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COMPLEX STRUCTURES ON $S^3 \times S^3$

HAJIME TSUJI

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0. Introduction. In the theory of complex manifolds, it is a fundamental problem to study complex structures on a given differentiable manifold. This problem is completely solved in the case of complex dimension 1. In the case of complex dimension 2, K. Kodaira completely classified complex structures on $S^1 \times S^3$ [10]. But little is known about such a problem in the case of complex dimensions greater than 2.

For the case of dimension greater than 2, E. Calabi and B. Eckmann constructed complex structures on the product of two odd dimensional spheres [4]. More general complex structures on the product of two odd dimensional spheres were constructed by E. Brieskorn and A. van de Ven [3].

In this paper, we study complex structures on $S^3 \times S^3$. In Section 1, we introduce a complex manifold $M^+(\alpha, A, m)$ (resp. $M^-(\alpha, A, m)$) which is diffeomorphic to a S^3-bundle over a lens space and which generalizes Calabi-Eckmann manifolds. To construct $M^+(\alpha, A, m)$, $M^-(\alpha, A, m)$, we use a surgery of new type. In Sections 2, 3, we study tubular neighbourhoods of a primary Hopf surface imbedded in a complex manifold of dimension 3. We show the existence of multiplicative holomorphic functions with the Hopf surface as divisor (Theorem 2.8) and the equivalence of the tubular neighbourhood of the Hopf surface with a tubular neighbourhood of the 0-section of the normal bundle of the Hopf surface (Theorem 2.48). In Section 3, we compute some local cohomologies and the irregularity of $M^{\pm}(\alpha, A, m)$ for general α. In Section 4, we characterize the complex structures $M^+(\alpha, A, m)$, $M^-(\alpha, A, m)$ by using the results of Sections 2, 3. The key point of the characterization is the possibility of the inversion of the surgery introduced in Section 1.

1. Constuction of $M^+(\alpha, A, m)$ and $M^-(\alpha, A, m)$. A compact complex manifold H is called a Hopf manifold, if its universal covering manifold is biholomorphic to $\boldsymbol{C}^n - O$ (O is the origin of $\boldsymbol{C}^n$, $n = \dim H$). Moreover if the fundamental group of H is an infinite cyclic group, we call H a primary Hopf manifold.

For Hopf surfaces i.e., Hopf manifolds of dimension 2, the following facts are well known [10].

(1.1) (1) Every primary Hopf surface S has the following normal form:

$$S = S_{\alpha,t} = C^2 - O/\langle g\rangle$$
$$g(z_1, z_2) = (\alpha_1 z_1 + t z_2^m, \alpha_2 z_2)$$

where $\alpha \in (\Delta^*)^2$ (Δ^* is the unit punctured disk in C), $t \in C$, $m \in Z^+$ satisfying $0 < |\alpha_1| \leqq |\alpha_2| < 1$, $(\alpha_1 - \alpha_2^m)t = 0$.

(2) Every Hopf surface S satisfies:

$$H^1(S, O_S) \cong H^1(S, C) \cong C\,, \qquad H^1(S, O_S^*) \cong H^1(S, C^*) \cong C^*\,.$$

In particular every complex line bundle on $S_{\alpha,t}$ has the following normal form for some $\beta \in C^*$.

$$L(\alpha, \beta)_t = (C^2 - O) \times C/\langle h_\beta\rangle\,, \qquad h_\beta(z_1, z_2, z_3) = (g(z_1, z_2), \beta z_3)$$

and the bundle projection $p: L(\alpha, \beta)_t \to S_{\alpha,t}$ is defined by

$$p([z_1, z_2, z_3]) = [z_1, z_2]$$

where [ ] denotes the class in the quotient spaces. We denote by $|L|$ the number $|\beta|$ for a line bundle $L \cong L(\alpha, \beta)_t$. And write $L^*(\alpha, \beta)_t$ for $L(\alpha, \beta)_t$ − (0-section).

LEMMA 1.2. *Let $E(\lambda)$ be a non-singular elliptic curve of the form:* $E(\lambda) = C/Z + \lambda Z$, $\lambda \in C$, $\operatorname{Im}\lambda > 0$. *Let $\pi: C \to E(\lambda)$ be the natural universal covering projection. Then for every multiplicative holomorphic function f on $E(\lambda)$, $f^* = \pi^* f$ is of the form: $f^*(z) = r\exp(sz)$ for some $r, s \in C$.*

PROOF. Let Z be a nowhere zero vector field on $E(\lambda)$. Since f and Zf are sections of a flat line bundle on $E(\lambda)$, they have no zero locus or they are identically zero. Suppose f is not constant. Let $\tilde{f}$ be a holomorphic function defined on C such that $f^*(z) = \exp(\tilde{f}(z))$. Since Zf has no zero locus, $\tilde{f}$ is an automorphism of C. This implies the lemma. q.e.d.

LEMMA 1.3. *Let $L^*(\alpha)$, $L^*(\beta)$ denote $L^*(\alpha_1, \alpha_2, \alpha_3)_0$, $L^*(\beta_1, \beta_2, \beta_3)_0$ for some $(\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3) \in (\Delta^*)^3$ respectively. Then $L^*(\alpha)$ is biholomorphic to $L^*(\beta)$, iff the following conditions are satisfied* (log *denotes the branch of logarithm on $C^* - R^-$ such that* $\log 1 = 0$). *Set* $\xi = (1/2\pi i)\log\alpha_3$, $\eta = (1/2\pi i)\log\beta_3$.

