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MINIMAL FOLIATIONS ON LIE GROUPS

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1. Introduction. The study of minimal foliations, totally geodesic foliations and foliations with bundle-like metrics may be of interest to us in the meaning that they are geometric properties combined foliated structures with Riemannian structures. The foliated Riemannian manifold with a bundle-like metric was defined by Reinhart [15] and is discussed by him and others ([7], [10], [11]). The Riemannian submersion ([2], [13]) is a special case of this conception. The foliated Riemannian manifold with totally geodesic leaves is discussed by Dombrowski [1], Ferus [3], Johnson and Whitt [8], Tanno [18] and others. This case often appears in the differential geometry. Recently, Haefliger [5], Kamber and Tondeur [9], Rummler [16], Sullivan [17] and many people discuss the foliated Riemannian manifold with minimal leaves.

In this paper we define a foliation on a Lie group. For a Lie group G, we take a Lie subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ associated to G and a left invariant Riemannian metric $\langle\ ,\ \rangle$. Then we have a foliated Riemannian manifold $(G, \langle\ ,\ \rangle, \mathscr{F}(\mathfrak{h}))$. On $(G, \langle\ ,\ \rangle, \mathscr{F}(\mathfrak{h}))$, we discuss the totally geodesicness and minimality of leaves and bundle-like-ness of the metric. We have many interesting examples, for instance, foliated Riemannian manifolds with minimal, not totally geodesic leaves. From these examples, we may remark that it is not able to extend Oshikiri's theorem [14] to the case of codimension $\geqq 2$.

2. Preliminaries. Let (M, g) be an n-dimensional Riemannian manifold M with a Riemannian metric g. The objects under consideration are of class C^∞. Let $\{e_A\}$ be a local orthonormal frame field in M and $\{w^A\}$ be the dual coframe field. Here and hereafter, indices $A, B, \cdots$ run from 1 to n. The connection forms w^A_B on M associated with $\{e_A\}$ are uniquely defined by

$$dw^A = -\sum_B w^A_B \wedge w^B\ , \quad w^A_B + w^B_A = 0\ .$$

We set $dw^A = -\sum_{B,C} \Gamma^A_{BC} w^B \wedge w^C$, $\Gamma^A_{BC} + \Gamma^A_{CB} = 0$, then w^A_B are given by

$$w^A_B = \sum_C (-\Gamma^A_{BC} + \Gamma^C_{AB} - \Gamma^A_{CA}) w^C\ . \tag{2.1}$$

The covariant derivative $\nabla_B e_A$ of a vector field e_A in the direction of e_B is defined by

$$\nabla_B e_A = \sum_C w_A^C(e_B)e_C .$$

The 2-forms Ω_B^A defined by

$$\Omega_B^A = dw_B^A + \sum_C w_C^A \wedge w_B^C$$

are called the curvature forms of M associated with $\{e_A\}$. The curvature tensor $R = (R_{BCD}^A)$ of M is defined by

$$\Omega_B^A = (1/2)\sum_{C\,D} R_{BCD}^A w^C \wedge w^D , \quad R_{BCD}^A + R_{BDC}^A = 0 .$$

The following properties of the curvature tensor R are well-known:

$$R_{BCD}^A + R_{ACD}^B = 0 , \quad R_{BCD}^A = R_{DAB}^C , \quad R_{BCD}^A + R_{CDB}^A + R_{DBC}^A = 0 .$$

The quantity R_{BAB}^A is called a sectional curvature with respect to the plane spanned by e_A and e_B. Let $S = (S_{AB} = \sum_C R_{ACB}^C)$ be called the Ricci tensor of M and S_{AA} be called a Ricci curvature in the direction of e_A.

Now let $(M, g, \mathscr{F})$ be an n-dimensional foliated Riemannian manifold, that is, an n-dimensional Riemannian manifold M with a Riemannian metric g admits a foliation $\mathscr{F}$. The foliation $\mathscr{F}$ is given by an integrable subbundle E of the tangent bundle TM over M. The maximal connected integral submanifolds of E are called leaves. Each leaf has the same dimension, say q. Then q is called the dimension of $\mathscr{F}$ and $p = n - q$ is called the codimension of $\mathscr{F}$ ([15]).

The following convention on the range of indices will be used throughout this paper unless otherwise stated:

$$\begin{aligned} &A, B, C, \cdots = 1, 2, \cdots, n \\ &i, j, k, \cdots = 1, 2, \cdots, q \\ &\alpha, \beta, \gamma, \cdots = q+1, q+2, \cdots, q+p = n . \end{aligned}$$

We may take a local orthonormal frame field $\{e_A\}$ in $(M, g, \mathscr{F})$ such that e_i are tangent to the leaves, hence e_α are orthogonal to the leaves. Such a frame field $\{e_A\}$ is called a local orthonormal adapted frame field with respect to $\mathscr{F}$ and g. Hereafter we consider a local orthonormal adapted frame field $\{e_A\}$ and the dual coframe field $\{w^A\}$ in $(M, g, \mathscr{F})$. Then $\mathscr{F}$ is locally given by $w^\alpha = 0$. We set

$$w_i^\alpha = \sum_j h_{ij}^\alpha w^j + \sum_\beta A_{i\beta}^\alpha w^\beta , \tag{2.2}$$

then the integrability of E implies that $h_{ij}^\alpha = h_{ji}^\alpha$. We remark that the distribution defined by $w^i = 0$ is integrable if and only if $A^\alpha_{i\beta} = A^\beta_{i\alpha}$. Thus the tensor $A = (A^\alpha_{i\beta})$ is called the integrability tensor of the orthogonal complement bundle $E^\perp$. The bundle $E^\perp$ is identified with the normal bundle $Q = TM/E$ of $\mathscr{F}$([13]).

