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LARGE TIME ASYMPTOTICS FOR FUNDAMENTAL SOLUTIONS OF DIFFUSION EQUATIONS

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1. Introduction. The purpose of the present paper is to give asymptotic expansions as $t \to \infty$ of the fundamental solution of a diffusion equation in R^n.

Let

$$A = \sum_{j,k=1}^{n} a_{jk}(x)\partial_j\partial_k + \sum_{j=1}^{n} b_j(x)\partial_j + c(x) \tag{1.1}$$

be an elliptic operator satisfying the following condition (A). Here $\partial_j = \partial/\partial x_j$.

(A. I) There exists a positive constant c_0 such that $\sum_{j,k} a_{jk}(x)\xi_j\xi_k \geqq c_0|\xi|^2$ for all $x, \xi \in R^n$.

(A. II) The functions $a_{jk}(x)$, $b_j(x)$, $c(x)$ are real-valued bounded functions on R^n which are uniformly Hölder continuous with exponent θ $(0 < \theta \leqq 1)$.

(A. III) There exist positive constants ρ and M such that for all $x \in R^n$

$$\sum_{j,k=1}^{n} |a_{jk}(x) - \delta_{jk}| + \sum_{j=1}^{n} \langle x\rangle |b_j(x)| + \langle x\rangle^2 |c(x)| \leqq M\langle x\rangle^{-\rho}, \tag{1.2}$$

where δ_{jk} is Kronecker's delta and $\langle x\rangle = (1 + |x|^2)^{1/2}$. Let $U(t, x, y)$ be the fundamental solution of the diffusion equation

$$\partial_t U(t, x, y) = AU(t, x, y) \text{ in } (0, \infty) \times \boldsymbol{R}^n, \quad U(0, x, y) = \delta(x - y), \tag{1.3}$$

where $\partial_t = \partial/\partial t$ and $\delta(z)$ is the delta function. For σ in R^1, we denote by $[\sigma]$ the largest integer smaller than or equal to σ. One of our main results is the following theorem.

THEOREM 1.1. *Let $c(x) \equiv 0$ and $U(t, x, y)$ be the corresponding fundamental solution. Then for any σ with $0 \leqq \sigma < \rho/2$ there hold the following formulas for all $t > 1$ and $(x, y) \in R^{2n}$:*

(i) *For n odd,*

$$U(t, x, y) = \sum_{j=0}^{[\sigma]} t^{-n/2-j} U_j(x, y) + \tilde{U}_\sigma(t, x, y), \tag{1.4}$$

$$|U_j(x, y)| \leqq M_j(\langle x\rangle + \langle y\rangle)^{2j}, \tag{1.5}$$

$$(1.6)\qquad |\partial_t^l \tilde{U}_\sigma(t, x, y)| \leqq M_{\sigma l} t^{-n/2-\sigma-l}(\langle x\rangle + \langle y\rangle)^{2\sigma}\,, \qquad l \geqq 0\,.$$

Here M_j and $M_{\sigma l}$ are positive constants independent of $t, x,$ and y.

(ii) *For n even,*

$$(1.7)\qquad U(t, x, y) = \sum_{j=0}^{[\sigma]} \sum_{k=0}^{[2j/n]} t^{-n/2-j}(\log t)^k U_{jk}(x, y) + \tilde{U}_\sigma(t, x, y)\,,$$

where $\tilde{U}_\sigma$ satisfies (1.6) *and U_{jk} satisfies the estimate*

$$(1.8)\qquad |U_{jk}(x, y)| \leqq M_j(\langle x\rangle + \langle y\rangle)^{2j-nk}\,.$$

Furthermore, $U_j(x, y)$ and $U_{jk}(x, y)$ are of the form $\sum_l f_l(x)g_l(y)$, where $\sum_l$ is a finite sum. In particular, $U_0(x, y)$ for odd $n \geqq 3$ or $U_{00}(x, y)$ for even n is equal to a uniformly Hölder continuous function $U_0(y)$ satisfying and determined uniquely by

$$(1.9)\qquad A^* U_0(y) \equiv \Big(\sum_{j,k=1}^{n} \partial_j\partial_k a_{jk}(y) - \sum_{j=1}^{n} \partial_j b_j(y)\Big) U_0(y) = 0 \quad in \quad R^n\,,$$

$$(1.10)\qquad U_0(y) = (4\pi)^{-n/2} + o(1) \quad \text{as} \quad |y| \to \infty\,,$$

$$(1.11)\qquad U_0(y) > 0\,.$$

Here (1.9) *must be considered in a distribution sense. For $n = 1$, $U_0(x, y)$ is equal to $U_0(y)$ defined by*

$$(1.12)\qquad U_0(y) = \pi^{-1/2} a(y)^{-1} \exp\Big[\int_y^\infty b(z)dz\Big]\Big(1 + \exp\Big[\int_{-\infty}^{\infty} b(z)dz\Big]\Big)^{-1}\,,$$

where $a(x) = a_{11}(x)$ and $b(x) = -b_1(x)/a_{11}(x)$.

Theorem 1.1 will be proved in Section 4. Asymptotic expansions of the fundamental solutions for the case $c(x) \not\equiv 0$ shall be given in Sections 5 and 6. We use the results there in [9] in order to solve a problem of Simon [10].

Theorem 1.1 is useful in obtaining limit theorems for the diffusion process X_t with the infinitesimal generator A. Here we give only one application.

APPLICATION 1.2. Let f be a bounded measurable function on R^2 which has compact support. Then Theorem 1.1 shows that for any $s > 0$

$$(1.13)\qquad \int_0^\infty e^{-st} e^{tA} f(x) dt = C_0 \log(1/s) + g(x) + \varepsilon(s, x)\,,$$

where $e^{tA}f(x) = \int U(t, x, y)f(y)dy$, $C_0 = \int U_0(y)f(y)dy$, and for some $\delta > 0$

$$\langle x\rangle^{-\delta} g(x) \in L_\infty\,, \quad \lim_{s\downarrow 0} \|\langle x\rangle^{-\delta}\varepsilon(s, x)\|_{L_\infty} = 0$$

Thus Theorem 1 in [1, p. 447] and Theorem 1 in [5, p. 804] yield the following limit theorems (i) and (ii), respectively.

(i) If $f \geqq 0$ and $C_0 > 0$, then

$$\lim_{t\to\infty} P\left\{(C_0 \log t)^{-1}\int_0^t f(X_\tau)d\tau > r\right\} = e^{-r}\,, \qquad r > 0\,. \tag{1.14}$$

(ii) If $C_0 = 0$ and $C_1 \equiv \int U_0(y)f(y)g(y)dy \neq 0$, then

$$\lim_{t\to\infty} P\left\{(C_1 \log t)^{-1/2}\int_0^t f(X_\tau)d\tau > r\right\} = 2^{-1}\int_r^\infty e^{-|u|}du\,, \quad r \in R^1\,. \tag{1.15}$$

Asymptotic behavior as $t\to\infty$ of solutions of diffusion equations in R^n with $c(x)\to 0$ as $|x|\to\infty$ has been investigated to some extent. Concerning the problem whether the diffusion process X_t for $A = \sum a_{jk}(x)\partial_j\partial_k + \sum b_j(x)\partial_j$ is recurrent or transient, some criteria of integrability near $t = \infty$ of the fundamental solution $U(t, x, y)$ were given in [3] and [4] (see also references there). Simon [10] gave the rate of divergence of the norm of e^{tA}, $A = \Delta + c(x)$ in $R^n (n \geqq 3)$, as a map from L_∞ to L_∞. These results are closely related to the problem of determining the leading term of the asymptotic expansion as $t\to\infty$ of the fundamental solution. As for the one dimensional case, the leading terms of the asymptotic formulas were given by many mathematicians (see [2], [4], [5], [11], and references there). Especially Éskin [2] gave the formula (1.12) for $b = 0$. Little attension, however, seems to have been paid to the higher dimensional case. The aim of this paper is to give complete asympotic expansions for the higher dimensional case. The complete asymptotic expansions given are new even for the one dimensional case.

The rest of this paper is organized as follows. In Section 2 we give some lemmas for the free resolvent $R_0(z) = (z - \Delta)^{-1}$. In Section 3 we investigate by modifying the method employed in [8] spectral properties of the resolvent $R(z) = (z - A)^{-1}$. Using the results in Section 3 we prove Theorem 1.1 in Section 4, where further properties of the fundamental solution are also given. The fundamental solutions for $c(x) \leqq 0$ are investigated in Section 5. When $n \leqq 2$, there is an essential difference between the expansions in Theorem 1.1 and those for $c(x) \leqq 0$ (see Theorems 5.4 and 5.5). Section 6 is devoted to the investigation of the case that $c(x) > 0$ in a non-empty open set.

2. The free resolvent.

We write $D = (-i\partial_1, \cdots, -i\partial_n)$ and $\langle D\rangle = (1 - \Delta)^{1/2}$. For $\tau, s \in R^1$ and $1 \leqq p \leqq \infty$,

$$W_p^{\tau,s} = \{f;\ \|f\|_{W_p^{\tau,s}} \equiv \|\langle x\rangle^s\langle D\rangle^\tau f(x)\|_{L_p(R^n)} < \infty\}\,. \tag{2.1}$$

We write $L_p^s = W_p^{0;s}$. For Banach spaces X and Y, $B(X, Y)$ and $C(X, Y)$ denote the Banach spaces of all bounded linear operators and compact ones from X to Y, respectively. We write $B(X) = B(X, X)$. For $0 < \delta \leqq \pi$, we put

$$\Sigma(\delta) = \{z \in \boldsymbol{C};\ |\arg z| < \delta\}\,. \tag{2.2}$$

Let $A_0 = \Delta$ and $R_0(z) = (z - A_0)^{-1}$ for $z \in \boldsymbol{C} \setminus (-\infty, 0]$. We first recall the well-known formula

$$R_0(z)g(x) = (2\pi)^{-n/2}\int (z^{1/2}|x-y|^{-1})^{n/2-1} K_{n/2-1}(z^{1/2}|x-y|)g(y)dy\,, \tag{2.3}$$

where $z^{1/2}|_{z=1} = 1$ and $K_{n/2-1}(\zeta)$ is the modified Bessel function of the second kind. That is, the function $L = (2\pi)^{-n/2}(z^{1/2}/w)^{n/2-1}K_{n/2-1}(z^{1/2}w)$ is given as follows:

(i) For $n = 1$,

$$L = \sum_{k=-1}^{\infty} d_{k/2} w^{k+1} z^{k/2}\,, \quad d_{k/2} = (-1)^{k+1}/2(k+1)!\,. \tag{2.4}$$

(ii) For odd $n \geqq 3$,

$$L = \sum_{k=0}^{\infty} d_{k/2} w^{k+2-n} z^{k/2}\,, \tag{2.5}$$

$$d_{k/2} = \begin{cases} (-1)^k/(4\pi k!)\,, & n = 3 \\ \dfrac{(-2)^{(n-3)/2}(-1)^k}{(4\pi)^{(n-1)/2}k!} \displaystyle\prod_{j=1}^{(n-3)/2}(k - 2j + 1)\,, & n \geqq 5\,. \end{cases}$$

(iii) For n even,

$$L = \sum_{j=0}^{n/2-2} d_j w^{2j+2-n} z^j + \sum_{j=n/2-1}^{\infty} e_j w^{2j+2-n}(\log w/2 + f_j) z^j + \sum_{j=0}^{\infty} c_j w^{2j} z^{n/2-1+j} \log z\,, \tag{2.6.1}$$

$$c_j = (-4\pi)^{-n/2} 4^{-j}/j!(n/2 - 1 + j)!\,, \tag{2.6.2}$$

$$d_j = (4\pi)^{-n/2} 2^{n-2} (-4)^{-j} (n/2 - 2 - j)!/j! \tag{2.6.3}$$

$$e_j = (-4\pi)^{-n/2} 2^{1-2k}/k!(n/2 - 1 + k)!\,, \qquad k = j - n/2 + 1\,, \tag{2.6.4}$$

$$f_j = \gamma - \frac{1}{2}\left(\sum_{m=1}^{k-1}\frac{1}{m} + \sum_{m=1}^{n/2-1+k}\frac{1}{m}\right), \tag{2.6.5}$$

where γ is Euler's constant and $\log z|_{z=1} = 0$. Here and everywhere else the convention is: $\sum_{m=j}^{k} a_m = 0$ when $k < j$.

The following lemma can be shown by usual calculations for pseudo-differential operators. (For pseudo-differential operators, see [6].)

LEMMA 2.1. *$R_0(z)$ is a holomorphic function on $\Sigma(\pi)$ to $B(W_p^{\tau,s}, W_p^{\tau+m,s})$ for any $\tau, s \in R^1$, $0 \leqq m \leqq 2$ when $1 < p < \infty$ and $0 \leqq m < 2$ when $p = 1$ or ∞.*

We have the following lemma for the operator $E(t)$ defined by

$$E(t)f(x) = (4\pi t)^{-n/2}\int \exp[-|x-y|^2/4t]f(y)dy \ . \tag{2.7}$$

LEMMA 2.2. (i) *$E(t)$ can be extended to a holomorphic function on $\Sigma(\pi/2)$ to $B(W_p^{\tau,s}, W_p^{\tau,s})$ for any $\tau, s \in R^1$ and $1 \leqq p \leqq \infty$.* (ii) *Let α be a multi-index, $\tau \geqq 0$, $1 \leqq p \leqq \infty$, $1/p' = 1 - 1/p$, $0 \leqq \sigma < n/2$, $2\sigma - n/p < s < n/p'$, and $0 < \delta < \pi/2$. Then there exists a constant M such that for all $t \in \Sigma(\delta)$*

$$\| \langle D\rangle^\tau D^\alpha E(t)\|_{B(p;s,s-2\sigma)} \leqq \begin{cases} M|t|^{-\sigma-|\alpha|/2}, & |t| > 1 \\ M|t|^{-(|\alpha|+\tau)/2}, & |t| \leqq 1, \end{cases} \tag{2.8}$$

where $B(p; s, s') = B(L_p^s, L_p^{s'})$.

PROOF. (i) is clear. We shall show (ii) only for $\alpha = 0$, $\tau = 0$, and $0 \leqq s \leqq 2\sigma$, since the proof for the other case is similar. With $g(t, x, y) = |(4\pi t)^{-n/2}\exp[-|x-y|^2/4t]|$, we obtain that for all $t \in \Sigma(\delta)$ and $x \in R^n$

$$\int g(t, x, y)dy \leqq M\ , \quad \sup_k \int_{\Omega_k} g(t, x, y)\langle y\rangle^{-n} dy \leqq M(1 + |t|)^{-n/2}\ ,$$

where $\Omega_k = \{y; 2^k \leqq \langle y\rangle < 2^{k+1}\}$, $k = 1, 2, \cdots$. This together with the interpolation theorem (cf. [p. 89, Proposition 3.1, 7]) shows that for any $0 \leqq r < n/2$ there exists a constant M_r such that

$$\int g(t, x, y)\langle y\rangle^{-2r}dy \leqq M_r(1 + |t|)^{-r}\ , \quad t \in \Sigma(\delta)\ , \quad x \in R^n\ . \tag{2.9}$$

With $s' = s - 2\sigma$, we have by (2.9)

$$\begin{aligned} \| E(t)f\|_{L_p^{s'}} &\leqq \left(\sup_y \int \langle x\rangle^{s'p} g(t, x, y)dx\right)^{1/p} \\ &\quad \times \left(\sup_x \int g(t, x, y)\langle y\rangle^{-sp'}dy\right)^{1/p'} \left(\int |\langle y\rangle^s f(y)|^p dy\right)^{1/p} \\ &\leqq M(1 + |t|)^{-\sigma}\| f\|_{L_p^s}\ . \end{aligned}$$

q.e.d.

As for the asymptotic expansion of $R_0(z)$ as $z \to 0$ with $z \in \Sigma(\delta)$, $0 < \delta < \pi$, we have the following lemmas.

LEMMA 2.3. (i) *For a multi-index α and σ with $|\alpha|/2 \leqq \sigma < (n + |\alpha|)/2$ and $|\alpha| \leqq 2$, one has*

$$D^\alpha R_0(z) = \sum_{j=0}^{[\sigma]-1} z^j D^\alpha G_j + O(z^{\sigma-1}) \quad as \quad z \to 0 \tag{2.10}$$

with $z \in \Sigma(\delta)$ *in* $B(W_p^{\tau,s}, W_p^{\tau',s'})$, *where* $\tau' - \tau \leqq 2 - |\alpha|$ *when* $1 < p < \infty$ *and* $\tau' - \tau < 2 - |\alpha|$ *when* $p = 1$ *or* ∞, *and*

$$(2.11)\quad \begin{aligned} &2\sigma - |\alpha| - n/p < s < n/p', \quad s' = s - 2\sigma + |\alpha|, \quad 1 < p < \infty; \\ &2\sigma - |\alpha| - n/p < s < n/p', \quad s' = s - 2\sigma + |\alpha|, \quad p = 1 \text{ or } \infty, \\ &\qquad\qquad \sigma > |\alpha|/2 \text{ or } |\alpha| < 2; \\ &-n/p < s < n/p', \quad s' < s, \quad p = 1 \text{ or } \infty, \quad \sigma = |\alpha|/2 = 1; \\ &s/p - s'/p' = 0 \text{ or } 2\sigma - |\alpha| - n, \quad s' < s - 2\sigma + |\alpha|, \quad p = 1 \text{ or } \infty. \end{aligned}$$

Here and in what follows p' *is the conjugate of* p: $1/p' = 1 - 1/p$. *Furthermore,* G_j $(0 \leqq j \leqq n/2 - 1)$ *is an integral operator with kernel* $d_j|x - y|^{2-n+2j}$, *where the constant* d_j *is given by* (2.5) or (2.6.3).

(ii) *For* $\sigma \geqq (n + |\alpha|)/2$, *one has*

$$(2.12)\quad D^\alpha R_0(z) = \sum_{j=0}^{[\sigma]-1} z^j D^\alpha G_j + \sum_{j=0}^{[\sigma-n/2]} z^{n/2-1+j}(\log z)^{\varepsilon(n)} D^\alpha F_j + o(z^{\sigma-1})$$

as $z \to 0$ *in* $B(W_p^{\tau,s}, W_p^{\tau',s'})$, *where* τ, τ', p *are the same as in* (i),

$$(2.13)\quad s > 2\sigma - |\alpha| - n/p, \quad s' < -2\sigma + |\alpha| + n/p',$$

$\varepsilon(n) = 1$ *for* n *even and* $\varepsilon(n) = 0$ *for* n *odd,* $z^{n/2}|_{z=1} = 1$ *and* $\log z|_{z=1} = 0$. *Here* F_j *is an integral operator with kernel* $c_j|x - y|^{2j}$, *where the constant* c_j *is given by* (2.6.2) *for* n *even, and is equal to* $d_{n/2-1+j}$ *given by* (2.4) *or* (2.5) *for* n *odd;* G_j $(j > n/2 - 1)$ *for* n *odd is an integral operator with kernel* $d_j|x - y|^{2-n+2j}$, *where the constant* d_j *is given by* (2.4) *or* (2.5); *and* G_j $(j \geqq n/2 - 1)$ *for* n *even is an integral operator with kernel*

$$e_j|x - y|^{2-n+2j}(\log|x - y|/2 + f_j),$$

where e_j *and* f_j *are the constants given by* (2.6).

