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ON NORMAL SUBGROUPS OF CHEVALLEY GROUPS OVER COMMUTATIVE RINGS

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1. Introduction. Let G be an almost simple Chevalley-Demazure group scheme with root system Φ (see, for example [1], [2], [6], [7], [8], [10], [17], [19], [20], [21], [24]). For any commutative ring R with 1, let $E(R)$ denote the subgroup of $G(R)$ generated by all elementary unipotent (root elements) $x_\varphi(r)$ with φ in Φ and r in R. Here is an example: $G = SL_n$, $G(R) = SL_nR$, $E(R) = E_nR$, $\Phi = A_{n-1}$.

As in [1], [2], we are interested in normal subgroups of $G(R)$. More precisely, we want to describe all subgroups of $G(R)$ which are normalized by $E(R)$.

The case when the rank of G is 1, i.e. G is of type A_1, i.e. G is isogenous to $SL_2 = Sp_2$, is known to be exceptional (see, for example, [9]). So for the rest of this paper we assume that the rank of G is at least 2.

When R is a field, it is known [21] that every non-central subgroup of $G(R)$ normalized by $E(R)$ contains $E(R)$, unless G is of type C_2 or G_2 and R consists of two elements. In particular, with these exceptions, $E(R)$ modulo its center is a simple (abstract) group.

When R is not a field, there are normal subgroups of $G(R)$ involving (proper) ideals J of R. For every ideal J of R we define $G(R, J)$ to be the inverse image of the center of $G(R/J)$ under the canonical homomorphism $G(R) \to G(R/J)$. The kernel of this homomorphism, i.e. the congruence subgroup of level J, is denoted by $G(J)$. Let $E(J)$ denote the subgroup of $E(R) \cap G(J)$ generated by all $x_\varphi(u)$ with φ in Φ and u in J. Let $E(R, J)$ be the normal subgroup of $E(R)$ generated by $E(J)$.

THEOREM 1. *For any ideal J of R, the subgroup $E(R, J)$ of $G(R)$ is normal, and it contains the mixed commutator subgroup* $[E(R), G(J)]$.

When $G = SL_n$, Sp_{2n}, or SO_{2n}, this statement was proved: by Klingenberg [14, 15, 16] for local rings R; by Bass [4] and Bak [3] under stable range or similar dimensional conditions on R, by Suslin [22], Kopeiko [18], and Suslin-Kopeiko [23] for any commutative R.

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The approach of [22], [18], [23] is based on [27, proof of Lemma 6.1 and Remark after Lemma 9.6]. A different approach, namely, localization and patching, was used in [27, Lemma 3.4] for a partial solution of Serre's problem on projective modules over polynomial rings and then by Suslin and Quillen for a complete solution of the problem, then in [22], [18], [23] for a similar stabilization problem at K_1-level, then in [25] for a description of normal subgroups of GL_nR, then by Taddei [24] to prove our statement in the case $J = R$ (i.e. that $E(R)$ is normal in $G(R)$). We use Taddei's result to obtain Theorem 1 for any J (see Section 2 below).

THEOREM 2. *For any ideal J of R, the group $E(R, J)$ is generated by elements of the form $x_\varphi(r)x_{-\varphi}(u)x_\varphi(-r)$ with φ in Φ, r in R, and u in J.*

THEOREM 3. *When G is of type B_2 or G_2, we assume that R has no factor rings of two elements. Then*

$$E(R, J) = [E(R), E(J)] = [E(R), G(R, J)]$$

for any ideal J of R. In particular, every subgroup of $G(R, J)$ containing $E(R, J)$ is normalized by $E(R)$.

Note that when R has a factor ring of two elements and G is of type $B_2 = C_2$ or G_2, then $E(R) \neq [E(R), E(R)]$ (see, for example, [7] or [21]).

Let now $e(\Phi)$ denote the ratio of the scalar squares of long and short roots in Φ. So $e(\Phi) = 1$ when $\Phi = A_n$, D_n, or E_n; $e(\Phi) = 2$ when $\Phi = B_n$, C_n, or F_4; $e(\Phi) = 3$ when $\Phi = G_2$.

