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A FORMULA IN SIMPLE JORDAN ALGEBRAS

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0. In this paper, we give a proof of a formula ((14) in 3) which gives a useful parametrization in reduced simple Jordan algebras. We also summarize some relevant facts on Jordan algebras. This formula and some of its consequences (e.g. Prop. 4, 5) were already used in [4b] and [5]. For basic facts on Jordan algebras, the reader is referred to [2], [3], [4a, c] and [6].

Let A be a Jordan algebra over a field F of characteristic zero. We use the following notation:

$$\{a, b, c\} = (ab)c + a(bc) - b(ac) ,$$
$$T_a(x) = ax , \qquad P_a(x) = \{a, x, a\} = (2T_a^2 - T_{a^2})x ,$$
$$(a \square b)x = \{a, b, x\} = (T_{ab} + [T_a, T_b])x \qquad (a, b, c, x \in A) .$$

It is well-known that A has a structure of "JTS" with respect to this triple product $\{\ \ \}$, i.e. one has

$$(1) \qquad \{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\} .$$

Throughout this paper, we assume that A is simple (and semi-simple). Then A has a unit element 1 and the following symmetric bilinear form on A is non-degenerate:

$$(2) \qquad \langle x, y\rangle = \kappa\, \mathrm{tr}(x \square y) = \kappa\, \mathrm{tr}(T_{xy}) \qquad (x, y \in A) ,$$

where κ is a fixed element in $F^{\times}(=F - \{0\})$.

1. Let e be an idempotent in A and let

$$A_\lambda = A_\lambda(e) = \{x \in A \mid ex = \lambda x\} \qquad \text{for} \quad \lambda \in F .$$

Then one has the direct sum decomposition ("Peirce decomposition")

$$A = A_0 + A_{1/2} + A_1$$

with

$$(3) \qquad \begin{cases} A_\lambda^2 = A_\lambda , \quad A_\lambda A_{1/2} \subset A_{1/2} \quad (\lambda = 0, 1) , \\ A_0 A_1 = 0 , \quad A_{1/2}^2 \subset A_0 + A_1 , \\ \{A_\lambda, A_\mu, A_\nu\} \subset A_{\lambda-\mu+\nu} . \end{cases}$$

Moreover, A_1 and A_0 are simple subalgebras with unit elements e and $1-e$, respectively, (which are central if A is so), and the map $a \mapsto 2T_a | A_{1/2}$ gives unital Jordan algebra representations of A_0 and A_1, which are mutually commutative. For $a \in A$, we denote by a_λ the A_λ-component of a in the above decomposition. Then one has

$$\begin{cases} a_1 = 2e(ea) - ea = P_e(a) , \\ a_{1/2} = 4(ea - e(ea)) , \\ a_0 = a - 3ea + 2e(ea) = P_{1-e}(a) . \end{cases}$$

Now let $x \in A_{1/2}$. Then by the definition one has

$$\{e, x, y\} = \begin{cases} 0 & \text{if } \ y \in A_1 , \\ e(xy) = (xy)_1 & \text{if } \ y \in A_{1/2} , \\ xy(\in A_{1/2}) & \text{if } \ y \in A_0 . \end{cases} \tag{4}$$

It follows that $e \square x$ is nilpotent and one has

$$\exp(e \square x)y = \begin{cases} y & \text{if } \ y \in A_1 , \\ y + e(xy) & \text{if } \ y \in A_{1/2} , \\ y + yx + \frac{1}{2}e(x(yx)) & \text{if } \ y \in A_0 . \end{cases} \tag{5}$$

An element $a \in A$ is called "invertible" if P_a is invertible. If a is invertible, then the inverse of a is given by $a^{-1} = P_a^{-1}(a)$ and one has $P_a P_{a^{-1}} = a \square a^{-1} = \mathrm{id}$; in particular, $aa^{-1} = 1$. For a given idempotent e, we say a is *invertible with respect to* e if $P_e(a)$ is invertible in $A_1(e)$.

LEMMA 1. *Let $a = a_0 + a_{1/2} + a_1$, $a_\lambda \in A_\lambda$, and suppose that a is invertible with respect to $1-e$. Then there exist uniquely determined elements $x \in A_{1/2}$ and $a_1' \in A_1$ such that*

$$a = \exp(e \square x)(a_0 + a_1') . \tag{6}$$

PROOF. In view of (5), it is enough to show that the following equations in x and a_1' have a unique solution:

$$a_0 x = a_{1/2} , \qquad a_1' + \frac{1}{2}(x(a_0 x))_1 = a_1 .$$

We denote the inverse of a_0 in A_0 by a_0^{-1}. Then, since $a_0 \mapsto 2T_{a_0} | A_{1/2}$ is a unital representation, we obtain a unique solution given by

$$\begin{cases} x = 4a_0^{-1} a_{1/2} \in A_{1/2} , \\ a_1' = a_1 - 2e(a_{1/2}(a_0^{-1} a_{1/2})) \in A_1 . \end{cases} \tag{6a}$$

q.e.d.