(iii) *Let* $\sigma \geqq (n + |\alpha|)/2$, $\sigma' = [\sigma]$ *for* $\sigma \notin \boldsymbol{Z}$ *and* $\sigma' = \sigma - 1$ *for* $\sigma \in \boldsymbol{Z}$, $\varepsilon(n, \sigma) = 1$ *for* n *even and* $\sigma \in \boldsymbol{Z}$, *and* $\varepsilon(n, \sigma) = 0$ *otherwise. Then one has*

$$(2.14)\quad D^\alpha R_0(z) = \sum_{j=0}^{\sigma'-1} z^j D^\alpha G_j + \sum_{j=0}^{[\sigma-n/2]} z^{n/2-1+j}(\log z)^{\varepsilon(n)} D^\alpha F_j + O(z^{\sigma-1}(\log z)^{\varepsilon(n,\sigma)})$$

as $z \to 0$ *in* $B(W_p^{\tau,s}, W_p^{\tau',s'})$, *where* $\tau' - \tau < 2 - |\alpha|$, $p = 1$ *or* ∞,

$$(2.15)\quad \begin{aligned} &s = 2\sigma - |\alpha| - n \quad \text{and} \quad s' < -2\sigma + |\alpha| \quad \text{for} \quad p = 1, \\ &s > 2\sigma - |\alpha| \quad \text{and} \quad s' = -2\sigma + |\alpha| + n \quad \text{for} \quad p = \infty. \end{aligned}$$

PROOF. The formulas for G_j and F_j follow from (2.3)~(2.6).

We first show (i) for $\sigma > [\sigma] \geqq 1$. With the notation (2.7) we have

$$(2.16)\quad R_0(z) = \int_0^\infty E(t)e^{-zt}dt$$

for $\operatorname{Re} z > 0$. Writing $k = [\sigma]$, we have

$$(2.17) \qquad R_0(z) = \sum_{j=0}^{k-1} z^j G_j + \int_0^\infty E(t) f_k(zt)dt ,$$

$$G_j = \int_0^\infty E(t)(-t)^j dt/j! , \quad f_k(zt) = e^{-zt} - \sum_{j=0}^{k-1} (-zt)^j/j! .$$

For $z = |z|e^{2i\phi}$ with $|\phi| < \delta/2 < \pi/2$ and $0 < |z| < 1$, we obtain that

$$\begin{aligned} &\langle D\rangle^{\tau'-\tau} D^\alpha \int_0^\infty E(t) f_k(zt)dt \\ &= \left(\int_0^1 + \int_1^{1/|z|} + \int_{1/|z|}^\infty\right) \langle D\rangle^{\tau'-\tau} D^\alpha E(te^{-i\phi}) f_k(|z|te^{i\phi})dt \\ &\equiv I_1 + I_2 + I_3 . \end{aligned}$$

By (2.8) and Taylor's remainder estimate,

$$\| I_1 \|_{B(p;s,s')} \leqq \int_0^1 Mt^{-(\tau'-\tau+|\alpha|)/2}(|z|t)^k dt \leqq M'|z|^k .$$

Similarly,

$$\| I_2 \|_{B(p;s,s')} \leqq \int_1^{1/|z|} Mt^{-\sigma}(|z|t)^k dt \leqq M|z|^{\sigma-1} \int_{|z|}^1 t^{k-\sigma} dt \leqq M'|z|^{\sigma-1} .$$

On the other hand,

$$\begin{aligned} \| I_3 \|_{B(p;s,s')} &\leqq \int_{1/|z|}^\infty Mt^{-\sigma}\left(1 + \sum_{j=0}^{k-1} (|z|t)^j/j!\right)dt \\ &\leqq M'|z|^{\sigma-1} \int_1^\infty t^{-(\sigma-k)-1} dt = M''|z|^{\sigma-1} . \end{aligned}$$

Hence

$$\int_0^\infty E(t) f_k(zt)dt = O(z^{\sigma-1}) \quad \text{as} \quad z \to 0 \quad \text{with} \quad z \in \Sigma(\delta)$$

in $B(W_p^{\tau,s}, W_p^{\tau',s'})$. The same argument as above shows that

$$\langle D\rangle^{\tau'-\tau} D^\alpha G_j \in B(p; s, s') , \quad j = 0, \cdots, k-1 ,$$

except for the case that $j = 0$, $\tau' - \tau + |\alpha| = 2$, and $1 < p < \infty$.

In order to complete the proof of (i) for σ with $\sigma > [\sigma] \geqq 1$ it is sufficient for us to show that

$$(2.18) \qquad \langle D\rangle^{\tau'-\tau} D^\alpha G_j \in B(p; s, s')$$

for $j = 0$, $\tau' - \tau + |\alpha| = 2$, $1 < p < \infty$, and $2 - |\alpha| - n/p < s < n/p'$. But let us show (2.18) for every τ', τ, α, p, s, s' satisfying the conditions in (i) with $\sigma = j + 1$, which is necessary for us to prove (i) for $\sigma = [\sigma]$. The proof is somewhat long.

Choose a C_0^∞-function ϕ such that $\phi(\xi)=1$ in a neighborhood of zero, and set

$$P=(1-\phi(D))\langle D\rangle^{\tau'-\tau}D^\alpha G_j\,,\quad Q=\phi(D)\langle D\rangle^{\tau'-\tau}D^\alpha G_j\,.$$

Then P is a classical pseudo-differential operator of order $\tau'-\tau+|\alpha|-2-2j$, and Q is a convolution operator with kernel which is a C^∞-function majorized by a constant multiple of $\langle x-y\rangle^{2j+2-(n+|\alpha|)}$. Since $\langle x\rangle^{s'}P\langle x\rangle^{-s}$ is also a classical pseudo-differential operator of order $\tau'-\tau+|\alpha|-2-2j$ ($\leqq 0$ if $1<p<\infty$, and <0 if $p=1$ or ∞), the L_p-boundedness theorem for pseudo-differential operators (cf. [6]) yields $P\in B(p;s,s')$. Thus we have only to show that there is a constant M such that

$$\left\|\int\langle x-y\rangle^{a-n}f(y)\,dy\right\|_{L_p^{s'}}\leqq M\|f\|_{L_p^s}\,,\qquad f\in L_p^s, \tag{2.18'}$$

where $a=2j+2-|\alpha|$.

First we show (2.18') for p, s, s' with $1<p\leqq 2$, $a-n/p<s<n/p'$, and $s'=s-a$. With $\Omega_j=\{y\in R^n;\ 2^j\leqq\langle y\rangle<2^{j+1}\}$, $j=0,1,\cdots$, we have by Hölder's inequality that

$$\begin{aligned}&\left|\langle x\rangle^{-n/p}\int_{\Omega_j}\langle x-y\rangle^{a-n}f(y)dy\right|^p\\&\leqq\langle x\rangle^{-n}\left(\int_{\Omega_j}\langle x-y\rangle^{(a-n)p'}\langle y\rangle^{(n/p-a)p'}dy\right)^{p/p'}\left(\int_{\Omega_j}|\langle y\rangle^{a-n/p}f(y)|^pdy\right).\end{aligned}$$

Noting that $p\leqq p'$ we have by Hölder's inequality that

$$\begin{aligned}I&\equiv\int_{\Omega_k}\langle x\rangle^{-n}\left(\int_{\Omega_j}\langle x-y\rangle^{(a-n)p'}\langle y\rangle^{(n/p-a)p'}dy^{p/p'}dx\right.\\&\leqq\left(\int_{\Omega_k}\langle x\rangle^{-n}dx\right)^{1-p/p'}\left(\int_{\Omega_k}\int_{\Omega_j}\langle x\rangle^{-n}\langle x-y\rangle^{(a-n)p'}\langle y\rangle^{(n/p-a)p'}dxdy\right)^{p/p'}.\end{aligned}$$

Split the domain of the above double integral into two parts: $\{|x-y|\leqq|y|/2\}$ and $\{|x-y|>|y|/2\}$, use the fact that if $|x-y|\leqq|y|/2$, then $\langle x\rangle^{-n}\leqq\langle y/2\rangle^{-n}$, and if $|x-y|>|y|/2$, then $\langle x-y\rangle^{(a-n)p'}<\langle y/2\rangle^{(a-n)p'}$, and reduce the double integral to single integrals. Then we obtain that I is estimated by a constant M^p independent of j and k. Hence

$$\left(\int_{\Omega_k}\left|\langle x\rangle^{s-a}\int_{\Omega_j}\langle x-y\rangle^{a-n}f(y)\,dy\right|^pdx\right)^{1/p}\leqq M\left(\int_{\Omega_j}|\langle y\rangle^s f(y)|^pdy\right)^{1/p},$$

where $s=a-n/p$. Similarly, we get the above estimate for $s=n/p'$. Thus the interpolation theorem ([7, Proposition 3.1]) yields (2.18') for $1<p\leqq 2$. This together with duality argument shows (2.18') for $2\leqq p<\infty$. Second we treat the case that $p=1$, $a>0$, $a-n<s<0$, and $s'=s-a$. Since $\langle x-y\rangle^{a-n}\leqq|x-y|^{a-n}$, Sobolev's inequality yields the estimate

$$\left\|\int_{R^n} \langle x-y\rangle^{a-n} f(y)dy\right\|_{L_{n/(n-a)}} \leqq M\|f\|_{L_1}, \qquad f \in L_1 .$$

This implies that

$$\int_{\Omega_k} \langle x\rangle^{-a}\left|\int \langle x-y\rangle^{a-n} f(y)\,dy\right| dx \leqq M'\|f\|_{L_1},$$

where M' is a constant independent of k. On the other hand,

$$\begin{aligned}\int_{\Omega_k} \langle x\rangle^{-n}&\left|\int_{R^n} \langle x-y\rangle^{a-n} f(y)\,dy\right| dx \\ &\leqq \int_{\Omega_k} \langle x\rangle^{-n}dx\int_{R^n} \langle y/2\rangle^{a-n}|f(y)|\,dy \\ &\quad + \int_{R^n}\left(\int_{|x|\leqq|y|/2} \langle x\rangle^{a-n}dx\right)\langle y/2\rangle^{-n}|f(y)|\,dy \\ &\leqq M''\|f\|_{L_1} .\end{aligned}$$

Thus the interpolation theorem shows (2.18′) for $p=1$ and $a>0$, from which (2.18′) for $p=\infty$ and $a>0$ is derived. Third we treat the case that $p=1$, $a=0$, $-n<s<0$ and $s'<s$. For any $\varepsilon>0$, we have by Hölder's inequality that

$$\sup_y \int_{R^n} \langle x\rangle^{-\varepsilon}\langle x-y\rangle^{-n}dx \leqq \int_{R^n} \langle x\rangle^{-n-\varepsilon}dx < \infty .$$

This implies (2.18′) for $p=1$, $a=0$, $s=0$, and $s'<0$. On the other hand, similar calculations yield (2.18′) for $p=1$, $a=0$, and $s=-n>s'$. This implies (2.18′) for every s with $-n<s<0$ when $p=1$ and $a=0$ from which (2.18′) for $p=\infty$ and $a=0$ follows. The estimate (2.18′) for the other cases can be shown similarly. This completes the proof of (2.18′), and so the proof of (i) for $\sigma>[\sigma]\geqq 1$.

Now let us show (i) for $0\leqq\sigma\leqq 1$. Choose a C_0^∞-function ψ such that $\hat{\psi}(0)=1$ and $D_\xi^\alpha\hat{\psi}(0)=0$ for $1\leqq|\alpha|\leqq 2n+1$, where $\hat{\psi}$ is the Fourier transform of ψ. Clearly, $(1-\hat{\psi}(D))\langle D\rangle^{\tau'-\tau}D^\alpha R_0(z)\in B(p;\,s,\,s')$. Elementary calculatians show that the operator $\hat{\psi}(D)\langle D\rangle^{\tau'-\tau}D^\alpha R_0(z)$ is a convolution operator with kernel which is a C^∞-function majorized by a constant multiple of

$$\exp(-\varepsilon|z|^{1/2}\langle x-y\rangle)\langle x-y\rangle^{2-n-|\alpha|},$$

where ε is a positive constant smaller than $((1+\cos\delta)/2)^{1/2}$. Thus the same argument as in the proof of (2.18′) shows that

$$\hat{\psi}(D)\langle D\rangle^{\tau'-\tau}D^\alpha R_0(z) = O(z^{\sigma-1}) \quad \text{as} \quad z\to 0 \quad \text{with} \quad z\in\Sigma(\delta)$$

in $B(p; s, s')$ This completes the proof of (i) for $0 \leqq \sigma \leqq 1$.

It remains to show (i) for σ with $2 \leqq \sigma = [\sigma] < (n + |\alpha|)/2$. Note that

$$\langle D\rangle^{\tau'-\tau}D^{\alpha}\Big(R_0(z) - \sum_{j=0}^{\sigma-2} z^j G_j\Big)f(x)$$
$$= (-z)^{\sigma-1}\int \langle\xi\rangle^{\tau'-\tau}\xi^{\alpha}|\xi|^{2-2\sigma}(z + |\xi|^2)^{-1}\hat{f}(\xi)e^{ix\xi}d\xi/(2\pi)^n$$

for all C_0^∞-function f. Thus the same argument as in the proof of (2.18) shows that the above operator belongs to $B(p; s, s')$ for p, s, s' satisfying (2.11). This together with (2.18) shows (i) for $\sigma = [\sigma] \geqq 2$. The proof of (i) is complete.

We now proceed to the proof of (ii). Put

$$E_{k+1}(t) = E(t) - \sum_{j=0}^{k} t^{-n/2-j}H_j\,, \quad k = 0, 1, \cdots,$$

where H_j is the integral operator with kernel $|x - y|^{2j}/[(4\pi)^{n/2}(-4)^j j!]$. Then an argument similar to that in the proof of (2.8) shows that for σ with $(n + |\alpha|)/2 + k \leqq \sigma < (n + |\alpha|)/2 + k + 1$ and s, s' satisfying (2.13) there exist positive constants ε and M such that for all $t \in \Sigma(\partial)$

$$\|\langle D\rangle^{\tau'-\tau}D^{\alpha}E_{k+1}(t)\|_{B(p;s,s')} \leqq \begin{cases} M|t|^{-a}\,, & |t| \leqq 1 \\ M|t|^{-\sigma-\varepsilon}\,, & |t| > 1\,, \end{cases} \tag{2.19}$$

where $a = \max((\tau' - \tau + |\alpha|)/2, n/2 + k)$. Let n be odd and $n/2 + k \leqq \sigma \leqq (n + 1)/2 + k$. Then we have, in view of (2.17),

$$R_0(z) - \sum_{j=0}^{(n-3)/2} z^j G_j = \int_0^\infty E(t)f_{(n-1)/2}(zt)dt$$
$$= \sum_{j=0}^{k}\int_0^\infty t^{-(n/2)-j}f_{(n-1)/2+j}(zt)dtH_j + \sum_{j=0}^{k-1}\int_0^\infty \frac{E_{j+1}(t)\,(-zt)^{(n-3)/2+j}}{((n-3)/2 + j)!}dt$$
$$+ \int_0^\infty E_{k+1}(t)f_{(n-1)/2+k}(zt)dt\,.$$

This together with (2.19) shows (ii) for odd n and σ with $n/2 + k \leqq \sigma < (n + 1)/2 + k$. In treating the case $(n + 1)/2 + k \leqq \sigma < n/2 + k + 1$, we have only to decompose $f_{(n-1)/2+k}(zt)$ into $(-zt)^{(n-3)/2+k}/((n - 3)/2 + k)! + f_{(n-1)/2+k+1}(zt)$. This completes the proof of (ii) for n odd. Let n be even and $n/2 + k \leqq \sigma < n/2 + k + 1$. With $E_0(t) = E(t)$, we have

$$R_0(z) - \sum_{j=0}^{n/2-2} z^j G_j$$
$$= \sum_{j=0}^{k}\Big\{\int_0^1 E_j(t)\frac{(-zt)^{n/2-1+j}}{(n/2 - 1 + j)!}dt + \int_1^\infty E_{j+1}(t)\frac{(-zt)^{n/2-1+j}}{(n/2 - 1 + j)!}dt\Big\}$$

$$+\sum_{j=0}^{k}\left\{\int_0^1 t^{-n/2-j}f_{n/2+j}(zt)dt+\int_1^\infty t^{-n/2-j}f_{n/2-1+j}(zt)\,dt\right\}H_j$$

$$+\int_0^\infty E_{k+1}(t)f_{n/2+k}(zt)dt\ .$$

This together with (2.19) shows (ii) for n even.

The assertion (iii) can be shown similarly. q.e.d.

For Banach spaces X_0 and X_1 imbedded in a Banach space X, X_0+X_1 denotes a Banach space defined by

$$\begin{aligned}(2.20)\qquad & X_0+X_1=\{x\in X;\ x=x_0+x_1,\ x_0\in X_0,\ x_1\in X_1\}\ ,\\ & \|x\|_{X_0+X_1}=\inf\{\|x_0\|_0+\|x_1\|_1;\ x=x_0+x_1\}\ ,\end{aligned}$$

where $\|\cdot\|_j$ stands for the the norm of X_j. In the sequel we shall use the spaces $B(L_1^s, L_\infty)+B(L_1, L_\infty^{-s})$ for $s\geqq 0$.

LEMMA 2.4. (i) *For a multi-index α and σ with $|\alpha|/2<\sigma<(n+|\alpha|)/2$ and $|\alpha|\leqq 2$, one has*

$$(2.21)\qquad D^\alpha R_0(z)-\sum_{j=0}^{[\sigma]-1} z^j D^\alpha G_j=O(z^{\sigma-1})\quad\text{as}\quad z\to 0$$

in $B(L_p, L_q)$, where $1\leqq p\leqq q\leqq\infty$ and

$$(2.22)\qquad n(1/p-1/q)=2\sigma-|\alpha|\ .$$

(ii) *Let $\sigma>(n+|\alpha|)/2$, or $\sigma=(n+|\alpha|)/2\notin \boldsymbol{Z}$. Let $\varepsilon(n,\sigma)=1$ for n even and $\sigma\in\boldsymbol{Z}$, and $\varepsilon(n,\sigma)=0$ otherwise, and $\sigma'=[\sigma]$ for $\sigma\notin\boldsymbol{Z}$ and $\sigma'=\sigma-1$ for $\sigma\in\boldsymbol{Z}$. Then one has*

$$(2.23)\qquad D^\alpha R_0(z)-\sum_{j=0}^{\sigma'-1} z^j D^\alpha G_j-\sum_{j=0}^{[\sigma-n/2]} z^{n/2-1+j}(\log z)^{\varepsilon(n)}D^\alpha F_j=O(z^{\sigma-1}(\log z)^{\varepsilon(n,\sigma)})$$

as $z\to 0$ in $B(L_1^s, L_\infty)+B(L_1, L_\infty^{-s})$, where $s=2\sigma-n-|\alpha|$.