(1.4) *There exist* $A = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL(2, Z)$ *and* $m = (m_1, m_2) \in Z^2$ *satisfying*

$$(+)\qquad \eta=\frac{a\xi+b}{c\xi+d}\,,\qquad \begin{aligned}\beta_1&=\exp((a-c\eta)\log\alpha_1+2\pi i m_1\eta)\\ \beta_2&=\exp((a-c\eta)\log\alpha_2+2\pi i m_2\eta)\end{aligned}$$

or

$$(-)\qquad \eta=\frac{a\xi+b}{c\xi+d}\,,\qquad \begin{aligned}\beta_1&=\exp((a-c\eta)\log\alpha_2+2\pi i m_1\eta)\\ \beta_2&=\exp((a-c\eta)\log\alpha_1+2\pi i m_2\eta)\end{aligned}\ .$$

PROOF. Let $p_1\colon (C^2-O)\times C\to L^*(\alpha)$ and $p_2\colon (C^2-O)\times C\to L^*(\beta)$ be the natural projections defined by: $(z_1, z_2, z_3)\in (C^2-O)\times C\to [z_1, z_2, \exp(2\pi i z_3)]\in L^*(\alpha)$ (resp. $L^*(\beta)$). Suppose that there exists a biholomorphic mapping $\phi\colon L^*(\beta)\to L^*(\alpha)$. Let $\Phi\colon (C^2-O)\times C\to (C^2-O)\times C$ be the lifting of ϕ. We set $\Phi(p)=(\Phi_1^1(p), \Phi_2^1(p), \Phi^2(p))=(\Phi^1(p), \Phi^2(p))$ for $p\in (C^2-O)\times C$. Let g_1, g_2 be the automorphisms of C^2-O defined by: $g_1(z_1, z_2)=(\alpha_1 z_1, \alpha_2 z_2)$, $g_2(z_1, z_2)=(\beta_1 z_1, \beta_2 z_2)$. Since ϕ is a biholomorphic mapping, we can find $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2, Z)$ such that:

$$(1.5)\qquad \begin{aligned}\Phi^1(g_2(z_1, z_2), z_3+\eta)&=g_1^a(\Phi^1(z_1, z_2, z_3))\\ \Phi^1(z_1, z_2, z_3+1)&=g_1^c(\Phi^1(z_1, z_2, z_3))\end{aligned}$$

and

$$(1.6)\qquad \begin{aligned}\Phi^2(g_2(z_1, z_2), z_3+\eta)&=\Phi^2(z_1, z_2, z_3)+a\xi+b\\ \Phi^2(z_1, z_2, z_3+1)&=\Phi^2(z_1, z_2, z_3)+c\xi+d\end{aligned}$$

hold. By Hartog's extension theorem, we can regard Φ as an automorphism of C^3. Since $\Phi(0, 0)\times C$ is an automorphism of $(0, 0)\times C$, we have that $\Phi^2(0, 0, z_3)$ is a linear function of z_3. So we have from (1.6) that $\eta=(a\xi+b)/(c\xi+d)$. Next differentiating (1.5) and setting $(z_1, z_2)=(0, 0)$, we have

$$(1.7)\qquad \begin{aligned}\beta_j\frac{\partial\Phi^1}{\partial z_j}(0, 0, z_3+\eta)&=\begin{pmatrix}\alpha_1^a & \\ & \alpha_2^a\end{pmatrix}\frac{\partial\Phi^1}{\partial z_j}(0, 0, z_3)\\ \frac{\partial\Phi^1}{\partial z_j}(0, 0, z_3+1)&=\begin{pmatrix}\alpha_1^c & \\ & \alpha_2^c\end{pmatrix}\frac{\partial\Phi^1}{\partial z_j}(0, 0, z_3)\ .\end{aligned}$$

Note that $(\partial\Phi_j^1/\partial z_3)(0, 0, z_3)=0$. Since the Jacobian matrix of Φ at $(0, 0, 0)$ is nondegenerate, we have that

$$(1.8)\qquad \frac{\partial\Phi_1^1}{\partial z_1}(0, 0, 0)\neq 0\ ,\qquad \frac{\partial\Phi_2^1}{\partial z_2}(0, 0, 0)\neq 0$$

or

$$(1.9)\qquad \frac{\partial\Phi_1^1}{\partial z_2}(0, 0, 0)\neq 0\ ,\qquad \frac{\partial\Phi_2^1}{\partial z_1}(0, 0, 0)\neq 0$$

holds. Suppose (1.8) holds. Since we can regard $(\partial\Phi_1^1/\partial z_1)(0, 0, z_3)$ and

$(\partial\Phi_2^1/\partial z_2)(0, 0, z_3)$ as multiplicative holomorphic functions on $E(\eta) = \boldsymbol{C}/\boldsymbol{Z} + \eta\boldsymbol{Z}$, by applying Lemma 1.2., we can find two complex numbers s_1, s_2 such that

$$\text{(1.10)} \qquad \exp(2\pi i s_j) = \alpha_1^c\ , \qquad \exp(2\pi i s_j \eta) = \alpha_1^a \beta_1^{-1}\ , \quad j = 1, 2\ .$$

From (1.10) by easy calculation, we obtain the condition (+). Suppose (1.9) holds. By using the same argument, we obtain the condition (−). The "if" part of this lemma can be proven by the reverse process of the proof of the "only if" part easily. q.e.d.

