Tôhoku Math. Journ.
37 (1985), 385-393.

ON ALGEBRAIC INDEPENDENCE OF SPECIAL VALUES OF GAP SERIES

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(Received July 17, 1984)

Introduction. In this paper we extend the result of Bundschuh and Wylegala [4].

Let $f(z) = \sum_{k=0}^{\infty} a(k)z^{e(k)}$ be a power series, where the $a(k)$ $(k \geqq 0)$ are non-zero algebraic numbers and where the $e(k)$ $(k \geqq 0)$ form an increasing sequence of non-negative integers. We denote by $A(f, n)$ the maximum of the $|\overline{a(k)}|$ $(0 \leqq k \leqq n)$, where for any k $(0 \leqq k \leqq n)$ the $|\overline{a(k)}|$ is the maximum of the absolute values of the conjugates of $a(k)$. We denote by $M(f, n)$ is the least positive integer d such that $d \cdot a(k)$ $(0 \leqq k \leqq n)$ are all algebraic integers, and by $S(f, n)$ is the degree of $\boldsymbol{Q}(a(k); 0 \leqq k \leqq n)$ over $\boldsymbol{Q}$. In [4], Bundschuh and Wylegala proved that $f(\alpha_1), \cdots, f(\alpha_m)$ are algebraically independent for any algebraic numbers $\alpha_1, \cdots, \alpha_m$ with $0 < |\alpha_1| < \cdots < |\alpha_m| < R(f)$, if the condition

$$\lim_{n\to\infty} S(f, n)(e(n) + \log A(f, n) + \log M(f, n))/e(n + 1) = 0$$

is satisfied. In §1, we extend this result as follows. Let $f_i(z) = \sum_{k=0}^{\infty} a(i, k)z^{e(i,k)}$ $(1 \leqq i \leqq m)$ be gap series, where the $a(i, k)$ $(1 \leqq i \leqq m, k \geqq 0)$ are non-zero algebraic numbers and where for any i $(1 \leqq i \leqq m)$ the $e(i, k)$ $(k \geqq 0)$ form an increasing sequence of non-negative integers, and let α_i $(1 \leqq i \leqq m)$ be algebraic numbers with $0 < |\alpha_i| < R(f_i)$. We put $A(n) = \max\{A(f_i, n); 1 \leqq i \leqq m\}$, $M(n) = \text{l.c.m.}\{M(f_i, n); 1 \leqq i \leqq m\}$, $S(n) = [\boldsymbol{Q}(a(i, k); 1 \leqq i \leqq m, 0 \leqq k \leqq n): \boldsymbol{Q}]$, $E(n) = \max\{e(i, n), 1 \leqq i \leqq m\}$, $e(n) = \min\{e(i, n); 1 \leqq i \leqq m\}$. Then we have the following.

THEOREM. *$f_1(\alpha_1), \cdots, f_m(\alpha_m)$ are algebraically independent if the following two conditions are satisfied*:

(i) $\lim_{n\to\infty} S(n)(E(n) + \log A(n) + \log M(n))/e(n + 1) = 0$;

(ii) $|a(i + 1, n)\alpha_{i+1}^{e(i+1,n)}| = o(|a(i, n)\alpha_i^{e(i,n)}|)$ as $n \to \infty$ $(1 \leqq i \leqq m - 1)$.

Our proof of this result is closely related to the proof of the result of Shiokawa [16].

For example, put $f(z) = \sum_{k=0}^{\infty} z^{k!}$. Let α_j $(1 \leqq j \leqq m)$ be algebraic numbers satisfying $0 < |\alpha_m| < \cdots < |\alpha_1| < 1$. Then the numbers $f^{(i)}(\alpha_j)$ $(0 \leqq i \leqq l,\ 1 \leqq j \leqq m)$ are algebraically independent.

In §1, we obtain another sufficient condition for the algebraic independence of given numbers.

For example, it will be proved that the m continued fractions $\xi_i = [i^{1!}, i^{2!}, i^{3!}, \cdots]$ $(2 \leqq i \leqq m+1)$ are algebraically independent.

