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CONTACT HYPERSURFACES OF A COMPLEX HYPERBOLIC SPACE

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0. Introduction. A differentiable manifold is said to be contact if it admits a linear functional f on the tangent bundle satisfying $f\wedge(df)^{n-1}\neq 0$. The investigation of this as an intrinsic condition has received considerable study, (see [1]). As a real hypersurface of a complex space form is almost contact, it is natural to ask: when is a real hypersurface of a complex space form extrinsically contact?

Such investigations have been carried out successfully for real hypersurfaces of complex Euclidean spaces, [6], and of complex projective space, [4], but until now not for real hypersurfaces of complex hyperbolic space. In this study, contact hypersurfaces of a complex hyperbolic space are classified using the congruence results of [7] in terms of the examples constructed in [7]. In brief: complete connected contact hypersurfaces of $CH^n(-4)$, $n\geqq 3$, are shown to be congruent to geodesic hyperspheres, horospheres or tubes of positive radii around totally geodesic n-dimensional real hyperbolic space forms imbedded in $CH^n(-4)$.

Along the way, a related condition, originally investigated in [5], is taken care of similarly: a complete connected real hypersurface of $CH^n(-4)$ whose induced almost contact structure commutes with its second fundamental form is congruent to a horosphere or a tube of radius $r>0$ around a totally geodesic $CH^p(-4)$, $0\leqq p\leqq n-1$.

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1. Real hypersurfaces of $CH^n(-4)$. Let $CH^n(-4)$, $n\geqq 2$, denote a complex hyperbolic space with the Bergman metric tensor, i.e., a complex space form of constant holomorphic sectional curvature -4. Let M^{2n-1} be a real hypersurface of CH^n, ∇ and $\bar{\nabla}$ be the metric connections on M and CH^n, respectively, so that the Gauss and Weingarten formulae can be written as:

(1.1) $\bar{\nabla}_X Y = \nabla_X Y + \langle HX, Y\rangle\xi$ and $\bar{\nabla}_X\xi = -HX$ for all $X, Y \in T(M)$,

where ξ is a unit normal field on M in CH^n and H denotes the second fundamental form (in this case the Weingarten map of ξ in End$[T(M)]$). We shall refer to the eigenvalues and eigenvectors of H in $\boldsymbol{R}$ and $T(M)$, respectively, as principal curvatures and principal directions.

If J is the complex structure of the ambient complex space form, it induces an endomorphism ϕ of rank $2n-2$ and a linear functional f on $T(M)$ given by setting at each point p of M

$$JX = \phi X + f(X)\xi \quad \text{for all} \quad X \text{ in } T_p(M) . \tag{1.2}$$

Set $U = -J\xi$. As M is of codimension one we have $U \in T(M)$. The following equations now hold for all X, Y in $T(M)$:

$$f(X) = \langle X, U\rangle \tag{1.3}$$

$$f(\phi X) = 0 \tag{1.4}$$

$$\phi U = 0 \tag{1.5}$$

$$\phi^2 X = -X + f(X)U \tag{1.6}$$

$$\langle X, \phi Y\rangle = -\langle \phi X, Y\rangle \tag{1.7}$$

$$\langle \phi X, \phi Y\rangle = \langle X, Y\rangle - f(X)f(Y) . \tag{1.8}$$

(ϕ, f, U) is an example of what is called an almost contact structure on M. The tensor fields ϕ and U have the following derivatives:

$$\nabla_X U = \phi HX \tag{1.9}$$

$$(\nabla_X\phi)Y = f(Y)HX - \langle HX, Y\rangle U . \tag{1.10}$$

We also have the usual Gauss and Codazzi equations for a real hypersurface of a complex space form (of holomorphic sectional curvature -4) in terms of ϕ and H:

$$\begin{aligned} R(X, Y)Z = {} & \langle X, Z\rangle Y - \langle Y, Z\rangle X + \langle \phi X, Z\rangle\phi Y - \langle \phi Y, Z\rangle\phi X \\ & - 2\langle X, \phi Y\rangle\phi Z + \langle HY, Z\rangle HX - \langle HX, Z\rangle HY \end{aligned} \tag{1.11}$$

$$(\nabla_X H)Y - (\nabla_Y H)X = -f(X)\phi Y + f(Y)\phi X - 2\langle X, \phi Y\rangle U \tag{1.12}$$

for all $X, Y, Z \in T(M)$, where R is the curvature tensor on M.