2. A non-zero idempotent e is called "primitive" if $A_1(e)$ does not contain any idempotent other than e and 0. We call e "absolutely primitive" if one has $A_1(e) = \{e\}_F$. A simple Jordan algebra A is called "reduced" if all primitive idempotents in A are absolutely primitive. In what follows, we assume that A is simple and reduced. (This implies that A is central.)

As is easily seen, there exists a "primitive decomposition" of 1, i.e. a set of (absolutely) primitive idempotents $\{e_1, \cdots, e_r\}$ such that

$$e_i e_j = \delta_{ij} e_i , \qquad \sum_{i=1}^{r} e_i = 1 .$$

The number r, which is uniquely determined, is called the *rank* of A. We set $\dim A = n$, $\operatorname{rank} A = r$, and use the inner product $\langle \ \rangle$ defined by (2) with $\kappa = r/n$.

Let $\{e_i \ (1 \leqq i \leqq r)\}$ be a fixed primitive decomposition of 1 in A and set

$$A_{ij} = \begin{cases} A_1(e_i) & \text{if} \quad i = j , \\ A_{1/2}(e_i) \cap A_{1/2}(e_j) & \text{if} \quad i \neq j . \end{cases} \tag{7}$$

Then one has the direct sum decomposition

$$A = \bigoplus_{1 \leqq i \leqq j \leqq r} A_{ij} .$$

From (3) one obtains multiplicative relations between the A_{ij}'s. In particular, one has $A_{ij} A_{kl} = 0$ if $\{i, j\} \cap \{k, l\} = \emptyset$, and $A_{ij} A_{jk} \subset A_{ik}$ if $i \neq k$. For $x_{ij}, y_{ij} \in A_{ij}$ $(i \neq j)$, one has

$$x_{ij} y_{ij} = \frac{1}{2} \langle x_{ij}, y_{ij} \rangle (e_i + e_j) . \tag{8}$$

When i, j, k, l are all distinct, one has

$$\begin{aligned} (x_{ij} y_{jk}) z_{kl} &= x_{ij} (y_{jk} z_{kl}) , \\ (x_{ij} y_{ij}) z_{jk} &= x_{ij} (y_{ij} z_{jk}) + y_{ij} (x_{ij} z_{jk}) \\ &= \frac{1}{4} \langle x_{ij}, y_{ij} \rangle z_{jk} , \end{aligned} \tag{9}$$

where $x_{ij} \in A_{ij}$, etc.

It is known that there exists a positive number d such that $\dim A_{ij} = d$ for all $i \neq j$. Thus one has

$$n = r + \frac{1}{2} r(r-1) d , \tag{10}$$

which implies $r \mid 2n$.

For $u \in A$, we write $u = \sum_{i \leqq j} u_{ij}$ with $u_{ij} \in A_{ij}$. In general, the symbols like a_{ij}, x_{ij} are meant to denote elements in A_{ij}. By (4) applied to $e = e_i$, we obtain

LEMMA 2. *Let* $x_{ij} \in A_{ij}$ $(i < j)$ *and* $y_{kl} \in A_{kl}$ $(k \leqq l)$. *Then one has*

$$\{e_i, x_{ij}, y_{kl}\} = \begin{cases} x_{ij}y_{kl} & \text{if } j = k \text{ or } l \text{ and } i \neq k, l, \\ e_i(x_{ij}y_{ij}) & \text{if } i = k \text{ and } j = l, \\ 0 & \text{otherwise}. \end{cases} \tag{11}$$

This lemma implies that if the set of pairs of indices $\{(k, l) \mid 1 \leqq k, l \leqq r\}$ is ordered in the lexicographical order, then $y' = \{e_i, x_{ij}, y_{kl}\} \neq 0$ $y' \in A_{k'l'}$ $(k' \leqq l')$ implies $(k', l') < (k, l)$. (In fact, if $j = k$ or if $j = l$ and $i < k$, then $k' = i < k$; and if $j = l$ and $i \geqq k$, then $k' = k$, $l' = i < l$.) It follows that $\sum_{i<j} e_i \square x_{ij}$ is nilpotent.