(iii) *Let $X=\{f\in L_1;\ F_0 f=0\}$. Then $R_0(z)|_X$ has a formula similar to* (2.23) *in $B(L_1^s\cap X, L_\infty)+B(X, L_\infty^{-(s-1)^+})$. Here $x^+=\max(x,0)$.*

(iv) *For an integer $\sigma=(n+|\alpha|)/2$,* (2.23) *holds in $B(W_1^{\tau,0}, W_\infty^{\tau',0})$, $\tau'<\tau$.*

PROOF. (i) follows from (2.17) and the inequality

$$(2.24)\qquad \|D^\alpha E(t)\|_{B(L_p,L_q)}\leqq M|t|^{-\alpha}\ ,\qquad t\in\Sigma(\delta)\ ,$$

for p and q satisfying (2.22). (ii) and (iii) follow from the decomposition of $E(t)$ used in the proof of Lemma 2.3 (ii) and the inequality $\langle x-y\rangle\leqq\langle x\rangle+\langle y\rangle$. (iv) is shown by the inequality

$$\| D^\alpha E(t) \|_{B(W_1^{\tau,0}, W_\infty^{\tau',0})} \leqq M|t|^{-(n+|\alpha|)/2} \min(|t|^{(\tau-\tau')/2}, 1) \,, \qquad t \in \Sigma(\delta) \,. \quad \text{q.e.d.}$$

We put

$$\tilde{R}_0(z) = R_0(z) - G_0 \quad \text{for} \quad n \geqq 3 \,, \tag{2.25}$$
$$\tilde{R}_0(z) = R_0(z) - z^{n/2-1}(\log z)^{\varepsilon(n)} F_0 - G_0 \quad \text{for} \quad n \leqq 2 \,.$$

LEMMA 2.5. *Let α be a multi-index with $|\alpha| \leqq 2$ and $\sigma > 1$. Then the following statements hold.*

(i) *For $\sigma < (n + |\alpha|)/2$, one has*

$$D^\alpha \tilde{R}_0(z) = \sum_{j=1}^{[\sigma]-1} z^j D^\alpha G_j + O(z^{\sigma-1}) \quad \text{as} \quad z \to 0 \tag{2.26}$$

in $B(W_p^{\tau,s} W_p^{\tau',s'})$, where $\tau' - \tau \leqq 4 - |\alpha|$ when $1 < p < \infty$ and $\tau' - \tau < 4 - |\alpha|$ when $p = 1$ or ∞, $s' = s - 2\sigma + |\alpha|$, and s is a constant satisfying (2.11).

(ii) *Let $m = 1$ for $n \leqq 2$ and $m = 0$ for $n \geqq 3$, and $\sigma \geqq (n + |\alpha|)/2$. Then one has*

$$D^\alpha \tilde{R}_0(z) = \sum_{j=1}^{[\sigma]-1} z^j D^\alpha G_j + \sum_{j=m}^{[\sigma-n/2]} z^{n/2-1+j}(\log z)^{\varepsilon(n)} D^\alpha F_j + o(z^{\sigma-1}) \tag{2.27}$$

as $z \to 0$ in $B(W_p^{\tau,s}, W_p^{\tau',s'})$, where $\tau' - \tau \leqq 4 - |\alpha|$ when $1 < p < \infty$ and $\tau' - \tau < 4 - |\alpha|$ when $p = 1$ or ∞, and s, s' satisfy (2.13).

(iii) *Lemma* 2.3(iii) *holds with obvious modifications.*

PROOF. Let ϕ be a C^∞-function on R^n such that $\phi(\xi) = 1$ for $|\phi| \geqq 2$ and $\phi(\xi) = 0$ for $|\phi| \leqq 1$. Then $\phi(D)\tilde{R}_0(z)$ is a pseudo-differential operator with symbol $-z\phi(\xi)[|\xi|^2(z + |\xi|^2)]^{-1}$, from which the lemma is derived.

q.e.d.

3. Resolvent expansion. In this section we give some results on the resolvent $R(z) = (z - A)^{-1}$. Throughout this section the operator

$$A = \sum_{j,k=1}^{n} a_{jk}(x)\partial_j\partial_k + \sum_{j=1}^{n} b_j(x)\partial_j$$

satisfies the assumption (A). Main results of this section are Theorems 3.10 and 3.12 below concerning asymptotic expansions of $R(z)$ as $z \to 0$.

We write $\Sigma(\delta, N) = \{z \in \boldsymbol{C};\ |\arg(z - N)| < \delta\}$ and $\Sigma(\delta) = \Sigma(\delta, 0)$. We start with following lemma.

LEMMA 3.1. *For any positive constant δ, θ', S with $\delta < \pi$ and $\theta' < \theta$ there exist $N \geqq 0$ and an operator-valued function $R(z)$ on $\Sigma(\delta, N)$ with the following properties*:

(i) *$R(z)$ is a holomorphic function on $\Sigma(\delta, N)$ to $B(W_p^{\tau,s}, W_p^{\tau+m,s})$ for any $|\tau| < \theta'$, $|s| \leqq S$, $0 \leqq m \leqq 2$ when $1 < p < \infty$ and $0 \leqq m < 2$ when*

$p=1$ or ∞, which satisfies

$$(z-A)R(z)f=f\,, \qquad f\in W_p^{\tau,s}\,, \tag{3.1}$$

$$R(z)(z-A)g=g\,, \qquad g\in W_p^{\tau+2,s}\,. \tag{3.2}$$

(ii) *For any $1<p<\infty$ there exists a constant M such that*

$$\|\langle D\rangle^m R(z)\|_{B(W_p^{\tau,s})}\leqq M|z|^{-1+m/2}\,, \qquad z\in\Sigma(\delta, N)\,, \tag{3.3}$$

for all $0\leqq m\leqq 2$, $|\tau|<\theta'$, and $|s|\leqq S$.

(iii) For any $\varepsilon>0$ there exists a constant M such that

$$\|\langle D\rangle^m R(z)\|_{B(W_p^{\tau,s})}\leqq M|z|^{-1+(m+\varepsilon)/2}\,, \quad p=1 \quad or \quad \infty\,, \quad z\in\Sigma(\delta, N)\,, \tag{3.4}$$

for all $0\leqq m\leqq 2-\varepsilon$, $|\tau|<\theta'$, and $|s|\leqq S$.

PROOF. Consider the operator $A^\sim=\Sigma_{j,k}a_{jk}^\sim(x, D)\partial_j\partial_k$, where $a_{jk}^\sim(x, D)$ is the regularizer of $a_{jk}(x)$ (see [6]). Then $A^\sim$ is an elliptic pseudo-differential operator whose symbol is estimated from below by $c_0|\xi|^2$. Thus we obtain by the standard calculus for pseudo-differential operators that there exist $N^\sim$ and $R^\sim(z)$ satisfying the properties in the lemma with A, $R(z)$, and N replaced by $A^\sim$, $R^\sim(z)$, and $N^\sim$, respectively. Since the order of the operator $A-A^\sim$ is less than 2, we can choose N so large that for all τ and s with $|\tau|<\theta'$ and $|s|\leqq S$, the norm of $(A-A^\sim)R^\sim(z)$ in $B(W_p^{\tau,s})$ is less than 1 if $z\in\Sigma(\delta, N)$. Hence $R(z)$ is given by

$$R(z)=\sum_{j=0}^{\infty}R^\sim(z)[(A-A^\sim)R^\sim(z)]^j$$

for $z\in\Sigma(\delta, N)$. q.e.d.

By virtue of this lemma the evolution operator e^{tA} for A is represented by

$$e^{tA}=\frac{1}{2\pi i}\int_{\gamma_N}R(z)e^{tz}dz\,, \tag{3.5}$$

where $\gamma_N=\{N+re^{i\phi};\ -\infty<r<0\}+\{N-re^{-i\phi};\ 0\leqq r<\infty\}$ for some $0<\phi<\pi/2$ and $N\geqq 0$. Theorem 1.1 will be shown by deformation of the contour in (3.5).

We write $A_0=\Delta$, $V=A-A_0$, and $R_0(z)=(z-A_0)^{-1}$.

LEMMA 3.2. *$R(z)$ is a meromorphic function on $\Sigma(\pi)$ to $B(W_p^{0,s}, W_p^{m,s})$ for any $s\in R^1$, $0\leqq m\leqq 2$ when $1<p<\infty$ and $0\leqq m<2$ when $p=1$ or ∞.*

PROOF. Lemma 3.1 shows that for any s and τ with $0<|\tau|<\theta$ there exists $N>0$ such that

$$(3.6)\quad (1+VR(N))(1-VR_0(N))f=(1-VR_0(N))(1+VR(N))f=f$$

for all $f\in W_p^{\tau,s}$. Since $R_0(z)=R_0(N)-(z-N)R_0(z)R_0(N)$,

$$(3.7)\qquad 1-VR_0(z)=(1-T(z))(1-VR_0(N))\,,$$
$$T(z)=(z-N)VR_0(z)R_0(N)(1+VR(N))\,.$$

By the assumption (A) and Lemma 2.1, $T(z)$ is a $C(L_p^s)$-valued holomorphic function on $\Sigma(\pi)$. Since $(1-T(N))^{-1}$ exists, this shows that $(1-T(z))^{-1}$ is a meromorphic function on $\Sigma(\pi)$ to $B(L_p^s)$. Hence

$$(3.8)\qquad R(z)=R_0(z)(1+VR(N))(1-T(z))^{-1}$$

is a meromorphic function on $\Sigma(\pi)$ to $B(W_p^{0,s}, W_p^{m,s})$. q.e.d.

LEMMA 3.3. *Every z with $\mathrm{Re}\,z\geqq 0$ and $z\neq 0$ is not a pole of $R(z)$.*

PROOF. We have only to show that if u in L_p^s satisfies $u=T(z)u$, then $u=0$. By the imbedding theorem, $u\in W_{p_1}^{\tau_1,s}$ for any τ_1 and p_1 with $\tau_1<\theta$ and $1/p_1>1/p-(2-\tau_1)/n$. Similarly, $u\in W_{p_2}^{\tau_2,s}$ for any τ_2 and p_2 with $\tau_2<\theta$ and $1/p_2>1/p_1-(2+\tau_1-\tau_2)/n$, and so on. Thus, $u\in W_\infty^{\theta',s}$ for any $0<\theta'<\theta$. Then we have by (A. III) that $u\in W_\infty^{\theta',s+\rho}$. Next, $u\in W_\infty^{\theta',s+2\rho}$, and so on. Hence we obtain that $u\in W_\infty^{\theta',\nu}$ for any $\nu>0$. Putting $v=R_0(z)(1+VR(N))u$, we have that $v\in C^2(R^n)$, $v(x)=o(1)$ as $|x|\to\infty$, and $Av=zv$. Setting $w(t,x)=\exp(\lambda t+i\mu t)v(x)$ with $z=\lambda+i\mu$, we have that

$$\partial_t w=\Big(\sum_{j,k}a_{jk}\partial_j\partial_k+\sum_j b_j\partial_j\Big)w\,,\quad w(0,x)=v(x)\,.$$

Since $\lambda\geqq 0$, the maximum principle implies that $\lambda=0$ if $v\neq 0$. (For the maximum principle, see, for example, [3].) Since $z\neq 0$, we have that $\mu\neq 0$ and $w(2\pi/\mu,x)=w(0,x)$. On the other hand, since $v(x)$ goes to zero as $|x|\to\infty$, $\mathrm{Re}\,w(0,x)$ attains the maximum or minimum at some point x^0. Thus the strong maximum principle for parabolic equations yields that $\mathrm{Re}\,w(t,x)=0$. Similarly, $\mathrm{Im}\,w(t,x)=0$. Hence $v(x)=0$, which implies that $u=0$. q.e.d.

LEMMA 3.4. $1-T(z)=1-T(0)+o(1)$ *as $z\to 0$ with $z\in\Sigma(\delta)$ in $B(L_p^s)$ for any $1\leqq p\leqq\infty$ and $1-n/p<s<n/p'+\rho$, where*

$$T(0)=NVG_0R_0(N)(1+VR(N))\,.$$

PROOF. Since $D^\alpha F_0=0$ for $|\alpha|\geqq 1$, the lemma follows from Lemma 2.3.

LEMMA 3.5. $(1-T(0))^{-1}$ *exists in $B(L_p^s)$, where*

$$1 - n/p < s < n/p' + \rho \quad \textit{when} \quad 1 < p \leqq \infty\,, \tag{3.9}$$
$$1 - n \leqq s < \rho \quad \textit{when} \quad p = 1\,.$$

PROOF. Since $T(0)$ is a compact operator by Lemmas 2.3 and 2.4, we have only to show that $1 - T(0)$ is injective. Let $u \in L_p^s$ satisfy

$$u = NVG_0R_0(N)(1 + VR(N))u\,. \tag{3.10}$$

The same argument as in the proof of Lemma 3.3 shows that $u \in W_\infty^{\theta',r}$ for any $0 < \theta' < \theta$ and $r < n + \rho$.

We first treat the case $n \geqq 2$. Putting $v = G_0(1 + VR(N))u$, we have that $v \in C^2(R^n)$ and $Av = 0$ on R^n. Furthermore, as $|x| \to \infty$

$$v(x) = O(|x|^{2-n}) \quad \text{for} \quad n \geqq 3\,, \tag{3.11}$$

$$v(x) = \lambda \log|x| + \mu + o(1) \quad \text{for} \quad n = 2\,, \tag{3.12}$$

where λ and μ are constants. Thus the maximum principle for elliptic equations shows that $v = 0$ when $n \geqq 3$, which implies that $u = 0$. This completes the proof for $n \geqq 3$. Next we consider the case $n = 2$. Since the coefficients of the operator A are real-valued, we may assume that v is real-valued and $\lambda, \mu \in R^1$. The maximum principle shows that for any $\varepsilon > 0$ there exists $r_0 > 0$ such that for all $r \geqq r_0$

$$\lambda \log r + \mu - \varepsilon < v(x) < \lambda \log r + \mu + \varepsilon\,, \qquad |x| < r\,.$$

This implies that $v(x) \equiv \mu$, from which we have that $u = (1 - VR_0(N))A_0\mu = 0$.

Finally we treat the case $n = 1$. Putting $v = 2(1 + VR(N))u$ and $w = (d/dx)G_0v$, we obtain that $v \in W_\infty^{\theta',r}$ for any $\theta' < \theta$ and $r < n + \rho$, and

$$w(x) = -\frac{1}{2}\int_{-\infty}^{x} v(y)\,dy + \frac{1}{2}\int_{x}^{\infty} v(y)dy\,, \tag{3.13}$$

$$w'(x) - b(x)w(x) = 0\,. \tag{3.14}$$

Here we have used the notation: $A = a(x)\{(d/dx)^2 - b(x)(d/dx)\}$. By (3.13),

$$\lim_{x\to\pm\infty} w(x) = \pm\lambda\,, \quad \lambda = -\frac{1}{2}\int_{-\infty}^{\infty} v(y)dy\,.$$

On the other hand, (3.14) yields $w(x) = -\lambda \exp\left[\int_{-\infty}^{x} b(y)\,dy\right]$. Thus

$$\lambda = -\lambda \exp\left[\int_{-\infty}^{\infty} b(y)dy\right].$$

This implies that $\lambda = 0$, for $b(y)$ is real-valued. Hence $w = 0$, from which we have that $u = (1 - VR_0(N))w'/2 = 0$. q.e.d.

We write $K=(1+VR(N))(1-T(0))^{-1}$. Obviously,

$$K=(1-VG_0)^{-1}\in B(L_p^s,\ W_p^{\tau,s})$$

where $\tau=0$ when $1<p<\infty$, $\tau<0$ when $p=1$ or ∞, $1-n/p<s<n/p'+\rho$ when $1<p\leqq\infty$, and $1-n\leqq s<\rho$ when $p=1$. With $\tilde{R}_0(z)$ defined by (2.25) we obtain the following lemma by Lemma 2.5.

LEMMA 3.6. *For any z in $\Sigma(\delta)$ with $|z|$ sufficiently small*

$$(1-V\tilde{R}_0(z)K)^{-1}=\sum_{j=0}^{\infty}(V\tilde{R}_0(z)K)^j \quad \text{in} \quad B(L_p^s)\ , \tag{3.15}$$

where $1\leqq p\leqq\infty$ and $1-n/p<s<n/p'+\rho$.

By virtue of this lemma $R(z)$ near $z=0$ is given by

$$R(z)=\sum_{j=0}^{\infty}R_0(z)K(V\tilde{R}_0(z)K)^j \tag{3.16}$$

in $B(L_p^s,\ W_p^{\tau,s})$, where $1\leqq p\leqq\infty$, $1-n/p<s<n/p'+\rho$, $\tau\leqq 2$ for $1<p<\infty$, and $\tau<2$ for $p=1$ or ∞. In order to get the asymptotic expansion of $R(z)$ as $z\to 0$ we need more precise information on $(1-V\tilde{R}_0(z)K)^{-1}$.

When $u=1$, K is given by

$$Kh(x)=\frac{h(x)}{a(x)}-b(x)\left\{H\exp\left[\int_{-\infty}^{x}b(y)dy\right]-\int_{-\infty}^{x}\exp\left[\int_{y}^{x}b(z)dz\right]\frac{h(y)}{a(y)}dy\right\}\ , \tag{3.17.1}$$

$$H=\int_{-\infty}^{\infty}\exp\left[\int_{y}^{\infty}b(z)dz\right]\frac{h(y)}{a(y)}dy\Big/\left\{1+\exp\left[\int_{-\infty}^{\infty}b(y)dy\right]\right\}\ . \tag{3.17.2}$$

Here we have used the notation: $A=a(x)\{(d/dx)^2-b(x)(d/dx)\}$. Since the formula can be shown by elementary computations, we omit the details. Using (3.17) we get:

LEMMA 3.7. *When $n=1$, $K\in B(L_p^s)$ for any p and s satisfying* (3.9).

LEMMA 3.8. *Let $n=1$. For $0<\sigma<\rho/2$ and $r<\rho$ one has*

$$(1-V\tilde{R}_0(z)K)^{-1}-1=\sum_{j=1}^{[2\sigma]}z^{j/2}C_j+O(z^\sigma) \quad \text{as} \quad z\to 0 \tag{3.18}$$

in $B(L_1^{2\sigma},\ L_1^r)+B(L_1,\ L_1^{r-2\sigma})$ (cf. (2.19) *and* (2.20)). *For $1<p\leqq\infty$,* (3.18) *holds in $B(L_p^s,\ L_p^r)+B(L_p,\ L_p^{r-s})$, $s>2\sigma+1/p'$, with $O(z^\sigma)$ replaced by $o(z^\sigma)$.*

PROOF. We have

$$VR_0(z)=1-a(x)+[z(a(x)-1)-a(x)b(x)(d/dx)]R_0(z)\ . \tag{3.19}$$

Thus, $V\tilde{R}_0(z)=z(a(x)-1)R_0(z)-a(x)b(x)(d/dx)(R_0(z)-G_0)$. Choosing τ so that $0<\tau\leqq\min(\sigma, 1/2)$, one has, in $B(L_1^{2\sigma}, L_1^r)+B(L_1, L_1^{r-2\sigma})$,

$$(V\tilde{R}_0(z)K)\sum_{j=[\sigma/\tau]+1}^{\infty}(V\tilde{R}_0(z)K)^j=o(z^\sigma) \quad \text{as} \quad z\to 0 .$$

On the other hand, inductive argument shows that for any $j\geqq 1$

$$(V\tilde{R}_0(z)K)^j=\sum_{k=0}^{[2\sigma]}z^{k/2}C_{jk}=O(z^\sigma) \quad \text{as} \quad z\to 0 \tag{3.20}$$

in $B(L_1^{2\sigma}, L_1^r)+B(L_1, L_1^{r-2\sigma})$. This proves (3.18). The estimate for $1<p\leqq\infty$ is derived similarly. q.e.d.