For each point of $(M, g, \mathscr{F})$, the quadratic form $\sum_{i,j} h^\alpha_{ij} w^i \cdot w^j$ is called the second fundamental form of the leaf through the point in the direction of e_α. Thus the tensor (h^α_{ij}) is called the second fundamental tensor on $(M, g, \mathscr{F})$. The vector field $H = \sum_{i,\alpha} h^\alpha_{ii} e_\alpha$ is called the mean curvature vector field on $(M, g, \mathscr{F})$, and $H^\alpha = \sum_i h^\alpha_{ii}$ is called the mean curvature in the direction of e_α. A leaf is *minimal* if $H = 0$ on the leaf, and $(M, g, \mathscr{F})$ is *minimal* if all leaves are minimal. A leaf is *totally geodesic* if

$$h^\alpha_{ij} = 0 \tag{2.3}$$

on the leaf, and $(M, g, \mathscr{F})$ is *totally geodesic* if all leaves are totally geodesic.

The Riemannian metric g on $(M, g, \mathscr{F})$ is called a bundle-like metric with respect to $\mathscr{F}$ if for each point $x \in M$ there exists a neighborhood U of x, a p-dimensional Riemannian manifold $(V, \tilde{g})$ and a Riemannian submersion $\varphi: (U, g\restriction_U) \to (V, \tilde{g})$ such that $\varphi^{-1}(y)$ is an intersection of U and a leaf ([15]). But the following fact is useful to us: The Riemannian metric g on $(M, g, \mathscr{F})$ is a *bundle-like metric with respect to* $\mathscr{F}$ if and only if

$$A^\alpha_{i\beta} + A^\beta_{i\alpha} = 0 \tag{2.4}$$

for a local orthonormal adapted frame field $\{e_A\}$ ([11], [16]).

3. Foliations on Lie groups. In this section, we construct a foliated Riemannian manifold $(G, \langle\, , \rangle, \mathscr{F}(\mathfrak{h}))$ and investigate the geometric properties of this foliated Riemannian manifold. We refer to [6] and [12] in this and the following sections.

Let G be an n-dimensional Lie group and $\mathfrak{g}$ be the associated Lie algebra consisting of all vector fields on G that are invariant under left translations. We take a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, then we have a foliated manifold $(G, \mathscr{F}(\mathfrak{h}))$. We denoted by L_x the left translation of G by $x \in G$. Let H be a connected subgroup of G whose Lie algebra is $\mathfrak{h}$. For each point $x \in G$, a submanifold $L_x(H)$ of G is the leaf through x of the foliation $\mathscr{F}(\mathfrak{h})$ of G. If we take a left invariant metric $\langle\, , \rangle$ on G, then we have a foliated Riemannian manifold $(G, \langle\, , \rangle, \mathscr{F}(\mathfrak{h}))$. We assume that the foliation $\mathscr{F}(\mathfrak{h})$ of G is of codimension p, and $n = p + q$.

We apply the discussion in the above section to $(G, \langle\, , \rangle, \mathscr{F}(\mathfrak{h}))$. Let

$\{e_A\}$ be an orthonormal basis for $\mathfrak{g}$ and $\{w^A\}$ be the dual basis. We denote by C^D_{AB} the structure constants of $\mathfrak{g}$ with respect to $\{e_A\}$, that is, $[e_A, e_B] = \sum_D C^D_{AB} e_D$. By the equation of Maurer-Cartan:

$$dw^A = (-1/2) \sum_{B,D} C^A_{BD} w^B \wedge w^D \tag{3.1}$$

and (2.1), the connection forms w^A_B are given by $w^A_B = (1/2)\sum_D (-C^A_{BD} + C^D_{AB} - C^B_{DA}) w^D$ ([12]). Hereafter, we take an orthonormal adapted basis $\{e_A\}$ for $\mathfrak{g}$ with respect to $\mathscr{F}(\mathfrak{h})$ and $\langle\ ,\ \rangle$, hence $\{e_i\}$ is a basis for $\mathfrak{h}$. Since $C^\alpha_{ij} = 0$, (2.2) and (3.1), we have

$$h^\alpha_{ij} = (-1/2)(C^j_{i\alpha} + C^i_{j\alpha}) \tag{3.2}$$

$$A^\alpha_{i\beta} = (-1/2)(C^\alpha_{i\beta} + C^\beta_{i\alpha} - C^i_{\alpha\beta}) . \tag{3.3}$$

PROPOSITION 3.1. *Let $(G, \langle\ ,\ \rangle, \mathscr{F}(\mathfrak{h}))$ be as above, and the metric $\langle\ ,\ \rangle$ is Ad(H)-invariant. Then the following holds:*

$$C^i_{jk} + C^k_{ji} = 0\ , \quad C^i_{j\alpha} = 0\ , \quad C^\alpha_{i\beta} + C^\beta_{i\alpha} = 0\ ,$$

so that all leaves are totally geodesic and $\langle\ ,\ \rangle$ is a bundle-like metric with respect to $\mathscr{F}(\mathfrak{h})$.

PROOF. By the assumption, for all $t \in \boldsymbol{R}$, $\langle \mathrm{Ad}(\exp(te_i))e_A, \mathrm{Ad}(\exp(te_i))e_B\rangle = \langle e_A, e_B\rangle = \delta_{AB}$.

Thus we have $\langle [e_i, e_A], e_B\rangle + \langle e_A, [e_i, e_B]\rangle = 0$, that is, $C^B_{iA} + C^A_{iB} = 0$. The second part follows from (2.3), (2.4), (3.2) and (3.3). q.e.d.

REMARK. If H is compact, then it is well known that there exists an Ad(H)-invariant and left invariant metric on G.

PROPOSITION 3.2. *Let $(G, \langle\ ,\ \rangle, \mathscr{F}(\mathfrak{h}))$ be as above. The following conditions are equivalent:*

(i) *The metric $\langle\ ,\ \rangle$ is a bundle-like metric with respect to $\mathscr{F}(\mathfrak{h})$.*

(ii) *For all X orthogonal to $\mathfrak{h}$, $\nabla_X X$ is orthogonal to $\mathfrak{h}$.*

PROOF. For all $X = \sum_\alpha f_\alpha e_\alpha$ orthogonal to $\mathfrak{h}$, we have

$$\begin{aligned}\nabla_X X &\equiv \sum_{\alpha,\beta,i} f_\alpha f_\beta w^i_\beta(e_\alpha) e_i \quad (\mathrm{mod}\ e_r) \\ &= (1/2) \sum_{\alpha,\beta,i} f_\alpha f_\beta (-C^i_{\beta\alpha} + C^\alpha_{i\beta} - C^\beta_{\alpha i}) e_i \\ &= (1/2) \sum_{\alpha,\beta,i} f_\alpha f_\beta (C^\alpha_{i\beta} + C^\beta_{i\alpha}) e_i\ .\end{aligned}$$

Thus, by (2.4) and (3.3), we have the equivalence. q.e.d.