Now we construct $M^{\pm}(\alpha, A, m)$. Let $H(\alpha)$ be a primary Hopf surface of dimension 3 of the form:

$$\text{(1.11)} \qquad H(\alpha) = \boldsymbol{C}^3 - O/\langle h\rangle\ , \qquad h(z_1, z_2, z_3) = (\alpha_1 z_1, \alpha_2 z_2, \alpha_3 z_3)\ ,$$

where $0 < |\alpha_1| \leqq |\alpha_2| < 1$, $0 < |\alpha_3| < 1$ and $\langle h\rangle$ denotes the group of automorphism of $\boldsymbol{C}^3$ generated by h. We set $S_0 = \{[z_1, z_2, z_3] \in H(\alpha); z_3 = 0\}$ and $C = \{[z_1, z_2, z_3] \in H(\alpha); z_1 = z_2 = 0\}$. Clearly S_0 is a primary Hopf surface and C is an elliptic curve. Let us consider an open complex manifold $W = H(\alpha) - S_0 - C$. It is clear that W is biholomorphic to $L^*(\alpha)$. Then for any element A of $SL(2, \boldsymbol{Z})$, if we take $m_1, m_2 \in \boldsymbol{Z}$ sufficiently large, there exists $L^*(\beta)$ satisfying the condition (1.4) (+) or (1.4) (−) with respect to $L^*(\alpha)$, A, $m = (m_1, m_2)$. Now we consider a compactification of $L(\beta)_0$ as a $\boldsymbol{P}^1$-bundle over $S_{(\beta_1,\beta_2),0}$. We denote it $P(\beta)$. Let S_∞ be the infinity section of $P(\beta)$. Note that $L^*(\alpha)$ and $L^*(\beta)$ have structures of rank 2 vector bundle over elliptic curves minus zero sections by the projections: $[z_1, z_2, z_3] \in L^*(\alpha)$ (resp. $L^*(\beta)) \to [z_3] \in \boldsymbol{C}^*/\langle\alpha_3\rangle$ (resp. $\boldsymbol{C}^*/\langle\beta_3\rangle$). By the proof of Lemma 1.3, we can choose a biholomorphic mapping ϕ^+ (or ϕ^-): $L^*(\beta) \to L^*(\alpha)$ of the form: ϕ^+ (resp. ϕ^-): $[z_1, z_2, z_3] \in L^*(\beta) \to [f_1(z_3)z_1, f_2(z_3)z_2, f_3(z_3)] \in L^*(\alpha)$ (resp. $[f_1(z_3)z_2, f_2(z_3)z_1, f_3(z_3)] \in L^*(\alpha)$), where $f_1(z_3)$, $f_2(z_3)$ are multiplicative holomorphic functions on the elliptic curve C. Then by identifying $L^*(\beta) \subset P(\beta) -$ (0-section) with $L^*(\alpha) \cong W \subset H(\alpha)$ by ϕ^+ or ϕ^-, we obtain a compact complex manifold. We denote the manifold by $M^+(\alpha, A, m)$ or $M^-(\alpha, A, m)$ according to the patching ϕ^+ or ϕ^-. $M^{\pm}(\alpha, A, m)$ has the following structure:

$$\text{(1.12)} \qquad M^{\pm}(\alpha, A, m) = (H(\alpha) - C) \cup U(S_\infty)\ ,$$

where $U(S_\infty)$ is a tubular neighbourhood of S_∞ in $P(\beta)$, i.e., $M^{\pm}(\alpha, A, m)$ is constructed from $H(\alpha)$ by the surgery which replaces the elliptic curve C with S_∞. We shall study the topology of $M^{\pm}(\alpha, A, m)$.

THEOREM 1.13. *$M^{\pm}(\alpha, A, m)$ is diffeomorphic to $S^3 \times S^3$ if and only if A is of the form:* $A = \begin{pmatrix} a & b \\ \pm 1 & d \end{pmatrix}$.

PROOF. Let us denote $M^{\pm}(\alpha, A, m)$ by M. Because of the construction of M, $M - S_0$ and $M - S_\infty$ have a structure of complex line bundle over S_∞ and S_0 respectively. We note that every primary Hopf surfacc is diffeomorphic to $S^1 \times S^3$ and in particular every complex line bundle over a primary Hopf surface is differentiably trivial. So we obtain that $M - S_0$ and $M - S_\infty$ are diffeomorphic to $S^1 \times S^3 \times \boldsymbol{C}$. This implies that M is diffeomorphic to a manifold constructed from two copies of $S^1 \times S^3 \times \boldsymbol{C}$ by gluing them along $S^1 \times S^3 \times \boldsymbol{C}^*$. We shall review the construction of M. We can naturally identify $M - S_\infty$ and $M - S_0$ with $L(\alpha)_0$ and $L(\beta)_0$ respectively. Review that $L^*(\alpha)$ and $L^*(\beta)$ have a structure of rank 2 vector bundle over an elliptic curve minus 0-section by the projection $[z_1, z_2, z_3] \in L^*(\alpha)$ (resp. $L^*(\beta)) \to [z_3] \in \boldsymbol{C}^*/\langle\alpha_3\rangle$ (resp. $\boldsymbol{C}^*/\langle\beta_3\rangle$). By using the definition of ϕ^+ (resp. ϕ^-), we see that ϕ^+ (resp. ϕ^-) is a restriction of an isomorphism between the above vector bundles over the elliptic curves. Let us identify $\boldsymbol{C}^*$ with $S^1 \times \boldsymbol{R}^+$ by the diffeomorphism: $z \in \boldsymbol{C}^* \to (z/|z|, |z|) \in S^1 \times \boldsymbol{R}^+$ and let us identify $S^3 \times \boldsymbol{R}^+$ with $\boldsymbol{R}^4 - O$ (O is the origin of $\boldsymbol{R}^4$) naturally. Then we can identify $S^1 \times S^3 \times \boldsymbol{C}^*$ with $S^1 \times S^1 \times (\boldsymbol{R}^4 - O)$. Then M is diffeomorphic to a manifold constructed from two copies of $S^1 \times S^3 \times \boldsymbol{C}$ by gluing them along $S^1 \times S^3 \times \boldsymbol{C}^* = (S^1 \times S^1) \times (\boldsymbol{R}^4 - O)$ by a diffeomorphism $u: (S^1 \times S^1) \times (\boldsymbol{R}^4 - O) \to (S^1 \times S^1) \times (\boldsymbol{R}^4 - O)$ of the form: $u(x, y) = (u_1(x), G(x)y)$, where $G(x)$ is a differentiable mapping from $S^1 \times S^1$ into $SO(4)$. This implies that M is diffeomorphic to a S^3-bundle over a manifold which is constructed from two solid torus by gluing their boundaries, i.e., a S^3-bundle over a lens space (c.f. [6]). Hence M is diffeomorphic to $S^3 \times S^3$ if and only if M is simply connected, because S^3-bundle over S^3 is differentiably trivial (c.f. [15]). By van Kampen theorem, one can easily see that M is simply connected, if and only if A is of the form: $\begin{pmatrix} a & b \\ \pm 1 & d \end{pmatrix}$. q.e.d.

