Tôhoku Math. Journ.
38 (1986), 343-350.

ON POSITIVE SOLUTIONS FOR A CLASS OF ELLIPTIC BOUNDARY VALUE PROBLEMS

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(Received May 7, 1985)

In this note we study the solvability of the Dirichlet problem

$$(1)\qquad Lu = -\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j u) = \lambda f(u) \quad \text{in} \quad Q\,,$$

$$(2)\qquad u(x) = \phi(x) \quad \text{on} \quad \partial Q\,,$$

where $\lambda > 0$, Q is a bounded domain in R_n with the boundary ∂Q, ϕ is a non-negative function in $L^2(\partial Q)$ and the nonlinearity f is a bounded and positive function on $[0, \infty)$. If L is an elliptic operator, then by the maximum principle any solution of the problem (1), (2) must be positive. In recent years there has been some interest in the class of semi-linear elliptic boundary value problems with positive solutions. The question of the existence of positive solutions (1), (2) arises from the theory of nonlinear heat generation, that is, positone problem (see [1], [2] and [16] for further historical comments). In these papers it is assumed that $\phi \equiv 0$ on ∂Q and the existence of solutions is established in the space $C^{2+\alpha}(Q)$. It is well known that under appropriate assumptions on f, there exists $\lambda^* \in (0, \infty)$ such that the problem (1), (2), with $\phi \equiv 0$, has for $\lambda \in (0, \lambda^*)$ multiple solutions and has a unique solution for $\lambda > \lambda^*$. The purpose of this note is to study the question of the existence and multiplicity of positive solutions of (1), (2) with $\phi \not\equiv 0$ and $\phi \in L^2(\partial Q)$ in a suitable Sobolev space $\tilde{W}^{1,2}(Q)$ defined in Section 1. The main difficulty in solving the problem (1), (2) with $\phi \in L^2(\partial Q)$ arises from the fact that not every function in $L^2(\partial Q)$ is the trace of an element from $W^{1,2}(Q)$. The earlier results [4], [5], [6], [13] and [14] for linear and semi-linear equations justify the choice of the space $\tilde{W}^{1,2}(Q)$ and the interpretation of the boundary condition adopted in this work.

1. Existence of a solution. Let $Q \subset R_n$ be a domain with the boundary ∂Q of class C^2. In Q we consider the Dirichlet problem (1), (2).

We begin by introducing some definitions. It follows from the regularity of boundary ∂Q that there is a number $\delta_0 > 0$ such that for $\delta \in (0, \delta_0)$ the domain

$$Q_\delta = Q \cap \{x; \min_{y \in \partial Q} |x-y| > \delta\}$$

with the boundary ∂Q_δ possesses the following property: to each $x_0 \in \partial Q$ there is a unique point $x_\delta(x_0) \in \partial Q_\delta$ such that $x_\delta(x_0) = x_0 - \delta\nu(x)$, where $\nu(x_0)$ is the outward normal to ∂Q at x_0. The above relation gives a one-to-one mapping, of class C^1, of ∂Q onto ∂Q_δ. According to the Lemma 1 in [8, p. 382], the distance $r(x)$ belongs to $C^2(\bar{Q} - Q_{\delta_0})$ if δ_0 is sufficiently small. Denote by $\rho(x)$ the extension of the function $r(x)$ into Q satisfying the following properties $\rho(x) = r(x)$ for $x \in \bar{Q} - Q_{\delta_0}$, $\rho \in C^2(\bar{Q})$, $\rho(x) \geqq 3\delta_0/4$ in Q_{δ_0}, $\gamma_1^{-1} r(x) \leqq \rho(x) \leqq \gamma_1 r(x)$ in Q for some positive constant γ_1, $\partial Q_\delta = \{x; \rho(x) = \delta\}$ for $\delta \in (0, \delta_0)$ and finally $\partial Q = \{x; \rho(x) = 0\}$.

Throughout this article we make the following assumptions:

(A) There exists a positive constant γ such that

$$\gamma^{-1}|\xi|^2 \leqq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leqq \gamma|\xi|^2$$

for all $x \in Q$ and $\xi \in R_n$, moreover $a_{ij} \in C^1(\bar{Q})$ and $a_{ij} = a_{ji}$ $(i, j = 1, \cdots, n)$.