In §2, we prove a result concerning the algebraic independence of special values $f_1(\alpha_1), \cdots, f_p(\alpha_p), f_{p+1}(\xi_1), \cdots, f_{p+q}(\xi_q)$, where $f_i(z)$ $(1 \leqq i \leqq p+q)$ are gap series with algebraic coefficients, α_i $(1 \leqq i \leqq p)$ are algebraic numbers and ξ_j $(1 \leqq j \leqq q)$ are transcendental numbers with certain conditions. This result contains a generalization of the theorem of Cijsouw and Tijdeman [5].

The author would like to express his gratitude to Professor Y. Morita for his precious advice. Thanks are due to the referee who gave the author valuable comments and pointed out the existence of [17] and [18].

NOTATION. For any power series $f(z) = \sum_{k=0}^{\infty} c_k z^k$, we denote by $R(f)$ the radius of convergence of $f(z)$.

For any polynomial $A(X) = A(X_1, \cdots, X_m)$ with arbitrary complex coefficients, we denote by $H(A(X))$ the maximum of the absolute value of the coefficients of $A(X)$ and by $L(A(X))$ the sum of the absolute values of the coefficients of $A(X)$. We put $\Lambda(A(X)) = 2^M L(A(X))$, where M is the total degree of $A(X)$.

For any algebraic number α with minimal polynomial $P(X)$, we put $\mathrm{H}(\alpha) = H(P(X))$, $L(\alpha) = L(P(X))$, $\deg(\alpha) =$ the degree of $P(X)$, $\overline{|\alpha|} = \max\{|\beta|; P(\beta) = 0\}$. Further, for any algebraic number α_i $(1 \leqq i \leqq m)$, we denote by $\mathrm{den}(\alpha_1, \cdots, \alpha_m)$ the least positive integer d such that $d\alpha_i$ $(1 \leqq i \leqq m)$ are all algebraic integers, and for any algebraic number α we put $\mathrm{size}(\alpha) = \max\{\log \mathrm{den}(\alpha), \log \overline{|\alpha|}\}$.

1. Algebraic independence of special values of gap series (I). In this section, we prove two theorems on the algebraic independence of certain numbers.

We need the following two lemmas.

LEMMA 1 (Cijsouw and Tijdeman [5]). *Let α be an algebraic number such that $H(\alpha) = h$, $\deg(\alpha) = n$ and $\mathrm{den}(\alpha) = d$. Then we have*

$$h \leqq (2d \cdot \max(1, \overline{|\alpha|}))^n .$$

LEMMA 2 (Güting [10]). *Let $A(X) = A(X_1, \cdots, X_m)$ be a polynomial of degrees $N(i)$ in X_i $(1 \leqq i \leqq m)$ with rational integral coefficients and with $L(A(X)) = q$. Let α_i $(1 \leqq i \leqq m)$ be algebraic numbers with $\deg(\alpha_i) = n(i)$ and $L(\alpha_i) = q(i)$, and let $s = [\boldsymbol{Q}(\alpha_1, \cdots, \alpha_m) : \boldsymbol{Q}]$. Then $A(\alpha_1, \cdots, \alpha_m) = 0$ or*

$$|A(\alpha_1, \cdots, \alpha_m)| \leqq q(q \cdot q(1)^{N(1)/n(1)} \cdots q(m)^{N(m)/n(m)})^{-s} .$$

Let $f_i(z) = \sum_{k=0}^{\infty} a(i, k) z^{e(i,k)}$ $(1 \leqq i \leqq m)$ be power series, where the $a(i, k)$ $(1 \leqq i \leqq m, k \geqq 0)$ are non-zero algebraic numbers and where for any i $(1 \leqq i \leqq m)$ the $e(i, k)$ $(k \geqq 0)$ form an increasing sequence of non-negative integers. We prove the following:

THEOREM 1. *Let $\alpha_i (1 \leqq i \leqq m)$ be algebraic numbers with $0 < |\alpha_i| < R(f_i)$. Then $f_1(\alpha_1), \cdots, f_m(\alpha_m)$ are algebraically independent over $\boldsymbol{Q}$ if the following two conditions are satisfied:*