REMARK. If curves $t \to \exp(tX)$ are geodesic in G for all X orthogonal to $\mathfrak{h}$, (ii) holds. We refer to Theorem 4.2 in [19].

Now, we will construct a new foliated Riemannian manifold $(G, \overline{\langle , \rangle}, \mathscr{F}(\mathfrak{h}))$ with $(G, \langle , \rangle, \mathscr{F}(\mathfrak{h}))$. Given a (p, q)-matrix (ξ^i_α) of real numbers, we set

$$\bar{e}_i = e_i \qquad (3.4)$$
$$\bar{e}_\alpha = \sum_j \xi^j_\alpha e_j + e_\alpha \, .$$

Then we may take a left invariant metric $\overline{\langle , \rangle}$ on G so that $\{\bar{e}_A\}$ is orthonormal. Thus we have a foliated Riemannian manifold $(G, \overline{\langle , \rangle}, \mathscr{F}(\mathfrak{h}))$. We investigate the relations between old and new foliated Riemannian manifolds. We denote by $\bar{C}^D_{AB}$ the struture constants of $\mathfrak{g}$ with respect to $\{\bar{e}_A\}$. By (3.4) and $C^\alpha_{ij} = 0$, we have

$$\bar{C}^j_{i\alpha} = C^j_{i\alpha} + \sum_k \xi^k_\alpha C^j_{ik} - \sum_\beta \xi^j_\beta C^\beta_{i\alpha} \qquad (3.5\text{-}1)$$

$$\bar{C}^\beta_{i\alpha} = C^\beta_{i\alpha} \qquad (3.5\text{-}2)$$

$$\bar{C}^k_{ij} = C^k_{ij} \qquad (3.5\text{-}3)$$

$$\bar{C}^k_{\alpha\beta} = \sum_{i,j} \xi^i_\alpha \xi^j_\beta C^k_{ij} + \sum_i \xi^i_\alpha C^k_{i\beta} - \sum_{i,\theta} \xi^i_\alpha \xi^k_\theta C^\theta_{i\beta} + \sum_i \xi^i_\beta C^k_{\alpha i} - \sum_{i,\theta} \xi^i_\beta \xi^k_\theta C^\theta_{\alpha i} - \sum_\theta \xi^k_\theta C^\theta_{\alpha\beta} + C^k_{\alpha\beta} \qquad (3.5\text{-}4)$$

$$\bar{C}^\theta_{\alpha\beta} = C^\theta_{\alpha\beta} + \sum_i \xi^i_\alpha C^\theta_{i\beta} + \sum_i \xi^i_\beta C^\theta_{\alpha i} \, . \qquad (3.5\text{-}5)$$

By (3.2), (3.3) and (3.5), the second fundamental tensor $(\bar{h}^\alpha_{ij})$ and the integrability tensor $(\bar{A}^\alpha_{i\beta})$ are given by

$$\begin{aligned} \bar{h}^\alpha_{ij} &= (-1/2)(\bar{C}^j_{i\alpha} + \bar{C}^i_{j\alpha}) \\ &= (-1/2)\Big(C^j_{i\alpha} + C^i_{j\alpha} + \sum_k \xi^k_\alpha C^j_{ik} + \sum_k \xi^k_\alpha C^i_{jk} - \sum_\beta \xi^j_\beta C^\beta_{i\alpha} - \sum_\beta \xi^i_\beta C^\beta_{j\alpha}\Big) \end{aligned} \qquad (3.6)$$

$$\begin{aligned} \bar{A}^\alpha_{i\beta} &= (-1/2)(\bar{C}^\alpha_{i\beta} + \bar{C}^\beta_{i\alpha} - \bar{C}^i_{\alpha\beta}) \\ &= (-1/2)\Big(C^\alpha_{i\beta} + C^\beta_{i\alpha} - C^i_{\alpha\beta} - \sum_{j,k} \xi^j_\alpha \xi^k_\beta C^i_{jk} - \sum_j \xi^j_\alpha C^i_{j\beta} \\ &\quad - \sum_j \xi^j_\beta C^i_{\alpha j} + \sum_{j,\theta} \xi^j_\alpha \xi^i_\theta C^\theta_{j\beta} + \sum_{j,\theta} \xi^j_\beta \xi^i_\theta C^\theta_{\alpha j} + \sum_\theta \xi^i_\theta C^\theta_{\alpha\beta}\Big) \, . \end{aligned} \qquad (3.7)$$

If the metric $\langle , \rangle$ is $Ad(H)$-invariant, we have, by Proposition 3.1, (3.6) and (3.7),

$$\bar{h}^\alpha_{ij} = (1/2)\Big(\sum_\beta \xi^j_\beta C^\beta_{i\alpha} + \sum_\beta \xi^i_\beta C^\beta_{j\alpha}\Big) \qquad (3.8)$$

$$\begin{aligned} \bar{A}^\alpha_{i\beta} &= (-1/2)\Big(-C^i_{\alpha\beta} - \sum_{j,k} \xi^j_\alpha \xi^k_\beta C^i_{jk} + \sum_{j,\theta} \xi^j_\alpha \xi^i_\theta C^\theta_{j\beta} + \sum_{j,\theta} \xi^j_\beta \xi^i_\theta C^\theta_{\alpha j} + \sum_\theta \xi^i_\theta C^\theta_{\alpha\beta}\Big) \\ &= -\bar{A}^\beta_{i\alpha} \, . \end{aligned} \qquad (3.9)$$

Thus, by (2.4), we have

LEMMA 3.3. *If the metric* $\langle\,,\rangle$ *is* $Ad(H)$*-invariant, then the metric* $\overline{\langle\,,\rangle}$ *is a bundle-like metric with respect to* $\mathscr{F}(\mathfrak{h})$.