(B) f is a bounded and non-negative function on $[0, \infty)$.

A function u is said to be a weak solution of equation (1) if $u \in W^{1,2}_{\mathrm{loc}}(Q)$ and u satisfies

$$\int_Q \sum_{i,j=1}^{n} a_{ij}(x) D_j u D_i \nu dx = \lambda \int_Q f(u)\nu dx \tag{3}$$

for every $\nu \in W^{1,2}(Q)$ with compact support in Q.

Since $\phi \in L^2(\partial Q)$, Proposition 1 and Theorem 1 from [5] justify the following definition of the Dirichlet problem.

Let $\phi \in L^2(\partial Q)$. A weak solution $u \in W^{1,2}_{\mathrm{loc}}(Q)$ of (1) is a solution of the Dirichlet problem with the boundary condition (2) if

$$\lim_{\delta\to 0} \int_{\partial Q} [u(x_\delta(x)) - \phi(x)]^2 dS_x = 0 .$$

The results for linear and semi-linear equations (see [4], [5], [6], [13] and [14]) also show that we can expect a solution to belong to the Sobolev space defined by

$$\tilde{W}^{1,2}(Q) = \left\{u;\, u \in W^{1,2}_{\mathrm{loc}}(Q),\ \int_Q u^2 dx + \int_Q |Du(x)|^2 r(x)\, dx < \infty\right\}$$

equipped with the norm

$$\|u\|^2_{\tilde{W}^{1,2}} = \int_Q u(x)^2 dx + \int_Q |Du(x)|^2 r(x) dx .$$

Let $F \in L^2(Q)$ and consider the Dirichlet problem with the boundary condition (2) for the equation (1)

$$(1') \qquad Lu = F(x) \quad \text{in} \quad Q\ .$$

It follows from the proofs of Lemmas 1 and (2) in [6] that the Dirichlet problem (1′), (2) has a unique solution in $\tilde{W}^{1,2}(Q)$ satisfying the estimate

$$(4) \qquad \int_Q |Du(x)|^2 r(x)\,dx + \int_Q u(x)^2 dx \leqq C\left[\int_{\partial Q} \phi(x)^2 dS_x + \int_Q F(x)^2 dx\right]$$

for some positive constant C. Using the local boundedness of a weak solution, the estimate (4) and the Riesz representation theorem of a linear and continuous functional on $L^2(\partial Q)$ we deduce the existence of a kernel function $K(x, \cdot) \in L^2(\partial Q)$, $x \in Q$, such that a solution of (1′), (2) is given by

$$(5) \qquad u(x) = \int_{\partial Q} K(x, y)\phi(y) dS_y + \int_Q G(x, y) F(y) dy$$

for $x \in Q$, where G denotes the Green function for the operator L (see [17]).

THEOREM 1. *Suppose that $\phi \in L^2(\partial Q)$ and $\phi \geqq 0$ on ∂Q. Then for each $\lambda \geqq 0$ the Dirichlet problem* (1), (2) *admits at least one solution in* $\tilde{W}^{1,2}(Q)$.

PROOF. The proof is based on the method of the successive approximations. Let $u_0 \equiv 0$ on Q and define

$$(1\text{m}) \qquad Lu_m = \lambda f(u_{m-1}) \quad \text{in} \quad Q\ ,$$

$$(2\text{m}) \qquad u_m = \phi \quad \text{on} \quad \partial Q$$

$m = 1, 2, \cdots$. By the remarks preceding Theorem 1 for every m there exists the unique solution $u_m \in \tilde{W}^{1,2}(Q)$. In view of (4) we have

$$\int_Q |Du_m|^2 r dx + \int_Q u_m^2 dx \leqq C\left[\int_{\partial Q} \phi^2 dS_x + \lambda^2 |Q| \cdot K^2\right],$$

