Tôhoku Math. Journ.
39 (1987), 223-229.

SOME REMARKS ON MINIMAL FOLIATIONS

Dedicated to Professor Itiro Tamura on his sixtieth birthday

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(Received January 24, 1986)

1. Introduction. A foliation $\mathscr{F}$ of a Riemannian manifold (M, g) is said to be minimal if every leaf of $\mathscr{F}$ is a minimal submanifold of (M, g). Sullivan [11], Rummler [9], and Harvey and Lawson [4] gave homological characterizations of minimal foliations. In [7], the author studied a Riemannian geometric aspect of minimal foliations and proved the following: If $(M, \mathscr{F}, g)$ is a codimension-one minimal foliation of a closed Riemannian manifold with non-negative Ricci curvature, then (M, g) is locally a Riemannian product of a leaf of $\mathscr{F}$ and a one-dimensional manifold perpendicular to $\mathscr{F}$. In [2], Brito gave partial extension of this result for codimension-two minimal foliations, and in [12], Takagi and Yorozu gave many interesting examples of minimal foliations of codimension greater than one and studied related topics. In this paper, we prove the following theorem which is an extension of Theorem in [7].

THEOREM 1. *Let $(M, \mathscr{F}, g)$ be a minimal foliation of a closed connected Riemannian manifold with non-negative Ricci curvature. Assume that the bundle $\mathscr{H}$ orthogonally complement to $\mathscr{F}$ is integrable and its normal connection is flat. Then (M, g) is locally a Riemannian product of a leaf of $\mathscr{F}$ and a leaf of $\mathscr{H}$.*

The proof will be given in § 3. In § 4, we give some examples and study related topics. In particular, we strengthen a result of Brito [2].

2. Notation and preliminary results. Let (M, g) be a Riemannian manifold. Denote by D the Riemannian connection of (M, g) and by R the curvature tensor of D. We also denote $g(u, v)$ by $\langle u, v\rangle$ for $u, v \in T_xM$, $x \in M$. Let $\mathscr{F}$ be a codimension-q foliation of M. A foliation $\mathscr{F}$ of (M, g) is said to be minimal (resp. totally geodesic) if every leaf of $\mathscr{F}$ is a minimal (resp. totally geodesic) submanifold of (M, g). Hereafter we shall identify a foliation $\mathscr{F}$ with its tangent bundle. Denote by $\mathscr{H}$ the bundle orthogonally complement to $\mathscr{F}$, that is, $\mathscr{H} = \{(x, v) \in T_xM; x \in M, v \perp T_x\mathscr{F}\}$. The normal connection of $\mathscr{H}$ is said to be flat if the bundle $\mathscr{H}$ locally admits an orthonormal frame field $\{X_1, \cdots, X_q\}$, where

$q = \dim \mathscr{H} = \operatorname{codim} \mathscr{F}$, such that $\langle D_V X_a, X_b \rangle = 0$ for $a, b = 1, \cdots, q$ and $V \in \mathscr{X}(M)$. Let U be an open subset of M. An orthonormal frame field $\{E_1, \cdots, E_p, X_1, \cdots, X_q\}$, where $p = \dim \mathscr{F}$, of $TM|U$ is said to be adapted if it satisfies the following properties:

$$\{E_i\}_{i=1,\cdots,p} \text{ gives an orthonormal frame field of } \mathscr{F}|U\,,$$
$$\{X_a\}_{a=1,\cdots,q} \text{ gives an orthonormal frame field of } \mathscr{H}|U\,, \quad \text{and}$$
$$\langle D_V X_a, X_b \rangle = 0 \quad \text{for} \quad a, b = 1, \cdots, q \text{ and } V \in \mathscr{X}(U)\,.$$

Therefore, if the normal connection of $\mathscr{H}$ is flat, then for any point x of M there is an open neighborhood U of x and an adapted frame field $\{E_i, X_a\}$ on U. Note that if the normal connection of $\mathscr{H}$ is flat and $\mathscr{H}$ is integrable, then the universal covering space of each leaf of $\mathscr{H}$ is diffeomorphic to $\boldsymbol{R}^q$, where $q = \dim \mathscr{H}$.