LEMMA 3.9. *Let n be odd and $0<\delta<\pi$. Then the following statements hold.*

(i) *For $0<\sigma<(\rho+n-1)/2$, one has*

$$(1-V\tilde{R}_0(z)K)^{-1}-1=\sum_{j=1}^{[2\sigma]}z^{j/2}C_j+o(z^\sigma) \quad as \quad z\to 0 \tag{3.21}$$

with $z\in\Sigma(\delta)$ in $B(L_p^s, L_p^r)$, where $C_j=0$ for j odd and $j/2<n/2-1$ and

$$1\leqq p\leqq\infty,\ s>2\sigma+1-n/p\ , \qquad r<\rho-2\sigma+\min(s, n/p') . \tag{3.22}$$

Furthermore, when $p=1, s=2\sigma+1-n$, and $r<\rho-2\sigma+\min(s, 0)$, (3.21) holds with $o(z^\sigma)$ replaced by $O(z^\sigma)$.

(ii) *For $0<\sigma<n/2$, one has*

$$(1-V\tilde{R}_0(z)K)^{-1}-\sum_{j=0}^{[2\sigma]}z^{j/2}C_j=O(z^\sigma) \quad as \quad z\to 0 \tag{3.23}$$

with $z\in\Sigma(\delta)$ in $B(L_1, L_q^r)$, where $C_0=1$, $n(1-1/q)=2$ and $r<\rho$.

(iii) *For $n/2\leqq\sigma<(\rho+n-1)/2$, one has (3.23) in $B(L_1^s, L_\infty^r)+B(L_1, L_\infty^{r-s})$, where $s=2\sigma-n$ and $r<\rho$.*

PROOF. (i) follows from (3.15) and Lemma 2.3 along the line given in the proof of Lemma 3.8. (ii) is derived by (2.21) for $|\alpha|=2$ and a similar estimate for $D^\alpha R_0(z)$, $|\alpha|=1$, in $B(L_1, L_q^{1+\varepsilon})$, $0<\varepsilon\ll 1$. (iii) is shown similarly. q.e.d.

THEOREM 3.10. *Let n be odd. Let $0<\delta<\pi$ and α be a multi-index with $|\alpha|\leqq 1$. Then the following statements hold.*

(i) *For $(1+|\alpha|)/2<\sigma<(\rho+n+|\alpha|)/2$, one has*

$$D^\alpha R(z)=\sum_{j=-1}^{[2\sigma]-2}z^{j/2}D^\alpha B_{j/2}+o(z^{\sigma-1}) \quad \text{as} \quad z\to 0 \tag{3.24}$$

with $z\in\Sigma(\delta)$ in $B(L_p^s, L_p^r)$, where

$$B_{n/2-1}=F_0K\ ,\quad B_{j/2}=0 \quad for \quad j \quad odd\ and \quad j/2<n/2-1\ , \tag{3.25}$$

$$1\leqq p\leqq\infty\ ,\quad s>2\sigma-|\alpha|-n/p\ ,\quad r<\min(s, n/p')-2\sigma+|\alpha| . \tag{3.26}$$

Furthermore, (3.24) *holds with* $o(z^{\sigma-1})$ *replaced by* $O(z^{\sigma-1})$ *in the following cases:*

(1) $(1+|\alpha|)/2<\sigma<(n+|\alpha|)/2$, $r=s-2\sigma+|\alpha|$, *and* s *satisfying*: $2\sigma-|\alpha|-n\leqq s<0$ *when* $p=1$, $2\sigma-|\alpha|<s\leqq n$ *when* $p=\infty$, $2\sigma-|\alpha|-n/p<s<n/p'$ *when* $1<p<\infty$.

(2) $(n+|\alpha|)/2\leqq\sigma<(\rho+n+|\alpha|)/2$, $p=1$, $s=2\sigma-|\alpha|-n$, $r<-2\sigma+|\alpha|$.

(3) $(n+|\alpha|)/2\leqq\sigma<(\rho+n+|\alpha|)/2$, $p=\infty$, $s>2\sigma-|\alpha|$, $r=-2\sigma+|\alpha|+n$.

(ii) *Let* $(n+|\alpha|)/2<\sigma<(\rho+n+|\alpha|)/2$, *or* $\alpha=0$ *and* $\sigma=n/2$. *Let* $s=2\sigma-n-|\alpha|$. *Then one has, in* $B(L_1^s, L_\infty)+B(L_1, L_\infty^{-s})$

$$D^\alpha R(z)-\sum_{j=-1}^{[2\sigma]-2} z^{j/2}D^\alpha B_{j/2}=O(z^{\sigma-1}) \quad as \quad z\to 0\,, \quad z\in\Sigma(\delta)\,. \tag{3.27}$$

(iii) *For* $\sigma=(n+1)/2$ *and* $|\alpha|=1$, (3.27) *holds in* $B(L_1, W_\infty^{\tau,0})$, *where* $\tau<0$ *for* $n\geqq 3$ *and* $\tau\leqq 0$ *for* $n=1$.

PROOF. The theorem except for $n=1$ and $\sigma=(1+|\alpha|)/2$ follows from (3.16), Lemmas 3.8, 3.9, 2.3, and 2.4 (ii & iv). Let us show (ii) for $n=1$ and $\sigma=(1+|\alpha|)/2$. By the resolvent equation,

$$R(z)=R_0(z)+R_0(z)(1-VR_0(z))^{-1}VR_0(z)\,. \tag{3.28}$$

(3.19) yields

$$VR_0(z)=O(1) \quad \text{as} \quad z\to 0 \quad \text{in} \quad B(L_1, L_1^r) \tag{3.29}$$

for any r with $0<r<\rho$. By Lemmas 3.8 and 2.4(ii),

$$D^\alpha R_0(z)(1-VR_0(z))^{-1}=O(z^{(|\alpha|-1)/2}) \quad \text{as} \quad z\to 0 \quad \text{in} \quad B(L_1^r, L_\infty)\,.$$

This together with (3.28) and (3.29) implies (ii) for $n=1$ and $\sigma=(1+|\alpha|)/2$. A similar argument shows the last half of (i). q.e.d.

We now proceed to investigate the case n is even.

LEMMA 3.11. *Let* n *be even. Then the following statements hold.*

(i) *For* $0<\sigma<(\rho+n-1)/2$, *one has*

$$(1-V\tilde{R}_0(z)K)^{-1}-1=\sum_{j=1}^{[\sigma]}\sum_{k=0}^{[2j/n]} z^j\log^k zC_{jk}+o(z^\sigma) \quad as \quad z\to 0 \tag{3.30}$$

in $B(L_p^s, L_p^r)$, *where* p, s, r *are the same as in* (3.22). *Furthermore,* (3.30) *holds with* $o(z^\sigma)$ *replaced by* $O(z^\sigma)$ *for* $p=1$, $s=2\sigma+1-n$, $r<\rho-2\sigma+\min(s,0)$, $\sigma-n/2\neq 0, 1, \cdots$, $0<\sigma<(\rho+n-1)/2$; *and for* $\sigma-n/2=0, 1, \cdots<(\rho-1)/2$

$$(1-V\tilde{R}_0(z)K)^{-1}-1=\sum_{j=1}^{\sigma}\sum_{k=(j-\sigma+1)^+}^{[2j/n]} z^j\log^k zC_{jk}+O(z^\sigma\log z) \tag{3.31}$$

as $z \to 0$ *in* $B(L_1^{2\sigma+1-n}, L_1^r)$, $r < \rho - 2\sigma$. *Here and everywhere else* $x^+ = \max(x, 0)$.

(ii) *For* $0 < \sigma \leqq n/2$, *one has*

$$(3.32)\quad (1 - V\tilde{R}_0(z)K)^{-1} - \sum_{j=0}^{[\sigma]} z^j C_{j0} = O(z^\sigma) \quad as \quad z \to 0 \quad in \quad B(L_1, L_q^r)\ ,$$

where $C_{00} = 1$, $n(1 - 1/q) = 2\sigma$, and $r < \rho$.

(iii) *Let* $n/2 \leqq \sigma < (\rho + n - 1)/2$, $\mu(\sigma) = 0$ *for* $\sigma \notin Z$ *and* $\mu(\sigma) = 1$ *for* $\sigma \in Z$, $\lambda(x) = 1$ *for* $x \geqq 0$ *and* $\lambda(x) = 0$ *for* $x < 0$, $s = 2\sigma - n$, *and* $r < \rho$. *Then one has, in* $B(L_1^s, L_\infty^r) + B(L_1, L_\infty^{r-s})$,

$$(3.33)\qquad (1 - V\tilde{R}_0(z)K)^{-1} - \sum_{j=0}^{[\sigma]} \sum_{k=\lambda(j-\sigma)}^{[2j/n]} z^j \log^k z C_{jk} = O(z^\sigma (\log z)^{\mu(\sigma)})$$

as $z \to 0$.

PROOF. Except for (ii) and (iii) for $\sigma = n/2$ the lemma can be shown in the same way as Lemma 3.9. In order to prove (ii) and (iii) for $\sigma = n/2$ we have only to note that Lemma 2.4 (ii) implies that for $|\alpha| = 2$

$$(3.34)\qquad \left[D^\alpha R_0(z) - \sum_{j=0}^{n/2-1} D^\alpha G_j\right] R_0(N)(1 + VR(N)) = O(z^{n/2} \log z)$$

as $z \to 0$ in $B(L_1, L_\infty)$. q.e.d.

Lemma 3.11 yields the following theorem.

THEOREM 3.12. *Let* n *be even. Let* $0 < \delta < \pi$ *and* α *be a multi-index with* $|\alpha| \leqq 1$. *Then the following statements hold.*

(i) *For* $(1 + |\alpha|)/2 < \sigma < (\rho + n + |\alpha|)/2$, *one has*

$$(3.35)\qquad D^\alpha R(z) = \sum_{j=0}^{[\sigma]-1} \sum_{k=0}^{[(2j+2)/n]} z^j \log^k z D^\alpha B_{jk} + o(z^{\sigma-1})$$

as $z \to 0$ *with* $z \in \Sigma(\delta)$ *in* $B(L_p^s, L_p^r)$ *with* p, s, r *satisfying* (1) *or* (2) *with* $\sigma \notin Z$ *or* (3) *with* $\sigma \notin Z$ *in Theorem* 3.10(i); *and for an integer* σ *and* p, s, r *satisfying* (2) *or* (3)

$$(3.36)\qquad D^\alpha R(z) = \sum_{j=0}^{\sigma-1} \sum_{k=(j-\sigma+2)^+}^{[(2j+2)/n]} z^j \log^k z D^\alpha B_{jk} + O(z^{\sigma-1} \log z)$$

as $z \to 0$ *in* $B(L_p^s, L_p^r)$.

(ii) *For any* $\varepsilon > 0$ *one has, in* $B(L_1, W_\infty^{-\varepsilon,0})$,

$$(3.37)\qquad R(z) - \sum_{j=0}^{n/2-2} z^j B_{j0} = O(z^{n/2-1} \log z) \quad as \quad z \to 0$$

with $z \in \Sigma(\delta)$.

(iii) *Let* $(n + |\alpha|)/2 \leqq \sigma < (\rho + n + |\alpha|)/2$, $\sigma \neq n/2$, $\mu(\sigma) = 1$ *for* $\sigma \in Z$ *and* $\mu(\sigma) = 0$ *for* $\sigma \notin Z$. *Then one has*

$$(3.38)\qquad D^\alpha R(z) - \sum_{j=0}^{[\sigma]-1} \sum_{k=[j-\sigma+2]^+}^{[(2j+2)/n]} z^j \log^k z D^\alpha B_{jk} = O(z^{\sigma-1}(\log z)^{\mu(\sigma)})$$

as $z \to 0$ *with* $z \in \Sigma(\delta)$ *in* $B(L_1^s, L_\infty) + B(L_1, L_\infty^{-s})$, $s = 2\sigma - n - |\alpha|$.

REMARK 3.13. Let us introduce the assumption

(A. III′): $\langle x\rangle^\rho(a_{jk}(x)-\delta_{jk})$, $\langle x\rangle^{\rho+1}b_j(x)$, and $\langle x\rangle^{\rho+2}c(x)$ are bounded functions on $\boldsymbol{R}^n$ which are uniformly Hölder continuous with exponent θ.

If we strengthen the assumption (A. III) to (A. III′), then Lemmas 3.2–3.9, 3.11, Theorems 3.10 and 3.12 hold also with L_p^s in the lemmas and theorems replaced by $W_p^{\tau,s}$ for any $|\tau| < \theta$.

REMARK 3.14. If an exact formula for $K = (1 - VG_0)^{-1}$ is obtained, then the coefficients B_j in Theorem 3.10 and B_{jk} in Theorem 3.12 are determined exactly by (3.16).

For $n = 1$, K is given by (3.17). For $A = a(x)\Delta$, we have that $K = a(x)^{-1}$.

4. The fundamental solution. In this section we prove Theorem 1.1 and investigate some properties of the fundamental solution $U(t, x, y)$:

$$\partial_t U(t, x, y) = (\Sigma_{j,k} a_{jk}(x)\partial_j\partial_k + \Sigma_j b_j(x)\partial_j) U(t, x, y) ,$$
$$U(0, x, y) = \delta(x - y) .$$

THEOREM 4.1. (i) *$U(t, x, y)$ is a Hölder continuous function on $(0, \infty) \times R^{2n}$ which is infinitely differentiable in t and twice differentiable in x. For any $k \geqq 0$, $|\alpha| \leqq 2$, and $t > 0$, $\partial_t^k\partial_x^\alpha U(t, x, y)$ is uniformly Hölder continuous in $(x, y) \in R^{2n}$.*

(ii) *For any $t > 0$, $s \in R^1$, and $m < 2 + \theta$, there exists a constant M such that*

$$(4.1)\qquad \|U(t, \cdot, y)\|_{W_1^{m,s}} \leqq M\langle y\rangle^s , \qquad y \in R^n .$$

(iii) $U(t, x, y) > 0$.

(iv) $\int U(t, x, y)dy = 1$ *for all* $(t, x) \in (0, \infty) \times R^n$.

(v) $\int U(t, x, y)dx$ *is bounded on* $(0, \infty)\times R^n$.

PROOF. Choose N so large that Lemma 3.1 holds for N replaced by $N - 1$. Since $(N - A)R(z) = (N - z)R(z) + 1$, we have by Lemma 3.1 that *for* $t > 0$

$$(4.2)\qquad U(t, x, y) = \frac{1}{2\pi i}\int_{\gamma_N} e^{tz}(N - z)^{2n}R(z)(N - A)^{-2n}\delta(x - y)dz ,$$

$$(4.3)\qquad \gamma_N = \{N + re^{i\phi}; -\infty < r < 0\} + \{N - re^{-i\phi}; 0 \leqq r < \infty\} ,$$

where ϕ is a number with $0 < \phi < \pi/2$. This implies (i).

Since for any $\varepsilon > 0$ and $s \in R^1$ there exists a constant M' such that

$$\|\delta(\cdot - y)\|_{W_1^{-\varepsilon,s}} \leqq M'\langle y\rangle^s, \qquad y \in R^n,$$

Lemma 3.1 and (4.2) show (ii).

By the maximum principle for parabolic equations, $U(t, x, y) \geqq 0$ for $(t, x, y) \in (0, \infty) \times R^{2n}$. We have that for and $0 < \tau < t$

$$U(t, x, y) = \int U(t - \tau, x, z) U(\tau, z, y) dz .$$

Since there exist $\tau \in (0, t)$ and an open ball B such that $\inf\{U(\tau, z, y); z \in B\} \geqq c > 0$,

$$U(t, x, y) \geqq c\int_B U(t - \tau, x, z) dz .$$

The strong maximum principle shows that the right hand side of this inequality is positive, which proves (iii).

(iv) is clear, since 1 is the solution of the equation: $(\partial_t - A)u(t, x) = 0$ and $u(0, x) = 1$.

It is known (see [3, Theorem 4.5, p. 141]) that for some positive constants m and M

(4.4) $\quad U(t, x, y) \leqq Mt^{-n/2}\exp[-m|x - y|^2/t], \qquad 0 < t \leqq 1, \quad (x, y) \in R^{2n}.$

Thus (v) follows, if we show that

(4.5) $$\|e^{tA}\|_{B(L_1)} \leqq M, \qquad t > 1,$$

since for every fixed $t > 0$ the function $\int U(t, x, y)dx$ is a positive Hölder continuous function of y. We first show (4.5) for $n \geqq 2$. By the resolvent equation,

(4.6) $$R(z) = R_0(z) + R_0(z)VR_0(z)K(1 - V\tilde{R}_0(z)K)^{-1} .$$

Lemmas 3.9 and 3.11 show that for a sufficiently small positive number ε

$$(1 - V\tilde{R}_0(z)K)^{-1} - 1 = O(z^\varepsilon) \quad \text{as} \quad z \to 0 \quad \text{in} \quad B(L_1) .$$

By Lemma 2.4(i), $VR_0(z) - VG_0 = O(z^\varepsilon)$ as $z \to 0$ in $B(L_1)$. Thus

$$R(z) - R_0(z)(1 + VG_0K) = O(z^{-1+\varepsilon}) \quad \text{as} \quad z \to 0 \quad \text{in} \quad B(L_1) .$$

Since $VG_0K \in B(L_1, W_1^{-\varepsilon})$, this implies (4.5) for $n \geqq 2$. Next let us show (4.5) for $n = 1$. Lemma 3.8 implies that for a sufficiently small positive number ε

$$(1 - VR_0(z))^{-1} - K = O(z^\varepsilon) \quad \text{as} \quad z \to 0 \quad \text{in} \quad B(L_1^{2\varepsilon}, L_1) .$$

This together with (3.19) and (3.29) yields

$$R(z) - R_0(z)[1 + K(1 - a(x))] + S(z) = O(z^{-1+\epsilon}) \tag{4.7}$$

as $z \to 0$ in $B(L_1)$, where $S(z) = R_0(z)Ka(x)b(x)(d/dx)R_0(z)$. We have that $R_0(z)$ and $(d/dx)R_0(z)$ are integral operators with kernel

$$(4z)^{-1/2} \exp[-z^{1/2}|x - y|]$$

and

$$(1/2 - H(x - y)) \exp[-z^{1/2}|x - y|],$$

respectively, where $H(w) = 1$ for $w \geqq 0$ and $H(w) = 0$ for $w < 0$. Calculating the kernel of the opertor

$$\int_{\gamma_0} S(z)e^{tz}dz/2\pi i$$

by using (3.17) and the formula

$$\int_{\gamma_0} (4z)^{-1/2} \exp[-z^{1/2}\lambda]e^{tz}dz/2\pi i = (4\pi t)^{-1/2} \exp[-\lambda^2/4t]$$

for any $\lambda \geqq 0$ and $t > 0$, we get

$$\left\| \int_{\gamma_0} S(z)e^{tz}dz/2\pi i \right\|_{B(L_1)} \leqq M, \qquad t > 0.$$

This together with (4.7) implies (4.5) for $n = 1$. The proof of (v) is now complete. q.e.d.

To prove Theorem 1.1 we need the following well-known lemma.

LEMMA 4.2. (i) *For* $\sigma > 0$ *and* $t > 0$,

$$\frac{1}{2\pi i}\int_{\gamma_0} e^{tz}z^{\sigma-1}dz = \pi^{-1} \sin \sigma\pi\, \Gamma(\sigma)t^{-\sigma}, \tag{4.8}$$

where $\Gamma(\sigma)$ *is the gamma function.*

(ii) *For* $t > 0$ *and nonnegative integers* j *and* k,

$$\frac{1}{2\pi i}\int_{\gamma_0} e^{tz}z^j \log^k z\, dz = \sum_{m=0}^{k-1} t^{-j-1}(-\log t)^m \binom{k}{m}\left(\frac{d}{d\sigma}\right)^{k-m} \frac{\sin \sigma\pi}{\pi}\Gamma(\sigma)\Big|_{\sigma=j+1}. \tag{4.9}$$

PROOF OF THEOREM 1.1. Lemmas 3.1, 3.3, Theorems 3.10, 3.12 and 4.1 yield

$$U(t, x, y) = \frac{1}{2\pi i}\int_{\gamma_0} e^{(t-\tau)z}(N - z)^l(N - A)^{-l}R(z)U(\tau, x, y)dz \tag{4.10}$$

for $t > 1$, $0 < \tau < 1$, $l \geqq 0$, and $N \gg 1$.