where $K = \sup_{t \geqq 0} f(t)$ and $|Q|$ denotes the measure of Q. Consequently there exists a subsequence denoted again by $\{u_m\}$, converging weakly in $\tilde{W}^{1,2}(Q)$ to a function u. By virtue of Theorem 4.11 in [12], $\tilde{W}^{1,2}(Q)$ is compactly embedded in $L^2(Q)$ and consequently we may assume that u_m converges to u in $L^2(Q)$ and a.e. on Q. It is obvious that u is a weak solution of (1). Since $\int_Q |Du(x)|^2 r(x) dx < \infty$, Proposition 1 and Theorem 1 in [5] imply the existence of a function $\zeta \in L^2(\partial Q)$ such that

$$\lim_{\delta \to 0} \int_{\partial Q} [u(x_\delta(x)) - \zeta(x)]^2 dS_x = 0\ .$$

It therefore remains to prove that $\zeta = \phi$ a.e. on ∂Q and this we accomplish by taking

$$v(x) = \begin{cases} \Psi(x)(\rho(x) - \delta) & \text{for} \quad x \in Q_\delta , \\ 0 & \text{for} \quad x \in Q - Q_\delta , \end{cases}$$

as a test function in (3), where Ψ is an arbitrary function in $C^1(\bar{Q})$. Applying Green's theorem and letting $\delta \to 0$ we obtain

$$\int_{\partial Q} \phi_m \Psi \sum_{i,j=1}^n a_{ij} D_i \rho D_j \rho dS_x$$
$$= \int_Q \Big[- \sum_{i,j=1}^n D_j(a_{ij} \Psi D_i \rho) u_m + \sum_{i,j=1}^n a_{ij} D_j u_m D_i \Psi \rho - \lambda f(u_{m-1}) \Psi \rho \Big] dx .$$

Similarly

$$\int_{\partial Q} \zeta \Psi \sum_{i,j=1}^n a_{ij} D_i \rho D_j \rho dS_x$$
$$= \int_Q \Big[- \sum_{i,j=1}^n D_j(a_{ij} \Psi D_i \rho) u dx + \sum_{i,j=1}^n a_{ij} D_j u D_i \Psi \rho - \lambda f(u) \Psi \rho \Big] dx .$$

Letting $m \to \infty$ we deduce from the last two equations that

$$\int_{\partial Q} \zeta \Psi \sum_{i,j=1}^n a_{ij} D_i \rho D_j \rho dS_x = \int_{\partial Q} \phi \Psi \sum_{i,j=1}^n a_{ij} D_i \rho D_j \rho dS_x$$

for every $\Psi \in C^1(\bar{Q})$ and this completes the proof.

2. Uniqueness of solutions. We commence with the following result.

THEOREM 2. *Suppose that f is a C^1-function on $[0, \infty)$ such that $|f'(t)| \leqq F(t)$ for $t \geqq 0$, where F is a decreasing function on $[0, \infty)$ with $\lim_{t\to\infty} F(t) = 0$. For each $\lambda > 0$ there exists a number $M > 0$ such that, if $\phi \in L^2(\partial Q)$ and $\phi \geqq M$ on ∂Q a.e., then the Dirichlet problem* (1), (2) *has a unique solution $\tilde{W}^{1,2}(Q)$.*

PROOF. It obviously suffices to prove the uniqueness. Let us assume that there exist two distinct solutions u_1 and u_2 of (1), (2). Let $w = u_1 - u_2$. Since $\phi \geqq M$ on ∂Q, it follows from (5) that $u_m(x) \geqq M$ a.e. on Q, $i = 1, 2$ and that

$$w(x) = \lambda \int_Q G(x, y)[f(u_1(y) - f(u_2(y))]dy . \tag{6}$$

Writing

$$w(x) = \lambda \int_Q G(x, y) \int_0^1 f'(su_1 + (1 - s)u_2) ds w(y) dy ,$$

we then have

$$\|w\|_\infty \leqq \lambda \sup_{y \in Q} \int_Q G(x, y) dx F(M) \|w\|_\infty .$$

If $w \not\equiv 0$ then $\lambda \sup_{y \in Q} \int_Q G(x, y)dxF(M) \geqq 1$ for all λ and M in $(0, \infty)$. Given $\lambda > 0$ we can find M sufficiently large so that $\lambda \sup_{y \in Q} \int_Q G(x, y)dxF(M) < 1$ and we get a contradiction.

