrak{g}$.

If $\tilde{\mathfrak{g}} = \mathfrak{gl}(n, \boldsymbol{R})$, then we get $\mathfrak{g} = \mathfrak{sl}(n, \boldsymbol{R})$.

If $\tilde{\mathfrak{g}} = \mathfrak{gl}(n, \boldsymbol{H})$, then $\mathfrak{g}$ is the Lie subalgebra consisting of all $X \in \mathfrak{gl}(n, \boldsymbol{H})$, which as a real linear transformation of the real $4n$-dimensional vector space has trace 0. This is $\mathfrak{sl}(n, \boldsymbol{H})$, known to be semisimple. See Hausner und Schwartz [6, p. 186]. $\square$

We now prove

LEMMA. *If $\mathfrak{g}$ is semisimple, then the associative algebra $\tilde{\mathfrak{g}}$ is semisimple, that is, $\tilde{\mathfrak{g}}$ does not contain any nonzero nilpotent two-sided ideal.*

PROOF. Let I be a nilpotent two-sided ideal of the associative algebra $\tilde{\mathfrak{g}}$. If $a \in I$, then left multiplication by a is a nilpotent linear transformation and has trace 0. Thus $a \in \mathfrak{g}$. Hence I is a nilpotent ideal of the Lie algebra $\mathfrak{g}$ and $I = \{0\}$, because $\mathfrak{g}$ is semisimple. $\square$

EXAMPLE. For $\mathfrak{g} = \mathfrak{sl}(n, \boldsymbol{R})$,

$$\nabla_X Y = XY - (\mathrm{trace}(XY)/n)I_n \quad (I_n\text{: identity matrix})$$

defines a biinvariant torsion-free affine connection which is projectively flat. The normalized Ricci tensor is given by

$$\gamma(Y, Z) = -\mathrm{trace}(YZ)/n$$

and the curvature tensor by

$$R(X, Y)Z = \gamma(Y, Z)X - \gamma(X, Z)Y,$$

as can be directly verified. Note that the Ricci tensor is a constant multiple of the Killing-Cartan form.

5. Bihomogeneous bounded domains in $\boldsymbol{RP}^n$.

In order to show the relationship of our work to that of Vinberg [12] we prove that the interior of a simplex is the only projectively bihomogeneous bounded domain in $\boldsymbol{R}^n$. The proof is more natural in terms of projective spaces. So we first give the following definitions.

By a *bounded domain Ω* in $\boldsymbol{RP}^n$ we mean a domain Ω which avoids a neighborhood of a hyperplane H. If H is taken as the hyperplane at infinity, then $\Omega \subset \boldsymbol{R}^n = \boldsymbol{RP}^n - H$ is bounded. A bounded domain Ω is called an *open simplex* if it is the interior of a simplex in $\boldsymbol{R}^n = \boldsymbol{RP}^n - H$. In suitable homogeneous coordinates it can be described as the set of all points $(x_0, \cdots, x_n)$, where $x_i > 0$ for all i.

We shall prove

THEOREM 5. *An open simplex is the only bihomogeneous bounded domain in $\boldsymbol{RP}^n$.*

We prove two lemmas.

LEMMA 1. *Let $\Omega \subset \boldsymbol{R}P^n$ be a bihomogeneous domain and $\pi: \boldsymbol{R}^{n+1} - \{0\} \to \boldsymbol{R}P^n$ be the canonical projection. Then $\boldsymbol{R}^{n+1}$ can be given the structure of an associative algebra in such a way that $\Omega = \pi(U_0)$, where U_0 is the identity component of the group of units.*

PROOF. Let f be an identifying map from the simply connected covering $\tilde{\Omega}$ of Ω onto the simply connected covering $\tilde{\hat{G}}_0$ of $\hat{G}_0 \subset P(\tilde{\mathfrak{g}})$ (in the manner of Theorem 3). If p_2 is the projection of $\tilde{\hat{G}}_0$ onto $\hat{G}_0$, then $p_2 \circ f$ is a development (i.e. a map which is locally a projective diffeomorphism) of $\tilde{\Omega}$ into $P(\tilde{\mathfrak{g}})$, as is the projection $p_1: \tilde{\Omega} \to \boldsymbol{R}P^n$. Thus there exists a projective transformation g of $\boldsymbol{R}P^n$ onto $P(\tilde{\mathfrak{g}})$ such that $p_2 \circ f = g \circ p_1$. So we get $g(\Omega) = \hat{G}_0$.