(i) $\lim_{n\to\infty} S(n)(E(n) + \log A(n) + \log M(n))/e(n+1) = 0$, *where* $A(n) = \max\{|\overline{a(i, k)}|; 1 \leqq i \leqq m, 0 \leqq k \leqq n\}$, $M(n) = \text{den}\{a(i, k); 1 \leqq i \leqq m, 0 \leqq k \leqq n\}$, $S(n) = [\boldsymbol{Q}(a(i, k); 1 \leqq i \leqq m, 0 \leqq k \leqq n) : \boldsymbol{Q}]$, $E(n) = \max\{e(i, n); 1 \leqq i \leqq m\}$ *and* $e(n) = \min\{e(i, n); 1 \leqq i \leqq m\}$;

(ii) $|a(i+1, n)\alpha_{i+1}^{e(i+1,n)}| = o(|a(i, n)\alpha_i^{e(i,n)}|$ *as* $n \to \infty$ $(1 \leqq i \leqq m-1)$.

PROOF. Suppose $\eta_1 = f_1(\alpha_1), \cdots, \eta_m = f_m(\alpha_m)$ are algebraically dependent. There is a non-zero polynomial $P(X_1, \cdots, X_m)$ with rational integral coefficients satisfying $P(\eta_1, \cdots, \eta_m) = 0$. We may assume $P(X_1, \cdots, X_m)$ has the least total degree. Put $p = \min\{i; X_i$ is actually contained in $P(X_1, \cdots, X_m)\}$ (hence $P(X_1, \cdots, X_m) \equiv P(X_p, \cdots, X_m)$ is a function of $X_p, \cdots, X_m$), and put $\eta_{i,n} = \sum_{k=0}^{n} a(i, k)\alpha_i^{e(i,k)}$ $(1 \leqq i \leqq m,\ n \geqq 1)$.

We denote by $c_0, c_1, \cdots$ positive constants which are independent of n.

By the assumption on $P(X_p, \cdots, X_m)$ and the condition (ii), we have

$$\begin{aligned} &|P(\eta_{p,n+1}, \cdots, \eta_{m,n+1}) - P(\eta_{p,n}, \cdots, \eta_{m,n})| \\ &\qquad \geqq c_0 |a(p, n+1)\alpha_p^{e(p,n+1)}| - o(|a(p, n+1)\alpha_p^{e(p,n+1)}|) > 0 \end{aligned}$$

as $n \to \infty$. Hence $P(\eta_{p,n}, \cdots, \eta_{m,n}) \neq 0$ or $P(\eta_{p,n+1}, \cdots, \eta_{m,n+1}) \neq 0$ for $n \gg 0$. Now let n be any suffix with $P(\eta_{p,n}, \cdots, \eta_{m,n}) \neq 0$. We obtain the inequality

$$(1) \qquad |P(\eta_{p,n}, \cdots, \eta_{m,n})| \geqq \exp\{-c_1 S(n)(E(n) + \log A(n) + \log M(n))\}$$

by Lemmas 1 and 2. On the other hand, we can obtain the inequality

$$(2) \qquad |P(\eta_p, \cdots, \eta_m) - P(\eta_{p,n}, \cdots, \eta_{m,n})| \leqq \exp\{-c_2 e(n+1)\} \quad \text{for} \quad n \gg 0\,.$$

We deduce from (1) and (2)

$$|P(\eta_p, \cdots, \eta_m)| \geqq |P(\eta_{p,n}, \cdots, \eta_{m,n})| - |P(\eta_p, \cdots, \eta_m) - P(\eta_{p,n}, \cdots, \eta_{m,n})| > 0$$

as $n \to \infty$. This is a contradiction, and therefore the theorem is proved.