3. For $x = \sum_{i<j} x_{ij}$, we set

$$T_x^{(+)} = \sum_{i<j} e_i \square x_{ij}, \qquad \nu(x) = \exp T_x^{(+)}, \tag{12}$$

$$\xi_{ij}(x) = \sum_{m=1}^{j-i} \frac{1}{m!} \sum_{i<k_1<\cdots<k_{m-1}<j} x_{ik_1}x_{k_1k_2}\cdots x_{k_{m-1}j} \qquad \text{for } i < j. \tag{13}$$

We are going to prove the following

PROPOSITION 1. *For* $x \in \sum_{i<j} A_{ij}$ *and* $t_i \in F$ $(1 \leqq i \leqq r)$ *one has*

$$\nu(x)\Big(\sum_{i=1}^{r} t_i e_i\Big) = \sum_{i=1}^{r} \Big(t_i + \frac{1}{4}\sum_{k>i} t_k \xi_{ik}(x)^2\Big)e_i + \frac{1}{2}\sum_{i<j}\Big(t_j\xi_{ij}(x) + \sum_{k>j} t_k\xi_{ik}(x)\xi_{jk}(x)\Big). \tag{14}$$

First, we apply the result in 1 to $e = 1 - e_r = \sum_{i=1}^{r-1} e_i$, setting $x_1 = \sum_{i<j\leqq r-1} x_{ij}$ and $x_{1/2} = \sum_{i=1}^{r-1} x_{ir}$. Then $x = x_1 + x_{1/2}$, $T_{x_{1/2}}^{(+)} = e \square x_{1/2}$ and one has by (4) $(e \square x_{1/2})A_1(e) = 0$. It follows that

$$\begin{aligned}\nu(x)\Big(\sum_{i=1}^{r-1} t_i e_i\Big) &= \exp(T_{x_1}^{(+)} + e \square x_{1/2})\Big(\sum_{i=1}^{r-1} t_i e_i\Big) \\ &= \nu(x_1)\Big(\sum_{i=1}^{r-1} t_i e_i\Big).\end{aligned}$$

Therefore, to prove (14) (by induction on r), it is enough to show

$$\nu(x)e_r = e_r + \frac{1}{4}\sum_{i<r} \xi_{ir}(x)^2 e_i + \frac{1}{2}\sum_{i<r} \xi_{ir}(x) + \frac{1}{2}\sum_{i<j<r} \xi_{ir}(x)\xi_{jr}(x). \tag{14a}$$

Now, since $T_{x_1}^{(+)} e_r = \sum_{i<j<r} \{e_i, x_{ij}, e_r\} = 0$ and $(e \square x_{1/2})e_r = (1/2)x_{1/2}$ by (4), one has

$$\begin{aligned}\nu(x)e_r &= \exp(T_{x_1}^{(+)} + e \,\square\, x_{1/2})e_r \\ &= e_r + \frac{1}{2}x_{1/2} + \frac{1}{2}\sum_{m=2}^{\infty}\frac{1}{m!}(T_{x_1}^{(+)} + e \,\square\, x_{1/2})^{m-1}x_{1/2} .\end{aligned}$$

In expanding $(T_{x_1}^{(+)} + e \,\square\, x_{1/2})^{m-1}x_{1/2}$, we denote by $X_\mu^{(m)}$ the sum of the terms containing $e \,\square\, x_{1/2}$ μ times. Then by (11) one has

$$\begin{aligned}X_0^{(m)} &= T_{x_1}^{(+)m-1}x_{1/2} = \sum_{i<k_1<\cdots<k_{m-1}<r} x_{ik_1}x_{k_1k_2}\cdots x_{k_{m-1}r} \quad (\in A_{1/2}(e)) , \\ X_1^{(m)} &= \sum_{\substack{m_1, m_2\geqq 1\\ m_1+m_2=m}} T_{x_1}^{(+)m_1-1}(e \,\square\, x_{1/2})T_{x_1}^{(+)m_2-1}x_{1/2} \quad (\in A_1(e)) ;\end{aligned}$$

hence also $X_\mu^{(m)} = 0$ for $\mu \geqq 2$ and $X_0^{(m)} = 0$ for $m \geqq r$. It follows that

$$x_{1/2} + \sum_{m=2}^{\infty}\frac{1}{m!}X_0^{(m)} = \sum_{i=1}^{r-1}\xi_{ir}(x) . \tag{15}$$

On the other hand, by (11) one has

$$(e \,\square\, x_{1/2})T_{x_1}^{(+)m_2-1}x_{1/2} = \sum_{\substack{i,j<r\\ i\neq j}} x_{ir}(X_0^{(m_2)})_{jr} + \sum_{i=1}^{r-1} e_i(x_{ir}(X_0^{(m_2)})_{ir}) . \tag{16}$$