THEOREM 4. *Under the condition of Theorem 3, assume additionally that for every z in R there are r, s in R such that $z = e(\Phi)rz + sz^{e(\Phi)}$ (for example, $e(\Phi)R = R$). Then:*

(a) *for every z in R and φ in Φ, the normal subgroup of $E(R)$ generated by $x_\varphi(z)$ coincides with $E(R, Rz)$;*

(b) *for any subgroup H of $G(R)$ which is normalized by $E(R)$ there is an ideal J of R such that $E(R, J) \subset H \subset G(R, J)$.*

When $G = SL_n$, Sp_{2n}, or SO_{2n}, this statement was proved: in [11], [12] for fields R; in [14, 15, 16] for local rings R; in [4] and [3] under stable range and similar conditions. The case $G = SL_n$ with any commutative R was done by Golubchik (see [25] for reference and another proof). Partial results for any Chevalley group G were obtained in [1], [2].

Note that when the additional condition of Theorem 3 does not hold, there are subgroups of $G(R)$ which are normalized by $E(R)$, but do not satisfy the ladder condition $E(R, J) \subset H \subset G(R, J)$ for any ideal J of R. Still it is possible to obtain a description of those H's using subgroups of $G(R)$ involving "special submodules associated with (G, J)" in the sense of [1]. This was done in [1], [2] under restrictions on R which, I believe, can be removed.

Any information on normal subgroup structure of groups $G(R)$ can be useful to describe automorphisms and homomorphisms of these groups. In this connection, we prove in Section 7 below the following theorem.

THEOREM 5. *Under the conditions of Theorem* 3, $E(R)$ *is a perfect characteristic subgroup of any larger subgroup of* $G(R)$.

2. Proof of Theorem 1.

Case 1. $J = R$. Then our statement was proved by Taddei [24].

General case. Let h be in $G(R)$ and g in $E(R, J)$. We consider the ring $R' := \{(r, s) \in R \times R: r - s \in J)\}$, its ideal $J' := (J, O)$, $h' := (h, h) \in G(R') \subset G(R) \times G(R)$, and $g' := (g, 1) \in E(R') \cap G(J') = E(R', J')$. The last equality holds, because R' is the semidirect product of its subring $\{(r, r): r \in R\}$ (which is isomorphic to R) and its ideal J' (which is isomorphic to J). Namely, let

$$g = \prod_{i=1}^{n} x_{\varphi_i}(t_i) \in E(R') \cap G(J')$$

with all t_i in R'. We express $t_i = s_i + u_i$ with $s_i = (r_i, r_i)$ in R' and u_i in J'. Set

$$h_k = \prod_{i=1}^{k} x_{\varphi_i}(s_i) \in E(R')$$

for $0 \leqq k \leqq n$. Then $h_0 = 1$ (by the definition), $h_n = 1$ (because $g \in G(J')$), and

$$g = \prod_{i=1}^{n} x_{\varphi_i}(s_i) x_{\varphi_i}(u_i) = \prod_{i=1}^{n} h_{i-1}^{-1} h_i x_{\varphi_i}(u_i) = \prod_{i=1}^{n} h_i x_{\varphi_i}(u_i) h_i^{-1} \in E(R', J') .$$

By Case 1 (applied to R' instead of R), $h'g'h'^{-1} \in E(R')$. On the other hand, evidently, $h'g'h'^{-1} = (hgh^{-1}, 1) \in G(J')$. So $h'g'h'^{-1} \in G(J') \cap E(R') = E(R', J')$, hence $hgh^{-1} \in E(R, J)$.

Thus, $E(R, J)$ is normal in $G(R)$.

Take now any h in $E(R)$ and g in $G(J)$. Define, as before, $h' = (h, h) \in E(R')$ and $g' = (g, 1) \in G(J')$. Then $[h', g'] \in E(R') \cap G(J') = E(R', J')$ by Case 1, hence $[h, g] \in E(R, J)$.

Thus, $E(R, J) \supset [E(R), G(J)]$.

3. Proof of Theorem 2. Let H be the subgroup of $E(R, J)$ generated by all $x_\varphi(r) x_{-\varphi}(u) x_\varphi(-r)$ with φ in Φ, r in R, and u in J. We want to prove that $H = E(R, J)$, i.e. that H is normalized by $E(R)$, i.e. that

$$g = x_\gamma(s)x_\varphi(r)x_{-\varphi}(u)x_\varphi(-r)x_\gamma(-s) \in H$$

for all φ, γ in Φ, r and s in R, and u in J. The case when $\gamma = \varphi$ is trivial, so we assume that $\gamma \neq \varphi$.