LEMMA 2. *If a bounded domain $\Omega \subset \boldsymbol{R}P^n$ is bihomogeneous, then the corresponding associative algebra A does not contain elements x with $x^2 = -1$ nor nonzero elements x with $x^2 = 0$.*

PROOF. If $x^2 = -1$, then $\lambda e + \mu x$ with $\lambda^2 + \mu^2 = 1$ has an inverse $\lambda e - \mu x$. So $\hat{G} = U/\boldsymbol{R}^* e \subset \boldsymbol{R}P^n$ contains a whole projective line through e, which is a contradiction because $g(\Omega) = \hat{G}_0$ by Lemma 1 and Ω is bounded. If $x^2 = 0$ and $x \neq 0$, then $e + \mu x$ has an inverse $e - \mu x$ for all μ. Thus $\hat{G}_0$ contains a line minus a point, which is again a contradiction.

PROOF OF THEOREM 5. The associative algebra A corresponding to a bihomogeneous bounded domain $\Omega \subset \boldsymbol{R}P^n$ (in the manner of Theorem 3) is semisimple, because if it had a nonzero nilpotent ideal, it would have an element $x \neq 0$ such that $x^2 = 0$. By the theorem of Wedderburn and by the fact that A contains no element x with $x^2 = -1$, we conclude that A is the direct sum $\boldsymbol{R} \oplus \boldsymbol{R} \oplus \cdots \oplus \boldsymbol{R}$. The identity component in the group of units in A consists of all points $(x_0, \cdots, x_n)$, where all $x_i > 0$. We thus obtain the conclusion that Ω is an open simplex. □

6. Projectively flat manifolds with transitive translation groups. Let M be a differentiable manifold with a projective structure P. A Lie group G acting freely on M is called a *translation group* for (M, P) if every orbit of each 1-parameter subgroup of G is a geodesic. We shall determine all projectively flat manifolds (M, P) which admit transitive translation groups.

LEMMA. *Suppose G is a translation group which is transitive on a manifold M with a projective structure P. Then (M, P) is projectively diffeomorphic to (G, P_0), where P_0 is the biinvariant projective structure on the Lie group G which is determined by the* (0)*-connection of G. Here* *the* (0)*-connection of* G *is the biinvariant affine connection on* G *which is given by*

$$\nabla^0_X Y = [X, Y]/2 .$$

For the (0)-connection, see Nomizu [8]. Every 1-parameter subgroup a_t of G has orbits $a_t x$ (left action) and $x a_t$ (right action) which are geodesics.

PROOF. Choose an arbitrary point x_0 as the origin of M and define a diffeomorphism $\phi: G \to M$ by $\phi(a) = a(x_0)$ for each $a \in G$. Also choose a torsion-free affine connection ∇ which determines P. Then the pull-back $\phi^*\nabla$ and the (0)-connection ∇^0 have the same geodesics and thus they are projectively equivalent. This means that ϕ is a projective diffeomorphism between (G, P_0) and (M, P). It also follows that the given translation group G is contained in $\mathrm{Aut}(M, P)$ and that (M, P) is bihomogeneous (corresponding to biinvariance of P_0 on G). □

Now E. Cartan [2] already considered the problem of determining all Lie algebras of Lie groups whose (0)-connections are projectively flat. We shall clarify and complete his work by using a method from §3. We shall prove the following theorem which is due to Cartan, except that his statement is somewhat vague on the last three cases.

THEOREM 6. *If the* (0)*-connection on a Lie group* G *is projectively flat, then the Lie algebra* $\mathfrak{g}$ *of* G *is isomorphic to one of the following Lie algebras:*

(1) $\mathfrak{g}$ *is abelian or* 2*-step nilpotent (this is the case if and only if the* (0)*-connection is flat).*

(2a) $\mathfrak{g}$ *has an abelian ideal* $\mathfrak{k}$ *of codimension* 1 *such that* $\mathfrak{g} = (X_0) \oplus \mathfrak{k}$, *where* $[X_0, X] = X$ *for all* $X \in \mathfrak{k}$.