LEMMA 3. *For $k < l < r$, one has*

$$\begin{aligned}&(e_k \,\square\, x_{kl})\Big(\sum_{\substack{i,j<r\\ i\neq j}} y_{ir}z_{jr} + \sum_{i=1}^{r-1} e_i(y_{ir}z_{ir})\Big) \\ &= \sum_{\substack{j<r\\ j\neq k}} ((x_{kl}y_{lr})z_{jr} + y_{jr}(x_{kl}z_{lr})) + e_k(y_{kr}(x_{kl}z_{lr}) + (x_{kl}y_{lr})z_{kr}) .\end{aligned} \tag{17}$$

PROOF. By (8), (9) and (11) one has for $i, j < r$, $i \neq j$

$$(e_k \,\square\, x_{kl})(y_{ir}z_{jr}) = \begin{cases} x_{kl}(y_{lr}z_{jr}) = (x_{kl}y_{lr})z_{jr} & \text{if } l = i, k \neq j , \\ x_{kl}(y_{ir}z_{lr}) = y_{ir}(x_{kl}z_{lr}) & \text{if } l = j, k \neq i , \\ e_k(x_{kl}(y_{kr}z_{lr})) = e_k(y_{kr}(x_{kl}z_{lr})) & \text{if } k = i, l = j , \\ e_k(x_{kl}(y_{lr}z_{kr})) = e_k((x_{kl}y_{lr})z_{kr}) & \text{if } k = j, l = i . \end{cases}$$

On the other hand, putting $\langle y_{ir}, z_{ir}\rangle = \alpha$, one has by (8), (9)

$$(e_k \,\square\, x_{kl})(e_i(y_{ir}z_{ir})) = \begin{cases} \frac{1}{4}\alpha x_{kl} & \text{if } l = i , \\ 0 & \text{if } l \neq i , \end{cases}$$

and for $l = i$

$$\frac{1}{4}\alpha x_{kl} = x_{kl}(y_{lr}z_{lr}) = (x_{kl}y_{lr})z_{lr} + y_{lr}(x_{kl}z_{lr}) .$$

Summing up, one obtains (17). q.e.d.

By an easy induction on m_1, one obtains by (16) and (17)

$$T_{x_1}^{(+)m_1-1}(e\,\square\,x_{1/2})T_{x_1}^{(+)m_2-1}x_{1/2} = \sum_{s=1}^{m_1}\binom{m_1-1}{s-1}\left\{\sum_{i\neq j}(X_0^{(s)})_{ir}(X_0^{(m-s)})_{jr}+\sum_{i=1}^{r-1}e_i((X_0^{(s)})_{ir}(X_0^{(m-s)})_{ir})\right\},$$

where $m=m_1+m_2$. Since $\sum_{m_1=s}^{m-1}\binom{m_1-1}{s-1}=\binom{m-1}{s}$, it follows that

$$\begin{aligned}\sum_{m=2}^{\infty}\frac{1}{m!}X_1^{(m)} &= \sum_{m_1,m_2=1}^{\infty}\frac{1}{(m_1+m_2)!}T_{x_1}^{(+)m_1-1}(e\,\square\,x_{1/2})T_{x_1}^{(+)m_2-1}x_{1/2}\\ &= \sum_{m=2}^{\infty}\frac{1}{m!}\sum_{s=1}^{m-1}\binom{m-1}{s}\left\{\sum_{i\neq j}(X_0^{(s)})_{ir}(X_0^{(m-s)})_{jr}+\sum_{i=1}^{r-1}e_i((X_0^{(s)})_{ir}(X_0^{(m-s)})_{ir})\right\}\\ &= \sum_{\substack{1\leqq i<j\leqq r-1\\ 1\leqq s,t\leqq r-1}}\gamma_{ij}^{(s,t)}(X_0^{(s)})_{ir}(X_0^{(t)})_{jr}+\sum_{\substack{1\leqq i\leqq r-1\\ 1\leqq s\leqq t\leqq r-1}}\gamma_{ii}^{(s,t)}e_i((X_0^{(s)})_{ir}(X_0^{(t)})_{ir}),\end{aligned}$$

where the coefficients $\gamma_{ij}^{(s,t)}$ are given as follows:

$$\gamma_{ij}^{(s,t)}=\begin{cases}\dfrac{1}{m!}\left(\dbinom{m-1}{s}+\dbinom{m-1}{m-s}\right)=\dfrac{1}{s!t!} & \text{if } i<j \text{ or } s<t,\\ \dfrac{1}{m!}\dbinom{m-1}{s}=\dfrac{1}{2}\dfrac{1}{(s!)^2} & \text{if } i=j \text{ and } s=t,\end{cases}$$

where $m=s+t$. Thus one obtains

$$\sum_{m=2}^{\infty}\frac{1}{m!}X_1^{(m)}=\sum_{1\leqq i<j\leqq r-1}\xi_{ir}(x)\xi_{jr}(x)+\frac{1}{2}\sum_{i=1}^{r-1}e_i(\xi_{ir}(x)^2),$$

which, together with (15), completes the proof of (14a) and Proposition 1.