COROLLARY 1.14. *$M^{\pm}(\alpha, A, m)$ is diffeomorphic to a S^3-bundle over a lens space. And there exists a complex structure on a S^3-bundle over any lens space.*

2. Neighbourhoods of a primary Hopf surface. In this section, we study complex analytic properties of tubular neighdourhoods of a primary Hopf surface imbedded in a complex manifold of dimension 3. We use the same notations as in Section 1.

DEFINITION 2.1. Let L be a line bundle over a primary Hopf surface S.

(1) L is said to be of infinite type, if $H^1(S, O_S(L^{-\nu})) = 0$ for $\nu \geqq 1$.

(2) L is said to be of tangentially infinite type, if $H^1(S, \Theta_S \otimes O_S(L^{-\nu})) = 0$ for $\nu \geqq 1$.

(3) L is said to be of smooth type, if $H^1(S, \Omega^1_S \otimes \Omega^2_S \otimes O_S(L^{-\nu})) = 0$ for $\nu > 1$.

(4) L is said to be torsion free, if for any curve C in S, $L^r_{|C} \otimes N^p_{C|S}$ is nontrivial for any $r \neq 0$, and any p.

REMARK 2.2. Let L be a line bundle over a primary Hopf surface S. If L is torsion free, L is also of infinite type, tangentially infinite type and smooth type.

PROOF. Let L be a torsion free line bundle over $S \cong S_{\alpha_1,\alpha_2,t}$ and let $L \cong L(\alpha_1, \alpha_2, \alpha_3)_t$. It is easy to verify that every curve in S is biholomorphic to $C^*/\langle \alpha_1^a \alpha_2^b \rangle$ and its normal bundle is the restriction of $L(\alpha_1, \alpha_2, \alpha_1^c \alpha_2^d)$ for some $a, b, c, d \in \boldsymbol{Z}$. Hence there exists no triple of integers (p, q, r) such that $\alpha_3^r = \alpha_1^p \alpha_2^q$ and $r \neq 0$. First we prove that L is of infinite type. Since S is diffeomorphic to $S^1 \times S^3$, Riemann-Roch theorem implies that

$$\dim H^1(S, O_S(L^{-\nu})) = \dim H^0(S, O_S(L^{-\nu})) + \dim H^2(S, O_S(L^{-\nu})) . \tag{2.3}$$

So it suffices to prove that $\dim H^0(S, O_S(L^{-\nu})) = \dim H^2(S, O_S(L^{-\nu})) = 0$ for $\nu \geqq 1$. Since every line bundle over S is flat, we can identify every global section of $O_S(L^{-\nu})$ with a multiplicative holomorphic function on S. Suppose that there exists a nontrivial section σ of $O(L^{-\nu})$ for some $\nu \geqq 1$. Since $L^{-\nu}$ is not trivial by the assumption, σ has zero locus. Let $C = \Sigma m_j C_j$ be the zero locus of σ, where C_j is an irreducible reduced curve in S. Since every line bundle over S is flat, there exists a multiplicative holomorphic function σ_j with divisor C_j for each j. Then $(\Pi \sigma_j^{m_j})^{-1}\sigma$ is a multiplicative holomorphic function with no zero locus. Hence it is a constant. Let us denote $L(\alpha_1, \alpha_2, \alpha_k)$ $(k = 1, 2)$ by L_k. Since $[C_j] \cong L_1$ or $\cong L_2$ [10], we conclude that $\alpha_3^{-\nu} = \alpha_1^p \alpha_2^q$ for some $p, q \geqq 0$. This contradicts the assumption. Hence we obtain that $H^0(S, O_S(L^{-\nu})) = 0$ for $\nu \geqq 1$. To prove that $H^2(S, O_S(L^{-\nu})) = 0$ for $\nu \geqq 1$, we note that $\Omega^2_S \cong L_1^* \otimes L_2^*$ [10]. Then by Serre duality, we have that $\dim H^2(S, O_S(L^{-\nu})) = \dim H^0(S, O_S(L^{\nu} \otimes L_1^* \otimes L_2^*))$. Then by the similar argument to the case of H^0, we can prove that $\dim H^2(S, O_S(L^{-\nu})) = 0$ for $\nu \geqq 1$.