Let $(M, \mathscr{F})$ and $(N, \mathscr{H})$ be two foliated manifolds and W be a manifold. We denote by $(M, \mathscr{F}) \times W$ the foliated manifold $(M \times W, \mathscr{F}')$ whose leaves are of the form $L \times W$ $(L \in \mathscr{F})$, and denote by $(M, \mathscr{F}) \times (N, \mathscr{H})$ the foliated manifold $(M \times N, \mathscr{F}')$ whose leaves are of the form $L \times H$ ($L \in \mathscr{F}$ and $H \in \mathscr{H}$). We also use the notation (W, pt), when we regard W as a foliated manifold with leaves consisting of points of W, that is, the point foliation.

A geodesic $c: \boldsymbol{R} \to M$ is said to be a line if any segment of c is a minimizing geodesic. A geodesic $c: [0, \infty) \to M$ is said to be a ray if any segment of c is a minimizing geodesic. For the structure of a Riemannian manifold with non-negative Ricci curvature, we have the following splitting theorem by Cheeger and Gromoll [3].

THEOREM A (Cheeger and Gromoll [3, Theorems 2 and 3]). *Let M be a closed Riemannian manifold with non-negative Ricci curvature. Then the universal covering space $\tilde{M}$ of M is the isometric product $\bar{M} \times \boldsymbol{R}^k$ where $\bar{M}$ is compact and $\boldsymbol{R}^k$ has its standard flat metric. Furthermore, if c is a line of the universal covering space $\tilde{M}$, then $\tilde{M}$ decomposes isometrically into a cross product $M' \times \boldsymbol{R}$, the second factor being represented by c.*

For the proof of our theorem, we need the following two theorems concerning totally geodesic foliations. Note that a smooth map $f: (M, g) \to (N, h)$ between Riemannian manifolds is said to be a Riemannian submersion if f is of maximal rank and f_* preserves the length of horizontal vectors, i.e., vectors orthogonal to the fiber $f^{-1}(x)$ for $x \in N$ (cf. O'Neill [6]).

THEOREM B (Blumenthal and Hebda [1]). *Let $(M, \mathscr{F}, g)$ be a totally* *geodesic foliation of a connected complete Riemannian manifold. Assume that the bundle $\mathscr{H}$ orthogonally complement to $\mathscr{F}$ is integrable. Then the universal covering space $\tilde{M}$ of M is topologically a product $L \times H$, where*

(1) *L (resp. H) is the universal covering space of the leaves of $\mathscr{F}$ (resp. $\mathscr{H}$),*

(2) *the canonical lifting $\tilde{\mathscr{F}}$ (resp. $\tilde{\mathscr{H}}$) of $\mathscr{F}$ (resp. $\mathscr{H}$) to $\tilde{M}$ is the foliation by leaves of the form $L \times \{h\}$, $h \in H$ (resp. $\{l\} \times H$, $l \in L$), and*

(3) *the projection $P: \tilde{M} \to L$ onto the first factor is a Riemannian submersion.*

THEOREM C (Oshikiri [8]). *Let (M, g) be a connected complete Riemannian manifold and $\mathscr{F}$ be a totally geodesic foliation of (M, g). Assume that the bundle orthogonally complement to $\mathscr{F}$ is also integrable. Then any Killing field Z on (M, g) with bounded length, i.e., $g(Z, Z) \leqq \text{const.} < \infty$ on M, preserves $\mathscr{F}$.*

3. Proof of Theorem 1. Let (M, g) be a Riemannian manifold as in Theorem 1. We continue to use the notation in § 2. For the proof, we may assume that the ambient manifold M and the foliation $\mathscr{F}$ are oriented.

LEMMA 1. *Let $(M, \mathscr{F}, g)$ be a minimal foliation of a connected closed Riemannian manifold with non-negative Ricci curvature. Assume that the induced connection of $\mathscr{H}$ is flat. Then the foliation $\mathscr{F}$ is totally geodesic.*