COROLLARY. *Let $f(z) = \sum_{k=0}^{\infty} z^{k!}$, and let α_j $(1 \leqq j \leqq m)$ be m algebraic numbers such that $0 < |\alpha_m| < |\alpha_{m-1}| < \cdots < |\alpha_1| < 1$, and let p be any natural number. Then the $m(p+1)$ numbers $f^{(i)}(\alpha_j)$, $(0 \leqq i \leqq p, 1 \leqq j \leqq m)$ are algebraically independent over $\boldsymbol{Q}$, where $f^{(i)}(z)$ is the i-th derived function of $f(z)$.*

PROOF. Put

$$f_i(z) = \sum_{k=0}^{\infty} (d/dz)^i z^{(p+k)!} = \sum_{k=0}^{\infty} a(i, k) z^{e(i,k)} \qquad (0 \leqq i \leqq p) .$$

Then $f_i(z) - f^{(i)}(z)$ $(0 \leqq i \leqq p)$ are polynomials with rational integral coefficients. Hence it is enough to prove that $f_i(\alpha_j)$ $(0 \leqq i \leqq p,\ 1 \leqq j \leqq m)$ are algebraically independent. We can easily show that these functions $f_i(z)$ $(0 \leqq i \leqq p)$ satisfy the condition (i) of Theorem 1. We can also show that

$$|a(i, n)\alpha_j^{e(i,n)}| = o(|a(i+1, n)\alpha_j^{e(i+1,n)}|) \qquad (0 \leqq i \leqq p) \quad \text{and}$$
$$|a(p, n)\alpha_{j+1}^{e(p,n)}| = o(|a(0, n)\alpha_j^{e(0,n)}|) \qquad (1 \leqq j \leqq m-1)$$

as $n \to \infty$. From these relations we obtain the required result.

Note that the number $\eta_i = f_i(\alpha_i)$ is the limit of the algebraic numbers $\eta_{i,n} = \sum_{k=0}^{n} a(i, k)\alpha_i^{e(i,k)}$ $(n \geqq 1)$. We may regard the conditions (i) and (ii) of Theorem 1 as conditions on the pairs $(\eta_i, \{\eta_{i,n};\ n \geqq 1\})$ $(1 \leqq i \leqq m)$. Then we can apply the method of the proof of Theorem 1 to prove the following:

THEOREM 2. *Let* ξ_i $(1 \leqq i \leqq m)$ *be the limits of the numbers* $\alpha_{i,n} (n \geqq 1)$: $\lim_{n\to\infty} \alpha_{i,n} = \xi_i$. *Then* $\xi_1, \cdots, \xi_m$ *are algebraically independent if the following conditions* (i), (ii), (iii) *or* (i), (ii), (iii)′ *are satisfied*:

(i) $\alpha_{i,n}$ *are algebraic numbers satisfying* $\alpha_{i,n} \neq \alpha_{i,n+1}$ *and* $\xi_i \neq \alpha_{i,n}$ $(1 \leqq i \leqq m,\ n \geqq 1)$;

(ii) $S(n) \cdot \max\{1, \text{size}(\alpha_{1,n}), \cdots, \text{size}(\alpha_{m,n})\} = o(\min\{-\log|\xi_1 - \alpha_{1,n}|, \cdots, -\log|\xi_m - \alpha_{m,n}|\})$ *as* $n \to \infty$, *where* $S(n) = [\boldsymbol{Q}(\alpha_{1,n}, \cdots, \alpha_{m,n}) : \boldsymbol{Q}]$;

(iii) $|\alpha_{i+1,n+1} - \alpha_{i+1,n}| = o(|\alpha_{i,n+1} - \alpha_{i,n}|)$ $(1 \leqq i \leqq m-1)$ *as* $n \to \infty$

(iii)′ *Put* $S(1, n) = [\boldsymbol{Q}(\alpha_{1,n}) : \boldsymbol{Q}]$, $S(i, n) = [\boldsymbol{Q}(\alpha_{1,n}, \cdots, \alpha_{i,n}) : \boldsymbol{Q}(\alpha_{1,n}, \cdots, \alpha_{i-1,n})]$ $(2 \leqq i \leqq m)$. *Then* $\lim_{n\to\infty} S(i, n) = +\infty$ $(1 \leqq i \leqq m)$.