The following theorem is a modification of the result due to Schuchman [15] (see also [18]):

THEOREM 3. *Let $\phi \in L^2(\partial Q)$ and $\phi \geqq 0$ a.e. on ∂Q. Suppose that f is a C^1-function on $[0, \infty)$ such that $f(0) > 0$ and $0 \leqq f'(t) \leqq K(1 + t)^{-1-\alpha}$, where K and α are positive constants. Then there exists a $\lambda_0 > 0$ such that the problem* (1), (2) *for $\lambda \geqq \lambda_0$ has a unique solution in $\tilde{W}^{1,2}(Q)$.*

PROOF. We follow the argument from the paper [15]. Without loss of generality we may assume that $f(0) \geqq 1$. Letting $w = u_1 - u_2$ we obtain as in the proof of Theorem 2 that

$$w(x) = \int_Q G(x, y) \int_0^1 f'(su_1 + (1 - s)u_2)dsw(y)dy .$$

Moreover it is clear from the representation (6) that $w \in L^\infty(Q)$. Let e be the solution of the Dirichlet problem

$$Le = 1 \quad \text{in} \quad Q , \tag{7}$$

$$e = 0 \quad \text{on} \quad \partial Q . \tag{8}$$

By the maximum principle $e > 0$ on Q. Since

$$\begin{aligned} u_i(x) &= \int_{\partial Q} K(x, y)\phi_i(y)dy + \lambda \int G(x, y)f(u_i)dy \geqq \lambda \int_Q G(x, y)f(0)dy \\ &\geqq \lambda \int_Q G(x, y)dy = \lambda e , \end{aligned}$$

we have

$$\|w\|_\infty \leqq \lambda K \sup_{x \in Q} \int_Q G(x, y)[1 + \lambda e(y)]^{-1-\alpha}dy\|w\|_\infty . \tag{9}$$

Let

$$F(\lambda) = \lambda K \sup_{x \in Q} \int_Q G(x, y)[1 + \lambda e(y)]^{-1-\alpha}dy .$$

It follows from (9) that $F(\lambda) \geqq 1$ if $w \not\equiv 0$. Now observe that Schuchman [15] proved that $\lim_{\lambda \to \infty} F(\lambda) = 0$ and this contradiction completes the proof of the uniqueness assertion.

3. Existence of multiple solutions. In this section we impose the following condition on f: there exists $c > 0$ such that f is increasing on $[0, c)$.

We obtain the existence of multiple solutions by using sub- and supersolution method.

Let $0 < l_1 < l_2 < c$. In the sequel we construct a supersolution V and a subsolution V_1 such that

$$\|V\|_\infty = l_1 < l_2 \leqq \|V_1\|_\infty \leqq c\,.$$

First we construct a supersolution. We follow the method from [1, p. 483–484]. Let e be a solution of the problem (7), (8) and put

$$V(x) = \frac{e(x)+1}{\|e\|_\infty + 1} l_1\,.$$

Then $L(V) = l_1/(\|e\|_\infty + 1)$ and choose λ so that

$$\lambda \leqq \frac{l_1}{f(l_1)} M_1\,, \tag{10}$$

where $M_1 = 1/(\|e\|_\infty + 1)$. Since f is increasing it follows from (10) that

$$L(V) = \frac{l_1}{\|e\|_\infty + 1} = \frac{l_1 f(l_1)}{(\|e\|_\infty + 1) f(l_1)} \geqq \lambda f(l_1) \geqq \lambda f\Big(\frac{e+1}{\|e\|_\infty + 1}\Big) = \lambda f(V)\,,$$

that is, V is a supersolution.