By [13], we can assume that $\gamma = -\varphi$. Indeed, if $\gamma \neq -\varphi$, then we have the commutator formula

$$[x_\varphi(-r), x_\gamma(s)] = \prod x_{i\varphi+j\gamma}(c_{i,j}r^i s^j) ,$$

where the product is taken over all natural numbers $i, j \geqq 1$ such that $i\varphi + j\gamma \in \Phi$ and $c_{i,j}$ are integers (which depend on φ, γ and the order in the product; and the signs of $c_{i,j}$ depend also on our choice of parametrizations x_α of root subgroups). Since no convex combination of $-\varphi, \gamma$ and the roots $i\varphi + j\gamma$ is 0, we have

$$g' := x_\varphi(-r)x_\gamma(s)x_\varphi(r)x_{-\varphi}(u)x_\varphi(-r)x_\gamma(-s)x_\varphi(r) \in E(J) ,$$

hence $g = x_\varphi(r)g'x_\varphi(-r) \in H$.

So let now $\gamma = -\varphi$, hence

$$g = x_{-\varphi}(s)x_\varphi(r)x_{-\varphi}(u)x_\varphi(-r)x_{-\varphi}(-s) .$$

We pick a connected subsystem $\Phi' \subset \Phi$ of rank 2 containing φ.

Case 1. $\Phi' = A_2$. Then $\psi - \varphi \in \Phi'$ for some ψ in Φ', hence $x_{-\varphi}(u) = [x_{-\psi}(u), x_{\psi-\varphi}(\pm 1)]$ and

$$\begin{aligned} g &= x_{-\varphi}(s)[x_{\varphi-\psi}(\pm ru)x_{-\psi}(u), x_\psi(\pm r)x_{\psi-\varphi}(\pm 1)]x_{-\varphi}(-s) \\ &= [x_{-\psi}(\pm rsu + u)x_{\varphi-\psi}(\pm ru), x_{\psi-\varphi}(\pm 1 \pm rs)x_\psi(\pm r)] \in E(A, J) \end{aligned}$$

(using, for example, the case $\gamma \neq -\varphi$ above).

For the remaining cases (namely, B_2 and G_2) we give a general argument (which works also for A_2) due to the referee rather than the original case by case computations which are almost as complicated for G_2 as in the general case.

We want to prove that the element g above belongs to the subgroup H of $E(R, J)$ defined above.

Let β in Φ' be such that (φ, β) is a base (fundamental system) of Φ'. Let Φ'_+ be the set of positive roots of Φ' with respect to the base, $\Phi'_- = \Phi'_+$, $\Phi''_+ = \{i\varphi + j\beta \in \Phi'_+ : j > 0\}$, $\Phi''_- = -\Phi''_+$, $U''_+(J)$ (resp. $U''_-(J)$) the subgroup of $E(R)$ generated by $x_\varphi(J)$ with φ in Φ''_+ (resp. in Φ''_-). Then $U''_+(J)$ and $U''_-(J)$ are subgroups of H.

Every element h of $U''_-(J)$ can be expressed uniquely as

$$h = x_{-a_1}(u_1)x_{-a_2}(u_2) \cdots x_{-a_n}(u_n)$$

with a_i in Φ''_+ and u_i in J. By induction on n, we can see that $[U''_-(J), U''_+(R)] \in H$. On the other hand, we have

$$x_{-\varphi}(u) = [x_{-(\varphi+\beta)}(u), x_\beta(\pm 1)]h' \text{ with } h' \text{ in } U''_-(J) ,$$
$$g_1 := x_{-\varphi}(s)x_\varphi(r)x_{-(\varphi+\beta)}(u)x_\varphi(-r)x_{-\varphi}(-s) \in U''_-(J) ,$$
$$g_2 := x_{-\varphi}(s)x_\varphi(r)x_\beta(\pm 1)x_\varphi(-r)x_{-\varphi}(-s) \in U''_+(R) ,$$
$$g_3 := x_{-\varphi}(s)x_\varphi(r)h'x_\varphi(-r)x_{-\varphi}(-s) \in U''_-(J) .$$

Therefore we conclude that $g = [g_1, g_2]g_3 \in H$.