PROOF. We can prove this theorem in the same way as Theorem 1. Suppose that $\xi_1, \cdots, \xi_m$ are algebraically dependent. Then we can take a non-zero polynomial $P(X_p, \cdots, X_m)$ with rational integral coefficients satisfying $P(\xi_p, \cdots, \xi_m) = 0$. Then we can obtain

$$(3) \qquad |P(\alpha_{p,n}, \cdots, \alpha_{m,n})| \geqq \exp\{-c_0 S(n) \cdot \max\{1, \text{size}(\alpha_{p,n}), \cdots, \text{size}(\alpha_{m,n})\}\}$$

and

$$(4) \qquad |P(\xi_p, \cdots, \xi_m) - P(\alpha_{p,n}, \cdots, \alpha_{m,n})| \leqq \exp\{-c_1 \cdot \min\{-\log|\xi_p - \alpha_{p,n}|, \cdots, -\log|\xi_m - \alpha_{m,n}|\}\}$$

for infinitely many n, where c_0 and c_1 are the constants which are independent of n. From (3), (4) and (ii), we obtain $P(\xi_p, \cdots, \xi_m) \neq 0$. This is a contradiction, and therefore the theorem is proved.

COROLLARY. *Let* $\xi_1 = [a_{i,0}, a_{i,1}, \cdots, a_{i,n}, \cdots]$ $(1 \leqq i \leqq m)$ *be* m *continued fractions, where the* $a_{i,n}$ *are all positive integers. We denote by* $\alpha_{i,n} = p_{i,n}/q_{i,n}$ $(1 \leqq i \leqq m,\ n \geqq 1)$ *the* n*-th principal convergents of* ξ_i. *Then the numbers* $\xi_1, \cdots, \xi_m$ *are algebraically independent if the following two conditions are satisfied*:

(ii)′ $\log q_{m,n} = o(\log q_{1,n+1})$ *as* $n \to \infty$;

(iii)″ $q_{i,n} = o(q_{i+1,n})$ *as* $n \to \infty$ $(1 \leqq i \leqq m-1)$.

Further, (ii)′ *and* (iii)″ *follow from the following condition*:

(iv) *There exist* $m-1$ *positive numbers* λ_i $(1 \leqq i \leqq m-1)$ *and a sequence* $\{\sigma_n;\ n \geqq 1\}$ *such that* $\lambda_1 > \lambda_2 > \cdots > \lambda_{m-1} > 1$, $\lim_{n\to\infty} \sigma_n = +\infty$, *and* $a_{m,n} > \lambda_{m-1}a_{m-1,n} > \lambda_{m-2}a_{m-2,n} > \cdots > \lambda_1 a_{1,n} > a_{m,n-1}^{\sigma_{n-1}}$ *for* $n \gg 0$.

PROOF. We claim that the conditions (i), (ii), (iii) of Theorem 2 are satisfied by ξ_i $(1 \leqq i \leqq m)$ and $\{\alpha_{i,n};\ n \geqq 1\}$ $(1 \leqq i \leqq m)$. Indeed (i) is trivially satisfied. We obtain the equality $\max\{1, \mathrm{size}(\alpha_{1,n}), \cdots, \mathrm{size}(\alpha_{m,n})\} = \log q_{m,n}$ $(n \gg 0)$ from the equality $\mathrm{size}(\alpha_{i,n}) = \log q_{i,n}$ $(1 \leqq i \leqq m,\ n \gg 0)$ and (iii)″. Further, we obtain the inequality $\min\{-\log|\xi_1 - \alpha_{1,n}|, \cdots, -\log|\xi_m - \alpha_{m,n}|\} \geqq \log q_{1,n+1}$ $(n \gg 0)$ from the inequality $|\xi_i - \alpha_{i,n}| < 1/q_{i,n}q_{i,n+1}$ and (iii)″. Hence (ii) follows from (ii)′. (iii) follows from the equality $|\alpha_{i,n+1} - \alpha_{i,n}| = 1/q_{i,n}q_{i,n+1}$ $(1 \leqq i \leqq m)$ and (iii)″.