4. Next we prove the following

PROPOSITION 2. *Let A be a reduced simple Jordan algebra over F and let $\{e_i\ (1\leqq i\leqq r)\}$ be a primitive decomposition of 1 in A. Then $u\in A$ can be expressed in the form*

$$u=\nu(x)\left(\sum_{i=1}^{r}t_ie_i\right) \tag{18}$$

with $x\in\sum_{i<j}A_{ij}$, $t_i\in F^{\times}$ $(1\leqq i\leqq r)$, if and only if u is invertible with respect to $e_{i+1}+\cdots+e_r$ for all $0\leqq i\leqq r-1$. When this condition is satisfied, x and t_i's in (18) are uniquely determined.

First, suppose u is expressed in the form (18) with $x\in\sum_{i<j}A_{ij}$, and $t_i\in F$ $(1\leqq i\leqq r)$. We observe that, since $\nu(x)$ belongs to the "structure group" of A (see 5), one has

$$P_u = \nu(x)P\Big(\sum_{i=1}^{r} t_i e_i\Big){}^t\nu(x) .$$

(We sometimes write $P(a)$ for P_a.) Since $\nu(x)$ is unipotent and so $\det(\nu(x)) = 1$, and since one has $P(\sum_k t_k e_k) \mid A_{ij} = (t_i t_j)\,\mathrm{id}$ for all i, j, it follows that

$$\det(P_u) = \Big(\prod_{i=1}^{r} t_i\Big)^{2n/r} . \tag{19}$$

Thus, if u is invertible, one has $t_i \in F^\times$ $(1 \leqq i \leqq r)$, and vice versa.

We put $P^{(i)} = P(e_{i+1} + \cdots + e_r)$ and

$$A_0^{(i)} = A_1(e_{i+1} + \cdots + e_r) = A_0(e_1 + \cdots + e_i) ;$$

$P^{(i)}$ is the projection operator onto $A_0^{(i)}$ in the corresponding Peirce decomposition. Then, in view of (14), it is clear that

$$P^{(i)}u = \nu(P^{(i)}x)\Big(\sum_{j=i+1}^{r} t_j e_j\Big)$$

and so by (19)

$$\det(P(P^{(i)}u) \mid A_0^{(i)}) = \Big(\sum_{j=i+1}^{r} t_j\Big)^{2+d(r-i-1)} .$$

Therefore, if $t_j \in F^\times$ $(1 \leqq j \leqq r)$, then $P^{(i)}u$ is invertible in $A_0^{(i)}$ for all $0 \leqq i \leqq r-1$. This proves the "only if" part of the Proposition.

Next, we prove the uniqueness of the expression (18) by induction on r. The case $r = 1$ being trivial, we assume $r > 1$. Using the notation in 1 relative to $e = e_1$, we write $u_1 = u_{11}$, $u_{1/2} = \sum_{j=2}^{r} u_{1j}$, $u_0 = P^{(1)}u$, and $x_0 = P^{(1)}x$. Then by (14) one has

$$\begin{cases} u_1 = t_1 e_1 + \dfrac{1}{4}\displaystyle\sum_{k>1} t_k e_1(\xi_{1k}(x)^2) , \\ u_{1j} = \dfrac{1}{2}\Big(t_j \xi_{1j}(x) + \displaystyle\sum_{k>j} t_k \xi_{1k}(x)\xi_{jk}(x)\Big) \qquad (2 \leqq j \leqq r) , \\ u_0 = \nu(x_0)\Big(\displaystyle\sum_{i=2}^{r} t_i e_i\Big) . \end{cases} \tag{20}$$

First, by the third equation in (20) and by the induction assumption applied to u_0, we see that x_0 (hence all x_{ij} with $2 \leqq i < j \leqq r$) and t_i $(2 \leqq i \leqq r)$ are uniquely determined. Then, by the second equation in (20), $\xi_{1r}(x)$, $\xi_{1,r-1}(x)$, $\cdots$, $\xi_{12}(x)$ are determined successively by u_{1r}, $u_{1,r-1}$, $\cdots$, u_{12}. Then x_{12}, x_{13}, $\cdots$, x_{1r} are determined successively by $\xi_{12}(x)$, $\xi_{13}(x)$, $\cdots$, $\xi_{1r}(x)$, and finally t_1 is determined by the first equation in (20). Thus all x_{ij} and t_i are uniquely determined.