4. **Proof of Theorem 3.** Let $\varphi \in \Phi$ and $u \in J$. We want to prove that $x_\varphi(u) \in [E(R), E(J)] =: H$. We include φ to a connected subsystem $\Phi' \subset \Phi$ of rank 2.

Case 1. $\Phi' = A_2$. Then we pick a root ψ in Φ' such that $\varphi + \psi \in \Phi'$ (i.e. φ and ψ make angle 120°; there are two such ψ). We have

$$x_\varphi(\pm u) = [x_{\varphi+\psi}(1), x_{-\psi}(u)] \in H ,$$

hence $x_\varphi(u) \in H$.

Case 2. $\Phi' = B_2 = \Phi$ and φ is long. Let ψ be a short root which makes angle 45° with φ (there are two of them). Then $y(r, s) := [x_\psi(r), x_{\varphi-2\psi}(su)] = x_{\varphi-\psi}(\pm rsu)x_\varphi(\pm r^2 su) \in H$ for all r, s in R, hence

$$y(r, s)y(1, rs)^{-1} = x_\varphi(\pm(r^2 - r)su) \in H .$$

By the condition of Theorem 3 in the case $\Phi = B_2$, 1 is the sum of elements of the form $(r^2 - r)s$ with r, s in R. So $x_\varphi(u) \in H$.

Case 3. $\Phi' = B_2$ and $H \supset x_\psi(J)$ for some ψ in Φ'. If φ and ψ make angle 45°, then we have

$$x_\varphi(\pm u)x_\psi(\pm u) = \begin{cases} [x_{\psi-\varphi}(1), x_{2\varphi-\psi}(u)] & \text{if } \psi \text{ is long} , \\ [x_{\varphi-\psi}(1), x_{2\psi-\varphi}(u)] & \text{if } \psi \text{ is short} , \end{cases}$$

hence $x_\varphi(u) \in H$.

In general, the angle between φ and ψ is $45°m$ with $m = 0, 1, 2, 3$, or 4. The case $m = 0$ is trivial, and the case $m = 1$ has been dealt with. When $m = 2, 3$, or 4, we find roots $\alpha(1), \cdots, \alpha(m)$ in Φ' such that $\alpha(1) = \psi$, $\alpha(m) = \varphi$, and $\alpha(i)$, $\alpha(i+1)$ make angle 45° for $i = 1, \cdots, m-1$. Then, as above, $x_{\alpha(i)}(J) \subset H$ for $i = 1, \cdots, m$.

Case 4. $\Phi' = B_2 = \Phi$. When φ is long, we are done by Case 2. When φ is short we done by Cases 2 and 3.

Case 5. $\Phi' = B_2 \neq \Phi$. Then there is a sequence $\alpha(1), \cdots, \alpha(m)$ of roots in Φ such that $\alpha(1)$ belong to a subsystem of type A_2, $\alpha(m) = \varphi$, and $\alpha(i)$, $\alpha(i+1)$ belong to a subsystem of type A_2 or B_2 for $i = 1, \cdots, m-1$. By Case 1 and Case 3, $x_{\alpha(i)}(J) \subset H$ for $i = 1, \cdots, m$.

Case 6. $\Phi' = G_2$ and φ is long. Then φ belongs to a subsystem of type A_2, so we are done by Case 1.

Case 7. $\Phi' = G_2$ and φ is short. Pick a root ψ in Φ' which makes angle 60° with φ. Then

$$H \ni [x_{\varphi-2\psi}(su), x_\psi(r)] = x_{\varphi-\psi}(\pm sur)x_\varphi(\pm sur^2)x_{\varphi+\psi}(\pm sur^3)x_{2\varphi-\psi}(\pm s^2u^2r^2)$$

hence (using Case 6) $H \ni y(r, s) := x_{\varphi-\psi}(\pm sur)x_\varphi(\pm sur^2)$. So

$$H \ni y(1, rs)^{-1}y(r, s) = x(\pm us(r^2 - r)) .$$

By the assumption of Theorem 3 in the case $\Phi = G_2$, we conclude that $x_\varphi(u) \in H$.