(2b) $\mathfrak{g}$ *has two ideals* $\mathfrak{k}^+$ *and* $\mathfrak{k}^-$ *such that* $\mathfrak{g} = (X_0) \oplus \mathfrak{k}^+ \oplus \mathfrak{k}^-$, *where*

$$\begin{aligned} &[X_0, X] = X && \textit{for all} \quad X \in \mathfrak{k}^+ \\ &[X_0, X] = -X && \textit{for all} \quad X \in \mathfrak{k}^- \\ &\mathfrak{k} = \mathfrak{k}^+ \oplus \mathfrak{k}^- && \textit{abelian} , \end{aligned}$$

and

$$0 < \dim \mathfrak{k}^- \leqq \dim \mathfrak{k}^+ < n - 1 .$$

(3) $\mathfrak{g} = (X_0) \oplus \mathfrak{k}$, *where* $\mathfrak{k}$ *is an abelian ideal with a basis* $\{X_1, \cdots, X_p, Y_1, \cdots Y_p\}$ *such that*

$$[X_0, X_i] = Y_i \quad \textit{and} \quad [X_0, Y_i] = -X_i \quad \textit{for} \quad 1 \leqq i \leqq p .$$

(4) $\mathfrak{g} = \mathfrak{sl}(2, \boldsymbol{R})$.

(5) $\mathfrak{g} = \mathfrak{o}(3)$.

Conversely, a Lie group with such a Lie algebra is projectively flat relative to the (0)*-connection.*

PROOF. We first note that the curvature tensor R of the (0)-connection is

$$R(X, Y) = -\mathrm{ad}([X, Y])/4$$

and the Ricci tensor is

$$\mathrm{Ric}(Y, Z) = -B(Y, Z)/4 ,$$

where B is the Killing-Cartan form $B(Y, Z) = \mathrm{trace}\,\mathrm{ad}(Y)\mathrm{ad}(Z)$. Thus Ric is symmetric. The flat case (1) is obvious.

Assume that the (0)-connection is projectively flat but not flat. The normalized Ricci tensor $\gamma = \mathrm{Ric}/(n-1)$ is not 0. Take X_0 such that $\gamma(X_0, X_0) = 1$ or -1. We first deal with the case $\gamma(X_0, X_0) = -1$.

By Theorem 2 we can construct the associative algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \boldsymbol{R}e$, in which we have

$$X_0^2 = \nabla_{X_0} X_0 - \gamma(X_0, X_0)e = e .$$

Let

$$\mathfrak{k} = \{X \in \mathfrak{g}; \gamma(X_0, X) = 0\} .$$

Since $\gamma = -B/4(n-1)$ is biinvariant, we have

$$\gamma([X_0, X], X_0) + \gamma(X, [X_0, X_0]) = 0$$

and hence $[X_0, X] \in \mathfrak{k}$ for every $X \in \mathfrak{g}$. In particular, we can consider the linear endomorphism θ of $\mathfrak{k}$ defined by

$$\theta(X) = [X_0, X]/2 = X_0 \cdot X .$$

Since $X_0^2 = e$ in $\tilde{\mathfrak{g}}$, we see that θ is involutive. Thus θ has eigenvalues 1 or -1 or both.

If 1 is the only eigenvalue, then we get $\mathfrak{g} = \boldsymbol{R}X_0 \oplus \mathfrak{k}$, where $[X_0, X] = 2X$ for every $X \in \mathfrak{k}$. By replacing X_0 by $X_0/2$, we get the Lie algebra described in (2a).

If -1 is the only eigenvalue, then replacing X_0 by $-X_0/2$, we get the same Lie algebra structure as above.

Now suppose that both 1 and -1 are eigenvalues of θ. Then changing X_0 to $X_0/2$ we get $\mathfrak{g} = \boldsymbol{R}X_0 \oplus \mathfrak{k}^+ \oplus \mathfrak{k}^-$, where $[X_0, X] = X$ for $X \in \mathfrak{k}^+$ and $[X_0, X] = -X$ for $X \in \mathfrak{k}^-$. Assume $X, Y \in \mathfrak{k}^+$. We show that $[X, Y] = 0$. We have

$$\begin{aligned}[X_0, [X, Y]] &= -[X, [Y, X_0]] - [Y, [X_0, X]] \\ &= -[X, -Y] - [Y, X] = 2[X, Y] .\end{aligned}$$

If we write $[X, Y] = \lambda X_0 + U + V$, with $U \in \mathfrak{k}^+$, $V \in \mathfrak{k}^-$, then

$$[X_0, [X, Y]] = [X_0, U] + [X_0, V] = U - V,$$

while the above shows that

$$[X_0, [X, Y]] = 2\lambda X_0 + 2U + 2V.$$

Hence $\lambda = 0$, $U = 0$, $V = 0$, that is, $[X, Y] = 0$ for $X, Y \in \mathfrak{k}^+$.