It remains to prove the "if" part of the Proposition. Suppose that $u \in A$ is invertible with respect to $e_{i+1} + \cdots + e_r$ for all $0 \leqq i \leqq r-1$. We will show by induction on r the existence of x and t_i $(1 \leqq i \leqq r)$ satisfying (18). The case $r = 1$ being trivial, we again assume $r > 1$ and define u_1, $u_{1/2}$, u_0 as above. Then u_0 satisfies the same condition as u for $1 \leqq i \leqq r-1$. Hence, by induction assumption, there exists (uniquely) $x'_0 \in \sum_{2 \leqq i < j \leqq r} A_{ij}$ and $t_i \in F^\times$ $(2 \leqq i \leqq r)$ such that

$$u_0 = \nu(x'_0)\Big(\sum_{i=2}^{r} t_i e_i\Big) .$$

Putting $y = 4u_0^{-1}u_{1/2}$, one has $u_{1/2} = u_0 y$, and by Lemma 1

$$u = \exp(e_1 \square y)(t_1 e_1 + u_0)$$

with some $t_1 \in F$. Since $\det(P_u) = t_1^2 \det(P_{u_0} | A_0(e_1)) \neq 0$ one has $t_1 \in F^\times$. Since $\{\nu(x) \,|\, x \in \sum_{i<j} A_{ij}\}$ is a group (see 5), there exists $x \in \sum_{i<j} A_{ij}$ such that

$$\nu(x) = \exp(e_1 \square y)\nu(x'_0) . \tag{21}$$

Then, since $\nu(x'_0)e_1 = 0$, one has

$$u = \nu(x)\Big(\sum_{i=1}^{r} t_i e_i\Big) ,$$

as desired.

REMARK. By the uniqueness of the expression (18), we see that $x'_0 = x_0 = P^{(1)}x$. By an explicit computation, it can be shown that

$$\nu(x)\Big(\sum_{i=1}^{r} t_i e_i\Big) = \exp\Big(-\sum_{j=2}^{r} e_1 \square \xi_{1j}(-x)\Big)\nu(x_0)\Big(\sum_{i=1}^{r} t_i e_i\Big) .$$

Hence, again by the uniqueness, we see that in (21) one has $y = -\sum_{j=2}^{r} \xi_{1j}(-x)$.

5. For a (semi-simple) Jordan algebra A, we define the "structure group" G and the "automorphism group" K as follows:

$$G = \operatorname{Str} A = \{g \in GL(A) \,|\, P(gx) = gP(x)^t g \text{ for all } x \in A\} ,$$

$$K = \operatorname{Aut} A = \{g \in GL(A) \,|\, g(xy) = (gx)(gy) \text{ for all } x, y \in A\} ,$$

where t denotes the adjoint with respect to the inner product $\langle\ \rangle$. These are algebraic groups defined over F acting on the underlying vector space of A.

We now assume, as always, that A is simple and reduced. Then one has

$$K = \{g \in G \,|\, g1 = 1\} . \tag{22}$$

In fact, it is clear that $g \in K$ implies $g1 = 1$, $g = {}^tg^{-1}$ and $g \in G$. Conversely, suppose that $g \in G$ and $g1 = 1$. Then one has

$$g(xy) = g\{x, y, 1\} = \{gx, {}^tg^{-1}y, g1\} = (gx)({}^tg^{-1}y) .$$

Putting $x = 1$, one has $g = {}^tg^{-1}$. Hence one has $g \in K$. Note that the condition $g \in G$ and ${}^tg^{-1} = g$ imply $P(g1) = 1$ and so $g1 = \pm 1$. Hence one has

$$(23) \qquad \{g \in G \mid {}^tg^{-1} = g\} = K \times \{\pm \mathrm{id}\} .$$

Let $\mathfrak{g} = \mathrm{Lie}\, G$ and $\mathfrak{k} = \mathrm{Lie}\, K$. By (1) one has $a \square b \in \mathfrak{g}$ for all $a, b \in A$; in particular, $T_a \in \mathfrak{g}$ and $[T_a, T_b] \in \mathfrak{k}$ for all $a, b \in A$. Actually, it is known that $\mathfrak{g}$ and $\mathfrak{k}$ coincide with the linear closure of $\{a \square b\ (a, b \in A)\}$ and $\{[T_a, T_b]\ (a, b \in A)\}$, respectively (see e.g. [4a]). For $x \in \sum_{i<j} A_{ij}$, $T_x^{(+)}$ is nilpotent element in $\mathfrak{g}$ and so one has $\nu(x) = \exp T_x^{(+)} \in G$.