Thus, $H = [E(R), E(J)] \supset E(R, J)$ in all cases.

Using Theorem 1, we conclude that

$$E(R, J) = [E(R), E(J)] = [E(R), G(J)] = [G(R), E(R, J)] = [G(R), E(J)] .$$

Therefore only the inclusion $E(R, J) \supset [E(R), G(R, J)]$ is left to prove. We fix an arbitrary g in $G(R, J)$. For each h in $E(R)$ we set

$$F(h) := [h, g]E(R, J) \in (E(R) \cap G(J))/E(R, J) .$$

Then $h \mapsto F(h)$ is a homomorphism from the perfect group $E(R)$ to a commutative group. So F is trivial, i.e. $[h, g] \in E(R, J)$ for all h in $E(R)$. Thus, $E(R, J) \supset [E(R), G(R, J)]$.

5. **Proof of Theorem** 4(**a**). Let H be the normal subgroup of $E(R)$ generated by $x_\varphi(z)$. We have to prove that $H \supset x_\psi(Rz)$ for every ψ in Φ. We include φ and ψ to a connected subsystem $\Phi' \subset \Phi$ of rank 2.

Case 1. $\Phi' = A_2$ and the angle between φ and ψ is 60°. Then $H \ni [x_\varphi(z), x_{\psi-\varphi}(r)] = x_\psi(\pm zr)$ for all r in R, so $H \supset x_\psi(Rz)$.

Case 2. $\Phi' = A_2$. We find a sequence $\alpha(1), \cdots, \alpha(m)$ in Φ' such that $2 \leqq m \leqq 6$, $\alpha(1) = \varphi$, $\alpha(m) = \psi$, and $\alpha(i)$, $\alpha(i + 1)$ make angle 60° for $i = 1, \cdots, m - 1$. Then, by Case 1, $x_{\alpha(i)}(Rz) \subset H$ for $i = 2, \cdots, m$.

Case 3. $\Phi' = \Phi = B_2$, φ is short, and ψ makes 45° angle with φ. Then $H \ni [x_\varphi(z), x_{\psi-2\varphi}(r)] = x_\psi(\pm 2rz)$ for all r in R, hence $H \supset x_\psi(2Rz)$. Moreover,

$$H \ni [x_\varphi(z), x_{\psi-2\varphi}(s)] = x_{\psi-\varphi}(\pm zs)x_\psi(\pm z^2s) =: y(s)$$

and

$$H \ni [y(s), x_{2\varphi-\psi}(r)] = x_\varphi(\pm zsr)x_\psi(\pm x^2s^2r) = y'(r, s)$$

for all r, s in R.

Therefore

$$H \ni y'(r, s)y'(sr, 1)^{-1} = x_\psi(\pm z^2s(r^2 - r)) .$$

Using the condition of Theorem 3, we conclude that $H \supset x_\psi(Rz^2)$.

Thus, $H \supset x_\psi(2Rz + Rz^2)$. By the condition of Theorem 4 (with $e(\Phi) = 2$), $H \supset x_\psi(Rz)$.

Case 4. $\Phi' = \Phi = B_2$, φ is long, and ψ makes angle 45° with φ. Then

$$H \ni y(r) := [x_\varphi(z), x_{\psi-\varphi}(r)] = x_\psi(\pm zr)x_{2\psi-\varphi}(\pm r^2 z)$$

and

$$H \ni y'(r, s) := [y(r), x_{\varphi-\psi}(s)] = x_\psi(\pm r^2 sz)x_\varphi(\pm s^2 r^2 z \pm 2rsz)$$

for all r, s in R, hence

$$H \ni y'(r, s)y'(1, rs)^{-1} = x_\psi(\pm(r^2 - r)sz) .$$

It follows from the condition of Theorem 3 that $H \supset x_\psi(Rz)$.

Case 5. $\Phi' = \Phi = B_2$. We find a sequence $\alpha(1), \cdots, \alpha(m)$ in Φ' such that $\alpha(1) = \varphi$, $\alpha(m) = \psi$, and $\alpha(i)$, $\alpha(i + 1)$ make angle 45° for $i = 1, \cdots, m - 1$. Then, by Cases 3 and 4, $H \supset x_{\alpha(i)}(Rz)$ for $i = 2, \cdots, m$.