We introduce a positive integer $I(G, \langle,\rangle, \mathscr{F}(\mathfrak{h}))$ associated with a foliated Rimannian manifold $(G, \langle\,,\rangle, \mathscr{F}(\mathfrak{h}))$. We assume that the metric $\langle\,,\rangle$ is $Ad(H)$-invariant. We consider the orthogonal decomposition of the Lie algebra $\mathfrak{g}$ of G with respect to the metric $\langle\,,\rangle$: $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. By Proposition 3.1, we have $C^j_{i\alpha} = 0$. Thus, for each $X \in \mathfrak{h}$, $ad(X)$ is a linear transformation from $\mathfrak{m}$ to itself, and we denote by $r(X)$ the rank of the linear transformation $ad(X)$. Then we define $I(G, \langle\,,\rangle \mathscr{F}(\mathfrak{h}))$ by

$$I(G, \langle\,,\rangle, \mathscr{F}(\mathfrak{h})) = \max_{\{e_i\} \in NB(\mathfrak{h})} \sum_i r(e_i) \tag{3.10}$$

where $NB(\mathfrak{h})$ denotes the set of all orthonormal bases $\{e_i\}$ for $\mathfrak{h}$.

THEOREM 3.4. *Let* $(G, \langle\,,\rangle, \mathscr{F}(\mathfrak{h}))$ *be a foliated Riemannian manifold with an* $Ad(H)$*-invariant and left invariant Riemannian metric* $\langle\,,\rangle$ *and a foliation* $\mathscr{F}(\mathfrak{h})$ *of codimension* $p = n - q$. *Let* $I(G, \langle\,,\rangle, \mathscr{F}(\mathfrak{h}))$ *be the positive integer defined by* (3.10). *Suppose that* $[\mathfrak{h}, \mathfrak{m}] = \mathfrak{m}$ *and* $p < I(G, \langle\,,\rangle, \mathscr{F}(\mathfrak{h}))$, *then* G *admits a left invariant Riemannian metric* $\overline{\langle\,,\rangle}$ *satisfying the following*:

(i) *The foliated Riemannian manifold* $(G, \overline{\langle\,,\rangle}, \mathscr{F}(\mathfrak{h}))$ *is not totally geodesic but minimal.*

(ii) *The metric* $\overline{\langle\,,\rangle}$ *is a bundle-like metric with respect to* $\mathscr{F}(\mathfrak{h})$.

Moreover, suppose that G *is compact and semisimple, and that* $[\mathfrak{h}, \mathfrak{m}] = \mathfrak{m}$ *and* $p < I(G, \langle\,,\rangle, \mathscr{F}(\mathfrak{h}))$ *with respect to a bi-invariant metric* $\langle\,,\rangle$ *on* G, *then* G *admits a left invariant Riemannian metric* $\overline{\langle\,,\rangle}$ *satisfying the above* (i), (ii) *and with positive Ricci curvatures in all directions.*

PROOF. We take an orthonormal basis $\{e_A\}$ for $\mathfrak{g}$ with respect to $\langle\,,\rangle$ such that $I(G, \langle\,,\rangle, \mathscr{F}(\mathfrak{h})) = \sum_i r(e_i)$. For a (p, q)-matrix (ξ^i_α) of real numbers, we take a new basis $\{\bar{e}_A\}$ for $\mathfrak{g}$ such that $\bar{e}_i = e_i$ and $\bar{e}_\alpha = \sum_j \xi^j_\alpha e_j + e_\alpha$. We may take a left invariant Riemannian metric $\overline{\langle\,,\rangle}$ on G which truns the basis $\{\bar{e}_A\}$ into an orthonormal basis for $\mathfrak{g}$. Since the metric $\langle\,,\rangle$ is $Ad(H)$-invariant, we have, by Lemma 3.3, that the metric $\overline{\langle\,,\rangle}$ is a bundle-like metric with respect to $\mathscr{F}(\mathfrak{h})$. By (3.8), we have $\sum_i \bar{h}^\alpha_{ii} = \sum_{i,\beta} \xi^i_\beta C^\beta_{i\alpha}$.

Let ψ_1 be a linear mapping from the space $\mathfrak{m}(p, q; \boldsymbol{R})$ of all (p, q)-matrices of real numbers to $\mathfrak{m}$ defined by

$$\psi_1((x^i_\alpha)) = \sum_{i,\alpha,\beta} x^i_\beta C^\beta_{i\alpha} e_\alpha\,.$$

Since $[\mathfrak{h}, \mathfrak{m}] = \mathfrak{m}$, we have $\dim \mathrm{Ker}(\psi_1) = pq - p$. Thus the set of all (ξ^i_α)

satisfying

$$\sum_{i,\beta} \xi^i_\beta C^\beta_{i\alpha} = \sum_i \bar{h}^\alpha_{ii} = 0 \tag{3.11}$$

is a vector space of dimension $pq - p$. On the other hand, let ψ_2 be a linear mapping from $\mathfrak{m}(p, 1; \boldsymbol{R})$ to $\mathfrak{m}$ defined by, for each i,

$$\psi_2((x^i_\alpha)) = \sum_{\alpha,\beta} x^i_\beta C^\beta_{i\alpha} e_\alpha .$$

Then we have $\dim \operatorname{Ker}(\psi_2) = p - r(e_i)$. Thus the set of all (ξ^i_α) satisfying

$$\sum_\beta \xi^i_\beta C^\beta_{i\alpha} = 0 \quad (\text{for each } i) \tag{3.12}$$

is a vector space of dimension $\sum_i (p - r(e_i))$. By the assumpsion, we have $pq - p > \sum_i (p - r(e_i))$. Thus there exists a solution (ξ^i_α) of (3.11), but not of (3.12). We take such a $(\xi^i_\alpha) \in \mathfrak{m}(p, q; \boldsymbol{R})$ so that the metric $\overline{\langle\ ,\ \rangle}$ has a property (i)