Next we shall prove that L is of tangentially infinite type. Since the tangent bundle T_S is of the form:

$$T_S = \boldsymbol{C}^2 \times (\boldsymbol{C}^2 - O)/\langle u \rangle \ , \quad u(t_1, t_2, z_1, z_2) = (\alpha_1 t_1 + mtz_2^{m-1} t_2, \alpha_2 t_2, g(z_1, z_2)) \tag{2.4}$$

by (1.1)(1), the following exact sequence holds:

(2.5) $$0 \to O_S(L_1) \to \Theta_S \to O_S(L_2) \to 0 .$$

Then we have the exact sequence of cohomology:

(2.6) $$\to H^1(S, O_S(L_1 \otimes L^{-\nu})) \to H^1(S, \Theta_S \otimes O(L^{-\nu})) \to H^1(S, O_S(L_2 \otimes L^{-\nu})) \to .$$

By the similar argument to the proof of the case of infinite type, we obtain that $H^1(S, O(L_k \otimes L^{-\nu})) = 0$ for $k = 1, 2$ and $\nu \geqq 1$. This completes the proof of the case of tangentially infinite type.

The proof of the case of smooth type is similar to that of the case of tangentially infinite type. Hence we omit it. q.e.d.

For the later use, we need the following lemma.

LEMMA 2.7. *Let $S \cong S_{\alpha,t}$ be a primary Hopf surface and let $\pi: C^2 - O \to S$ be the natural covering projection. Then we have the following table*:

(2.8)

type	(α_1, α_2, t)	basis of $\pi^* H^0(S, \Theta_S)$	$\dim H^1(S, \Theta_S)$
I	$(\alpha, \alpha, 0)$	$z_1\, \partial/\partial z_1$, $z_2\, \partial/\partial z_2$ $z_2\, \partial/\partial z_1$, $z_1\, \partial/\partial z_2$	4
II	$(\alpha^m, \alpha, 0)$ $m > 1$	$z_1\, \partial/\partial z_1$, $z_2\, \partial/\partial z_2$ $z_2^m\, \partial/\partial z_1$	3
III	(α^m, α, t) $t \neq 0$	$m z_1\, \partial/\partial z_1 + z_2\, \partial/\partial z_2$ $z_2^m\, \partial/\partial z_1$	2
VI	otherwise	$z_1\, \partial/\partial z_1$, $z_2\, \partial/\partial z_2$	2

PROOF. The proof of this lemma is easy calculation. Hence we omit it. q.e.d.

Now we study tubular neighbourhoods of a primary Hopf surface imbedded in a complex manifold of dimension 3.

THEOREM 2.8. *Let S be a primary Hopf surface imbedded in a complex manifold M of dimension 3 and let N be the normal bundle of S. If N is of infinite type and $|N| < 1$, then there exists a multiplicative holomorphic function u defined on some tubular neighbourhood of S with divisor S.*

PROOF. We divide the proof of this theorem into several steps.

Step 1. We choose a biholomorphic mapping $i(t): S_{\alpha,t} \to S$ and identify S with $S_{\alpha,t}$. If $t \neq 0$, since for any $\varepsilon < 0$, $i_\varepsilon: S_{\alpha,t} \to S_{\alpha,\varepsilon^m t}$ defined by

$[z_1, z_2] \to [z_1, \varepsilon z_2]$ is a biholomorphic mapping, we can choose t arbitrary near 0. Let $\pi: C^2 - O \to S$ be the natural covering projection defined by $\pi(z_1, z_2) = [z_1, z_2]$. Then there exist $r_i, r_i' > 0$ $(i = 1, 2, 3, 4)$ and $\delta > 0$ satisfying the following conditions:

$0 < r_1 < r_2 < r_3 < r_4$, $0 < r_1' < r_2' < r_3' < r_4'$, $r_1 = |\alpha_1| r_4$, $r_1' = |\alpha_2| r_4'$ and the domains U_i $(1 \leqq i \leqq 6)$ in C^2 defined by:

$$(2.9) \quad \begin{aligned} U_i &= \left\{(z_1, z_2) \in C^2;\ r_i - \delta < |z_1| < r_{i+} + \delta,\ |z_2| < r_4' + \frac{\delta}{2}\right\}, \\ &\quad i = 1, 2, 3, \\ U_i &= \left\{(z_1, z_2) \in C^2;\ |z_1| < r_4 + \frac{\delta}{2},\ r_{i-3}' - \delta < |z_2| < r_{i-2}' + \delta\right\}, \\ &\quad i = 4, 5, 6 \end{aligned}$$

satisfy

(1) U_i is biholomorphic onto its image by π for each i,

(2) $\pi(U_1) \cap \pi(U_2) \cap \pi(U_3) = \phi$, $\pi(U_4) \cap \pi(U_5) \cap \pi(U_6) = \phi$.

By the property (2.9) (1), we can identify each U_i with its image $\pi(U_i)$. Hereafter we denote U_i instead of $\pi(U_i)$. Clearly $\mathscr{U} = \{U_i\}$ is a Stein covering of S. By replacing δ by slightly larger one, we obtain another Stein covering $\mathscr{U}^* = \{U_i^*\}$ of S such that U_i is a relatively compact subdomain of U_i^* for each i.