Case 6. φ is long and Φ is of type B_n, $n \geqq 3$, or F_4. Then the long roots in Φ form a connected subsystem, so $H \supset x_\gamma(Rz)$ for every long root γ by Case 1. If ψ is short, it makes angle 45° with a long γ in Φ', hence

$$x_\psi(u) = [x_\gamma(u), x_{\psi-\gamma}(\pm 1)]x_{2\psi-\gamma}(\pm u) \in H$$

for all u in Rz.

Case 7. φ is short and Φ is of type C_n, $n \geqq 3$, or F_4. Then, by Case 1, $H \supset x_\gamma(Rz)$ for every short root γ in Φ. If ψ is long, it makes angle 45° with a short root γ in Φ', hence

$$x_\psi(u) = [x_\gamma(u), x_{\psi-\gamma}(\pm 1)]x_{\psi+\gamma}(\pm u^2) \in H$$

for all u in Rz.

Case 8. φ is long and $\Phi = C_n$ with $n \geqq 3$. Let $\alpha \in \Phi'$ make angle 45° with φ and $\beta \in \Phi$ make angle 120° with α. Then $H \ni g := [x_\varphi(z), x_{\alpha-\varphi}(1)] = x_\alpha(\pm z)x_{2\alpha-\varphi}(\pm r^2 z)$ and

$$H \ni [g, x_\beta(1)] = x_{\alpha+\beta}(z) .$$

By Case 1, $H \supset x_\gamma(Rz)$ for all short roots γ in Φ. If ψ is long, we conclude that $H \supset x_\psi(Rz)$ as in Case 7.

Case 9. φ is short and $\Phi = B_n$ with $n \geqq 3$. Let $\alpha \in \Phi'$ make angle 45° with φ and $\beta \in \Phi$ make angle 120° with α. Then

$$H \ni [x_\varphi(z), z_{\alpha-\varphi}(r)] = x_\alpha(\pm 2rz)$$

and

$$H \ni y(s) := [x_\varphi(z), x_{\alpha-2\varphi}(s)] = x_{\alpha-\varphi}(\pm zs)x_\alpha(\pm z^2 s) ,$$

hence

$$H \ni [y(s), x_\beta(1)] = x_{\alpha+\beta}(\pm z^2 s)$$

for all r, s in R.

By Case 1, $H \supset x_\gamma(2Rz + Rz^2)$ for all long roots γ in Φ. By the condition of Theorem 4 (with $e(\Phi) = 2$), $H \supset x_\gamma(Rz)$ for all long γ. If ψ is short, we find a long γ in Φ' which makes angle 45° with ψ and obtain, as in Case 6, that $H \supset x_\psi(Rz)$.

Case 10. $\Phi' = G_2$ and φ is long. By Case 1, $H \supset x_\alpha(Rz)$ for all long roots α in $\Phi' = \Phi$. If ψ is short, let α make angle 150° with ψ. Then

$$H \ni [x_{\alpha+2\psi}(r), x_{-2\alpha-3\psi}(sz)] = x_{-\alpha-\psi}(\pm rsz)x_\psi(\pm r^2sz)x_{\alpha+3\psi}(\pm r^3sz)x_{-\alpha}(\pm r^3s^2z^2)$$

for all r, s in R, hence

$$H \ni y(r, s) := x_{-\alpha-\psi}(\pm rsz)x_\psi(\pm r^2sz) .$$

Therefore $H \ni y(r, s)y(1, rs)^{-1} = x_\psi(\pm(r^2 - r)sz)$. By the condition of Theorem 3, it follows that $H \supset x_\psi(Rz)$.

Case 11. $\Phi' = G_2$ and φ is short. Let α make angle 30° with φ. Then

$$H \ni [x_\varphi(z), x_{\alpha-\varphi}(r)] = x_\alpha(\pm 3zr)$$

for all r in R, hence $x_\alpha(3Rz) \subset H$. By Case 10, it follows that $x_\gamma(3Rz) \subset H$ for all roots γ in $\Phi' = \Phi$.