PROOF. By assumption, for each point of M there is an adapted frame field $\{E_i, X_a\}$ on a neighborhood of the point. Denote by $\mathrm{Ric}(X, X)$ the Ricci curvature in the direction of X. Set $X = X_1$. Note that $\mathrm{div}(D_X X) = \sum_{i=1}^{p} \langle D_{E_i} D_X X, E_i\rangle + \sum_{a=1}^{q} \langle D_{X_a} D_X X, X_a\rangle$, where $p = \dim \mathscr{F}$ and $q = \dim \mathscr{H}$, and $\langle X, \sum_{i=1}^{p} D_{E_i} E_i\rangle = 0$ by the minimality of $\mathscr{F}$. It follows that

$$\begin{aligned}
\mathrm{Ric}(X, X) &= \sum_{i=1}^{p} (\langle D_{E_i} D_X X, E_i\rangle - \langle D_X D_{E_i} X, E_i\rangle - \langle D_{[E_i, X]} X, E_i\rangle) \\
&\quad + \sum_{a=2}^{q} (\langle D_{X_a} D_X X, X_a\rangle - \langle D_X D_{X_a} X, X_a\rangle - \langle D_{[X_a, X]} X, X_a\rangle) \\
&= \mathrm{div}(D_X X) + \sum_{a=1}^{q} \langle D_{X_a} X, D_X X_a\rangle + X\left\langle X, \sum_{i=1}^{p} D_{E_i} E_i\right\rangle \\
&\quad + \sum_{i,j=1}^{p} 2\langle D_{E_i} X, E_j\rangle\langle D_X E_i, E_j\rangle - \sum_{i,j=1}^{p} \langle D_{E_i} X, E_j\rangle^2 \\
&\quad + \sum_{i=1}^{p} \sum_{a=1}^{q} \langle D_X E_i, X_a\rangle\langle D_{X_a} X, E_i\rangle
\end{aligned}$$

$$= \operatorname{div}(D_X X) - \sum_{i,j=1}^{p} \langle D_{E_i} X, E_j \rangle^2 .$$

Thus we have $\operatorname{div}(D_{X_a} X_a) = \operatorname{Ric}(X_a, X_a) + \sum_{i,j=1}^{p} \langle D_{E_i} X_a, E_j \rangle^2$ for $a = 1, \cdots, q$. As the normal connection is flat, the vector field $W = \sum_{a=1}^{q} D_{X_a} X_a$ is globally well-defined on M. Hence we have $\operatorname{div}(W) = \sum_{a=1}^{q} \operatorname{Ric}(X_a, X_a) + \sum_{i,j=1}^{p} \sum_{a=1}^{q} \langle D_{E_i} X_a, E_j \rangle^2 \geqq 0$ on M. As $\int_M \operatorname{div}(W) = 0$, it follows that $\langle D_{E_i} X_a, E_j \rangle = 0$ for $a = 1, \cdots, q$ and $i, j = 1, \cdots, p$, which means that $\mathscr{F}$ is totally geodesic.

REMARK. Under the hypotheses of Theorem 1, the fact that the foliation $\mathscr{F}$ is totally geodesic is a direct consequence of the main theorem in Sawada [10].

Now assume that the bundle $\mathscr{H}$ is integrable. Let $\tilde{M}$ be the universal covering space of M and $\tilde{\mathscr{F}}$ (resp. $\tilde{\mathscr{H}}$) be the canonical lifting of $\mathscr{F}$ (resp. $\mathscr{H}$) to $\tilde{M}$. Then, by Theorem B, the projection $P: \tilde{M} \to L$ is a Riemannian submersion.

LEMMA 2. *If a leaf $L \times \{h\}$, $h \in H$, of $\tilde{\mathscr{F}}$ admits a line c, then $(\tilde{M}, \tilde{\mathscr{F}})$ is the isometric product $(M', \mathscr{F}') \times \boldsymbol{R}$, where $(M', \mathscr{F}')$ is a totally geodesic foliation satisfying the hypotheses in Theorem* 1 *except the compactness of M', while $\boldsymbol{R}$ has its standard flat metric.*

PROOF. First note that c is a line of $\tilde{M}$. Indeed, as $P: \tilde{M} \to L$ is a Riemannian submersion, the curve $P \circ c$ is a line of L. If c were not a line of $\tilde{M}$, then there would be a geodesic segment $\bar{c}$ joining $c(s)$ to $c(t)$ for some $s, t \in \boldsymbol{R}$ and $\operatorname{len}(\bar{c}) < \operatorname{len}(c[s, t])$, where $\operatorname{len}(c)$ is the length of c. As P is the projection of the Riemannian submersion, it would follow that $\operatorname{len}(P \circ \bar{c}) \leqq \operatorname{len}(\bar{c})$. Thus $\operatorname{len}(P \circ \bar{c}) < \operatorname{len}(P \circ c[s, t])$ which contradicts the fact that $P \circ c$ is a line of L. By Theorem A, there is a parallel vector field X on $\tilde{M}$ with $X|_c = c'$. By Theorem C, the vector field X preserves $\tilde{\mathscr{F}}$ and $\tilde{\mathscr{H}}$. Thus X is tangent to $L \times \{h\}$ everywhere on $L \times \{h\}$ and there is a vector field Y on L with $P_* X = Y$, because the flow of X preserves the fibers of P and can be projected to the flow on L generating Y. As $|X|$ is constant, $|P_* X| \leqq |X|$ and $P_* | T(L \times \{h\})$ is an isometry, it follows that X is tangent to $\tilde{\mathscr{F}}$ everywhere on $\tilde{M}$. Now Lemma 2 follows.