Let n be odd and $n/2 \leqq r < (\rho + n)/2$. Then Theorem 3.10, (4.1), (4.8), and (4.10) show that

$$U(t, x, y) = \sum_{k=0}^{[r-n/2]} \pi^{-1}(-1)^{(n-1)/2+k}\Gamma(n/2 + k)(t-\tau)^{-n/2-k} \times [B_{n/2-1+k}U(\tau, x, y)] + W_r(t, x, y) , \tag{4.11}$$

$$|W_r(t, x, y)| \leqq M_r t^{-r}(\langle x\rangle + \langle y\rangle)^{2r-n} . \tag{4.12}$$

This yields (1.4)~(1.6). Since $B_{n/2-1+k}$ are clearly of finite rank, $U_j(x, y)$ is a function of the form $\Sigma_l f_l(x)g_l(y)$. We have from (4.11) that

$$U_0(x, y) = \pi^{-1}(-1)^{(n-1)/2}\Gamma(n/2)B_{n/2-1}U(\tau, x, y) .$$

By (3.25), $B_{n/2-1} = F_0 K$ with $K = (1 - VG_0)^{-1}$. Since $F_0 = d_{n/2-1}\langle\cdot, 1\rangle$ and $e^{tA_0} = (4\pi t)^{-n/2}\langle\cdot, 1\rangle + o(t^{-n/2})$, we have that

$$\pi^{-1}(-1)^{(n-1)/2}\Gamma(n/2)d_{n/2-1} = (4\pi)^{-n/2} .$$

Hence $U_0(x, y) = U_0(y)$ with

$$U_0(y) = (4\pi)^{-n/2}\int KU(\tau, x, y)dx \tag{4.13}$$

for any $0 < \tau < 1$. This together with (3.17) yields (1.12). Next let us show (1.10). Since $K = 1 + VG_0K$, we have by (4.13) that

$$(4\pi)^{n/2}U_0(y) = \int U(\tau, x, y)\,dx + \int VG_0KU(\tau, x, y)dx . \tag{4.14}$$

We claim that

$$\left|\int U(\tau, x, y)dx - 1\right| \leqq M\tau^{\theta/2} , \qquad 0 < \tau < 1 , \qquad y \in R^n . \tag{4.15}$$

With $[a^{jk}(y)]_{j,k} = [a_{jk}(y)]_{j,k}^{-1}$ and $a(y) = \det[a^{jk}(y)]_{j,k}$, put

$$H(t, x, y) = a(y)^{1/2}(4\pi t)^{-n/2}\exp\Big[-\sum_{j,k=1}^{n} a^{jk}(y)(x_j - y_j)(x_k - y_k)/4t\Big] , \tag{4.16}$$

$$J(t, x, y) = (\partial_t - A_x)H(t, x, y) . \tag{4.17}$$

Then elementary calculations show that

$$H(t, x, y) \to \delta(x - y) \quad \text{as} \quad t \to 0 , \tag{4.18}$$

$$\int H(t, x, y)dx = 1 , \tag{4.19}$$

$$\int |J(t, x, y)|\,dx \leqq Mt^{\theta/2-1} , \qquad 0 < t < 1 , \qquad y \in R^n . \tag{4.20}$$

By (4.17) and (4.18), we have that

$$U(\tau, x, y) = H(\tau, x, y) - \int_0^\tau \int U(\tau - t, x, z) J(t, z, y) dt dz .$$

This together with (4.19), (4.20), and (4.4) show the claim (4.15). Given $\varepsilon > 0$, we can choose $\tau > 0$ so that

$$\left| \int U(\tau, x, y) dx - 1 \right| < \varepsilon , \quad y \in R^n . \tag{4.21}$$

Since $VG_0K \in B(W_1^{m,s}, L_1)$ for some $m > 0$ and s with $\max(1-n, -\rho) < s < 0$, (4.1) shows that for any y with $|y|$ sufficiently large

$$\left| \int VG_0KU(\tau, x, y) dx \right| \leqq M\langle y \rangle^s < \varepsilon .$$

This together with (4.21) and (4.14) shows (1.10). We have that for any $\phi \in C_0^\infty(R^n)$

$$\int U_0(y) A\phi(y) dy = \lim_{t\to\infty} t^{n/2} \int U(t, x, y) A\phi(y) dy = \lim_{t\to\infty} t^{n/2} \partial_t e^{tA} \phi(x) = 0 .$$

This proves (1.9). Since $U(t, x, y) = \int U(t - \tau, x, z) U(\tau, z, y) dz$ for $t > \tau > 0$, we have that

$$U_0(y) = \int U_0(z) U(\tau, z, y) dz . \tag{4.22}$$

By (1.10), there exists $N > 0$ such that $U_0(z) > (4\pi)^{-n}$ for all z with $|z| > N$. Thus (4.22) shows that

$$U_0(y) > (4\pi)^{-n} \int_{|z|>N} U(\tau, x, y) dz > 0 ,$$

which proves (1.11). It remains to show the uniqueness: if uniformly Hölder continuous functions $U_0(y)$ and $U_0'(y)$ satisfy (1.9) and (1.10), then $U_0(y) = U_0'(y)$. Put $w(y) = U_0(y) - U_0'(y)$. Then, $A^*w(y) = 0$ on R^n and $w(y) = o(1)$ as $|y| \to \infty$. We have

$$\Delta(w(y) - G_0 V^* w(y)) = 0 \quad \text{on} \quad R^n . \tag{4.23}$$

Choose $0 < \delta < \theta$ and $\phi_N(\xi)$ in $C^\infty(R^n)$ such that $\phi_N(\xi) = 1$ for $|\xi| \geqq N + 1$ and $\phi_N(\xi) = 0$ for $|\xi| \leqq N$. With $\tilde{a}_{jk}(y) = a_{jk}(y) - \delta_{jk}$ we have that

$$\begin{aligned} G_0 V^* w(y) = & \sum_{j,k} \phi_N(D) G_0 \partial_j \partial_k \langle D \rangle^{-\delta} [\langle D \rangle^{-\delta} (\tilde{a}_{jk}(y) w(y))] \\ & + \sum_{j,k} (1 - \phi_N(D)) G_0 \partial_j \partial_k (\tilde{a}_{jk}(y) w(y)) + \sum_j G_0 \partial_j (b_j(y) w(y)) . \end{aligned}$$

Given $\varepsilon > 0$, we can choose N so large that the L_∞-norm of the first term of the right hand side of the above equality is smaller than ε, for

$$\phi_N(D)G_0\partial_j\partial_k\langle D\rangle^{-\delta} = o(1) \quad \text{as} \quad N \to \infty \quad \text{in} \quad B(L_\infty) .$$

Since the second and third terms belong to L_∞^r for some $0 < r < \min(\rho, n-1)$, we get

$$|G_0 V^* w(y)| < 2\varepsilon \quad \text{for} \quad |y| \gg 1 .$$

Hence $w(y) - G_0 V^* w(y) = o(1)$ as $|y| \to \infty$, which together with (4.23) implies that $w = G_0 V^* w$. Since $1 - VG_0$ is a bijection on $L_2^{(n+\rho)/2}$ and $L_\infty \subset L_2^{-(n+\rho)/2}$, this shows that $w = 0$. That is, $U_0(y) = U_0'(y)$. This completes the proof of Theorem 1.1 for odd n.

Let n be even and $n/2 \leqq r < (\rho + n)/2$. Then Theorem 3.12, (4.1), (4.9), and (4.10) show that

$$\begin{aligned} U(t, x, y) = \sum_{j=0}^{[r-n/2]} \sum_{k=0}^{[2j/n]} (t-\tau)^{-n/2-j}(-\log(t-\tau))^k \\ \times \sum_{l=k+1}^{[2j/n]+1} c_{jkl} B_{n/2-1+jl} U(\tau, x, y) + W_r(t, x, y) , \end{aligned} \tag{4.24}$$

$$c_{jkl} = \binom{l}{k}\left(\frac{d}{dr}\right)^{l-k} \frac{\sin \pi r}{\pi} \Gamma(r)\Big|_{r=n/2+j} ,$$

where $W_r(t, x, y)$ is a function satisfying (4.12). Since $U_{00}(x, y) = U_0(y)$ with $U_0(y)$ defined by (4.13) holds also for n even, (1.9)$\sim$(1.11) have been shown already. It remains to prove (1.8). We have by (3.16) that the operator $B_{n/2-1+j}$ is a sum of operators of the form

$$\begin{gathered} F_\mu K \prod_{i=1}^{l-1} VF_{\mu_i} K \prod_{j=1}^{m} VG_{\nu_j} K,\ \mu \geqq 0,\ \mu_i \geqq 1,\ \nu_j \geqq 1,\ m \geqq 0 , \\ (n/2-1)l + \mu + \mu_1 + \cdots + \mu_{l-1} + \nu_1 + \cdots + \nu_m = n/2 - 1 + j , \end{gathered}$$

and operators of the form

$$\begin{gathered} G_\nu K \prod_{i=1}^{l} VF_{\mu_i} K \prod_{j=1}^{m} VG_{\nu_j} K,\ \nu \geqq 0,\ \mu_i \geqq 1,\ \nu_j \geqq 1,\ m \geqq 0 , \\ (n/2-1)l + \mu_1 + \cdots + \mu_l + \nu + \nu_1 + \cdots + \nu_m = n/2 - 1 + j , \end{gathered}$$

and operators which we obtain by changing the order of products in the above ones with fixing $F_\mu K$ and $G_\nu K$. Note that the maximum of the above indices $\mu, \nu, \mu_1, \cdots, \nu_1, \cdots, \nu_m$ is $j - (l-1)n/2$, which is less than or equal to $j - nk/2$ if $l \geqq k+1$. This together with (4.24) implies (1.8). q.e.d.

REMARK 4.3. By Remark 3.13, if we strengthen the assumption (A. III) to (A. III′), we can calculate the asymptotic expansion of $U(t, x, y)$ by the formula more direct than (4.10):

$$U(t, x, y) = \frac{1}{2\pi i} \int_{\gamma_0} e^{tz} R(z) \delta(x-y) dz . \tag{4.25}$$

Hence the formulas (4.11) and (4.24) hold with $t-\tau$ and $U(\tau, x, y)$ replaced by t and $\delta(x-y)$, respectively.

REMARK 4.4. By (4.11), (4.24), and Remark 3.14, the functions $U_j(x, y)$ and $U_{jk}(x, y)$ are determined exactly if an exact formula for K is given.

EXAMPLE 4.5. (i) for $n=1$, $U_0(y)$ is given by (1.12). (ii) Let $A=a(x)$. Then we see from the characterization (1.9)~(1.11) of $U_0(y)$ that $U_0(y)=(4\pi)^{-n/2}a(y)^{-1}$. (iii) For $A=\sum_{j,k}\partial_j a_{jk}(x)\partial_k$, $U_0(y)=(4\pi)^{-n/2}$.

The following theorem can be shown in the same way as Theorem 1.1.

THEOREM 4.6. *Let α be a multi-index with $|\alpha|=1$, and $1/2\leqq\sigma<(\rho+1)/2$. Then the following asymtotic formulas as $t\to\infty$ hold.*

(i) *For n odd,*

$$D_x^\alpha U(t, x, y)=\sum_{j=1}^{[\sigma]} t^{-n/2-j}D_x^\alpha U_j(x, y)+D_x^\alpha \tilde{U}_\sigma(t, x, y)\ , \tag{4.26}$$

$$|D_x^\alpha U_j(x, y)|\leqq M_j(\langle x\rangle+\langle y\rangle)^{2j-1}\ , \tag{4.27}$$

$$|\partial_t^l D_x^\alpha \tilde{U}_\sigma(t, x, y)|\leqq M_{\sigma l}t^{-n/2-\sigma-l}(\langle x\rangle+\langle y\rangle)^{2\sigma-1}\ , \qquad l\geqq 0\ . \tag{4.28}$$

(ii) For n even,

$$D_x^\alpha U(t, x, y)=\sum_{j=1}^{[\sigma]}\sum_{k=0}^{[2j/n]} t^{-n/2-j}\log^k t D_x^\alpha U_{jk}(x, y)+D_x^\alpha \tilde{U}_\sigma(t, x, y)\ , \tag{4.29}$$

where $D_x^\alpha \tilde{U}_\sigma(t, x, y)$ satisfies (4.28) and

$$|D_x^\alpha U_{jk}(x, y)|\leqq M_j(\langle x\rangle+\langle y\rangle)^{2j-nk-1}\ . \tag{4.30}$$

THEOREM 4.7. *There exists a constant M such that for any multi-index α with $|\alpha|\leqq 1$ and $1\leqq p\leqq q\leqq\infty$*

$$\|D^\alpha e^{tA}\|_{B(L_p, L_q)}\leqq Mt^{-(1/p-1/q)n/2-|\alpha|/2}\ , \qquad t>0\ . \tag{4.31}$$

PROOF. We first treat the case $\alpha=0$. The inequality (4.31) for $p=q=\infty$ follows from Theorem 4.1(iv), and that for $p=q=1$ follows from (v). Thus the interpolation theorem shows (4.31) for $1\leqq p=q\leqq\infty$. The estimate for $p=1$ and $q=\infty$ follows from (4.4) and Theorem 1.1 for $\sigma=0$. Hence the interpolation theorem shows (4.31) for $\alpha=0$ and $1\leqq p\leqq q\leqq\infty$. In the same way as in the proof of Theorem 4.1(v) we obtain that

$$\int|D_x^\alpha U(t, x, y)|dx\leqq Mt^{-1/2}\ , \qquad \int|D_x^\alpha U(t, x, y)|dy\leqq Mt^{-1/2} \tag{4.32}$$

for $|\alpha|=1$, where M is a constant independent of (t, x, y) in $(0, \infty)\times R^{2n}$ (in proving the second inequality of (4.32), we use $B(L_\infty)$ instead of

$B(L_1)$). Thus (4.31) for $|\alpha|=1$ is derived in the same way as above. q.e.d.

THEOREM 4.8. *Let n be odd. Then the following statements hold.*

(i) *Let α be a multi-index with $|\alpha| \leqq 1$ and $|\alpha|/2 \leqq \sigma < (\rho + |\alpha|)/2$. Then, for all $t>1$ and $f \in L_1^{2\sigma}$*

$$(4.33) \qquad \left| D_x^\alpha e^{tA} f(x) - \sum_{j=0}^{[\sigma]} t^{-n/2-j} \int D_x^\alpha U_j(x, y) f(y) dy \right| \leqq M_\sigma t^{-n/2-\sigma} \int (\langle x \rangle + \langle y \rangle)^{2\sigma - |\alpha|} |f(y)| dy .$$

(ii) *Let $0 \leqq \sigma < (\rho + 1)/2$. Then, for all $t > 1$ and $f \in L_1^{2\sigma}$ satisfying*

$$(4.34) \qquad \int U_0(y) f(y) dy = 0$$

(4.33) *for $\alpha = 0$ holds with $(\langle x \rangle + \langle y \rangle)^{2\sigma}$ replaced by $\langle x \rangle^{(2\sigma-1)^+} + \langle y \rangle^{2\sigma}$.*

PROOF. (i) follows from Theorems 1.1 and 4.6. If f satisfies (4.34), then it follows from (4.13) that

$$\int K \left[\int U(\tau, x, y) f(y) dy \right] dx = 0 .$$

Using (4.10) and (3.16) we thus get (ii) for $\sigma \geqq 1/2$. Since (ii) holds for $\sigma = 0$ by (i), the interpolation method shows (ii) for $0 \leqq \sigma \leqq 1/2$. q.e.d.

The same argument as above yields:

THEOREM 4.9. *Results similar to Theorem* 4.8 *hold also for n even.*

5. The case that $c(x) \leqq 0$ and $c(x) \not\equiv 0$. *Let* $A = \sum_{j,k} a_{jk}(x)\partial_j\partial_k + \sum_j b_j(x)\partial_j + c(x)$ be an operator satisfying the assumption (A) and the condition

$$c(x) \leqq 0 \quad \text{and} \quad c(x) \not\equiv 0 .$$

Let $U(t, x, y)$ be the fundamental solution for A.

THEOREM 5.1. *$U(t, x, y)$ has the properties* (i), (ii), (iii), (v) *in Theorem* 4.1 *and* (iv'): $\int U(t, x, y) dy \leqq 1$ *on* $(0, \infty) \times R^n$.

PROOF. (i)~(iii) is shown in the same way as Theorem 4.1 (i)~(iii). The comparison theorem together with (iv) and (v) in Theorem 4.1 yields (iv') and (v). q.e.d.

THEOREM 5.2. *The inequality* (4.31) *for $\alpha = 0$ holds for any $1 \leqq p \leqq q \leqq \infty$.*

PROOF. The comparison theorem and Theorem 4.7 show the theorem. q.e.d.

THEOREM 5.3. *For any* $0 \leqq \sigma < \rho/2$ *there hold the following formulas for all* $t > 1$ *and* $(x, y) \in R^{2n}$:

(i) *For odd* $n \geqq 3$, (1.4)$\sim$(1.6) *hold. Furthermore,* $U_j(x, y)$ *is a finite sum of functions of the form* $f(x)g(y)$. *In particular,*

$$U_0(x, y) = \chi(x)\chi_*(y) \tag{5.1}$$

with $\chi(x)$ *and* $\chi_*(y)$ *satisfying and determined uniquely by* (χ) *and* (χ_*), *respectively*:

(χ) $\chi(x)$ *is a* C^2*-function such that*

$$0 < \chi(x) < 1 \,, \qquad A\chi(x) = 0 \quad \text{on} \quad R^n \,,$$
$$\chi(x) = 1 + O(|x|^{2-n}) \quad \text{as} \quad |x| \to \infty \,.$$

(χ_*) $\chi_*(y)$ *is a uniformly Hölder continuous function such that*

$$\chi_*(y) > 0,\ A^*\chi_*(y) = 0 \quad \text{on} \quad R^n \,,$$
$$\chi_*(y) = (4\pi)^{-n/2} + o(1) \quad \text{as} \quad |y| \to \infty \,.$$

(ii) *For even* $n \geqq 4$,

$$U(t, x, y) = \sum_{j=0}^{[\sigma]} \sum_{k=0}^{[2j/(n-2)]} t^{-n/2-j} \log^k t\, U_{jk}(x, y) + \tilde{U}_\sigma(t, x, y) \,, \tag{5.2}$$

$$|U_{jk}(x, y)| \leqq M_j(\langle x\rangle + \langle y\rangle)^{2j-(n-2)k} \,, \tag{5.3}$$

$$|\partial_t^l \tilde{U}_\sigma(t, x, y)| \leqq M_{\sigma l} t^{-n/2-\sigma-l}(\langle x\rangle + \langle y\rangle)^{2\sigma} \,, \qquad l \geqq 0 \,. \tag{5.4}$$

Furthermore, $U_{jk}(x, y)$ *is a finite sum of functions of the form* $f(x)g(y)$. *In particular,* $U_{00}(x, y) = \chi(x)\chi_*(y)$ *with* $\chi(x)$ *and* $\chi_*(y)$ *having the same properties as in* (i).