All hypersurfaces of CH^n studied in the succeeding sections will have U as a principal direction on M. So we should review some general facts about such hypersurfaces set forth in [7].

Suppose that U is a principal direction on M with principal curvature

α. Then

$$2(H\phi H + \phi) = \alpha(\phi H + H\phi) \tag{1.13}$$

on M. If λ is a principal curvature on M, let D_λ denote the distribution of principal directions on M with principal curvature λ. If $X \in D_\lambda \cap \ker(f)$ and $\lambda^2 - 1 \neq 0$, then ϕX is also principal with principal curvature

$$\gamma = (\alpha\lambda - 2)/(2\lambda - \alpha)\,. \tag{1.14}$$

In case $\lambda = -1$, then $\gamma = 1$. If $\lambda = 1$, then (1.13) reduces to an identity on $\mathrm{span}\{X, \phi X, U\}$.

There are two classes of real hypersurfaces in CH^n that have U principal with all the principal curvatures constant and these are the subjects of the next two sections.

2. Contact hypersurfaces: Algebraic consequences of the contact condition. Let M be a $2n-1$ dimensional Riemannian manifold that admits a triple of tensor fields (ϕ, f, U), (where $\phi \in \mathrm{End}[T(M)]$, f is a linear functional on $T(M)$ and $U \in T(M)$), satisfying (1.3) and (1.6). As remarked in the preceding section, such a triple (ϕ, f, U) forms an *almost contact structure* on M. In general, a Riemannian manifold that admits an almost contact structure also admits a metric satisfying (1.8). From these formulae, (1.4), (1.5) and (1.7) can be obtained. This is a generalization of an intrinsic condition that can be imposed on a Riemannian manifold: M is said to be a *contact manifold* if it admits a linear functional f on $T(M)$ that is compatible with the Riemannian metric satisfying $f \wedge (df)^{n-1} \neq 0$. Such a manifold also admits an almost contact structure (ϕ, f, U), [1].

In Section 1, we saw that a real hypersurface M of CH^n (in fact of any complex space form) automatically admits an almost contact structure that is compatible with the metric induced from the ambient space. In [6], Okumura showed that if M^{2n-1} is a contact real hypersurface of a complex space form of complex dimension n, then

$$\phi H + H\phi = 2\rho\phi \tag{2.1}$$

on M, where ρ can be shown to be a constant. By selecting an appropriate orientation of M, ρ may be assumed to be positive. (In fact, (2.1) is equivalent to $\rho^{-n} f \wedge (df)^{n-1} \neq 0$). Contact metric hypersurfaces are contact hypersurface on which $\rho = 1$.

In the following discussion we shall assume that M is a complete, connected, contact hypersurface of $CH^n(-4)$, with $n \geqq 3$, and obtain algebraic consequences of (2.1). In particular, the principal curvatures and directions on M will be determined using (2.1).

Combining (1.5) and (2.1) we have $\phi HU = 0$, which shows that $HU \in \text{span}\{U\}$, i.e., U is a principal direction. Set $HU = \alpha U$. We recall some fundamental formulae for contact hypersurfaces:

LEMMA 1 (cf. [6]). *On a contact hypersurface of CH^n, $H|_{\ker(f)}$ satisfies the polynomial*

$$\lambda^2 - 2\rho\lambda + \alpha\rho - 1 = 0 . \tag{2.2}$$

LEMMA 2 (cf. [4]). *α is constant on M, if M is contact.*

From (2.2), M has at most three distinct principal curvatures, all of which must be constant by (2.2) and Lemma 2. Since $CH^n(-4)$ has no complete totally umbillic hypersurfaces (see [2] and [3]), we are left with only two cases to consider:

(A) (2.2) has two distinct solutions $\lambda_1 \neq \lambda_2$, or

(B) (2.2) has only one solution $\lambda \neq \alpha$.