Now we consider a Stein covering of S in M. Since every Stein submanifold admits a Stein neighbourhood [15], we can find a Stein neighbourhood V_i^* of U_i^* in M for each i. We define complex manifolds U_{ijk}^* (resp. V_{ijk}^*) for $(i, j, k) = (1, 2, 3), (4, 5, 6)$ by gluing disjoint unions of U_i^*, U_j^*, U_k^* (resp. V_i^*, V_j^*, V_k^*) naturally on $U_i^* \cap U_j^*$ and $U_j^* \cap U_k^*$ (resp. $V_i^* \cap V_j^*$ and $V_j^* \cap V_k^*$). Clearly U_{ijk}^* is a closed Stein submanifold of V_{ijk}^*. We take a Stein neighbourhood V_{ijk}^{**} of U_{ijk}^* in V_{ijk}^*. Replacing V_i^*, V_j^*, V_k^* by $V_i^* \cap V_{ijk}^*$, $V_j^* \cap V_{ijk}^*$, $V_k^* \cap V_{ijk}^*$ respectively, we may assume that V_{123}^* and V_{456}^* are Stein manifolds. Then we can find defining equations $w_{ijk} \in H^0(V_{ijk}, O)$ of U_{ijk}^* in V_{ijk}^* for $(i, j, k) = (1, 2, 3), (4, 5, 6)$. We set $w_h = w_{ijk} | V_h^*$ for $h \in \{i, j, k\}$. Then $t_{ij} = (w_i/w_j) | U_i^* \cap U_j^*$, $1 \leqq i, j \leqq 6$ $(i \neq j)$ define the normal bundle of S. Since $H^1(S, O_S^*) \cong H^1(S, C^*)$, modifying w_{ijk}, if necessary, we may assume that t_{ij} is constant for each (i, j) from the beginning.

Next we construct local coordinates. We note that we can naturally identify U_{ijk}^* with a domain in C^2 by its construction. We restrict the standard coordinate of C^2 to U_{ijk}^* and obtain a local coordinate z_{ijk} of U_{ijk}^*. Since V_{ijk}^* is a Stein neighbourhood of U_{ijk}^*, we can extend z_{ijk} to a vector valued holomorphic function defind on V_{ijk}^* and we denote it by z_{ijk} again. We set $z_h = z_{ijk}|V_h^*$ for $h \in \{i, j, k\}$. Shrinking V_i^* (without shrinking U_i^*), if necessary, we may assume that $(z_i, w_i)\colon V_i^* \to C^3$ is a local coordinate of V_i^* for each i. Again we shrink U_i^* and V_i^* so that the following conditions are satisfied:

(2.10)
(1) U_i^* is of the form (2.9), if we identify U_i^* with a domain in C^2 naturally. In particular U_i^* is a Stein manifold.
(2) V_i^* is a Stein neighbourhood of U_i^*.
(3) U_i^* contains U_i as a relatively compact subset.
(4) (z_i, w_i) is defined on the closure of V_i^*.
(5) $z_i(V_i^*) = z_i(U_i^*)$.

The existence of such shrinking is clear. So we may assume the condition (2.10) from the beginning.

LEMMA 2.11. *If we choose t sufficiently close to 0 at the beginning of this step, we may assume that the Stein coverings $\mathscr{U} = \{U_i\}$ and $\mathscr{U}^* = \{U_i^*\}$ satisfy the following condition*:

(2.12) *Every holomorphic function defind on $W_{ij} = (U_i^* \cap U_j) \cup (U_i \cap U_j^*)$ has an analytic continuation to a holomorphic function defined on a domain which contains $U_i \cap U_j$ as a relatively compact except for $(i, j) = (1, 2), (2, 3), (4, 5), (5, 6)$.*

PROOF. First we consider the case: $t = 0$. Since each W_{ij} is identified with a Reinhaldt domain in C^2 in this case, every holomorphic function defined on W_{ij} can be expanded into a Laurent power series by the theorem of *H.* Cartan (c.f. [7]). Since the domain of convergence of a Laurent power series is logarithmically convex (c.f. [7]), we can prove this lemma only by writing the figure of W_{ij}. Details are left to readers.

In the case: $t \neq 0$, W_{ij} is not necessary a Reinhaldt domain. But as t goes 0, every W_{ij} approaches to a Reinhaldt domain. Then it is clear that, if we take t sufficiently near 0, the same assertion as in the case: $t = 0$ holds. q.e.d.

We set $W_{ij}^* =$ (the holomorphic envelope of W_{ij}) except for $(i, j) = (1, 2), (2, 3), (4, 5), (5, 6)$ and $W_{ij}^* = U_i^* \cap U_j^*$ for $(i, j) = (1, 2), (2, 3), (4, 5), (5, 6)$.

Step 2. First we construct the desired multiplicative holomorphic function as a formal power series. We write the transformation of local coordinates on $V_i^* \cap V_j^*$ as follows:

$$(2.13)\qquad \begin{aligned} w_j(p) &= \phi_{ji}(z_i(p), w_i(p)) = t_{ji}w_i(p) + \sum_{\nu=2}^{\infty} \phi_{ji|\nu}(z_i(p))w_i(p) \\ z_j(p) &= \psi_{ji}(z_i(p), w_i(p)) . \end{aligned}$$

The construction of the formal power series given below in entirely the same as in [17]. But for the next step, we repeat the construction.