PROOF. The proof is similar to that of Theorem 1.1. The results analogous to Theorems 3.10 and 3.12 hold, and the coefficient of $z^{n/2-1}(\log z)^{\epsilon(n)}$ is equal to

$$(1 - G_0 V)^{-1} F_0 (1 - V G_0)^{-1} \,;$$

from which (5.1) is derived. Here and in what follows

$$V = \sum_{j,k} (a_{jk}(x) - \delta_{jk})\partial_j \partial_k + \sum_j b_j(x)\partial_j + c(x) \,. \qquad \text{q.e.d.}$$

THEOREM 5.4. *Let* $n = 2$ *and* $0 \leqq \sigma < \rho/2$. *Then one has*

$$U(t, x, y) = \sum_{j=0}^{[\sigma]} \sum_{(k,l)\in I(j,\sigma)} t^{-1-j} \Phi_{jkl}(t) U_{jkl}(x, y) + \tilde{U}_\sigma(t, x, y) \,, \tag{5.5}$$

$$I(j, \sigma) = \{(k, l);\ 0 \leqq k \leqq j,\ 0 \leqq l \leqq j+1\} \setminus \{(0, 0)\} \quad \text{for} \quad j < \sigma \,,$$
$$I(j, \sigma) = \{(k, l);\ 0 \leqq k \leqq j,\ 0 \leqq l \leqq j+1,\ k - l \geqq 2\} \quad \textit{for} \quad j = \sigma \,,$$

for all $t > 1$ *and* $(x, y) \in R^{2n}$. *Here* $\tilde{U}_\sigma(t, x, y)$ *is a function satisfying*

(5.4), $U_{jkl}(x, y)$ *is a finite sum of functions of the form* $f(x)g(y)$ *and satisfies the estimate*

$$(5.6) \qquad |U_{jkl}(x, y)| \leqq M_j(\langle x\rangle + \langle y\rangle)^{2j-2(k-l-1)^+} \log 2\langle x\rangle \log 2\langle y\rangle ,$$

and

$$(5.7) \quad \Phi_{jkl}(t) = \int_0^\infty \frac{z^j e^{-z} \operatorname{Im}[(\log t - \log z - d + \pi i)^l(-\log t + \log z + \pi i)^k]dz}{\pi[(\log t - \log z - d)^2 + \pi^2]^l} ,$$

$$(5.8) \qquad d = 2\gamma + \left(4\pi - \pi^{-1/2}\int V_x\left[\int \log\frac{|x-y|}{2} V_y \chi(y)dy\right]dx\right)\left(\int c(y)dy\right)^{-1} ,$$

where γ *is Euler's constant and* χ *is a* C^2*-function on* R^2 *satisfying and determined uniquely by*

$$(5.9) \qquad \chi(x) > 0 , \qquad A\chi(x) = 0 \quad \text{on} \quad R^2 ,$$

$$(5.10) \qquad \chi(x) = \pi^{-1/2}(\log|x|/2 + \gamma) - (4\pi)^{-1/2}d + o(1) \quad as \quad |x| \to \infty .$$

Furthermore,

$$(5.11) \qquad U_{001}(x, y) = \chi(x)\chi_*(y) ,$$

where $\chi_*(y)$ *is a uniformly Hölder continuous function on* R^2 *satisfying and determined uniquely by*

$$(5.12) \qquad \chi_*(y) > 0 . \qquad A^*\chi_*(y) = 0 \quad on \quad R^2 ,$$

$$(5.13) \qquad \chi_*(y) = \pi^{-1/2}(\log|y|/2 + \gamma) - (4\pi)^{-1/2}d_* + o(1) \quad \text{as} \quad |y| \to \infty ,$$

$$(5.14) \quad d_* = 2\gamma + \left(4\pi - \pi^{-1/2}\iint c(x) \log\frac{|x-y|}{2} V_y^* \chi_*(y)dxdy\right)\left(\int c(y)dy\right)^{-1} .$$

REMARK. If $d = 0$, $\Phi_{jkk}(t) = 0$ for any j and k, and $\Phi_{jkl}(t) = O(\log^{k-l-1} t)$ as $t \to \infty$.

The proof of this theorem and the following one will be given after the proof of Theorem 5.6.

THEOREM 5.5. *Let* $n = 1$. *Then the following statements hold.*

(i) *For any* s *with* $0 \leqq s \leqq 1$ *and* $s < (\rho + 1)/2$ *and a nonnegative integer* l *there exists a constant* M *such that for all* $t > 1$ *and* $(x, y) \in R^{2n}$

$$(5.15) \qquad |\partial_t^l U(t, x, y)| \leqq M t^{-1/2-s-l} m_s(x, y) ,$$

where $m_s(x, y) = \min(\langle x\rangle^{2s}, \langle y\rangle^{2s})$ *for* $0 \leqq s \leqq 1/2$ *and* $m_s(x, y) = (\langle x\rangle + \langle y\rangle)^{2s-1} \min(\langle x\rangle, \langle y\rangle)$ *for* $1/2 \leqq s \leqq 1$.

(ii) *For any* σ *with* $0 \leqq \sigma < (\rho - 1)/2$, $t > 1$, *and* $(x, y) \in R^{2n}$,

$$(5.16) \qquad U(t, x, y) = \sum_{j=0}^{[\sigma]} t^{-3/2-j} U_j(x, y) + \tilde{U}_\sigma(t, x, y) ,$$

(5.17) $$|U_j(x, y)| \leqq M_j(\langle x\rangle + \langle y\rangle)^{2j}\langle x\rangle\langle y\rangle ,$$

(5.18) $$|\partial_t^l \tilde{U}_\sigma(t, x, y)| \leqq M_{\sigma l} t^{-3/2-\sigma-l}(\langle x\rangle + \langle y\rangle)^{2\sigma+1} \min(\langle x\rangle, \langle y\rangle) , \qquad l \geqq 0 ,$$

where M_j *and* $M_{\sigma l}$ *are constants independent of* t, x, y. *Furthermore,* $U_j(x, y)$ *is a finite sum of functions of the form* $f(x)g(y)$. *In particular*

(5.19) $$U_0(x, y) = (4\pi)^{-1/2}(\chi(x)\chi_*(y) + \psi(x)\psi_*(y)) > 0 ,$$

where $\chi(x)$ *is a* C^2*-function on* R^1 *satisfying and determind uniquely by*

(5.20) $$A\chi(x) = 0 \quad on \quad R^1 ,$$

(5.21) $$\chi(x) = 2^{-1/2}(|x| + d) + (x/2|x|)\int y V\chi(y)dy + o(1) \quad as \quad |x| \to \infty ,$$

(5.22) $$d = -\left(2 - 2^{-1/2}\iint V_x|x - y|V_y\chi(y)dxdy\right)\left(\int c(y)dy\right)^{-1} ,$$

$\chi_*(y)$ *is a uniformly Hölder continuous function satisfying and determined uniquely by*

(5.23) $$A^*\chi_*(y) = 0 \quad on \quad R^1 ,$$

(5.24) $$\chi_*(y) = 2^{-1/2}(|y| + d_*) + (y/2|y|)\int x V^*\chi_*(x)dx + o(1) \quad \text{as} \quad |y| \to \infty ,$$

(5.25) $$d_* = -\left(2 - 2^{-1/2}\iint c(x)|x - y|V_y\chi_*(y)\,dxdy\right)\left(\int c(y)dy\right)^{-1} ,$$

$\psi(x)$ *is a* C^2*-function on* R^1 *satisfying and determined uniquely by* (5.20) *and*

(5.26) $$\psi(x) = 2^{-1/2}x + (x/2|x|)\int y V\psi(y)dy + \frac{1}{2}\int \chi_*(y)\,Vydy + o(1) \quad as \quad |x| \to \infty ,$$

and $\psi_*(y)$ *is a uniformly Hölder continuous function on* R^1 *satisfying and determined uniquely by* (5.23) *and*

(5.27) $$\psi^*(y) = 2^{-1/2}y + (y/2|y|)\int x V^*\psi_*(x)dx + \frac{1}{2}\int \chi(x)\,V^*xdx + o(1) \quad as \quad |y| \to \infty .$$

THEOREM 5.6. *Let* α *be a multi-index with* $|\alpha| = 1$. *Then* $\langle x\rangle\partial_x^\alpha U_j(x, y)$ *and* $\langle x\rangle\partial_x^\alpha \tilde{U}_\sigma(t, x, y)$ *for odd* $n \geqq 3$ *satisfy the same estimates as* $U_j(x, y)$ *and* $\tilde{U}_\sigma(t, x, y)$.

The same statement holds also for even $n \geqq 4$, $n = 2$, *and* $n = 1$.

PROOF. With $A_1 = \sum_{j,k} a_{jk}(x)\partial_j\partial_k + \sum_j b_j(x)\partial_j$ and $R_1(z) = (z - A_1)^{-1}$, we have that

(5.28) $$R(z) = R_1(z) + R_1(z)c(x)R(z) .$$

This together with Theorems 5.3∼5.5, 3.10, and 3.12 implies the theorem.

q.e.d.

The rest of this section is devoted to the proof of Theorems 5.4 and 5.5.

DEFINITION 5.7. A C^2-function u on R^n is said to be a generalized eigenfunction for A when $u \not\equiv 0$, $Au(x) = 0$ on R^n,

$$u(x) = O(|x|^{2-n}) \quad \text{as} \quad |x| \to \infty , \quad \text{for} \quad n \geqq 3 , \tag{5.29.1}$$

$$u(x) = \lambda + o(1) \quad \text{as} \quad |x| \to \infty , \quad \text{for} \quad n = 2 , \tag{5.29.2}$$

$$u(x) = \lambda + \mu x/|x| + o(1) \quad \text{as} \quad |x| \to \infty , \quad \text{for} \quad n = 1 , \tag{5.29.3}$$

where λ and μ are some constants, and $D^\alpha u \in L_p^s$ for any $1 \leqq |\alpha| \leqq 2$ and

$$1 < p < \infty , \qquad s < \begin{cases} n - 2 - n/p + |\alpha| & (n \geqq 3) \\ -n/p + |\alpha| + \min(1, \rho) & (n \leqq 2) . \end{cases} \tag{5.30}$$

The importance of generalized eigenfunctions is seen from the following theorem, which is shown in the same way as Theorem 7.2 in [8].

THEOREM 5.8. *There are no generalized eigenfunctions for A if and only if*

$$R(z) = O(1) \quad as \quad z \to 0 \quad with \quad z \in \Sigma(\delta) \quad in \quad B(L_p^s, W_p^{\tau, r}) ,$$

where $0 < \delta < \pi$, $\tau = 2$ *when* $1 < p < \infty$ *and* $\tau < 2$ *when* $p = 1$ *or* ∞, *and*

$$2 - n/p < s < n/p' \quad and \quad r < s - 2 \quad when \quad n \geqq 3 ,$$
$$s > 2/p' \quad and \quad r < -2/p \quad when \quad n = 2 ,$$
$$s > 2 - 1/p \quad and \quad r < -1/p \quad when \quad n = 1 .$$

LEMMA 5.9. *There are no generalized eigenfunctions for A.*

PROOF. The maximum principle implies that if $u(x) \to 0$ as $|x| \to \infty$, then $u = 0$, which shows the lemma for $n \geqq 3$.

Let $n = 2$ and $u(x)$ be a C^2-function on R^2 satisfying (5.29.2) and $Au(x) = 0$ on R^2. We may assume that $\lambda \geqq 0$. Since $u(x)$ does not attain the positive maximum or negative minimum, we have that $0 \leqq u(x) \leqq \lambda$. With the same notation as in the proof of Theorem 5.6,

$$u(x) = R_1(z)(c(x)u(x)) + zR_1(z)u(x) = \log z F_0 K(c(x)u(x)) + O(1)$$

as $z \to 0$ with $z \in \Sigma(\delta)$. By (4.13), $\int U_0(y)c(y)u(y)dy = 0$. This implies that $u(y) = 0$ on $\{y;\, c(y) \neq 0\}$, since $U_0(y) > 0$, $c(y) \leqq 0$, and $u(y) \geqq 0$. Thus, $A_1 u(x) = 0$ on R^2. Hence, $u(x) = \lambda$, from which we obtain by (5.1) that $u = 0$.

Let $n = 1$ and $u(x)$ be a C^2-function on R^1. If $u(x) \geqq 0$ or $u(x) \leqq 0$

on R^1, the above argument shows that $u=0$. Thus we may assume that $u(-\infty)=\lambda-\mu<0<\lambda+\mu=u(\infty)$. Let $x_0=\sup\{y; u(x)\leqq 0$ for any $x\leqq y\}$. By the uniqueness theorem, $u(x_0)=0$ and $u'(x_0)>0$. We may assume that $u''-bu'+cu=0$ on R^1. We have that for $x>x_0$

$$u'(x)=u'(x_0)-\int_{x_0}^{x}\exp\left(\int_{y}^{x}b(s)ds\right)c(y)u(y)dy\ ,$$

$$u(x)=\int_{x_0}^{x}u'(y)dy\ .$$

Since $c(y)\leqq 0$, we obtain that for any $x\geqq x_0$

$$u'(x)\geqq u'(x_0)>0\ ,\quad u(x)\geqq u'(x_0)(x-x_0)\ .$$

This implies that $u(\infty)=\infty$, which is a contradictions. q.e.d.

We now proceed to the proof of Theorem 5.4. Recalling (3.7), put

$$T_0(z)=NV(\log zF_0+G_0)R_0(N)(1+VR(N))\ . \tag{5.31}$$

Then Lemma 2.3 yields:

LEMMA 5.10. *$1-T(z)=1-T_0(z)+o(1)$ as $z\to 0$ with $z\in\Sigma(\delta)$ in $B(L_p^s)$, where $1\leqq p\leqq\infty$ and $2/p'<s<2/p'+\rho$.*

We get by Lemma 5.9 the following lemma in the same way as Theorem 7.2 in [8].

LEMMA 5.11. *For any z in $\Sigma(\delta)$ with $|z|$ sufficiently small*

$$(1-T_0(z))^{-1}=\sum_{k=0}^{\infty}\log^{-k}zC_k\quad\text{in}\quad B(L_p^s)\ , \tag{5.32}$$

where $1\leqq p\leqq\infty$ and $2/p'<s<2/p'+\rho$.

With $S(z)=-\log zVF_0+1-VG_0$, we have that

$$S(z)^{-1}=(1+VR(N))(1-T_0(z))^{-1}\quad\text{in}\quad B(L_p^s,\ W_p^{\tau,s})\ , \tag{5.33}$$

where $1\leqq p\leqq\infty$, $2/p'<s<2/p'+\rho$, $\tau\leqq 0$ when $1<p<\infty$, and $\tau<0$ *when $p=1$ or ∞*. More precisely, we obtain:

LEMMA 5.12. *Let d be the constant given by* (5.8) *and $J=-\langle\cdot,\chi_*\rangle\chi$, where χ and χ_* are the functions determined by* (5.9), (5.10), *and* (5.12), (5.13), *respectively. Then, for any z in $\Sigma(\delta)$ with $|z|$ sufficiently small*

$$S(z)^{-1}=K-(\log z+d)^{-1}VJ\ , \tag{5.34}$$

where K is an operator satisfying the equality $F_0K=0$ and the following equalities in $B(L_p^s)$, $1\leqq p\leqq\infty$ and $2/p'<s<2/p'+\rho$:

(5.35) $$VF_0VJ + (1 - VG_0)K = VJVF_0 + K(1 - VG_0) = 1\ ,$$

(5.36) $$KVF_0 = VJ(1 - VG_0)K = K(1 - VG_0)VJ = 0\ .$$

PROOF. By (5.32) and (5.33), $S(z)^{-1} = \sum_{j=0}^{\infty} \log^{-j} z S_j$. Since $S(z) = (1 - T_0(z))(1 - VR_0(N))$, $S(z)S(z)^{-1} = S(z)^{-1}S(z) = 1$ in $B(L_p^s)$. Thus

(5.37) $$VF_0S_0 = S_0VF_0 = 0\ ,$$

(5.38) $$-VF_0S_1 + (1 - VG_0)S_0 = -S_1VF_0 + S_0(1 - VG_0) = 1\ ,$$

(5.39) $$-VF_0S_j + (1 - VG_0)S_{j-1} = -S_jVF_0 + S_{j-1}(1 - VG_0) = 0 \text{ for } j \geqq 2\ .$$

Calculating

$$S_1[-VF_0S_j + (1 - VG_0)S_{j-1}] + S_0[-VG_0S_{j+1} + (1 - VG_0)S_j]\ ,$$

we obtain that $S_j + S_1(1 - VG_0)S_{j-1} = 0$ for $j \geqq 2$, which yields

(5.40) $$S(z) = S_0 + \sum_{j=1}^{\infty} \log^{-j} z[-S_1(1 - VG_0)]^{j-1}S_1\ .$$

Similarly, $S_1(1 - VG_0)S_0 = S_0(1 - VG_0)S_1 = 0$. Putting

(5.41) $$J = -F_0S_2 - G_0S_1\ ,$$

we have by (5.39) that $S_1 = -VJ$. Thus (5.35) and (5.36) have already been shown with $S_0 = K$ and $S_1 = -VJ$. The equality $F_0K = 0$ follows from the equality $0 = VF_0K = c(x)F_0K$. It remains to prove (5.34) and the properties of J. By (5.35), (5.36), and $F_0K = 0$,

(5.42) $$F_0VJVF_0 = F_0 \quad \text{and} \quad VJ = VJVF_0VJ\ ,$$

which implies that $\operatorname{rank} VJ = 1$. Thus we can write

$$J = -\langle \cdot, \chi_* \rangle \chi\ .$$

It follows from (5.41), $J^* = -S_2^*F_0^* - S_1^*G_0^*$, and $S_1 = -VJ$ that $\chi \in W_p^{2,r}$, $V\chi \in L_p^{r+\rho}$, and $\chi_* \in L_p^s$ for and $1 < p < \infty$, $r < 2/p'$, and $s < -2/p$. Since $F_0 = -(4\pi)^{-1}\langle \cdot, 1 \rangle$, we obtain by (5.42) that

$$\langle V\chi, 1 \rangle \langle V1, \chi_* \rangle = 4\pi\ .$$

Consequently we can choose χ and χ_* so that

(5.43) $$\int V\chi(x)dx = \int c(y)\chi_*(y)dy = -(4\pi)^{1/2}\ .$$

Since $S_1 = -VJ$, we have, for some constant d',

(5.44) $$-S_2 = S_1(1 - VG_0)S_1 = d'S_1\ .$$

This together with (5.41) and (5.43) implies that

(5.45) $$\chi(x) - G_0V\chi(x) = -(4\pi)^{-1/2}d'\ .$$

Since $G_0f(x) = -(2\pi)^{-1}\int(\gamma + \log|x-y|/2)f(y)dy$, (5.43) and (5.45) imply that $d' = d$. This together with (5.40) shows (5.34). Furthermore, (5.43) and (5.45) yield (5.10). The equality (5.45) also shows that $A\chi(x) = 0$ on R^2, which implies that χ is a C^2-function. Since $A\chi(x) = 0$ and $\chi(x) \to \infty$ as $|x| \to \infty$, the maximum principle shows that $\chi(x) \geqq 0$. We have that $\chi(x) = \int U(t, x, y)\chi(y)dy$ for any $t > 0$, which implies that $\chi(x) > 0$ because $U(t, x, y) > 0$. This completes the proof of (5.9) and (5.10). The properties of $\chi_*(y)$ can be shown similarly. q.e.d.