Now we show (ii)′ and (iii)″ follow from (iv). We note that $\lim_{n\to\infty} a_{i,n} = +\infty$ $(1 \leqq i \leqq m)$ because of the condition (iv). We denote by $c_0, c_1, \cdots$ positive constants which are independent of n. Then we have the following inequalities:

$$q_{i+1,n} \geqq \prod_{k=1}^{n} a_{i+1,k} \quad (n \geqq 1)\,, \qquad \prod_{k=1}^{n} (a_{i,k}+1) \geqq q_{i,n} \quad (n \geqq 1)\,, \quad \text{and}$$

$$\lambda_{i+1}a_{i+1,n} > \lambda_i a_{i,n} > \lambda_{i'}(a_{i,n}+1) \qquad (1 \leqq i \leqq m-1,\ n \gg 0)\,,$$

where $\lambda_{i'}$ $(1 \leqq i \leqq m-1)$ are positive numbers such that $\lambda_{i'} > \lambda_{i+1}$ and $\lambda_m = 1$. Hence we obtain $c_0 q_{i+1,n} > (\lambda_{i'}/\lambda_{i+1})^n q_{i,n}$ $(1 \leqq i \leqq m-1,\ n \gg 0)$, namely, $q_{i,n} = o(q_{i+1,n})$ as $n \to \infty$. We also have the following inequality from the condition (iv)

$$q_{1,n+1} \geqq \prod_{k=1}^{n+1} a_{1,k} \geqq c_1 \cdot \prod_{k=1}^{n} a_{m,k}^{\sigma_k} \geqq c_2 \cdot \prod_{k=1}^{n} (a_{m,k}+1)^{\sigma_k/2}\,.$$

Hence we obtain $\log q_{m,n} = o(\log q_{1,n+1})$ as $n \to \infty$. This completes the proof.

For example, the $m-1$ continued fractions $\xi_i = [i^{1!}, i^{2!}, i^{3!}, \cdots, i^{n!}, \cdots]$ $(2 \leqq i \leqq m)$ are algebraically independent.

The above corollary of Theorem 2 is a generalization of the result of Bundschuh [3]. He proved the algebraic independence of the numbers $\xi_i = [a_{i,0}, a_{i,1}, a_{i,2}, \cdots]$ $(i = 1, 2)$ satisfying the condition (iv) of the corollary.

2. Algebraic independence of special value of gap series (II). We recall Mahler's definition of the order function. Let $\xi_1, \cdots, \xi_q$ be q complex numbers. Then the *order function* $O(u|\xi_1, \cdots, \xi_q)$ of a positive integral variable u is defined by

$$O(u|\xi_1, \cdots, \xi_q) = -\log(\min\{|P(\xi_1, \cdots, \xi_q)| > 0; \Lambda(P(X)) \leqq u\})$$

where $P(X) = P(X_1, \cdots, X_q)$ runs through polynomials with rational integral coefficients. Fundamental properties of the function $O(u|\xi_1, \cdots, \xi_q)$ were investigated by Mahler [12] and by Durand [6] in case of $q = 1$, and by Durand [7] in the general case.

Let $f_i(z) = \sum_{k=0}^{\infty} a(i, k)z^{e(i,k)}$ $(1 \leqq i \leqq m = p + q;\ p \geqq 0,\ q \geqq 1)$ be such power series as the power series defined in §1. Now we prove the following:

THEOREM 3. *Let α_i $(1 \leqq i \leqq p)$ be algebraic numbers with $|\alpha_i| < R(f_i)$ and let ξ_j $(1 \leqq j \leqq q)$ be transcendental numbers with $|\xi_j| < R(f_{p+j})$. Then the $m = p + q$ numbers $f_1(\alpha_1), \cdots, f_p(\alpha_p)$, $f_{p+1}(\xi_1), \cdots, f_m(\xi_q)$ are algebraically independent over $\boldsymbol{Q}$ if the following three conditions are satisfied*:

(i) *There exists a positive number $b \geqq 1$ such that*

$$\lim_{n\to\infty} (S(n)(E(n) + \log A(n) + \log M(n)))^b/e(n + 1) = 0\ ,$$

where $A(n)$, $M(n)$, $S(n)$, $E(n)$ and $e(n)$ are the constants defined in Theorem 1.