Case (B) is the easiest to analyse. Let D_λ and D_α denote the eigendistributions of λ and α on M. Of course, D_λ is of dimension $2n-2$ and D_α has dimension 1. It is immediate that ϕ acts as a complex structure on D_λ. Requiring (2.2) to have only one solution forces $\lambda = \rho$ and $\rho^2 - \alpha\rho + 1 = 0$. The latter equation has real solutions only when $\alpha^2 - 4 \geqq 0$. In case $\alpha^2 - 4 > 0$, we may regard α as a parameter. By selecting the orientation of M appropriately, we may assume that $\alpha = 2\coth(2r)$ and $\lambda = \rho = \tanh(r)$ or $\coth(r)$, for some $r > 0$. Otherwise, set $\alpha = 2$ and $\lambda = \rho = 1$. So with respect to the frame consisting of principal directions $\{X_1, \cdots, X_{n-1}, \phi X_1, \cdots, \phi X_{n-1}, U)$ of $T(M)$, (where for each $i = 1, \cdots, n-1$, $HX_i = \lambda X_i$ and $H\phi X_i = \lambda\phi X_i$), H has only three possible matrix representations:

(i) $\text{diag}(\tanh(r)I_{2n-2}, 2\coth(2r))$,

(ii) $\text{diag}(\coth(r)I_{2n-2}, 2\coth(2r))$, or

(iii) $\text{diag}(I_{2n-2}, 2)$.

Notice that in each of these cases, $HX = \lambda X + \alpha f(X)U$, i.e., M is *totally U-umbillic*. These hypersurfaces also satisfy the condition $\phi H = H\phi$. In fact, it is not hard to show that a contact hypersurface is totally U-umbillic if and only if $\phi H = H\phi$. Real hypersurfaces satisfying this condition in CH^n have been classified in [5]. However, the classification for hypersurfaces satisfying (B) in this paper will involve a new geometric characterization that utilizes [7].

The analysis of Case (A) is considerably more laborious and basically new. We first notice that if $\alpha^2 - 4 < 0$, then M must satisfy (A). In the following we will show that if M satisfies (A) and $n \geqq 3$, then

$\alpha^2 - 4 < 0$.

Let D_1 and D_2 be the eigendistributions of λ_1 and λ_2, respectively. Since (2.2) has two distinct solutions $\lambda_1 \neq \lambda_2$, we have $\lambda_1 + \lambda_2 = 2\rho$. Now if $X \in D_1$, (2.1) shows that $\phi X \in D_2$. Therefore, ϕ interchanges the distributions D_1 and D_2 from which it follows that each distribution is of dimension $n - 1$.

LEMMA 3. *If M satisfies* (A) *and $n \geqq 3$, then $\lambda_1\lambda_2 = 1$.*

PROOF. This is established in a series of steps. First assume that $\alpha \neq \lambda_i$ for $i = 1, 2$.

Step 1. If X and Y are in D_i then so is $\nabla_X Y$.

Proof of Step 1. We shall prove this in the case $i = 1$. Let $X, Y \in D_1$. Then

$$\langle \nabla_X Y, U \rangle = X\langle Y, U\rangle - \langle Y, \nabla_X U\rangle = -\langle Y, \phi HX\rangle = -\lambda_1\langle Y, \phi X\rangle = 0 .$$

This shows that $\nabla_X Y \in \ker(f)$ so that $H\nabla_X Y \in \ker(f)$.

Let $Z \in \ker(f)$. Then

$$\begin{aligned}\langle H\nabla_X Y, Z\rangle &= \langle \nabla_X HY, Z\rangle - \langle (\nabla_X H)Y, Z\rangle = \lambda_1\langle \nabla_X Y, Z\rangle - \langle Y, (\nabla_X H)Z\rangle \\ &= \lambda_1\langle \nabla_X Y, Z\rangle - \langle Y, (\nabla_Z H)X - f(X)\phi Z + f(Z)\phi X - 2\langle X, \phi Z\rangle U\rangle \\ &= \lambda_1\langle \nabla_X Y, Z\rangle - \langle Y, (\nabla_Z H)X\rangle \\ &= \lambda_1\langle \nabla_X Y, Z\rangle - \langle Y, \nabla_Z HX\rangle + \langle Y, H\nabla_Z X\rangle \\ &= \lambda_1\langle \nabla_X Y, Z\rangle - \lambda_1\langle Y, \nabla_Z X\rangle + \lambda_1\langle Y, \nabla_Z X\rangle = \lambda_1\langle \nabla_X Y, Z\rangle .\end{aligned}$$

Thus, $H\nabla_X Y = \lambda_1 \nabla_X Y$ which completes the proof of Step 1.