If G is compact, then G admits a left invariant (and in fact a bi-invariant) metric $\langle\ ,\ \rangle$ so that all sectional curvatures are non-negative ([12]). The sectional curvature K_{AB} associated by e_A and e_B $(A \neq B)$ is given by $K_{AB} = (1/4)\| [e_A, e_B] \|^2$, where $\| \cdot \|^2 = \langle \cdot\ ,\ \cdot \rangle$, and the Ricci curvature S_{AA} in the direction e_A is given by

$$S_{AA} = (1/4) \sum_B \| [e_A, e_B] \|^2 . \tag{3.13}$$

Since $\mathfrak{g}$ is semisimple, we may prove that the Ricci curvatures in all directions are positive. If $S_{AA} = 0$ for some A, then (3.13) implies $[e_A, e_B] = 0$ for all B. This means that e_A belongs to the center of $\mathfrak{g}$. This contradicts that $\mathfrak{g}$ is semisimple. Under that $[\mathfrak{h}, \mathfrak{m}] = \mathfrak{m}$ and $p < I(G, \langle\ ,\ \rangle, \mathscr{F}(\mathfrak{h}))$, we may construct a metric $\overline{\langle\ ,\ \rangle}$ as above. The metric $\overline{\langle\ ,\ \rangle}$ is a continuous function of (ξ^i_α). Hence the Ricci curvatures in all directions with respect to $\overline{\langle\ ,\ \rangle}$ are positive for (ξ^i_α) sufficiently near the zero matrix.
q.e.d.

REMARK. As an example satisfying the condition in Theorem 3.4, we take a symmetric pair (G, H). But this example is a case of $q \geqq 3$. Examples of case of $q = 2$ are given in section 5.

4. Minimal foliations on non-compact Lie groups. Let $\mathfrak{g}$ be a non-compact simple Lie algebra and θ be an involutive automorphism of $\mathfrak{g}$. We set $\mathfrak{k} = \{X \in \mathfrak{g};\ \theta(X) = X\}$, $\mathfrak{p} = \{X \in \mathfrak{g},\ \theta(X) = -X\}$, then we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ (direct sum) with $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. Let B denote the Killing form of $\mathfrak{g}$, and we define $\langle\ ,\ \rangle$ by

$$\langle X, Y \rangle = -B(X, \theta(Y))$$

for all $X, Y \in \mathfrak{g}$. Then the metric $\langle\ ,\ \rangle$ is a left invariant metric on G and, for all $X \in \mathfrak{p}$, $ad(X)$ is a symmetric linear transformation of $\mathfrak{g}$ with respect to $\langle\ ,\ \rangle$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{a}^*$ be the dual space. For $\lambda \in \mathfrak{a}^*$, we set

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g};\ [A, X] = \lambda(A)X \quad \text{for} \quad A \in \mathfrak{a}\}\ . \tag{4.1}$$

Then λ is called a root if $\mathfrak{g}_\lambda \neq 0$, and let Δ denote the set of all roots. We have

$$\mathfrak{g} = \sum_{\lambda \in \Delta} \mathfrak{g}_\lambda \qquad \mathfrak{g}_0 \supset \mathfrak{a} \tag{4.2}$$
$$[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu} \quad \text{for} \quad \lambda, \mu \in \Delta\ .$$

We take an ordering in $\mathfrak{a}^*$. Let Δ^+ denote the set of positive roots. Next we consider two subsets Δ_1 and Δ_2 of Δ^+ satisfying

(i) $\Delta_1 \supset \Delta_2$

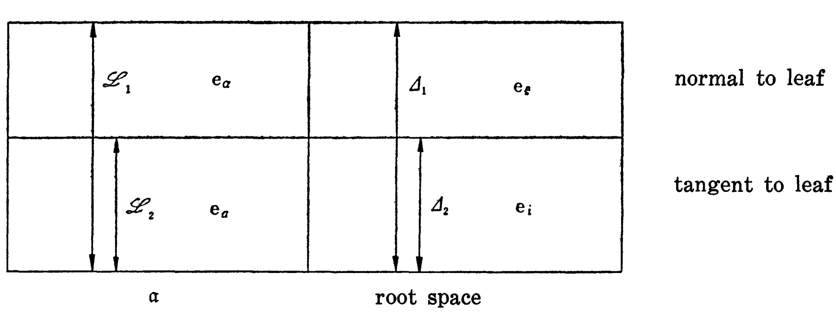
(ii) $\lambda, \mu \in \Delta_r$, $\lambda + \mu \in \Delta^+$ implies $\lambda + \mu \in \Delta_r$ $(r = 1, 2)$.

Moreover, let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two subspaces of $\mathfrak{a}$ such that $\mathscr{L}_1 \supsetneqq \mathscr{L}_2$. We set $\mathfrak{n}_r = \mathscr{L}_r + \sum_{\lambda \in \Delta_r} \mathfrak{g}_\lambda$ $(r = 1, 2)$, then $\mathfrak{n}_1$ is an algebra and $\mathfrak{n}_2$ is a subalgebra of $\mathfrak{n}_1$ Let N_1 (resp. N_2) be a connected Lie subgroup of G with the Lie algebra $\mathfrak{n}_1$ (resp. $\mathfrak{n}_2$). Then we may have a foliated Riemannian manifold $(N_1, \langle\ ,\ \rangle, \mathscr{F}(\mathfrak{n}_2))$, where $\langle\ ,\ \rangle$ denotes a suitable left invariant metric on N_1 as below. We now compute the second fundamental tensor (h^α_{ij}) and the integrability tensor $(A^\alpha_{i\beta})$ of the orthogonal complement bundle of $\mathscr{F}(\mathfrak{n}_2)$. Here we use the range of indices as follows:

$$1 \leqq a, b \leqq \dim \mathscr{L}_2$$
$$\dim \mathscr{L}_2 + 1 \leqq i, j \leqq \dim \mathscr{L}_2 + \sharp\Delta = \dim N_2$$
$$\dim N_2 + 1 \leqq \alpha, \beta \leqq \dim N_2 + \dim \mathscr{L}_1 - \dim \mathscr{L}_2$$
$$\dim N_2 + \dim \mathscr{L}_1 - \dim \mathscr{L}_2 + 1 \leqq \xi, \eta \leqq \dim N_1\ .$$

We set

$\{e_a, e_i\}$: basis for $\mathfrak{n}_2$



$$\{e_a, e_\alpha\}\colon \text{ basis for } \mathscr{L}_1$$
$$\{e_i, e_\xi\}\colon \text{ basis for } \sum_{\lambda\in\Delta_1} \mathfrak{g}_\lambda .$$