(ii) *The q numbers $\xi_1, \cdots, \xi_q$ are algebraically independent and there exist a positive number γ and a positive integer u_0 such that*

$$O(u|\xi_1, \cdots, \xi_q) \leqq \gamma(\log u)^b \quad for \quad u \geqq u_0\ ,$$

where b is the number given in (i);

(iii) *The p numbers $f_1(\alpha_1), \cdots, f_p(\alpha_p)$ are algebraically independent.*

PROOF. Suppose $\eta_i = f_i(\alpha_i)$ $(1 \leqq i \leqq p)$, $\eta_{p+j} = f_{p+j}(\xi_j)$ $(1 \leqq j \leqq q)$ are algebraically dependent. There is a non-zero polynomial $P(X) = P(X_1, \cdots, X_m)$ with rational integral coefficients satisfying $P(\eta_1, \cdots, \eta_m) = 0$. We may assume $P(X)$ has the least total degree. We denote by $c_0, c_1, \cdots$ positive constants which are independent of n.

Put $K_n = \boldsymbol{Q}(\alpha_1, \cdots, \alpha_p, a(i, k);\ 1 \leqq i \leqq m,\ 0 \leqq k \leqq n)$ and $S(n)' = [K_n : \boldsymbol{Q}]$. We denote by $\tau(l, n)$ $(1 \leqq l \leqq S(n)')$ all non-equivalent embeddings of K_n

into $\bar{Q}$ (we assume $\tau(1, n)$ is the identity map). Put

$$\eta_{i,l}^{(n)} = \sum_{k=0}^{n} a(i, k)^{\tau(l,n)}(\alpha_i^{\tau(l,n)})^{e(i,k)} \qquad (1 \leqq i \leqq p,\ n \geqq 1) ,$$

$$\eta_{p+j,l}^{(n)} = \sum_{k=0}^{n} a(p + j, k)^{\tau(l,n)}\xi_j^{e(p+j,k)} \qquad (1 \leqq j \leqq q,\ n \geqq 1) .$$

Further, we put

$$\Gamma_n = d^{E(n)S(n)'}\cdot M(n)^{MS(n)'}\prod_{l=1}^{S(n)'} P(\eta_{1,l}^{(n)}, \cdots, \eta_{m,l}^{(n)}) ,$$

$$\Gamma'_n = d^{E(n)S(n)'}\cdot M(n)^{MS(n)'}\cdot P(\eta_1, \cdots, \eta_m)\prod_{l=2}^{S(n)'} P(\eta_{1,l}^{(n)}, \cdots, \eta_{m,l}^{(n)}) ,$$

$$\Gamma_n(Y) = \Gamma_n(Y_1, \cdots, Y_q) = d^{E(n)S(n)'}\cdot M(n)^{MS(n)'}\prod_{l=1}^{S(n)'} P(\eta_{1,l}^{(n)}, \cdots, \eta_{p,l}^{(n)} ,$$

$$\sum_{k=0}^{n} a(p + 1, k)^{\tau(l,n)}Y_1^{e(p+1,k)}, \cdots, \sum_{k=0}^{n} a(m, k)^{\tau(l,n)}Y_q^{e(m,k)}) ,$$

where $d = \operatorname{den}(\alpha_1, \cdots, \alpha_p)$ and M is the total degree of $P(X)$. By applying the fundamental theorem on symmetric functions, we can easily show that $\Gamma_n(Y)$ is the polynomial in $Y_1, \cdots, Y_q$ with rational integral coefficients. We have $\Gamma_n = \Gamma_n(\xi_1, \cdots, \xi_q)$, and $\Gamma_n \neq 0$ for $n \gg 0$ by (ii), (iii) and the assumption on $P(X)$.

Now we fix some notations. Let $A(X) = A(X_1, \cdots, X_m) = \sum a_I X^I$ and $B(X) = B(X_1, \cdots, X_m) = \sum b_I X^I$ ($b_I \geqq 0$) be polynomials, where $I = (i(1), \cdots, i(m))$ and $X^I = X_1^{i(1)} \cdots X_m^{i(m)}$. Then we denote $A(X) \prec B(X)$ if the inequalities $|a_I| \leqq b_I$ are satisfied for any I.