Step 2. If $X \in D_1$ and $Y \in D_2$, then $\nabla_X Y \in D_2 \oplus \mathrm{span}\{U\}$ and $\nabla_Y X \in D_1 \oplus \mathrm{span}\{U\}$.

Proof of Step 2. Let $Z \in D_1$. Then,

$$\langle Y, Z\rangle = 0 \Rightarrow 0 = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \Rightarrow \langle \nabla_X Y, Z\rangle = 0 \quad \text{by Step 1.}$$

This yields the first inclusion. The second follows in exactly the same way by choosing $Z \in D_2$.

Step 3. If $X \in D_1$ and $Y \in D_2 \cap \{\phi X\}^\perp$, then $\nabla_X Y \in D_2$ and $\nabla_Y X \in D_1$. (Here we see the reason for the stipulation $n \geqq 3$.)

Proof of Step 3. From the hypotheses,

$$\langle \nabla_X Y, U\rangle = X\langle Y, U\rangle - \langle Y, \nabla_X U\rangle = -\langle Y, \phi HX\rangle = -\lambda_1\langle Y, \phi X\rangle = 0 .$$

Combining this with Step 2 we have Step 3.

Let $X \in D_1$ and $Y \in D_2 \cap \{\phi X\}^\perp$ with $\|X\| = \|Y\| = 1$. Applying Steps 1, 2 and 3 we find that

$$R(X, Y)Y = \nabla_X(\nabla_Y Y) - \nabla_Y(\nabla_X Y) - \nabla_{[X,Y]} Y \in D_2 \oplus \mathrm{span}\{U\}$$

by writing $[X, Y] = \nabla_X Y - \nabla_Y X$. However, a direct computation using the Gauss equation reveals that $R(X, Y)Y = (\lambda_1\lambda_2 - 1)X \in D_1$. Therefore, $R(X, Y)Y = 0$ and since X is nontrivial, we must have $\lambda_1\lambda_2 - 1 = 0$.

Now if $\alpha = \lambda_i$ for $i = 1$ or 2, the same statements hold if D_1 and D_2 are replaced by $D_1 \cap \ker(f)$ and $D_2 \cap \ker(f)$. q.e.d.

Because of (2.2) we must have $\lambda_1\lambda_2 = \alpha\rho - 1$. So by Lemma 3, $\alpha\rho = 2$ when $n \geqq 3$ in Case (A). (2.2) can now be written as $\lambda^2 - 2\rho\lambda + 1 = 0$. This has two distinct solutions only when $4\rho^2 - 4 > 0$, i.e., when $\alpha^2 - 4 < 0$. Notice that $\alpha = 0$ is ruled out by $\alpha\rho = 2$.

Hence, if M is a contact hypersurface satisfying (A), α can be viewed as a parameter. So set $\alpha = 2\tanh(2r)$, $r > 0$. Then the solutions of (2.2) are $\lambda_1 = \tanh(r)$ and $\lambda_2 = \coth(r)$. (Note that $\rho = (\tanh(r) + \coth(r))/2$ in this case.) So with respect to a suitably chosen basis of $T(M) = D_1 \oplus D_2 \oplus \mathrm{span}\{U\}$, H has the matrix representation:

$$\mathrm{diag}(\tanh(r)I_{n-1}, \coth(r)I_{n-1}, 2\tanh(2r)) .$$

Notice that all the preceding hypersurfaces are isoparametric and have the direction U as a principal direction. Hence, Theorem 2 of [7] may be applied to classify these hypersurfaces, which is the purpose of Section 4.

Remark. Okumura in [6] treats the case $\alpha = \lambda_i$ for $i = 1$ or 2 separately. But from our work so far we see that this occurs in Case (A) for a specific r, namely $r = \ln(2 + \sqrt{3})$ in which case $\alpha = \lambda_2 = \sqrt{3}$ and $\lambda_1 = 1/\sqrt{3}$.