We may take e_i (resp. e_ξ) in the root space $\mathfrak{g}_{\lambda_i}$ (resp. $\mathfrak{g}_{\lambda_\xi}$), and it may happen that $\lambda_s = \lambda_t$ for $s \neq t$. We take a left invariant metric $\langle\ ,\ \rangle$ on N_1 so that $\{e_a, e_i, e_\alpha, e_\xi\}$ is orthonormal. It is obvious that the structure constants $C^i_{ai}, C^\xi_{a\xi}, C^i_{\alpha i}$ and $C^\xi_{\alpha\xi}$ of N_1 (with respect to the basis $\{e_a, e_i, e_\alpha, e_\xi\}$) are not necessarily zero. By (3.2) and (4.1), the mean curvature H^α in the direction of e_α is given by

$$H^\alpha = \sum_i C^i_{\alpha i} = \sum_i \lambda_i(e_\alpha) ,$$

and the mean curvature H^ξ in the direction of e_ξ satisfies $H^\xi = 0$. By (3.3), (4.1) and (4.2), we have

$$A^\xi_{a\xi} = -\lambda_\xi(e_a)$$
$$A^\xi_{i\eta} = (-1/2)(C^\xi_{i\eta} + C^\eta_{i\xi} - C^i_{\xi\eta})$$

the others vanish.

Then we have the following:

$\langle\ ,\ \rangle$ is a bundle-like metric with respect to $\mathscr{F}(\mathfrak{n}_2)$.

$\iff \lambda_\xi(e_a) = 0 ,\quad C^\xi_{i\eta} + C^\eta_{i\xi} = 0 \quad$ for all $\ a, i, \xi, \eta$.

$\iff \lambda_\xi(e_a) = 0 ,\quad \lambda_i + \lambda_\xi \notin \Delta_1 \setminus \Delta_2 \quad$ for all $\ a, i, \xi$.

The orthogonal complement bundle of $\mathscr{F}(\mathfrak{n}_2)$ is integrable.

$\iff C^i_{\xi\eta} = 0 \quad$ for all $\ i, \xi, \eta$.

$\iff \lambda_\xi + \lambda_\eta \notin \Delta_2 \quad$ for all $\ \xi, \eta$.

Thus we have

PROPOSITION 4.1. *Let $(N_1, \langle\ ,\ \rangle, \mathscr{F}(\mathfrak{n}_2))$ be as above. Then*

(i) *$(N_1, \langle\ ,\ \rangle, \mathscr{F}(\mathfrak{n}_2))$ is minimal if and only if $\sum_i \lambda_i(e_\alpha) = 0$ for all α,*

(ii) *$\langle\ ,\ \rangle$ is a bundle-like metric with respect to $\mathscr{F}(\mathfrak{n}_2)$ if and only if $\lambda_\xi(e_a) = 0$ for all a, ξ and $((\Delta_1 \setminus \Delta_2) + \Delta_2) \cap (\Delta_1 \setminus \Delta_2) = \varnothing$,*

(iii) *the orthogonal complement bundle of $\mathscr{F}(\mathfrak{n}_2)$ is integrable if and only if $((\Delta_1 \setminus \Delta_2) + (\Delta_1 \setminus \Delta_2)) \cap \Delta_2 = \varnothing$. And the above three conditions are independent of each others.*

We may show an example of minimal foliated Riemannian manifold $(N_1, \langle\ ,\ \rangle, \mathscr{F}(\mathfrak{n}_2))$ such that the metric $\langle\ ,\ \rangle$ is not a bundle-like metric with respect to $\mathscr{F}(\mathfrak{n}_2)$ and the orthogonal complement bundle of $\mathscr{F}(\mathfrak{n}_2)$ is not integrable:

EXAMPLE 4.2. Let $\mathfrak{g} = \mathfrak{sl}(n, \boldsymbol{R})$ and $\theta: \mathfrak{g} \to \mathfrak{g}$ defined by $\theta(X) = -{}^tX$ for all $X \in \mathfrak{g}$. Then we have

$$\mathfrak{k} = \mathfrak{so}(n)$$
$$\mathfrak{p} = \{X \in \mathfrak{gl}(n, \boldsymbol{R}) \mid {}^tX = X, \quad \text{Trace } X = 0\}$$
$$\mathfrak{a} = \left\{ \begin{pmatrix} H_1 & & 0 \\ & \ddots & \\ 0 & & H_n \end{pmatrix} \middle| H_1 + \cdots + H_n = 0 \right\}.$$

We define $\lambda_i \in \mathfrak{a}^*$ by

$$\lambda_i\left(\begin{pmatrix} H_1 & & 0 \\ & \ddots & \\ 0 & & H_n \end{pmatrix}\right) = H_i$$

Then we have $\Delta = \{\lambda_i - \lambda_j \mid 1 \leqq i, j \leqq n\}$. If we take an ordering of Δ such that $\lambda_1 > \cdots > \lambda_n$, then $\Delta^+ = \{\lambda_i - \lambda_j \mid 1 \leqq i < j \leqq n\}$.

In the case of that $n = 4$, we set

$$\Delta_1 = \{\lambda_1 - \lambda_2, \lambda_1 - \lambda_3, \lambda_1 - \lambda_4, \lambda_2 - \lambda_3, \lambda_2 - \lambda_4\}$$
$$\Delta_2 = \{\lambda_1 - \lambda_3, \lambda_2 - \lambda_4\}$$
$$\mathscr{L}_1 = \boldsymbol{R} \cdot \begin{bmatrix} 2 & & & 0 \\ & -2 & & \\ & & 1 & \\ 0 & & & -1 \end{bmatrix}, \quad \mathscr{L}_2 = \{0\},$$

and then we have a foliated Riemannian manifold $(N_1, \langle\ ,\ \rangle, \mathscr{F}(\mathfrak{n}_2))$ as above. Then we have the following. (i) $(N_1, \langle\ ,\ \rangle, \mathscr{F}(\mathfrak{n}_2))$ is a minimal, not totally geodesic because $(\lambda_1 - \lambda_3 + \lambda_2 - \lambda_4)(\mathscr{L}_1) = 0$ and $(\lambda_1 - \lambda_3)(\mathscr{L}_1) \neq 0$. (ii) The metric $\langle\ ,\ \rangle$ is not a bundle-like metric with respect to $\mathscr{F}(\mathfrak{n}_2)$ because $\lambda_1 - \lambda_2 \in \Delta_1 \setminus \Delta_2$, $\lambda_2 - \lambda_4 \in \Delta_2$ and $(\lambda_1 - \lambda_2) + (\lambda_2 - \lambda_4) = \lambda_1 - \lambda_4 \in \Delta_1 \setminus \Delta_2$. (iii) The orthogonal complement bundle $\mathscr{F}(\mathfrak{n}_2)$ is not integrable because $\lambda_1 - \lambda_2 \in \Delta_1 \setminus \Delta_2$, $\lambda_2 - \lambda_3 \in \Delta_1 \setminus \Delta_2$ and $(\lambda_1 - \lambda_2) + (\lambda_2 - \lambda_3) = \lambda_1 - \lambda_3 \in \Delta_2$.