Since $|\eta_{i,l}^{(n)}| \leqq (1+n)A(n)c_0^{E(n)} \leqq A(n)c_1^{E(n)}$ ($1 \leqq i \leqq p$, $1 \leqq l \leqq S(n)'$, $n \geqq 1$), we obtain

$$\Gamma_n(Y) \prec d^{E(n)S(n)'}\cdot M(n)^{MS(n)'}(A(n)c_1^{E(n)})^{MS(n)'}(p+\sum_{k=0}^{n} Y_1^{e(p+1,k)} + \cdots + \sum_{k=0}^{n} Y_q^{e(m,k)})^{MS(n)'} .$$

Hence

$$L(\Gamma_n(Y)) \leqq d^{E(n)S(n)'}\cdot M(n)^{MS(n)'}(A(n)c_1^{E(n)})^{MS(n)'}(p + (n + 1)q^{MS(n)'} \quad \text{and}$$
$$\text{tot.deg.}\, \Gamma_n(Y) \leqq qME(n)S(n)' .$$

From the above inequalities we obtain

$$\Lambda(\Gamma_n(Y)) \leqq (c_2^{E(n)}A(n)M(n))^{c_3 S(n)} .$$

Then we deduce from (ii) that

$$(5) \qquad |\Gamma_n| \geqq \exp\{-O(\Lambda(\Gamma_n(Y))|\xi_1, \cdots, \xi_q)\}$$
$$\geqq \exp\{-c_4(S(n)(E(n) + \log A(n) + \log M(n)))^b\} \quad \text{for} \quad n \gg 0 .$$

On the other hand, we have

$$(6)\qquad |\Gamma_n - \Gamma'_n| \leqq (c_5^{E(n)} A(n) M(n))^{c_6 S(n)} \cdot c_7 \cdot \sum_{i=1}^{m} |\eta_i - \eta_{i,1}^{(n)}|$$
$$\leqq \exp\{c_8 S(n)(E(n) + \log A(n) + \log M(n)) - c_9 e(n+1)\} \quad \text{for} \quad n \gg 0 .$$

From (5), (6) and (i), we obtain $|\Gamma'_n| \geqq |\Gamma_n| - |\Gamma_n - \Gamma'_n| > 0$ as $n \to \infty$. It follows that $P(\eta_1, \cdots, \eta_m) \neq 0$. This is a contradiction, and therefore the theorem is proved.

For example, put $e_0 = 1$, $e(k) = 2^{e(k-1)}$ $(k \geqq 1)$ and define $f(z) = \sum_{k=0}^{\infty} z^{e(k)}$. Then for any $b \geqq 1$, the condition (i) of Theorem 3 is satisfied. Hence the numbers $f(\xi_j)$ $(1 \leqq j \leqq q)$ are algebraically independent for any transcendental numbers ξ_j $(1 \leqq j \leqq q)$ satisfying $|\xi_j| < 1$ and the condition (ii) of Theorem 3.

COROLLARY. *Let $f(z) = \sum_{k=0}^{\infty} a(k) z^{e(k)}$ be a power series and let ξ be a complex number with $0 < |\xi| < R(f)$. Then $f(\xi)$ is a transcendental number if the following two conditions are satisfied:*

(i)′ *There exists a positive number $b \geqq 1$ such that*

$$\lim_{n\to\infty} (S(n)(e(n) + \log A(n) + \log M(n)))^b / e(n+1) = 0 ;$$

(ii)′ *There exist a positive number γ and a positive integer u_0 such that*

$$O(u|\xi) \leqq \gamma (\log u)^b \quad for \quad u \geqq u_0 ,$$

where b is the number given in (i).

PROOF. If ξ is algebraic $f(\xi)$ is transcendental by the theorem of Cijsouw and Tijdeman [5]. If ξ is transcendental, $f(\xi)$ is also transcendental by Theorem 3. This completes the proof.

REMARK. For any algebraic ξ, the condition (ii)′ is satisfied by $b = 1$. Hence the above corollary may be regarded as a generalization of the theorem of Cijsouw and Tijdeman.

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