Using this with $\gamma = \alpha$ and $\gamma = 2\alpha - 3\varphi$, it follows from

$$H \ni [x_\varphi(z), x_{\alpha-2\varphi}(r)] = x_{\alpha-\varphi}(\pm 2rz)x_\alpha(\pm 3z^2r)x_{2\alpha-3\varphi}(\pm 3r^2z)$$

that $H \ni x_{\alpha-\varphi}(\pm 2rz)$ for all r in R. So $H \supset x_{\alpha-\varphi}(2Rz)$. Rotating this by 30°, we obtain that $H \supset x_{\alpha-2\varphi}(4Rz)$.

Using these inclusions and that

$$H \ni [x_\varphi(z), x_{\alpha-3\varphi}(4r)] = x_{\alpha-2\varphi}(\pm 4rz)x_{\alpha-\varphi}(\pm 4rz^2)x_\alpha(\pm 4rz^3)x_{2\alpha-3\varphi})(\pm 16r^2z^3)$$

we conclude that

$$H \ni x_\alpha(\pm 4rz^3)x_{2\alpha-3\varphi}(\pm 16r^2z^3) =: g$$

for all r in R. Therefore

$$H \ni [g, x_{\alpha-3\varphi}(1)] = x_{2\alpha-3\varphi}(\pm 4rz^3) ,$$

hence $H \supset x_{2\alpha-3\varphi}(4Rz^3)$. By Case 10, $H \supset x_\gamma(4Rz^3)$ for all roots γ in Φ.

Thus, $H \supset x_\gamma(3Rz + 4Rz^3)$ for all γ. By the condition of Theorem 4 (with $e(\Phi) = 3$), $3Rz + 4Rz^3 = 3Rz + Rz^3 = Rz$.

6. Proof of Theorem 4(b).

LEMMA 6. *Under the condition of Theorem* 3, *assume that* H *is a* *non-central subgroup of* $G(R)$ *normalized by* $E(R)$. *Then* $H \ni x_\varphi(z)$ *for some* φ *in* Φ *and a non-zero* z *in* R.

PROOF. We pick a non-central element h in H. There is a finitely generated subring R' of R such that $1 \in R'$ and $h \in G(R')$. Let $p_1, \cdots, p_m$ be the minimal prime ideals of R' (where $m \geqq 1$). Consider the images H_i in $G(R'/p_i)$ of $H \cap G(R')$. The subgroup H_i of $G(R'/p_i)$ is normalized by $E(R'/p_i)$. By [26, Theorem 10.1 with $A = B = R'/p_i$], either H_i is central or $H \supset E(J_i)$ for a non-zero ideal J_i of R'/p_i.

Suppose first that H_i is not central for some i, say, for $i = 1$. Then we pick: a subsystem Φ' of Φ of type A_2 or B_2; a long root φ in Φ'; a root ψ in Φ' which makes angle 60° or 45° with φ; a non-zero u_1 in J_1; some u in R' with $u_1 = u + p_1$; g in $H \cap G(R')$ with image $x_\varphi(u_1)$ in H_1; an element t in R' outside p_1 which belongs to all p_i with $i = 2, \cdots, m$; an ordering on Φ such that φ and, when $\Phi' = B_2$, $2\psi - \varphi$ are positive. Then $gx_\varphi(-u) \in G(p_1)$.

We have

$$H \ni [g, x_{\psi-\varphi}(t)] = \begin{cases} x_\psi(\pm ut)g_0 & \text{when} \quad \Phi' = A_2 , \\ x_\psi(\pm ut)x_{2\psi-\varphi}(\pm ut^2)g_0 & \text{when} \quad \Phi' = B_2 , \end{cases}$$

with $g_0 \in G(R'ut) \subset G(\text{rad}(R')) \subset G(\text{rad}(R))$, where rad means the Jacobson radical. By [1], [2], $G(\text{rad}(R)) = U(\text{rad}(R))T(\text{rad}(R))V(\text{rad}(R))$, where U is the subgroup of G generated by positive roots, V is the subgroup of G generated by negative roots, and T is the torus.

Thus, H contains a non-central element (namely, $[g, x_{\psi-\varphi}(t)]$) of $U(R)T(R)V(\text{rad}(R))$, assuming that H_i is not central for some i. If H_i is central for all i, then $g \in G(\text{rad}(R')) \subset G(\text{rad}(R))$ is a non-central element of $U(R)T(R)V(\text{rad}(R))$. Now the conclusion of Lemma 6 follows from [2].