5. Further examples of minimal foliations.

EXAMPLE 1. Let $\mathfrak{g}$ be a 3-dimensional unimodular Lie algebra. There exists a basis $\{e_1, e_2, e_3\}$ for $\mathfrak{g}$ such that

$$[e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad [e_1, e_2] = \lambda_3 e_3,$$

where $\lambda_s = 0, \pm 1$ $(s = 1, 2, 3)$. There exist just six distinct cases on the signs of λ_1, λ_2 and λ_3. By changing signs if necessary, we may assume that at most one of the structure constants $\lambda_1, \lambda_2, \lambda_3$ is negative. We have the following table ([12]):

Signs of $\lambda_1, \lambda_2, \lambda_3$	Associated Lie group	Description
$+ \quad + \quad +$	$SU(2)$ or $SO(3)$	compact, simple
$+ \quad + \quad -$	$SL(2, \boldsymbol{R})$ or $O(1, 2)$	noncompact, simple
$+ \quad + \quad 0$	$E(2)$	solvable
$+ \quad - \quad 0$	$E(1, 1)$	solvable
$+ \quad 0 \quad 0$	Heisenberg group	nilpotent
$0 \quad 0 \quad 0$	$\boldsymbol{R} \oplus \boldsymbol{R} \oplus \boldsymbol{R}$	commutative

Let G denote a connected Lie group whose Lie algebra is $\mathfrak{g}$.

EXAMPLE 1-1. We consider the cases of (i) $\lambda_1 = \lambda_2 = 1, \lambda_3 = 0$, (ii) $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 0$, (iii) $\lambda_1 = 1, \lambda_2 = \lambda_3 = 0$, (iv) $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Let $\mathfrak{h}$ be an abelian subalgebra of $\mathfrak{g}$ generated by e_1 and e_2. We take a left invariant metric $\langle\ ,\ \rangle$ on G so that $\{e_1, e_2, e_3\}$ is orthonormal. Then we have foliated Riemannian manifolds $(G, \langle\ ,\ \rangle, \mathscr{F}(\mathfrak{h}))$ of the type (i) $\sim$ (iv). The above four foliated Riemannian manifolds are minimal because $H = -(C^1_{13} + C^2_{23})e_3 = 0$. Since $h^3_{12} = (-1/2)(C^2_{13} + C^1_{23}) = (-1/2)(-\lambda_2 + \lambda_1)$, the foliated Riemannian manifold of type (i) and (iv) are totally geodesic, and ones of type (ii) and (iii) are not totally geodesic. The metrics $\langle\ ,\ \rangle$ on the above four types are bundle-like with respect to the foliation $\mathscr{F}(\mathfrak{h})$ because $A^3_{13} = A^3_{23} = 0$.

EXAMPLE 1-2. We consider the case of that $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = -1$. By changing $e'_1 = e_1$, $e'_2 = e_2 - e_3$ and $e'_3 = e_2 + e_3$, we may assume that the basis $\{e_1, e_2, e_3\}$ already satisfies the following:

$$[e_2, e_3] = 2e_1\ , \quad [e_3, e_1] = e_3\ , \quad [e_1, e_2] = e_2\ .$$

Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ generated by e_1 and e_2. We take a left invariant metric $\langle\ ,\ \rangle$ on G so that $\{e_1, e_2, e_3\}$ is orthonormal. Then we have a foliated Riemannian manifold $(G, \langle\ ,\ \rangle \mathscr{F}(\mathfrak{h}))$. This is minimal, not totally geodesic because $h^3_{11} = h^3_{22} = 0$ and $h^3_{12} = -1$. The metric $\langle\ ,\ \rangle$ is not bundle-like with respect to $\mathscr{F}(\mathfrak{h})$ because $A^3_{13} = 1$. This example is due to Roussarie ([4]).

EXAMPLE 1-3. We consider the case of that $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Then $\mathfrak{g} = \mathfrak{so}(3)$. Let $\mathscr{A}$ denote the Lie algebra of $SO(2)$ and e_0 basis for $\mathscr{A}$. Let $\mathfrak{h}$ be a subalgebra of $\mathscr{A} \times \mathfrak{g}$ generated by e_0 and e_1. By changing

$e'_0 = e_0$, $e'_1 = e_1$, $e'_2 = ae_0 + e_2$ and $e'_3 = e_3$ for any real number a, we may assume that the basis $\{e_0, e_1, e_2, e_3\}$ already satisfies the following:

$$[e_1, e_2] = e_3\ , \quad [e_2, e_3] = e_1\ , \quad [e_3, e_1] = -ae_0 + e_2\ ,$$
$$[e_0, e_s] = 0 \quad (s = 1, 2, 3)\ .$$

We take a left invariant metric $\langle\ ,\ \rangle$ on $SO(2) \times SO(3)$ so that $\{e_0, e_1, e_2, e_3\}$ is orthonormal. Then we have a foliated Riemannian manifold $(SO(2) \times SO(3), \langle\ ,\ \rangle, \mathscr{F}(\mathfrak{h}))$. We have

$$h^2_{00} = h^2_{11} = h^2_{01} = h^3_{00} = h^3_{11} = 0\ , \quad h^3_{01} = -a/2$$

and

$$A^2_{02} = A^2_{03} = A^3_{02} = A^3_{03} = A^2_{12} = A^3_{13} = 0\ , \quad A^2_{13} = -A^3_{12} = 1/2\ .$$

Thus $(SO(2) \times SO(3), \langle\ ,\ \rangle, \mathscr{F}(\mathfrak{h}))$ is minimal, and if $a \neq 0$ then it is not totally geodesic. And the metric $\langle\ ,\ \rangle$ is bundle-like with respect to $\mathscr{F}(\mathfrak{h})$.