To prove Theorem 2.8, it suffices to construct a system $\{u_i\}$ of holomorphic functions defined respectively on a neighbourhood $V_i' \ (\subset V_i^*)$ of U_i^* satisfying the conditions:

(i) Each u_i is of the form:

$$\begin{aligned} u_i(p) &= g_i(z_i(p), w_i(p)) \\ &= w_i(p) + (\text{terms of order} \geqq 2 \text{ with respect to } w_i) \end{aligned}$$

(ii) $u_i = t_{ij}u_j$ on $V_i \cap V_j$.

We shall determine each $u_i = g_i(z_i, w_i)$ as an implicit function defined by the equation:

$$(2.14)\qquad w_i = f_i(z_i, u_i) = u_i + \sum_{\nu=2}^{\infty} f_{i|\nu}(z_i)u_i ,$$

where $f_i(z_i, u_i)$ is a power series in u_i whose coefficients $f_{i|\nu}(z_i)$ are holomorphic functions of the variable z_i. By (2.13) the condition (ii) is equivalent to:

$$(2.15)\qquad \phi_{ji}(z_i, f_i(z_i, u_i)) = f_j(\psi_{ji}(z_i, f_i(z_i, u_i)), t_{ji}u_i) .$$

We expand the left-hand side of (2.15) into the power series:

$$(2.16)\qquad \phi_{ji}(z_i, f_i(z_i, u_i)) = t_{ji}\Big(u_i + \sum_{\nu=2}^{\infty} f_{i|\nu}(z_i)u_i^\nu\Big) + t_{ji}\sum_{\nu=2}^{\infty} h'_{ij|\nu}(z_i)u_i^\nu$$

where

$$(2.17)\qquad t_{ji}\sum_{\nu=2}^{\infty} h'_{ij|\nu}(z_i)u_i^\nu = \sum_{\nu=2}^{\infty} \phi_{ji|\nu}(z_i)\Big(u_i + \sum_{\mu=2}^{\infty} f_{i|\mu}(z_i)u_i^\mu\Big)^\nu .$$

The right-hand side of (2.15) is expanded into the form

$$t_{ji}u_i + \sum_{\nu=2}^{\infty} f_{j|\nu}(\psi_{ji}(z_i, f_i(z_i, u_i)))(t_{ji}u_i)^\nu .$$

Letting

$$f_{j|\nu}(\psi_{ji}(z_i, w_i)) = f_{i|\nu}(\psi_{ji}(z_i, 0)) + \sum_{\mu=1}^{\infty} f_{ji|\nu\mu}(z_i)w_i^\mu$$

we have

$$(2.18)\qquad \begin{aligned} &f_j(\psi_{ji}(z_i, f_i(z_i, u_i)), t_{ji}u_i) \\ &\qquad = t_{ji}u_i + \sum_{\nu=2}^{\infty} f_{j|\nu}(\psi_{ji}(z_i, 0))(t_{ji}u_i)^\nu + t_{ji}\sum_{\nu=2}^{\infty} h''_{ij|\nu}(z_i)u_i^\nu \end{aligned}$$

where

$$(2.19)\qquad t_{ji}\sum_{\nu=2}^{\infty} h''_{ij|\nu}(z_i)u_i^{\nu} = \sum_{\nu=2}^{\infty}\left[\sum_{\mu=1}^{\infty} f_{ji|\nu\mu}(z_i)\Big(u_i + \sum_{\lambda=2}^{\infty} f_{i|\lambda}(z_i)u_i^{\lambda}\Big)^{\mu}\right](t_{ji}u_i)^{\nu} .$$

We infer from (2.17) and (2.19) that if $f_{i|2}, \cdots, f_{i|\nu}$ $(1 \leqq i \leqq 6)$ are determined, then $h'_{ij|\nu+1}$ and $h''_{ij|\nu+1}$ are determined independently of $f_{i|\nu+1}$, $f_{i|\nu+2}, \cdots$. The proof of the following lemma is [17].

LEMMA 2.20. (1) *$f_i(z_i, u_i)$ $(1 \leqq i \leqq 6)$ satisfy* (2.15) *as formal power series, if and only if the equations*

$$(2.21)_{\nu}\qquad f_{i|\nu+1}(z_i(p)) - t_{ij}^{-\nu} f_{j|\nu+1}(\psi_{ji}(z_i(p), 0)) = h_{ij|\nu}(z_i(p)) \quad \textit{for} \quad p \in U_i^* \cap U_j^*$$

are satisfied for any $1 \leqq i, j \leqq 6$ $(i \neq j)$ *and* $\nu \geqq 1$, *where we have set* $h_{ij|\nu} = -h'_{ij|\nu+1} - h''_{ij|\nu+1}$.

(2) *Suppose that* $f_{i|2}, \cdots, f_{i|\nu}$ *satisfying* $(2.21)_1, \cdots, (2.21)_{\nu-1}$ *respectively are already determined. Then* $\{h_{ij|\nu}\}$ *is an element of* $Z^1(\mathscr{U}^*, O_S(N^{-\nu}))$.

Since N is of infinite type, Lemma 2.20 completes the construction of the formal power series.