PROOF OF THEOREM 5.4. With $\tilde{R}_0(z)$ defined by (2.25) we have by (5.32) and (5.33) that

$$R(z) = R_0(z)S(z)^{-1}\sum_{j=0}^{\infty}(V\tilde{R}_0(z)S(z)^{-1})^j \tag{5.46}$$

provided $|z|$ sufficiently small. This together with (5.34), (5.41), and (5.44) yields

$$R(z) = G_0K - F_0VJ + (\log z + d)^{-1}\langle\cdot, \chi_*\rangle\chi + O(z^\varepsilon) \quad \text{as} \quad z \to 0 \tag{5.47}$$

for some $\varepsilon > 0$. Thus the same argument as in the proof of Theorem 1.1 shows Theorem 5.4. The uniqueness of χ follows from Lemma 5.9 q.e.d.

For $n = 1$, we have the following lemma.

LEMMA 5.13. *Assume that* $\rho > 1$. *Let* d *be the constant given by* (5.22) *and* $J = \langle\cdot, \chi_*\rangle\chi$, *where* χ *and* χ_* *are the functions determined by* (5.20), (5.21), *and* (5.23), (5.24), *respectively. Then the following statements hold.*

(i) *Put* $S(z) = -z^{-1/2}VF_0 + 1 - VG_0$. *Then for any* z *in* $\Sigma(\delta)$ *with* $|z|$ *sufficiently small*

$$S(z)^{-1} = K - z^{1/2}(1 + dz^{1/2})^{-1}VJ \tag{5.48}$$

in $B(L_1^s)$, $1 \leqq s < \rho$, *where* K *satisfies* (5.35), (5.36), *and* $F_0K = 0$. *Furthermore,* K *belongs to* $B(L_1^r)$ *for any* r *with* $1 \leqq r < \rho + 1$.

(ii) *Put* $S'(z) = -z^{-1/2}F_0V + 1 - G_0V$. *Then for any* z *in* $\Sigma(\delta)$ *with* $|z|$ *sufficiently small*

$$S'(z)^{-1} = K' - z^{1/2}(1 + dz^{1/2})^{-1}JV \tag{5.49}$$

in $B(W_\infty^{2,s})$, $-\rho < s \leqq -1$, *where* K' *is an operator satisfying*

$$F_0VJV + (1 - G_0V)K' = JVF_0V + K'(1 - G_0V) = 1\,, \tag{5.50}$$

(5.51) $K'F_0 = 0,\ F_0VK' = JV(1 - G_0V)K' = K'(1 - G_0V)JV = 0$.

Furthermore, K' belongs to $B(W^{2,r}_\infty)$ for any r with $-\rho - 1 < r \leqq -1$.

PROOF. The first half of (i) is shown in the same way as Lemma 5.12; instead of (5.43) we choose χ and χ_* so that

$$\int V\chi(x)dx = \int c(y)\chi_*(y)dy = -2^{1/2}\ , \tag{5.52}$$

for $F_0 = 2^{-1}\langle\cdot, 1\rangle$. Since $K = 1 + VG_0K - VF_0VJ$ and $F_0K = 0$, we obtain that $K \in B(L_1^r)$ for any r *with* $1 \leqq r < \rho + 1$.

(ii) can be shown similarly (cf. [8, Theorem 7.2]). q.e.d.

PROOF OF THEOREM 5.5. We first assume that $\rho > 1$. By Lemma 5.13(i), the formula (5.46) holds for $|z|$ sufficiently small. Making use of $F_0K = 0$, we obtain by Lemma 2.4 that

$$R(z) = \sum_{j=0}^{[2\sigma]-2} z^{j/2}B_{j/2} + O(z^{\sigma-1}) \tag{5.53}$$

in $B(L_1^1, L_\infty^{-2\sigma+2}) + B(L_1^{2\sigma-1}, L_\infty)$, where $1 \leqq \sigma < \rho/2 + 1$. On the other hand, we can construct $R(z)$ also by the formula

$$R(z) = \sum_{j=0}^{\infty} (S'(z)^{-1}\tilde{R}_0(z)V)^j S'(z)^{-1}R_0(z)\ . \tag{5.54}$$

This together with Lemma 5.13(ii) shows that (5.53) holds also in $B(L_1, L_\infty^{-2\sigma+1}) + B(L_1^{2\sigma-2}, L_\infty^{-1})$. Hence we get (5.15) for $s \geqq 1/2$, (5.16) and (5.18). Furthermore, elementary calculations show that

$$U_j(x, y) = f_1(x) + f_2(x, y) = g_1(y) + g_2(x, y)\ ,$$
$$|f_2(x, y)|,\ |g_2(x, y)| \leqq M(\langle x\rangle + \langle y\rangle)^{2j}\langle x\rangle\langle y\rangle\ .$$

Since $f_1(x) = g_1(0) + g_2(x, 0) - f_2(x, 0)$, this yields (5.17). It follows from Theorem 1.1 and the comparison theorem that $U(t, x, y) \leqq Mt^{-1/2}$, which yields (5.15) for $0 \leqq s \leqq 1/2$. It remains to prove (5.19)$\sim$(5.27). In the same way as in $n = 2$ we have that $J = -dF_0VJ + G_0VJ$. By (5.46),

$$R(z) = G_0K - F_0VJ + z^{1/2}B_{1/2} + o(z^{1/2}) \quad \text{as} \quad z \to 0\ , \tag{5.55}$$

$$B_{1/2} = -J + (1 + G_0KV - F_0VJV)F_1K\ . \tag{5.56}$$

Since $F_0K = 0$ and $(1 + G_0KV - F_0VJV)F_0 = 0$, we obtain that

$$(1 + G_0KV - F_0VJV)F_1Kh(x) = -2^{-1}(1 + G_0KV - F_0VJV)x\int yKh(y)dy\ .$$

Putting

$$\psi(x) = 2^{-1/2}(1 + G_0KV - F_0VJV)x\ , \tag{5.57}$$

(5.58) $$\psi_*(y) = 2^{-1/2}K^*y\ ,$$

we thus obtain (5.19)~(5.27). This completes the proof of Theorem 5.5 for $\rho > 1$.

Let $0 < \rho \leqq 1$. Choose a C_0^∞-function $c_0(x)$ such that $c_0(x) \geqq 0$ and $c_0(x) \not\equiv 0$, and put

$$A_1 = (d/dx)^2 - c_0(x)\ , \quad R_1(z) = (z - A_1)^{-1}\ , \quad V_1 = V + c_0(x)\ .$$

In the following we shall consider A_1 as an unperturbed operator. We have already shown (see (5.53)) that

$$R_1(z) = C + O(z^\delta) \quad \text{as} \quad z \to 0 \quad \text{in} \quad B(L_1^{1+2\delta}, L_\infty^{-2\delta})\ , \qquad 0 \leqq \delta < 1/2\ .$$

By Lemma 5.9, there exists the inverse $(1 - V_1C)^{-1}$ of $1 - V_1C$ in $B(L_1^s)$, $1 \leqq s < \rho + 1$. With $K_1 = (1 - V_1C)^{-1}$ and $R_1(z) = \tilde{R}_1(z) - C$, we obtain that

(5.59) $$R(z) = R_1(z)K_1 \sum_{j=0}^{\infty} (V_1\tilde{R}_1(z)K_1)^j\ ,$$

provided $|z|$ is sufficiently small. Thus

(5.60) $$R(z) = B_0 + O(z^{\sigma-1}) \quad \text{as} \quad z \to 0 \quad \text{in} \quad B(L_1^1, L_\infty^{-2\sigma+2}) + B(L_1^{2\sigma-1}, L_\infty)\ ,$$

where $1 \leqq \sigma < \rho/2 + 1$. Similarly, (5.60) holds also in $B(L_1, L_\infty^{-2\sigma+1}) + B(L_1^{2\sigma-2}, L_\infty^{-1})$. Hence we get (5.15) for s with $1/2 \leqq s < (\rho + 1)/2$. On the other hand, the comparison theorem shows (5.15) for $s = 0$. This together with (5.15) for $s = 1/2$ yields (5.15) for $0 \leqq s \leqq 1/2$. q.e.d.

6. The case that $c(x) > 0$ on a non-empty open set. In this section we deal with the operator $A = \sum_{j,k} a_{jk}(x)\partial_j\partial_k + \sum_j b_j(x)\partial_j + c(x)$ satisfying the assumption (A) and the condition:

(6.1) $$c(x) > 0 \text{ on a non-empty open set}\ .$$

$U(t, x, y)$ stands for the fundamental solution for A.

Generalized eigenfunctions for A are also defined by Definition 5.7, and Theorem 5.8 holds also.

REMARK 6.1. If there are no generalized eigenfunctions for A, then results similar to Theorems 5.3~5.5 hold. For example, $U(t, x, y)$ for odd $n \geqq 3$ has the asymptotic formula

(6.2) $$U(t, x, y) = \sum_{j,k,l} e^{\lambda_j t}t^k\phi_{jkl}(x)\psi_{jkl}(y) + \sum_{j=0}^{[\sigma]} t^{-n/2-j}U_j(x, y) + O(t^{-n/2-\sigma})$$

as $t \to \infty$, where the first summation on the right hand side of (6.2) is a finite sum and $\mathrm{Re}\,\lambda_j \geqq 0$ and $\lambda_j \neq 0$.

In order to treat the case where there are generalized eigenfunctions for A, we further assume:

(A. IV) The inequality (1.2) holds with $\rho > 2$ for $n \leqq 2$, $\rho > 1$ for $n = 3$, and $\rho > 0$ for $n \geqq 4$.

(A. V) $A^* = A$.

From here on, we assume that A satisfies (A. I) $\sim$ (A. V) and (6.1), and that there are generalized eigenfunctions for A.

Let us denote by A_2 the self-adjoint realization of A in $L_2(R^n)$.

LEMMA 6.2. (i) *A_2 has at most finite positive eigenvalues with finite multiplicity.* (ii) *Every eigenfunction associated with a positive eigenvalue of A_2 is a C^2-function which decays exponentially as $|x| \to \infty$.*

PROOF. It is well known that each positive eigenvalue has finite multiplicity. Finiteness of positive eigenvalues is seen from the proof of Theorems 6.5$\sim$6.9 below. Let $\lambda > 0$, $u \in W_2^2$, and $Au = \lambda u$. Since Lemma 3.1 holds also for A, we obtain by the imbedding theorem that $u \in W_p^2$ for any $2 \leqq p \leqq \infty$. Choose a C^∞-function ϕ on R^n such that $\phi(x) = 1$ for $|x| \geqq N+1$ and $\phi(x) = 0$ for $|x| \leqq N$. We have that

$$(1 - R_0(\lambda)\phi V)u = R_0(\lambda)(1-\phi)Vu .$$

Fix p and a so that $2 \leqq p < \infty$ and $0 < a < \lambda^{1/2}$, and choose N so large that

$$\|R_0(\lambda)\phi V\| + \|e^{a\langle x\rangle}R_0(\lambda)\phi V e^{-a\langle x\rangle}\| < 1/2 ,$$

where $\|\ \ \|$ is the norm in $B(W_p^2)$. Thus

$$u = (1 - R_0(\lambda)\phi V)^{-1}R_0(\lambda)(1-\phi)Vu . \tag{6.3}$$

Since $(1-\phi)Vu(x)$ has compact support, $e^{a\langle x\rangle}D^\alpha R_0(\lambda)(1-\phi)Vu \in L_p$ for $|\alpha| \leqq 2$. Hence $e^{a\langle x\rangle}u(x) \in W_p^2$ for any $p < \infty$ and $a < \lambda^{1/2}$, which yields the exponential decay of $u(x)$ as $|x| \to \infty$. q.e.d.

LEMMA 6.3. (i) *The zero eigenvalue of A_2 has at most finite multiplicity.* (ii) *Every eigenfunction for the zero eigenvalue is a C^2-function which decays like $|x|^{-k}$ as $|x| \to \infty$, where $k = 2 - n$ for $n \geqq 5$, $k = 3$ for $n = 4$, $k = 2$ for $n = 3, 2$, and k is any positive number for $n = 1$.*

PROOF. (i) is clear. (ii) follows from the equality: $(1 - G_0 V)u = 0$ for any eigenfunction u (cf. [8, Lemma 3.2]). q.e.d.

We call a generalized eigenfunction not in L_2 a resonance state.

LEMMA 6.4. (i) *If $n \geqq 5$, there are no resonance states.* (ii) *For*

$n=4, 3, 1$, the dimension of the linear hull of resonance states is at most one; and for $n=2$, the dimension is at most three. (iii) *A resonance state ψ is a C^2-function having the following asymptotic formula as $|x|\to\infty$: When $n=3, 4$,*

$$\psi(x)=\lambda|x|^{2-n}+o(|x|^{2-n}) \text{ for some } \lambda\neq 0\ ; \tag{6.4}$$

when $n=2$,

$$\psi(x)=\lambda+o(1) \quad \text{for} \quad \text{some} \quad \lambda\neq 0 \tag{6.5.1}$$

or for some μ, ν with μ or $\nu\neq 0$

$$\psi(x)=(\mu x_1+\nu x_2)|x|^{-2}+o(|x|^{-1})\ ; \tag{6.5.2}$$

when $n=1$,

$$\psi(x)=\mu+\nu x/|x|+o(1) \quad \text{for some } (\mu, \nu)\neq(0, 0)\ . \tag{6.6}$$

(iv) *Every resonance state ψ is orthogonal to all eigenfunctions χ associated with positive eigenvalues of A_2: $\langle\psi, \chi\rangle\equiv\int\psi\bar{\chi}dx=0$.*

PROOF. (i) is clear. (iv) follows from Lemma 6.2(ii). Let $n=3, 4$, and ψ be a resonance state. Then we obtain that $(1-G_0V)\psi=0$ and $\int V\psi(x)dx\neq 0$, which implies (ii) and (iii) for $n=3, 4$. For $n=2$, we have that $(1-G_0V)\psi=\text{constant}$, $\int V\psi(x)dx=0$, $\int x_jV\psi(x)dx\neq 0$ for $j=1$ or 2, which shows (ii) and (iii) for $n=2$. The formula (6.6) is shown similarly. It remains to prove (ii) for $n=1$. To this end we have only to show that there is an unbounded solution of $Au=0$. We may assume that $Au=u''-bu'-cu$, from which we obtain the integral equation

$$u(x)=u(N)+u'(N)\int_N^x B(y)dy+\int_N^x\left(\int_y^x B(t)dt\right)B(y)^{-1}c(y)u(y)dy\ ,$$
$$B(y)=\exp\left[\int_N^y b(z)dz\right].$$

Choose N so large that $\int_N^x\left|\int_y^x B(t)dtB(y)^{-1}c(y)y\right|dy<x/4$ for all $x\geqq N$. Solving the integral equation with $u(N)$ and $u'(N)$ sufficiently large, we get a solution u on (N, ∞) which grows to infinity. Extending this solution to the left we get an unbounded solution u of $Au(x)=0$ on R^1. q.e.d.

REMARK. We see from the proof that when $n=1$ the dimension of the linear hull of generalized eigenfunction is at most 1.

Let λ_j ($j=1, \cdots, M$) be the repeated positive eigenvalues of A_2. Let χ_j and ϕ_k ($k=1, \cdot\cdot, N$) be real-valued eigenfunctions for the eigenvalues λ_j and 0, respectively, such that $\{\chi_j, \phi_k\}_{j,k}$ forms an orthonormal basis of the liner hull of all eigenspaces for the nonnegative eigenvalues of A_2. Put

$$(6.7)\qquad E(t, x, y) = U(t, x, y) - \sum_{j=1}^{N} \phi_j(x)\phi_j(y) - \sum_{j=1}^{M} e^{\lambda_j t}\chi_j(x)\chi_j(y) .$$

Here and in what follows the convention is: $\sum_j \phi_j(x)\phi_j(y) = 0$ if such functions ϕ_j do not exist. We note that $N \leqq 1$ when $n = 1$ by the remark after Lemma 6.3.

THEOREM 6.5. *For any* $-1 < \sigma < \rho/2 - 1$ *there hold the following formulas for all* $t > 1$ *and* $(x, y) \in R^{2n}$.

(i) *For odd* $n \geqq 5$,

$$(6.8)\qquad E(t, x, y) = \sum_{j=-2}^{[\sigma]} t^{-n/2-j} U_j(x, y) + \tilde{U}_\sigma(t, x, y) ,$$

$$(6.9)\qquad |\partial_t^l \tilde{U}_\sigma(t, x, y)| \leqq M_{\sigma l} t^{-n/2-\sigma-l} m(\sigma; x, y) , \qquad l \geqq 0 ,$$

$$(6.10)\qquad |U_j(x, y)| \leqq M_j m(j; x, y) , \qquad j \geqq 0 ,$$

where $m(\sigma; x, y) = (\langle x\rangle + \langle y\rangle)^{2\sigma} + \langle x\rangle^{2-n}\langle y\rangle^{2\sigma+2} + \langle x\rangle^{2\sigma+2}\langle y\rangle^{2-n}$, *and*

$$(6.11)\qquad |U_{-1}(x, y)| \leqq M_{-1}(\langle x\rangle^{2-n} + \langle y\rangle^{2-n}) ,$$

$$(6.12)\qquad U_{-2}(x, y) = \sum_{j,k=1}^{N} \frac{4\int V\phi_j(z)dz \int V\phi_k(z)dz}{(4\pi)^{n/2}(n-2)(n-4)} \phi_j(x)\phi_k(y) .$$

Furthermore, $U_j(x, y)$ *is a finite sum of functions of the form* $f(x)g(y)$ *with* $\langle f, \chi_j\rangle = \langle g, \chi_j\rangle = 0$ *for* $j = 1, \cdots, M$, *and* $U_j(y, x) = U_j(x, y)$.