Similarly, $[X, Y] = 0$ for $X, Y \in \mathfrak{k}^-$.

Now let $X \in \mathfrak{k}^+$ and $Y \in \mathfrak{k}^-$. From

$$[X_0, [X, Y]] = -[X, [Y, X_0]] - [Y, [X_0, X]] = -[X, Y] - [Y, X] = 0$$

we get $[X, Y] = \lambda X_0$ for some $\lambda \in \boldsymbol{R}$. We now repeat Cartan's argument. Let $\{X_1, \cdots, X_m\}$ be a basis of $\mathfrak{k}^+$ and $\{X_{m+1}, \cdots, X_{n-1}\}$ a basis of $\mathfrak{k}^-$. From what we have seen already, we have the following information about the structure constants of the Lie algebra $\mathfrak{g}$ $(0 \leqq \sigma, \rho \leqq n-1)$:

$$[X_i, X_j] = c_{ij}^0 X_0 \quad \text{for} \quad 1 \leqq i \leqq m, \quad m+1 \leqq j \leqq n-1;$$

$$c_{0i}^i = 1, \quad c_{0i}^\sigma = 0 \quad \text{for} \quad \sigma \neq i, \quad c_{ij}^\sigma = 0 \quad \text{for all } \sigma, \text{ if} \quad 1 \leqq i, j \leqq m;$$

$$c_{0j}^j = -1, \quad c_{0j}^\sigma = 0 \quad \text{for} \quad \sigma \neq j, \quad c_{ij}^\sigma = 0$$
$$\text{for all } \sigma, \text{ if} \quad m+1 \leqq i, j \leqq n-1;$$

$$c_{ij}^\sigma = 0 \quad \text{for all} \quad \sigma \neq 0, \quad \text{if} \quad 1 \leqq i \leqq m \quad \text{and} \quad m+1 \leqq j \leqq n-1.$$

Using all this information we can compute the Killing-Cartan form B for X_i and X_j, $1 \leqq i \leqq m$, $m+1 \leqq j \leqq n-1$:

$$\begin{aligned} B(X_i, X_j) &= \operatorname{trace}(\operatorname{ad}(X_i)\operatorname{ad}(X_j)) = \sum_{\sigma,\rho} c_{i\rho}^\sigma c_{j\sigma}^\rho \\ &= c_{ij}^0 c_{j0}^j + c_{i0}^i c_{ji}^0 = c_{ij}^0 - c_{ji}^0 = 2c_{ij}^0 \end{aligned}$$

For the normalized Ricci tensor we have

$$\gamma(X_i, X_j) = -c_{ij}^0/2(n-1).$$

On the other hand

$$\begin{aligned} R(X_0, X_i)X_j &= \gamma(X_i, X_j)X_0 = -c_{ij}^0 X_0/2(n-1) \\ &= -[[X_0, X_i], X_j]/4 = -[X_i, X_j]/4 = -c_{ij}^0 X_0/4. \end{aligned}$$

Hence $c_{ij}^0/2(n-1) = c_{ij}^0/4$. If $n \neq 3$, this implies $c_{ij}^0 = 0$, that is, $[X_i, X_j] = 0$ for $1 \leqq i \leqq m$, $m+1 \leqq j \leqq n-1$. If $n = 3$, we might still have $c_{ij}^0 = 0$, but c_{ij}^0 may be a nonzero number.

When $c_{ij}^0 = 0$, we have the Lie algebra structure described in (2b), including the case $n = 3$. We may assume $\dim \mathfrak{k}^- \leqq \dim \mathfrak{k}^+$ without loss of generality.

When $n = 3$ and $c_{ij}^0 \neq 0$, this means we have a basis $\{X_0, X_1, X_2\}$ of

$\mathfrak{g}$ such that $[X_0, X_1] = X_1$, $[X_0, X_2] = -X_2$ and $[X_1, X_2] = X_0$. Or by taking $Y_1 = (X_1 + X_2)/\sqrt{2}$ and $Y_2 = (X_1 - X_2)/\sqrt{2}$, we have

$$[X_0, Y_1] = Y_2\ , \quad [Y_1, Y_2] = -X_0\ , \quad [Y_2, X_0] = Y_1\ .$$

Thus we get $\mathfrak{g} = \mathfrak{sl}(2, \boldsymbol{R})$ in (4).