We set

$$(24) \qquad \mathfrak{p} = \{T_a\ (a \in A)\} , \qquad \mathfrak{a} = \{T_{e_i}\ (1 \leqq i \leqq r)\}_F .$$

Then it is easy to see that one has $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ (direct sum). Clearly, $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{g}$ and, as is easily seen, it is "relatively maximal" in $\mathfrak{p}$, i.e. there exists no (abelian) subalgebra $\mathfrak{a}'$ of $\mathfrak{g}$ such that $\mathfrak{a} \subsetneqq \mathfrak{a}' \subset \mathfrak{p}$. Let $\mathfrak{a}^*$ denote the dual space of $\mathfrak{a}$ and let (ξ_i) be a basis of $\mathfrak{a}^*$ dual to (T_{e_i}). Then, by an easy computation, it can be shown that for any pair (i, j) with $i < j$

$$\{T_x^{(+)} \mid x \in A_{ij}\}$$

is the "root space" in $\mathfrak{g}$ relative to $\mathfrak{a}$ corresponding to the root $(1/2)(\xi_i - \xi_j)$ (see [1], [4c]). Therefore

$$\mathfrak{n}_+ = \{T_x^{(+)} \mid x \in \sum_{i<j} A_{ij}\}$$

is a nilpotent subalgebra of $\mathfrak{g}$ normalized by $\mathfrak{a}$, and so

$$\exp \mathfrak{n}_+ = \{\nu(x) = \exp T_x^{(+)} \mid x \in \sum_{i<j} A_{ij}\}$$

is a unipotent subgroup of G normalized by the subgroup corresponding to $\mathfrak{a}$. (It is clear that $\mathfrak{a}$ and $\mathfrak{n}_+$ are algebraic subalgebras of $\mathfrak{g}$.)

6. In this section, we consider the case where $F = \boldsymbol{R}$. A Jordan algebra A over $\boldsymbol{R}$ is called "formally real", if $a^2 + b^2 = 0$ $(a, b \in A)$ implies $a = b = 0$, or equivalently, if the inner product $\langle\ \ \rangle$ (with $\kappa > 0$) is positive definite (see [2]). This condition implies that A is semi-simple and does not contain any (non-zero) nilpotent element. It follows that, for any primitive idempotent e, one has $A_1(e) = \{e\}_{\boldsymbol{R}}$. Thus any formally real simple Jordan algebra is reduced.

LEMMA 4. *Let A be a formally real simple Jordan algebra over $\boldsymbol{R}$. Then, for any $a \in A$, there exists a primitive decomposition $\{e'_i\}$ of 1 in A and $\alpha_i \in \boldsymbol{R}$ $(1 \leqq i \leqq r)$ such that $a = \sum_{i=1}^{r} \alpha_i e'_i$, $\alpha_1 \geqq \cdots \geqq \alpha_r$. The α_i's are uniquely determined by a.*

This follows immediately from the fact that the minimal polynomial of a in A has only simple real roots (see [4c]).

Now let A be a formally real simple Jordan algebra over $\boldsymbol{R}$. We denote by G° and K° the identity connected components of G and K. From the definition it is easy to see that $g \in G$ implies ${}^t g \in G$ and, for $g \in G^\circ$, one has $g \in K^\circ$ if and only if ${}^t g^{-1} = g$. Therefore, by a theorem of Mostow, G is reductive and K° is a maximal compact subgroup of G°. Note that G and K themselves may not be connected even in the sense of Zariski topology.

In the present case, the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$ associated to K° and hence $r = \dim \mathfrak{a}$ coincides with the (real) rank of $\mathfrak{g}$. (Thus r is certainly independent of the choice of $\{e_i\}$.) Moreover, the conjugacy of relatively maximal subalgebras in $\mathfrak{p}$ implies

LEMMA 5. *Let $\{e_i\}$ and $\{e'_i\}$ be two primitive decompositions of 1 in A. Then there exists $k \in K^\circ$ such that $e'_i = ke_i$ $(1 \leqq i \leqq r)$.*

From Lemmas 4 and 5 one obtains the following

PROPOSITION 3. *Let A be a formally real simple Jordan algebra over $\boldsymbol{R}$ and let $\{e_i\}$ be a (fixed) primitive decomposition of 1 in A. Then for every $u \in A$ there exist $k \in K^\circ$ and $\alpha_i \in \boldsymbol{R}$ such that*

$$u = k\Big(\sum_{i=1}^{r} \alpha_i e_i\Big) , \qquad \alpha_1 \geqq \cdots \geqq \alpha_r . \tag{25}$$

The α_i's in this expression are uniquely determined by u.

It is clear that u is invertible if and only if in (25) one has $\alpha_i \in \boldsymbol{R}^\times$ for all i. We say that the *signature* of u is $(p, r-p)$ if $\alpha_p > 0$ and $\alpha_{p+1} < 0$. We denote by $A^\times$ the set of all invertible elements in A and by $A^{(i)}$ the set of all elements of signature $(r-i, i)$ in A. Then one has

$$A^\times = \coprod_{i=0}^{r} A^{(i)} , \tag{26}$$

Since $A^\times$ is stable under G° and all $A^{(i)}$'s are open, it is clear that each $A^{(i)}$ is also stable under G°.