(ii) *For even* $n \geqq 6$,

$$(6.13)\qquad E(t, x, y) = \sum_{j=-2}^{[\sigma]} \sum_{k=0}^{K(j)} t^{-n/2-j} \log^k t\, U_{jk}(x, y) + \tilde{U}_\sigma(t, x, y) ,$$

where $K(j) = [(j+2)/(n/2-2)]$, $\tilde{U}_\sigma(t, x, y)$ *satisfies* (6.9), $U_{jk}(x, y)$ *for* $j \geqq 0$ *and* $U_{-1k}(x, y)$ *satisfy the same estimates as* (6.10) *and* (6.11), *respectively, and* $U_{-20}(x, y)$ *is equal to the right hand side of* (6.12). *Furthermore,* $U_{jk}(x, y) = U_{jk}(y, x)$ *is a finite sum of function of the form* $f(x)g(y)$ *with* $\langle f, \chi_j\rangle = \langle g, \chi_j\rangle = 0$ *for* $j = 1, \cdots, M$. *In particular, when* $n = 6$

$$(6.14)\qquad U_{-11}(x, y) = 4^{-7}\pi^{-6} \sum_{j,k,l=1}^{N} c_j c_k c_l^2 \phi_j(x)\phi_k(y) , \qquad c_j = \int V\phi_j(z)dz .$$

PROOF. We only give a sketch of the proof of (i). Let ψ be a C^∞-function on R^1 such that $\psi(t) = 1$ for $t \geqq 1$ and $\psi(t) = 0$ for $t \leqq 0$, and set $g_N(x) = \psi(|x| - N)$ for $N \gg 1$. Choose a $C_0^\infty(R^n)$-function $c_0(x)$ such that

$c_0(x) \geqq 0$ and $c_0(x) \not\equiv 0$, and put

$$A_N = \sum_{j,k} a_{jk}(x)\partial_j\partial_k + \sum_j b_j(x)\partial_j - c_0(x) + g_N(x)c(x) . \tag{6.15}$$

We can choose N so large that there are no generalized eigenfunctions for A_N. Then we have that for any $1 < \sigma < n/2 + \rho/2 - 1$

$$R_N(z) \equiv (z - A_N)^{-1} = \sum_{j=0}^{[\sigma]} z^{j/2} C_{j/2} + O(z^\sigma) \quad \text{as} \quad z \to 0 ,$$
$$C_{j/2} = 0 \quad \text{for} \quad j \quad \text{odd and} \quad j/2 < n/2 - 1 .$$

Put $V_N = [c_0(x) + (1 - g_N(x))c(x)]\cdot$, $\tilde{R}_N(z) = R(z) - C_0 - zC_1$, and $S(z) = 1 - V_N C_0 - z V_N C_1$. Then along the line given in the proof of Theorems 4.1 and 4.2 in [8] we construct $R(z)$ in the form

$$R(z) = R_N(z)S(z)^{-1} \sum_{j=0}^{\infty} (V_N \tilde{R}_N(z) S(z)^{-1})^j$$

as operators between weighted L_2-spaces, and get the asymptotic formula as $z \to 0$: For $1 < \sigma < n/2 + \rho/2 - 1$,

$$R(z) - \sum_{j=-2}^{[2\sigma]-2} z^{j/2} B_{j/2} = O(z^{\sigma-1}) , \tag{6.16}$$

where $B_{-2} = P \equiv \sum_{j=1}^N \langle \cdot, \phi_j \rangle \phi_j$, $B_{j/2} = 0$ for j odd and $1 \leqq j < n/2 - 3$, and $B_{n/2-3} = PVF_0VP$, which one derives by taking the limit as $N \to \infty$ of the coefficient of $z^{n/2-3}$ in the expansion of $R(z)$ calculated for $A_N = \Delta + g_N(x) V g_N(x)$ instead of (6.15). (For more precise information on $B_{j/2}$, see [8, Theorems 4.1~4.3].)

By using the equality

$$R(z) = R_N(z) + R_N(z) V_N R_N(z) + R_N(z) V_N R(z) V_N R_N(z) , \tag{6.17}$$

we easily see that (6.16) for $n/2 - 1 < \sigma < n/2 + \rho/2 - 1$ holds also as operators from a weighted L_1-space to a weighted L_∞-space, which implies (6.8) ~ (6.12). The equality $U_j(y, z) = U_j(x, y)$ is shown by $A^* = A$. It follows from the expression of $B_{j/2}$ for j odd that $U_j(x, y)$ is a finite sum of functions of the from $f(x)g(y)$. Computing $e^{t\Delta}\chi_k$ by (6.8) and Lemma 6.2(ii), we have

$$e^{\lambda_k t}\chi_k = e^{\lambda_k t}\chi_k + \sum_{j=-2}^{[\sigma]} t^{-n/2-j} \int U_j(x, y)\chi_k(y)dy + O(t^{-n/2-\sigma}) \quad \text{as} \quad t \to \infty .$$

Thus $\int U_j(x, y)\chi_k(y)dy = 0$. Hence $\langle f, \chi_k \rangle = \langle g, \chi_k \rangle = 0$. q.e.d.

REMARK. There are differences between the notations in this paper and [8]. If we denote A, V, and $R(z)$ in [8] by A_S, V_S, and $R_S(z)$, respectively, then the correspondence is: $A = -A_S$, $V = -V_S$ and $R(z) = R_S(-z)$.

REMARK. In deriving (6.14) we use the formula (4.31) in [8, Theorem 4.3(i)]. But the formula is incorrect for the space dimension $n = 6$. When $n = 6$, the term

$$-z \log^2 z(PVF_0VP)^2$$

must be added before $o(z^{n/2-2})$ in (4.31) in order for the formula to be correct.

When $n = 3, 4$, the following asymptotic formulas hold for $t > 1$ and $(x, y) \in R^{2n}$. The formulas can be shown in the same way as Theorem 6.5.

THEOREM 6.6. *Let $n = 3$. For any $0 < \sigma < (\rho - 1)/2$ one has*

$$E(t, x, y) = \sum_{j=-1}^{[\sigma]} t^{-3/2-j} U_j(x, y) + \tilde{U}_\sigma(t, x, y) , \tag{6.18}$$

$$|\partial_t^l \tilde{U}_\sigma(t, x, y)| \leqq M_{\sigma l} t^{-3/2-\sigma-l} m'(\sigma; x, y) , \quad l \geqq 0 , \tag{6.19}$$

$$|U_j(x, y)| \leqq M_j m'(j; x, y) , \qquad j \geqq 0 , \tag{6.20}$$

where $m'(\sigma; x, y) = (\langle x\rangle + \langle y\rangle)^{2\sigma} + \langle x\rangle^{-1}\langle y\rangle^{2\sigma+1} + \langle x\rangle^{2\sigma+1}\langle y\rangle^{-1}$, *and*

$$U_{-1}(x, y) = \sum_{j,k=1}^{N} \sum_{l=1}^{3} \int \phi_j(z) V z_l dz \int \phi_k(z) V z_l dz \frac{\phi_j(x)\phi_k(y)}{12\pi^{1/2}} + \pi^{-1/2}\psi(x)\psi(y) , \tag{6.21}$$

where $\psi(x)$ is a resonance state determined by

$$\begin{cases} \psi(x) = (4\pi)^{-1/2}|x|^{-1} + o(|x|^{-1}) \quad as \quad |x| \to \infty , \\ \int \Delta\psi(x)\left(\int (4\pi|x - y|)^{-1}\phi_j(y)dy\right)dx = 0 , \quad j = 1, \cdots, N . \end{cases} \tag{6.22}$$

Furthermore, $U_j(x, y)$ is a finite sum of function of the form $f(x)g(y)$ with $\langle f, \chi_j\rangle = \langle g, \chi_j\rangle = 0$ for $j = 1, \cdots, M$, and $U_j(y, x) = U_j(x, y)$.

THEOREM 6.7. *Let $n = 4$ and $-1 < \sigma < \rho/2 - 1$. Then one has*

$$E(t, x, y) = \Psi(t)\psi(x)\psi(y) + \sum_{j=-1}^{[\sigma]} \sum_{(k,l)\in I(j,\sigma)} t^{-2-j}\Psi_{jkl}(t) U_{jkl}(x, y) + \tilde{U}_\sigma(t, x, y) , \tag{6.23}$$

$$I(j, \sigma) = \{(k, l); k, l \leqq j + 3, -j - 2 \leqq k + l\}\setminus\{(0, 0)\} \qquad for \quad j < \sigma ,$$

$$I(j, \sigma) = \{(k, l); k, l \leqq j + 3, -j - 2 \leqq k + l \leqq -2\} \qquad for \quad j = \sigma ,$$

where $\tilde{U}_\sigma(t, x, y)$ satisfies (6.9) with $n = 4$, $\psi(x)$ is a resonance state determined by

$$(6.24)\quad \begin{cases} \psi(x) = \pi^{-1}|x|^{-2} + o(|x|^{-2}) \quad as \quad |x| \to \infty , \\ \int \Delta\psi(x)\left(\int \pi^{-1}|x-y|^{-2}\phi_j(y)dy\right)dx = 0 , \qquad j = 1, \cdots, N . \end{cases}$$

and

$$(6.25)\qquad \Psi(t) = \int_0^\infty \frac{e^{-z}dz}{z[(\log z - \log t + d)^2 + \pi^2]} ,$$

$$(6.26)\qquad d = 2\gamma - 1 + (8\pi^2)^{-1}\iint V\psi(x)V\psi(y)\log|x-y|/2dxdy ,$$

where γ *is Euler's constant, and*

$$(6.27)\quad \Psi_{jkl}(t) = -\int_0^\infty \frac{(-z)^{j+1}e^{-z}\operatorname{Im}(\log z - \log t - \pi i)^k(\log z - \log t + d - \pi i)^l]dz}{\pi[(\log z - \log t)^2 + \pi^2]^k[(\log z - \log t + d)^2 + \pi^2]^l} ,$$

$$(6.28)\quad \begin{aligned} |U_{jkl}(x, y)| \leqq M_j\{(\langle x\rangle + \langle y\rangle)^{2m}\log(\langle x\rangle + \langle y\rangle) \\ + \langle x\rangle^{-2}\langle y\rangle^{2m+2}\log\langle y\rangle + \langle y\rangle^{-2}\langle x\rangle^{2m+2}\log\langle x\rangle\} , \\ m = j + \min(k + l + 1, 0) . \end{aligned}$$

Furthermore, $U_{jkl}(x, y)$ *is a finite sum of functions of the form* $f(x)g(y)$ *with* $\langle f, \chi_j\rangle = \langle g, \chi_j\rangle = 0$ *for* $j = 1, \cdots, M$ *and* $U_{jkl}(x, y) = U_{jkl}(y, x)$.

THEOREM 6.8. *Let* $n = 1$. *For any* $1 < \sigma < \rho/2$ *one has*

$$(6.29)\qquad E(t, x, y) = \sum_{j=0}^{[\sigma]} t^{-1/2-j}U_j(x, y) + \tilde{U}_\sigma(t, x, y) ,$$

$$(6.30)\qquad |\partial_t^l\tilde{U}_\sigma(t, x, y)| \leqq M_{\sigma l}t^{-1/2-\sigma-l}(\langle x\rangle + \langle y\rangle)^{2\sigma} , \qquad l \geqq 0 ,$$

$$(6.31)\qquad |U_j(x, y)| \leqq M_j(\langle x\rangle + \langle y\rangle)^{2j} .$$

Furthermore, $U_j(x, y)$ *is a finite sum of functions of the form* $f(x)g(y)$ *with* $\langle f, \chi_j\rangle = \langle g, \chi_j\rangle = \langle f, \phi_k\rangle = \langle g, \phi_k\rangle$ *for* $j = 1, \cdots, M, k = 1, \cdots, N$, *and* $U_j(x, y) = U_j(y, x)$. *In particular,* $U_0(x, y) = \psi(x)\psi(y)$ *for a resonance state* $\psi(x)$ *when there is a resonace state, and otherwise* $U_0(x, y) \equiv 0$.

PROOF. The theorem except for the last assertion can be shown in the same way as Theorem 6.5. Let P be the orthogonal projection onto the zero eigenspace, and put $Qf(x) = \pi^{1/2}\int U_0(x, y)f(y)dy$. Then

$$R(z) = z^{-1}P + z^{-1/2}Q + B_0 + o(1) \quad \text{as} \quad z \to 0 .$$

By the resolvent equation $(1 - R_0(z)V)R(z) = R_0(z)$,

$$(1 - G_0V)Q = F_0VB_0 + F_1VP + F_0 , \quad F_0VP = F_0VQ - (1 - G_0V)P = 0 .$$

This implies that $U_0(x, y) = \sum_k g_k(x)g_k(y)$ for some generalized eigenfunctions $g_k(x)$ or zero (cf (4.20) and (4.42) in [8]). Now assume that there is a generalized eigenfunction $g(x)$. Then the remark after Lemma 6.4 shows that $U_0(x, y) = \lambda g(x)g(y)$ for some constant λ. First, assuming that $g \notin L_2$, we show that $\lambda \neq 0$. Suppose that $\lambda = 0$. Then $R(z) = O(1)$ as $z \to 0$, for there are no eigenfunctions for zero. Thus the same argument as in the proof of Lemma 5.13 shows that there are operators J and K' such that $JVF_0V + K'(1 - G_0V) = 1$ and $K'F_0 = 0$. Since $F_0Vg = 0$ and $(1 - G_0V)g = F_0h$ for some function h, we get $g = 0$. This is a contradiction. Hence we obtain that $U_0(x, y) = \psi(x)\psi(y)$ with $\psi(x) = \lambda^{1/2}g(x) \not\equiv 0$. Next assume that $g \in L_2$. Then we have by Lemma 6.3(ii) that

$$g = e^{tA}g = g + \lambda t^{-1/2}\|g\|^2 g + o(t^{-1/2}) \quad \text{as} \quad t \to \infty .$$

This implies that $\lambda = 0$. Thus $U_0(x, y) = 0$. This completes the proof of the last assertion. q.e.d.

THEOREM 6.9. *Let $n = 2$. Let $0 < \sigma < \rho/2 - 1$ and σ' be the largest integer smaller than σ. Then one has*

$$E(t, x, y) = \sum_{k=1}^{N(-1)} \sum_{l=1}^{N(-1)} \Phi_{-1k}(t)g_{kl}(x)g_{kl}(y) + \sum_{k=-m}^{0} \sum_{l=1}^{N(0)} t^{-1}\log^{-k}t\, g_{kl}(x)\, g_{kl}(y)$$
$$+ \sum_{j=0}^{\sigma'} \sum_{k=2}^{N(j)} \sum_{l=1}^{N(j)} t^{-1-j}\Phi_{jk}(t)\, U_{jk}(x, y)$$
$$+ \sum_{j=1}^{[\sigma]} \sum_{k=-(j+1)m}^{0} \sum_{l=1}^{N(j)} t^{-1-j}\log^{-k}t\, U_{jk}(x, y) + \tilde{U}_\sigma(t, x, y) . \tag{6.32}$$

Here $N(j)$ is an integer depending on j, m is an integer with $0 \leqq m \leqq 6$,

$$\Phi_{jk}(t) = \sum_{i=k}^{\nu-1} c_{ijk}\log^{-i}t + O(\log^{-\nu}t) \quad as \quad t \to \infty\ , \quad \nu > k\ , \tag{6.33}$$

where the c_{ijk} are numerical constants and the asymptotic expansion (6.33) *is termwise differentiable, the functions $g_{kl}(x)$ are zero or generalized eigenfunctions such that*

$$|g_{kl}(x)| \leqq M_{-1}\langle x\rangle^{-1}\ , \qquad k \neq 0\ , \tag{6.34}$$

$$|g_{0l}(x)| \leqq M_{-1}\ , \tag{6.35}$$

and

$$|U_{jk}(x, y)| \leqq M_j(\langle x\rangle + \langle y\rangle)^{2j-1}\log 2\langle x\rangle \log 2\langle y\rangle\ , \qquad k \leqq -1\ , \tag{6.36}$$

$$|U_{j0}(x, y)| \leqq M_j(\langle x\rangle + \langle y\rangle)^{2j}\ , \tag{6.37}$$

$$|U_{jk}(x, y)| \leqq M_j\{(\langle x\rangle + \langle y\rangle)^{2j}\log 2\langle x\rangle \log 2\langle y\rangle + \langle x\rangle^{-1}\langle y\rangle^{2j+1}\log 2\langle y\rangle + \langle y\rangle^{-1}\langle x\rangle^{2j+1}\log 2\langle x\rangle\}\ , \quad k \geqq 2\ , \tag{6.38}$$

$$|\partial_t^l \tilde{U}_\sigma(t, x, y)| \leqq M_{\sigma l}t^{-1-\sigma-l}\{(\langle x\rangle + \langle y\rangle)^{2\sigma} + \langle x\rangle^{-1}\langle y\rangle^{2\sigma+1} + \langle y\rangle^{-1}\langle x\rangle^{2\sigma+1}\}\ , \quad l \geqq 0\ . \tag{6.39}$$

Furthermore, $U_{jk}(x, y)$ is a finite sum of functions of the form $f(x)g(y)$ with $\langle f, \chi_j\rangle = \langle g, \chi_j\rangle = 0$, $j = 1, \cdots, M$, and $U_{jk}(y, x) = U_{jk}(x, y)$.

PROOF. The expansions (6.32) and (6.33) can be shown in the same way as Theorem 6.5 (cf. [8, Theorem 4.1]). We give here only the outline of the proof of (6.34) $\sim$ (6.38). For $\sigma > 0$ with $\sigma \notin \boldsymbol{Z}$, we obtain that

$$R(z) = \sum_{j=-1}^{[\sigma]} \sum_{k=-(j+1)m-1}^{\infty} z^j \log^{-k} z\, B_{j,k} + O(z^\sigma) \quad \text{as} \quad z \to 0\,, \tag{6.40}$$

where $B_{-1,-1} = 0$ and $B_{-1,0} = P \equiv \sum_{j=1}^{N} \langle \cdot, \phi_j\rangle \phi_j$. By the resolvent equation $(1 - R_0(z) V)R(z) = R(z)(1 - VR_0(z)) = R_0(z)$, we have that

$$(1 - G_0 V)B_{-1,k} = F_0 VB_{-1,k+1}\,, \qquad k \geqq 0\,, \tag{6.41}$$

$$B_{-1,k}(1 - VG_0) = B_{-1,k+1} VF_0\,, \qquad k \geqq 0\,, \tag{6.41'}$$

and for $j \geqq 0$

$$(1 - G_0 V)B_{j,k} = \sum_{l=0}^{j+1} F_l VB_{j-l,k+1} + \sum_{l=1}^{j+1} G_l VB_{j-l,k} + \delta_{k+1} F_j + \delta_k G_j\,, \tag{6.42}$$

$$B_{j,k}(1 - VG_0) = \sum_{l=0}^{j+1} B_{j-l,k+1} VF_l + \sum_{l=1}^{j+1} B_{j-l,k} VG_l + \delta_{k+1} F_j + \delta_k G_j\,, \tag{6.42'}$$

where $\delta_l = 1$ for $l = 0$ and $\delta_l = 0$ for $l \neq 0$. We first show by induction that for any $k \geqq 0$

$$(1 - G_0 V)B_{-1,k} = 0\,, \qquad F_0 VB_{-1,k} = 0\,. \tag{6.43}$$

Since $B_{-1,0} = P$, (6.43) holds for $k = 0$. Suppose that (6.43) holds for k. By (6.41), $F_0 VB_{-1,k+1} = 0$. This together with (6.41′) implies that $F_0 VB_{-1,k+2} VF_0 = 0$. Since $A^* = A$, there are real-valued functions $\{\psi_l\}_l$ such that $B_{-1,k+2} = \sum_l \langle \cdot, \psi_l\rangle \psi_l$. Thus

$$0 = F_0 VB_{-1,k+2} VF_0 = -\sum_l \left(\int V\psi_l dx\right)^2 F_0\,.$$

This implies that $F_0 V\psi_l = 0$ for any l. Hence $F_0 VB_{-1,k+2} = 0$, which implies that $(1 - G_0 V)B_{-1,k+1} = 0$. This completes the proof of (6.43). Similarly,

$$(1 - G_0 V)B_{0,k} = 0 \quad \text{and} \quad F_0 VB_{0,k} = 0\,, \qquad k \leqq -2\,, \tag{6.44}$$

$$(1 - G_0 V)B_{0,-1} = F_0 VB_{0,0} + F_1 VP + F_0 \quad \text{and} \quad F_0 VB_{0,-1} = 0\,. \tag{6.45}$$

It follows from (6.43) $\sim$ (6.45) that $g_{kl}(x)$ are zero or generalized eigenfunctions satisfying (6.34) and (6.35). We have by (6.42) and (6.42′) that for $j \geqq 0$ and $k \geqq 1$

$$\begin{aligned} B_{j,k} = \sum_{\substack{i+l \leqq j+1 \\ i,l \geqq 0}} (&F_i VB_{j-i-l,k+2} VF_l + F_i VB_{j-i-l,k+1} VG_l \\ &+ G_i VB_{j-i-l,k+1} VF_l + G_i VB_{j-i-l,k} VG_l)\,. \end{aligned} \tag{6.46}$$

This implies (6.38). Similar argument shows (6.36) and (6.37). q.e.d.

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