Next, we compute the Ricci tensor S. We easily have the following:

(i) $dw^0 = -aw^1 \wedge w^3\ , \quad dw^1 = -w^2 \wedge w^3\ , \quad dw^2 = w^1 \wedge w^3\ ,$
$dw^3 = -w^1 \wedge w^2\ .$

(ii) $w^0_1 = (-a/2)w^3\ , \quad w^0_2 = 0\ , \quad w^0_3 = (a/2)w^1\ , \quad w^1_2 = (-1/2)w^3\ ,$
$w^1_3 = (a/2)w^0 + (1/2)w^2\ , \quad w^2_3 = (-1/2)w^1\ .$

(iii) $R^0_{101} = a^2/4\ , \quad R^0_{112} = a/4\ , \quad R^0_{303} = a^2/4\ ,$
$R^0_{323} = -a/4\ , \quad R^1_{212} = 1/4\ , \quad R^1_{313} = (1 - 3a^2)/4\ ,$
$R^2_{323} = 1/4$ (the others vanish) .

By these, we have

$$S_{00} = a^2/2\ , \quad S_{02} = -a/2\ , \quad S_{11} = (1 - a^2)/2\ ,$$
$$S_{22} = 1/2\ , \quad S_{33} = (1 - a^2)/2 \quad \text{(the others vanish)}\ .$$

Thus the Ricci curvature S_X in the direction of $X = \sum_A x_A e_A$ is given by

$$\begin{aligned} S_X &= x_0 x_0 S_{00} + 2x_0 x_2 S_{02} + x_1 x_1 S_{11} + x_2 x_2 S_{22} + x_3 x_3 S_{33} \\ &= (ax_0 - x_2)^2/2 + (1 - a^2)((x_1)^2 + (x_3)^2)/2\ . \end{aligned}$$

If $|a| \leqq 1$, then the Ricci curvatures in all directions are non-negative.

REMARK. Oshikiri [14] proved the following: If $(M, \langle\ ,\ \rangle, \mathscr{F})$ is a compact Riemannian manifold with a minimal foliation $\mathscr{F}$ of codimension one and with non-negative Ricci curvature with respect to the metric $\langle\ ,\ \rangle$, then all leaves of $\mathscr{F}$ are totally geodesic and the metric $\langle\ ,\ \rangle$ is bundle-like with respect to $\mathscr{F}$. But, by the above example, we can not extend this result to the case of codimension $\geqq 2$.

EXAMPLE 2. Let $\boldsymbol{R}^{q+p} = \boldsymbol{R}^q \times \boldsymbol{R}^p$ be a foliated manifold each leaf of which is given by $\boldsymbol{R}^q \times \{m\}$ for each $m \in \boldsymbol{R}^p$. For positive valued functions $f^1, \cdots, f^q: \boldsymbol{R}^p \to \boldsymbol{R}$, we may define a metric $\langle\ ,\ \rangle$ on $\boldsymbol{R}^{q+p}$ by

$$\langle\ ,\ \rangle = \sum_i (f^i)^2(dx^i)^2 + \sum_\alpha (dx^\alpha)^2 .$$

Then we have a foliated Riemannian manifold $(\boldsymbol{R}^{q+p}, \langle\ ,\ \rangle, \mathscr{F}(\boldsymbol{R}^q))$. A frame field $\{e_i = (f^i)^{-1}\partial/\partial x^i, e_\alpha = \partial/\partial x^\alpha\}$ in $\boldsymbol{R}^{q+p}$ is an orthonormal adapted frame field with respect to $\mathscr{F}(\boldsymbol{R}^q)$ and $\langle\ ,\ \rangle$. The dual coframe field $\{w^i, w^\alpha\}$ is given by $w^i = f^i dx^i$ (not sum) and $w^\alpha = dx^\alpha$. Since it holds that

$$\begin{aligned} dw^i &= df^i \wedge dx^i \\ &= \sum_\alpha (\partial f^i/\partial x^\alpha) dx^\alpha \wedge dx^i \\ &= \sum_\alpha (\partial f^i/\partial x^\alpha)(f^i)^{-1} w^\alpha \wedge w^i \\ &= -\sum_\alpha (\partial \log f^i/\partial x^\alpha) w^i \wedge w^\alpha , \end{aligned}$$

we have $w^i_j = 0$, $w^\alpha_\beta = 0$ and $w^i_\alpha = -w^\alpha_i = (\partial \log f^i/\partial x^\alpha) w^i$. By (2.2), we have

$$h^\alpha_{ij} = -(\partial \log f^i/\partial x^\alpha)\delta_{ij} , \quad A^\alpha_{i\beta} = 0 .$$

Thus we have that

(i) The metric $\langle\ ,\ \rangle$ is bundle-like with respect to $\mathscr{F}(\boldsymbol{R}^q)$.

(ii) $(\boldsymbol{R}^{q+p}, \langle\ ,\ \rangle, \mathscr{F}(\boldsymbol{R}^q))$ is minimal if and only if $f^1 \times \cdots \times f^q =$ constant.

(iii) If $(\boldsymbol{R}^{q+p}, \langle\ ,\ \rangle, \mathscr{F}(\boldsymbol{R}^q))$ is minimal, then the Ricci curvature $S_{\alpha\alpha}$ in the direction of e_α is non-positive, that is $S_{\alpha\alpha} = -\sum_i (\partial \log f^i/\partial x^\alpha)^2$.

REMARK. If each f^i satisfies that $f^i(x^\alpha + n^\alpha) = f^i(x^\alpha)$ for some $n^\alpha \in \boldsymbol{Z}$, then we may define a minimal foliated Riemannian manifold $(\boldsymbol{T}^{q+p}, \langle\ ,\ \rangle, \mathscr{F}(\boldsymbol{T}^q))$.

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