LEMMA 3. *If a leaf $L \times \{h\}$ of $\tilde{\mathscr{F}}$ does not admit a line, then $L \times \{h\}$ is compact.*

PROOF. Note that all leaves of $\tilde{\mathscr{F}}$ are isometric to the manifold L by the projection $P: \tilde{M} \to L$. As $\tilde{\mathscr{F}}$ and $\tilde{\mathscr{H}}$ are canonical liftings of $\mathscr{F}$

and $\mathscr{H}$ on $\tilde{M}$, the action of $\pi_1(M)$ preserves $\tilde{\mathscr{F}}$ and $\tilde{\mathscr{H}}$. Thus it preserves the product structure $L\times H$, and the induced action of $\pi_1(M)$ to L consists of isometries of L. Suppose that $L\times\{h\}$ is not compact. Then there is a ray c of $L\times\{h\}$ starting from a point $x\in L\times\{h\}$. Let K be a compact fundamental domain for $\pi_1(M)$ which exists by the compactness of M. For each positive integer n, there is an element $g_n\in\pi_1(M)$ such that $P(g_n^{-1}(c(n)))\in P(K)$. By compactness, we find a subsequence g_{n_i} such that $P(g_{n_i}^{-1}(c(n_i)))\to y\in P(K)$ and $d(P\circ g_{n_i}^{-1})(c'(n_i))\to v\in T_yL$ as $n_i\to\infty$. If $\bar{c}:(-\infty,\infty)\to L$ is the geodesic with $\bar{c}(0)=y$ and $\bar{c}'(0)=v$, then $\bar{c}$ is easily seen to be a line of L, a contradiction.

By Lemmas 2 and 3, we may assume that the foliation $\tilde{\mathscr{F}}$ consists of compact and simply-connected leaves.

LEMMA 4. *The foliated Riemannian manifold* $(\tilde{M}, \tilde{\mathscr{F}})$ *is isometric to the foliated Riemannian product* $(\bar{M}, \bar{\mathscr{F}})\times(\boldsymbol{R}^k, \text{pt})$, *where* $\bar{M}\times\boldsymbol{R}^k$ *is as in Theorem* A.

PROOF. Let X be a parallel vector field of $\tilde{M}$ and Y be the orthogonal projection of X to a leaf L of $\tilde{\mathscr{F}}$. Then Y is a parallel vector field on L. As a leaf L is compact and simply-connected, L does not admit any non-trivial parallel vector field. Thus $Y=0$ and X is orthogonal to $\tilde{\mathscr{F}}$, and the lemma follows.

Except in Lemma 1, we do not use the assumption that the normal connection of $\mathscr{H}$ is flat. What we use below is the fact that the leaves of $\tilde{\mathscr{H}}$ are diffeomorphic to $\boldsymbol{R}^q$ (see § 2).

We now finish the proof of Theorem 1. By Lemma 4, $\bar{M}$ has a totally geodesic foliation such that the bundle orthogonally complement to it is integrable. If $\bar{M}\neq L$, then by Theorem B and the fact that $\bar{M}$ is closed and simply-connected, $\bar{M}$ is topologically a product $L'\times H'$ of compact simply-connected manifolds L' and H' with $\dim H'\geqq 2$. Thus H is homeomorphic to $H'\times\boldsymbol{R}^k$ which is not contractible, a contradiction.

4. Concluding remarks. In [2], Brito proved the following among others.

THEOREM (Brito [2]). *Let* $(M, \mathscr{F}, g)$ *be a codimension-*2 *minimal foliation of a closed connected Riemannian manifold with non-negative Ricci curvature. If the bundle* $\mathscr{H}$ *orthogonally complement to* $\mathscr{F}$ *is integrable and trivial, then* $\mathscr{F}$ *is totally geodesic.*

As a corollary to the proof of Theorem 1, we can strengthen the above theorem and, consequently, we obtain a more natural extension of

Theorem in [7] as follows.