We now proceed to the case where we have $\gamma(X_0, X_0) = 1$. In $\mathfrak{g}$ we have

$$X_0^2 = \nabla_{X_0} X_0 - \gamma(X_0, X_0)e = -e\ .$$

If $\mathfrak{k} = \{X \in \mathfrak{g}; \gamma(X_0, X) = 0\}$, then $\mathfrak{k}$ is invariant by the endomorphism θ induced by $u \to X_0 \cdot u$ as before. Since θ^2 is $-I$, we have a basis in $\mathfrak{k}$ of the form $\{X_1, \cdots, X_p, Y_1, \cdots, Y_p\}$ where $Y_j = \theta X_j$, $1 \leqq j \leqq p$.

We complexify $\mathfrak{g}$ and write $\mathfrak{g}^C = CX_0 \oplus \mathfrak{k}^C$

$$Z_j = X_j - iY_j \quad \text{and} \quad \bar{Z}_j = X_j + iY_j\ , \quad 1 \leqq j \leqq p\ .$$

Extending θ to $\mathfrak{k}^C$, we have $\theta(Z_j) = iZ_j$ and $\theta(\bar{Z}_j) = -i\bar{Z}_j$. We may write $\mathfrak{k}^C = \mathfrak{k}^{+i} \oplus \mathfrak{k}^{-i}$, the direct sum of the eigenspaces $\mathfrak{k}^{+i}$ for i and $\mathfrak{k}^{-i}$ for $-i$. We have $\dim \mathfrak{k}^{+i} = \dim \mathfrak{k}^{-i}$. We can now repeat the argument in the case of $\mathfrak{g} = \boldsymbol{R}X_0 + \mathfrak{k}^+ + \mathfrak{k}^-$ essentially in the same way in order to conclude that unless $n = 3$, $\mathfrak{k}^C$ is abelian, so that $\mathfrak{k}$ is abelian. We get the Lie algebra structure described in (3) by simply replacing X_0 by $X_0/2$.

If $n = 3$ and $\mathfrak{k}$ is not abelian, we have a basis $\{X_0, X_1, X_2\}$ in $\mathfrak{g}$ such that

$$[X_0, X_1] = X_2\ , \quad [X_0, X_2] = -X_1 (\text{so } [X_2, X_0] = X_1)\ , \quad [X_1, X_2] = X_0\ .$$

Thus $\mathfrak{g} = \mathfrak{o}(3)$ as in (5). This completes the proof of Theorem 5. □

REMARK. For the Lie algebras in (2a), (2b) and (3) we can determine all biinvariant affine connections and show that they are projectively flat. On the other hand, each of $\mathfrak{sl}(2, \boldsymbol{R})$ and $\mathfrak{o}(3)$ has a unique biinvariant torsion-free affine connection.

7. Geometric models for Theorem 6. We now wish to construct geometric models for all projectively flat manifolds admitting a transitive translation group. According to Theorem 3 we can proceed in the following two steps:

(1) Determine the group of units U in the associative algebra $\tilde{\mathfrak{g}}$ corresponding to each Lie algebra listed in Theorem 6 and consider the identity component $\hat{G}_0$ of its projectivization $\hat{G} = U/\boldsymbol{R}^*e$.

(2) Determine the discrete central subgroups Γ of the universal covering group $\tilde{G}$ of $\hat{G}$. Every Lie group with Lie algebra $\mathfrak{g}$ is then

obtained as $G = \widetilde{G}/\Gamma$.

We distinguish the following cases as in Theorem 6.

Case (1): $\mathfrak{g}$ is abelian or 2-step nilpotent. Since the Ricci form γ vanishes identically on $\mathfrak{g}$, the associative product $X \cdot Y$ in $\tilde{\mathfrak{g}}$ for $X, Y \in \mathfrak{g}$ is simply given by $[X, Y]/2 \in \mathfrak{g}$. An element $z = \lambda e + X$, $X \in \mathfrak{g}$, in $\tilde{\mathfrak{g}}$ is a non-unit if and only if there exists $\mu e + Y$, $Y \in \mathfrak{g}$, in $\tilde{\mathfrak{g}}$ such that

$$0 = (\lambda e + X)(\mu e + Y) = \lambda\mu e + \lambda Y + \mu X + [X, Y]/2 .$$

Considering the cases $\mu = 0$ and $\mu \neq 0$ separately we see that λ has to be 0. Conversely, $\lambda = 0$ clearly implies that z is a non-unit. Thus the units in $\tilde{\mathfrak{g}}$ are exactly the elements $z = \lambda e + X$ with $\lambda \neq 0$. The projection $\hat{G}$ of the group of units U to the projective space $P(\tilde{\mathfrak{g}})$ is the complement of a hyperplane. $\hat{G}$ is simply connected and we therefore obtain the most general Lie group with Lie algebra $\mathfrak{g}$ by passing to the quotient $G = \hat{G}/\Gamma$, where Γ is a discrete central subgroup of $\hat{G}$.