3. Hypersurfaces of CH^n satisfying $\phi H = H\phi$. A quick glance at the classification results in [5] will convince the reader that not all hypersurfaces satisfying

$$\phi H = H\phi \tag{3.1}$$

are contact. Yet, these related hypersurfaces yield to the same algebraic techniques and fall into the same type of classification as do the contact hypersurfaces of the preceding section. So we should spend a little time on algebraic consequences of (3.1). (For more detail see [5].)

Combining (3.1) with (1.4) yields U principal; say with principal curvature α. An argument similar to that of Lemma 4 shows that α is constant. By combining (1.13) with (3.1) we have

$$(H|_{\ker(f)})^2 - \alpha H|_{\ker(f)} + I_{\ker(f)} = 0 .$$

That is, H satisfies the polynomial

(3.2) $$\lambda^2 - \alpha\lambda + 1 = 0$$

on $\ker(f)$.

This equation has real solutions only if $\alpha^2 - 4 \geqq 0$. In case $\alpha^2 - 4 > 0$ we can again regard α as a parameter and set $\alpha = 2\coth(2r)$, $r > 0$. The solutions of (3.2) are now $\lambda = \tanh(r)$ or $\coth(r)$. In case $\alpha = \pm 2$ set $\lambda = \pm 1$.

If D_λ is a proper subspace of $\ker(f)$, then (3.1) ensures that D_λ is ϕ-invariant. Since $\ker(f) = D_\lambda \oplus D_{1/\lambda}$, $D_{1/\lambda}$ is also ϕ-invariant so that ϕ acts as a complex structure on each of the even dimensional distributions D_λ and $D_{1/\lambda}$. From our analysis of Case (A) for contact hypersurfaces we see that a hypersurface satisfying (3.1) is not in general contact.

The possible matrix representations for the second fundamental form of a real hypersurface satisfying (3.1) with respect to a suitably chosen basis of $T(M) = D_\lambda \oplus D_{1/\lambda} \oplus \operatorname{span}\{U\}$ are now

(i) $\operatorname{diag}(\tanh(r)I_{2p}, \coth(r)I_{2n-2-2p}, 2\coth(2r))$, $p = 0, \cdots, n-1$, or

(ii) $\operatorname{diag}(I_{2n-2}, 2)$.

Of course, if $p = 0$ or $p = n-1$ in (i) then M is contact. (ii) is obviously contact.

4. Classifications. We invoke Theorem 2 of [7] to classify contact hypersurfaces of CH^n in terms of the families of isoparametric hypersurfaces constructed in [7]:

THEOREM 1. *Let M be a complete connected contact hypersurface of $CH^n(-4)$, $n \geqq 3$. Then M is congruent to one of the following*:

(i) *A tube of radius $r > 0$ around a totally geodesic, totally real hyperbolic space form $H^n(-1)$,*

(ii) *A tube of radius $r > 0$ around a totally geodesic complex hyperbolic space form $CH^{n-1}(-4)$,*

(iii) *A geodesic hypersphere of radius $r > 0$, or*

(iv) *A horosphere.*

PROOF. Apply Theorem 2 of [7], the matrix representation of a contact hypersurface from Section 2 and the examples in [7]. q.e.d.

We thus have:

COROLLARY. *Horospheres are the only contact metric hypersurfaces of $CH^n(-4)$.*

The same techniques can be applied to obtain a new classification of hypersurfaces satisfying (3.1):

THEOREM 2. *Let M be a complete connected real hypersurface of*

$CH^n(-4)$ that satisfies (3.1). *Then M is congruent to one of the following*:

(i) *A tube of radius $r > 0$ around a totally geodesic $CH^p(-4)$, $0 \leqq p \leqq n-1$, or*

(ii) *A horosphere.*

Only one case remains to analyse: that of a real hypersurface satisfying $\phi H = -H\phi$ (i.e., $\alpha = 2$, $\lambda_1 = 1$ and $\lambda_2 = -1$). However, I have found no examples of such hypersurfaces so I will defer that investigation.

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