PROPOSITION 4. *Let A be a formally real simple Jordan algebra over $\boldsymbol{R}$. Then the G°-orbit decomposition of $A^\times$ is given by* (26).

It is enough to show that, for each i, G° is transitive on $A^{(i)}$. Let $u \in A^{(i)}$; then by proposition 3 one has the expression (25) with $\alpha_{r-i} > 0 > \alpha_{r-i+1}$. Hence it is clear that there exists $g \in G^\circ$ such that $u = g(\sum_{j=0}^{r-i} e_j - \sum_{j=r-i+1}^{r} e_j)$. This proves our assertion.

7. In this section, we assume $F = \boldsymbol{C}$. Then all simple Jordan algebra is reduced. By the classification theory, one has

LEMMA 6. *All simple Jordan algebra A over $\boldsymbol{C}$ has a real form which is formally real.*

It follows, in particular, that G is reductive, since it has a reductive real form. Therefore, the same is also true over any field F of characteristic zero. (An analogue of the "unitary trick" in the theory of Lie algebras.)

LEMMA 7. *Let A be a simple Jordan algebra over $\boldsymbol{C}$. Then, for any invertible element u in A, there exists a primitive decomposition $\{e_i'\}$ of 1 in A such that u is invertible with respect to $e_{i+1}' + \cdots e_r'$ for all $0 \leqq i \leqq r-1$.*

PROOF. We prove the Lemma by induction on r. The case $r = 1$ being trivial, we assume $r > 1$. Take a real structure on A such that $A_{\boldsymbol{R}}$ is formally real (Lemma 6) and write $u = u' + \sqrt{-1}u''$ with $u', u'' \in A_{\boldsymbol{R}}$; then one has u' or $u'' \neq 0$. By Lemma 4 there exists a primitive idempotent e_r' in $A_{\boldsymbol{R}}$ such that u' or u'', and hence u, is invertible with respect to e_r'. By Lemma 1 there exist $y' \in A_{1/2}(1 - e_r')$, $u_1' \in A_1(1 - e_r')$ and $\alpha' \in \boldsymbol{C}^\times$ such that

$$u = \exp((1 - e_r') \square y')(u_1' + \alpha' e_r') .$$

Since u is invertible and $\alpha' \neq 0$, one has that u_1' is invertible in $A_1' = A_1(1 - e_r')$. By induction assumption, there exists a primitive decomposition $\{e_i'\ (1 \leqq i \leqq r-1)\}$ of $1 - e_r'$ in A_1' such that u_1' is invertible with respect to $e_{i+1}' + \cdots + e_{r-1}'$ for $0 \leqq i \leqq r-2$. Then by Proposition 2 there exist $x' \in \sum_{i<j<r} A_{ij}$, $\alpha_i' \in \boldsymbol{C}^\times (1 \leqq i \leqq r-1)$ such that

$$u_1' = \nu'(x_1')\left(\sum_{i=1}^{r-1} \alpha_i' e_i'\right) ,$$

where ν' denotes the mapping ν defined with respect to $\{e_i'\}$. By what we mentioned in 5, there exists $x' \in \sum_{i<j} A_{ij}$ such

$$\nu'(x') = \exp((1 - e_r') \square y')\nu'(x_1') .$$

Then one has

$$u = \exp((1 - e'_r) \square y')\nu'(x'_1)\Big(\sum_{i=1}^{r-1} \alpha'_i e'_i + \alpha' e'_r\Big)$$
$$= \nu'(x')\Big(\sum_{i=1}^{r-1} \alpha'_i e'_i + \alpha' e'_r\Big) .$$

Therefore, again by Proposition 2, u is invertible with respect to $\sum_{j=i+1}^{r} e'_j$ for $0 \leqq i \leqq r-1$. q.e.d.

PROPOSITION 5. *Let A be a simple Jordan algebra over **C**. Then $G^\circ = (\mathrm{Str}\, A)^\circ$ is transitive on $A^\times$.*

PROOF. For a primitive decomposition of unity $E = \{e_1, \cdots, e_r\}$ (considered as an ordered set), we denote by $A_E^\times$ the set of all elements in A which are invertible with respect to $e_{i+1} + \cdots + e_r$ for all $0 \leqq i \leqq r-1$. Then Proposition 2 implies that, for a given E, the group $(\exp \mathfrak{a})(\exp \mathfrak{n})(\subset G^\circ)$ is transitive on $A_E^\times$. Clearly, for any two primitive decompositions of unity E, E', one has $A_E^\times \cap A_{E'}^\times = \varnothing$, and by Lemma 7 one has $A^\times = \cup_{E'} A_{E'}$. Hence one can conclude that G° is transitive on $A^\times$. q.e.d.

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