THEOREM 2. *Let* $(M, \mathscr{F}, g)$ *be as in Brito's theorem. Then* (M, g) *is locally a Riemannian product of a leaf of* $\mathscr{F}$ *and a leaf of* $\mathscr{H}$.

Furthermore, as a corollary to the proof of Theorem 1, we obtain the following foliated splitting theorem for such totally geodesic foliations as in Theorems 1 and 2.

THEOREM 3. *Let* $(M, \mathscr{F}, g)$ *be a totally geodesic foliation of a closed connected Riemannian manifold with non-negative Ricci curvature. Assume that the bundle orthognally complement to* $\mathscr{F}$ *is integrable. Then the universal covering space* $(\tilde{M}, \tilde{g})$ *splits as a foliated Riemanninan product* $(\bar{M}, \bar{\mathscr{F}}) \times (\boldsymbol{R}^s, \mathrm{pt}) \times \boldsymbol{R}^t$, *where* $\bar{M} \times (\boldsymbol{R}^s \times \boldsymbol{R}^t)$ *is as in Theorem* A *and* $\bar{\mathscr{F}}$ *is a totally geodesic foliation of* $\bar{M}$ *consisting of compact and simply-connected leaves.*

Finally we give a few examples.

EXAMPLE 1. Let $\boldsymbol{E}^3$ be the flat Euclidean space with coordinates (x, y, z). Define $\mathscr{F}$ to be the orbits of the vector field $\sin(2\pi z)\partial/\partial x + \cos(2\pi z)\partial/\partial y$. Then $\mathscr{F}$ is a one-dimensional totally geodesic foliation of $\boldsymbol{E}^3$. As the natural action of $\boldsymbol{Z} + \boldsymbol{Z} + \boldsymbol{Z}$ on $\boldsymbol{E}^3$ preserves $\mathscr{F}$, we have a one-dimensional totally geodesic foliation of the flat torus $\boldsymbol{T}^3$. Note that the bundle $\mathscr{H}$ orthogonally complement to $\mathscr{F}$ is not integrable and that the normal connection of $\mathscr{H}$ is flat.

EXAMPLE 2 (cf. Meyer [5]). Consider a warped product $(S^2 \times S^2, g_0 \times h^2 g_0)$, where h is a non-constant positive smooth function on the first factor and g_0 is the standard metric of S^2. If h is sufficiently near the constant function 1, then the Ricci curvature is positive. Thus we get a codimension-two totally geodesic foliation on an irreducible Riemannian manifold with positive Ricci curvature. Note that the bundle orthogonally complement to this foliation is integrable. If we replace the second factor S^2 by S^3, then we get a codimension-three totally geodesic foliation on an irreducible Riemannian manifold with positive Ricci curvature. Note that the bundle orthogonally complement to this foliation is integrable and trivial. Thus, Theorem 2 cannot be extended to the cases of codimension greater than two without further assumptions.

EXAMPLE 3. We give a metric on $S^2 \times S^2$, which satisfies: (1) $\mathrm{Ric} > 0$, (2) $\mathscr{F} = \{S^2 \times (y)\}$ is a minimal foliation, but not a totally geodesic foliation, and (3) $\mathscr{F}^{\perp} = \{(x) \times S^2\}$. Let x_0 be a point of S^2. Then, there exist a neighborhood U of x_0 and an isothermal coordinate (u, v) on U

with $ds^2 = G(x)^2(du^2 + dv^2)$. Let V be a neighborhood of x_0 with $\bar{V} \subset U$ and f be a non-constant positive smooth function on S^2 with $f(x) = 1$ for $x \in S^2 - \bar{V}$. Let h be a non-constant positive smooth function on S^2. Define a Riemannian metric for $S^2 \times S^2$ by $g_0 \times g_0$ on $(S^2 - \bar{V}) \times S^2$ and $G(x)^2(du^2/k(x, y)^2 + k(x, y)^2 dv^2) \times g_0$ on $U \times S^2$, where $k(x, y) = 1 + (f(x) - 1)h(y)$. If we choose f and h sufficiently close to the constant function 1, then we have the desired metric.

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