To construct a subsolution we choose an open subset $\Omega \subset Q$ with $\operatorname{dist}(\bar{\Omega}, \partial Q) > 0$ and put

$$M_2 = \Big[\min_{x\in\Omega} \int_\Omega G(x, y)dy\Big]^{-1}\,,$$

$$M_3 = \Big[\max_{x\in Q} \int_Q G(x, y)dy\Big]^{-1}$$

and

$$V_1(x) = \lambda \int_Q G(x, y) f(l_2 X_\Omega(y))dy\,,$$

where X_Ω is a characteristic function of Ω. It follows from [1, p. 484] that if

$$M_2 \frac{l_2}{f(l_2)} \leqq \lambda \leqq M_3 \frac{c}{f(l_2)} \tag{11}$$

then $LV_1 \leqq \lambda f(V_1)$ and $l_2 \leqq \|V_1\|_\infty \leqq c$, that is, V_1 is a subsolution.

We are now in a position to establish the existence of multiple solutions.

Theorem 4. *Suppose that $\phi \in L^\infty(\partial Q)$, $\phi \geqq 0$ a.e. on ∂Q and that $\|\phi\|_\infty < l_1/(\|e\|_\infty + 1)$. Let M_1, M_2 and M_3 be defined as above. Then for each*

$$M_2\frac{l_2}{f(l_2)} \leqq \lambda \leqq \min\left(M_1\frac{l_1}{f(l_1)},\ M_3\frac{c}{f(l_3)}\right) \tag{12}$$

the problem (1), (2) *has at least two distinct solutions.*

PROOF. Let ϕ_m be a sequence of $C^1(\partial Q)$-functions such that

$$0 \leqq \phi_m(x) \leqq \frac{l_1}{\|e\|_\infty + 1} \quad \text{on} \quad \partial Q \quad \text{and} \quad \lim_{m\to\infty}\int_{\partial Q}[\phi_m(x) - \phi(x)]^2 dS_x = 0 .$$

We now observe that $W \equiv 0$ and V are a subsolution and supersolution, respectively, for the Dirichlet problem in $W^{1,2}(Q)$ for (1) with the boundary condition

$$u(x) = \phi_m \quad \text{on} \quad \partial Q . \tag{2m}$$

It follows from [7] that for each m there exists a solution u_m in $W^{1,2}(Q)$ of the problem (1), (2m) satisfying the estimate

$$0 \leqq u_m(x) \leqq V(x) \quad \text{on} \quad Q$$

for each m. It is clear that $u_m \in C(\bar{Q})$ for each m. By virtue of (4) we have the estimate

$$\int_Q |Du_m(x)|^2 r(x)\, dx + \int_Q u_m(x)^2 dx \leqq C\left[\sup_{m\geqq 1}\int_{\partial Q}\phi_m(x)^2 dS_x + |Q|K^2\lambda^2\right]$$

for each m. Repeating the argument of the proof of Theorem 1 we show that there exists a subsequence of $\{u_m\}$ converging weakly in $\tilde{W}^{1,2}(Q)$ to a solution u of (1), (2). To construct a second solution we consider a solution V_2 of the problem

$$LV_2 = \lambda K \quad \text{in} \quad Q ,$$
$$V_2 = \frac{l_1}{\|e\|_\infty + 1} \quad \text{on} \quad \partial Q .$$

Since $f(t) \leqq K$ on $[0, \infty)$, V_2 is a supersolution. On the other hand

$$L(V_2 - V_1) = \lambda(K - f(V_1)) \geqq 0 \quad \text{on} \quad Q ,$$
$$V_2 - V_1 = \frac{l_1}{\|e\|_\infty + 1} \quad \text{on} \quad \partial Q$$

and by the maximum principle $V_2 \geqq V_1$ on Q. Applying now the result of [7] to the Dirichlet problem (1), (2m) with V_1 and V_2 as a subsolution and a supersolution, respectively, we obtain a subsequence converging weakly in $\tilde{W}^{1,2}(Q)$ to a solution v of (1), (2). Finally the inequalities

$$0 \leqq u(x) \leqq V(x) \leqq V_1(x) \leqq v(x) \leqq V_2(x) \quad \text{on} \quad Q ,$$
$$l_1 = \|V\|_\infty < l_2 \leqq \|V_1\|_\infty$$

show that the solutions u and v are different.

Examples of functions f showing that the condition (12) is non-vacuous can be found in [1].

Finally we point out that the boundary condition (2) can also be expressed in terms of the non-tangential limit (see [2], [6] and [10]).

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