Now we can conclude our proof of Theorem 4(b). By Theorem 4(a), there is an ideal J of R such that $H \cap x_\alpha(R) = x_\alpha(J)$ for every root α in Φ. Applying Lemma 6 to the ring R/J and the image H' of H in $G(R/J)$, we conclude that either H' is central (i.e. $H \subset G(R, J)$ and we are done) or $H' \ni x_\varphi(z')$ for some non-zero z' in R/J.

In the latter case we are going to obtain a contradiction with our choice of J. Applying Theorem 4(a), we have $H' \ni x_\varphi(z')$ for all φ in Φ. We pick z in R such that $z + J = z'$.

If Φ contains a subsystem Φ' of type A_2, we pick roots φ, ψ in Φ' such that $\varphi - \psi \in \Phi'$, and we pick g in H such that $gx_\varphi(-z) \in G(J)$. Then $H \ni [g, x_{\psi-\varphi}(1)] = x_\psi(\pm z)g_0$ with $g_0 \in E(R, J) \subset H$, using Theorem 1. Therefore $x_\psi(z) \in H$ which contradicts our choice of J.

If Φ does not contain a subsystem of type A_2, then $\Phi = B_2$. We pick a long root φ and a short root ψ such that $\varphi - \psi \in \Phi$. For every r in R we pick $g(r)$ in H such that $g(r)x_\varphi(-zr) \in G(J)$. Then, for every s in R,

$$H \ni [g(r), x_{\psi-\varphi}(s)] = x_\psi(\pm urs)x_{2\psi-\varphi}(\pm urs^2)g_0$$

with $g_0 \in E(R, J) \subset H$, hence

$$H \ni y(r, s) := x_\psi(\pm urs)x_{2\psi-\varphi}(\pm urs^2) .$$

Therefore $H \ni y(r, s)y(rs, 1)^{-1} = x_{2\psi-\varphi}(ur(s^2 - s))$ for all r, s in R. In view of the condition of Theorem 3, this contradicts our choice of J.

7. Proof of Theorem 5. The group $E(R)$ is perfect by Theorem 3 with $J = R$.

Let H be a subgroup $G(R)$ containing $E(R)$ and $f: H \to H$ an automorphism. By Theorem 1, $E(R)$ is normal in H, so $f(E(R))$ is normal in $f(H) = H$. By Theorem 4(b), $E(R, J) \subset f(E(R)) \subset G(R, J)$ for an ideal J of R.

The main step in our proof is to show that $J = R$. We assume that $J \neq R$ and will obtain a contradiction.

When G is not of type B_2 or G_2, let R' denote the subring of R generated by 1. When G is of type B_2 or G_2, we use the condition of Theorem 3 to write $1 = \sum s_i(r_i^2 - r_i)$, and we denote by R' the subring of R generated by these s_i and r_i. Then R' is a finitely generated ring with 1. By Theorem 3, $E(R')$ is perfect; from the proof of the theorem it is easy to see that the group $E(R')$ is finitely generated.

Therefore there is a finitely generated ideal J' of R' such that $f(E(R')) \subset G(J')$, $J' \subset J$, and $J'J' = J'$, where $J'J'$ is the additive subgroup of J' generated by all rs with r, s in J'. By the Nakayama lemma, $sJ' = 0$ for some $s \in R' \setminus J'$.

Therefore $E(sR)$ commutes with $f(E(R'))$, so the centralizer of $f(E(R'))$ in H is not commutative. On the other hand, the centralizer of $E(R')$ in $G(R)$ is commutative. This contradiction proves that $J = R$.

Thus, $f(E(R)) \supset E(R)$. Since f^{-1} is also an automorphism of H, we have $f^{-1}(E(R)) \supset E(R)$. So $f(E(R)) = E(R)$. That is, $E(R)$ is a characteristic subgroup of H.

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of two elements in the case when G is of type B_2 or G_2 (which in fact is a necessary and sufficient condition for the conclusions of Theorems 3 and 4 to be true). For the types other than B_2 and G_2 no assumptions on R are needed, and proofs can be simplified.

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