Cases (2a), (2b) and (3): $\mathfrak{g}$ is one of these in Theorem 6. Using the notation in the proof of Theorem 5 any element $z \in \tilde{\mathfrak{g}}$ can be written as $z = \lambda e + \mu X_0 + X$, with $X \in \mathfrak{k}$. A computation similar to that in Case (1) shows that z is a unit if and only if

$$\lambda^2 - \mu^2 \neq 0 \quad \text{in case (2a) and (2b)}$$

or

$$\lambda^2 + \mu^2 \neq 0 \quad \text{in case (3)} .$$

Thus the group of units in $\tilde{\mathfrak{g}}$ projects to the complement $\hat{G}^2$ of two hyperplanes in $P(\tilde{\mathfrak{g}})$ in case (2a) and (2b) and to the complement $\hat{G}^3$ of a subspace of codimension 2 is case (3).

$\hat{G}^2_0$ is the unique connected Lie group with Lie algebra $\mathfrak{g}$, because it is simply connected and has trivial center. To see that $\hat{G}^2_0$ has trivial center one can use the following matrix representation of the Lie algebra $\tilde{\mathfrak{g}}$:

$$\begin{pmatrix} \lambda+\mu & & & & & & x_1 \\ & \ddots & & & 0 & & \\ & & \lambda+\mu & & & & \vdots \\ & & & \lambda-\mu & & & \\ & 0 & & & \ddots & & \\ & & & & & \lambda-\mu & x_n \\ & & & & & & 1 \end{pmatrix}$$

Similarly, $\hat{G}^3$ has trivial center as can be seen from the following representation of the Lie algebra $\mathfrak{g}$ of case (3):

$$\begin{pmatrix} \lambda & -\mu & & & & & & x_1 & -z_1 \\ \mu & \lambda & & & & & & z_1 & x_1 \\ & & \lambda & -\mu & & & 0 & x_2 & -z_2 \\ & & \mu & \lambda & & & & z_2 & x_2 \\ & & & & \ddots & & & \vdots & \\ & & 0 & & & \lambda & -\mu & x_n & -z_n \\ & & & & & \mu & \lambda & z_n & x_n \\ & & & & & & & 1 & 0 \\ & & & & & & & 0 & 1 \end{pmatrix}$$

Thus the center of the universal covering group $\tilde{G}$ of $\hat{G}^3$ corresponds exactly to the fundamental group of $\hat{G}^3$, which is isomorphic to $\boldsymbol{Z}$. Therefore the most general group with Lie algebra $\mathfrak{g}$ in case (3) is given by an arbitrary convering of $\hat{G}^3$.

Case (4): $\mathfrak{g} = \mathfrak{sl}(2, \boldsymbol{R})$. Here $\tilde{\mathfrak{g}}$ is isomorphic to $\mathfrak{gl}(2, \boldsymbol{R})$ and $z \in \mathfrak{gl}(2, \boldsymbol{R})$ is a unit if and only if $\det z \neq 0$. The determinant is a quadratic polynomial on $\mathfrak{gl}(2, \boldsymbol{R})$ of signature $(-, -, +, +)$. Thus the image $\hat{G}$ of the group of units is the complement of a quadratic in $P(\tilde{\mathfrak{g}}) = \boldsymbol{R}P^3$.

As in case (3) we can use the fact that the identity component of $\hat{G}_0$ has trivial center to show that an arbitrary connected Lie group with Lie algebra $\mathfrak{sl}(2, \boldsymbol{R})$ is a covering of $\hat{G}_0 = PSL(2, \boldsymbol{R})$.

Case (5): $\mathfrak{g} = \mathfrak{o}(3)$. Here $\tilde{\mathfrak{g}}$ is the quaternion algebra and every nonzero element is a unit. The image of all the units in $P(\tilde{\mathfrak{g}})$ is therefore the whole of $P(\tilde{\mathfrak{g}}) = \boldsymbol{R}P^3$. The only further connected group with Lie algebra $\mathfrak{o}(3)$ is S^3.

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