Step 3. Let $a(u) = \sum_{\nu=0}^{\infty} a_{\nu}u^{\nu}$ and $A(u) = \sum_{\nu=0}^{\infty} A_{\nu}u^{\nu}$, $A_{\nu} \geqq 0$ be two power series of u. We write $a(u) \ll A(u)$, when $|a_{\nu}| \leqq A_{\nu}$ hold for all $\nu \geqq 0$. To prove the convergence of the power series $f_i(z_i, u_i) = u_i + \sum_{\nu=2}^{\infty} f_{i|\nu}(z_i)u_i^{\nu}$ on some neighbourhood V_i' $(\subset V_i^*)$ of U_i^* respectively, we shall show that there exists a power series $A(u) = u + \sum_{\nu=2}^{\infty} A_{\nu}u^{\nu}$ with constant coefficients and positive radius of convergence satisfying:

$$(2.22)\qquad f_i(z_i(p), u_i) \ll A(u_i) \quad \text{for} \quad p \in U_i^* \quad (1 \leqq i \leqq 6) .$$

If we write $f_i^{\nu}(z_i, u_i) = u_i + \sum_{\mu=2}^{\nu} f_{i|\mu}(z_i)u_i^{\mu}$ and $A^{\nu}(u) = u + \sum_{\mu=2}^{\nu} A_{\mu}u^{\mu}$, then (2.22) is equivalent to the conditions:

$$(2.23)_{\nu}\qquad f_i^{\nu}(z_i(p), u_i) \ll A^{\nu}(u_i) \quad \text{for} \quad p \in U_i^* \quad (1 \leqq i \leqq 6) , \quad \nu = 1, 2, \cdots .$$

Suppose that $f_i^{\nu}(z_i, u_i)$ and $A^{\nu}(u)$ satisfying $(2.23)_{\nu}$ are already determined. We shall estimate $|h'_{ij|\nu+1}|$, $|h''_{ij|\nu+1}|$ in terms of $A_2, \cdots, A_{\nu}$.

Let R be a sufficiently large number such that $|\phi_{ji|\mu}(z_i(p))| \leqq R^{\mu}$ for $p \in U_i^* \cap U_j^*$, $1 \leqq i, j \leqq 6$ $(i \neq j)$, $\mu = 2, 3, \cdots$. From (2.17), we obtain

$$t_{ji}\sum_{\mu=2}^{\nu+1} h'_{ij|\mu}(z_i(p))u_i^{\mu} \ll \frac{R^2(A^{\nu}(u_i))^2}{1 - RA^{\nu}(u_i)} \quad \text{for} \quad p \in U_i^* \cap U_j^* .$$

Let C be $\max_{(i,j)} \{|t_{ij}|\}$. Then we have:

$$(2.24)\qquad \sum_{\mu=2}^{\nu+1} h'_{ij|\mu}(z_i(p))u_i^{\mu} \ll C\frac{R^2(A^{\nu}(u_i))^2}{1 - RA^{\nu}(u_i)} \quad \text{for} \quad p \in U_i^* \cap U_j^* .$$

Since each U_i is relatively compact in U_i^*, we can choose sufficiently large number Q such that, for every point p in $U_i^* \cap U_j$, the closed disk; $D_p = \{q \in V_i^*;\ z_i(q) = z_i(p),\ |w_i(q)| \leqq 1/Q\}$ is contained in V_j^*. By the assumption and the condition (2.10) (5), we have:

$$(2.25)\qquad |f_{j|\mu}(\psi_{ji}(z_i(q),\, w_i(q)))| \leqq A_\mu\ , \quad q \in D_p \subset V_i^* \cap V_j^*\ , \quad \mu = 2, 3, \cdots .$$

Then we have:

$$(2.26)\qquad |f_{ij|\mu\lambda}(z_i(p))| \leqq A_\mu Q^\lambda\ , \quad p \in U_i^* \cap U_j\ ,\quad \mu = 2, 3, \cdots, \nu\ , \quad \lambda = 1, 2, \cdots .$$

Therefore, by (2.19), we have:

$$(2.27)\qquad t_{ji}\sum_{\mu=2}^{\nu+1} h''_{ij|\mu}(z_i(p))u_i^\mu \ll \sum_{\mu=2}^{\nu}\left[\sum_{\lambda=1}^{\infty} A_\mu Q^\lambda (A^\nu(u_i))^\lambda\right](t_{ji}u_i)^\mu \quad \text{for} \quad p \quad U_i^* \cap U_j\ .$$

Then from (2.24) and (2.27), we get the following estimate:

$$(2.28)\qquad \min\{|h_{ij|\nu}(z_i(p))|,\ |h_{ji|\nu}(z_j(p))|\} \leqq \left[C\frac{R^2(A^\nu(u))^2}{1 - RA^\nu(u)} + C\frac{Q(A^\nu(u))^2}{1 - QA^\nu(u)}\right]_{\nu+1} \quad \text{for} \quad p \in W_{ij}$$

where $W_{ij} = (U_i^* \cap U_j) \cup (U_i \cap U_j^*)$ and $[\quad]_{\nu+1}$ denotes the coefficient of $u^{\nu+1}$ in the power series. Let P be $\max\{R, R^2, Q\}$. Then we have:

$$(2.29)\qquad \min\{|h_{ij|\nu}(z_i(p))|,\ |h_{ji|\nu}(z_j(p))|\} \leqq \left[\frac{2CP(A^\nu(u))^2}{1 - PA^\nu(u)}\right]_{\nu+1} \quad \text{for} \quad p \in W_{ij}\ .$$

Let L be a line bundle over S. We introduce a norm on $Z^1(\mathscr{U}^*, O_S(L))$ as follows:

$$(2.30)\qquad |\{\gamma_{ij}\}|_m = \max_{(i,j)} \sup\{\min\{|\gamma_{ij}|\,|\gamma_{ji}|:\ W_{ij}^*\}\} \quad \text{for} \quad \{\gamma_{ij}\} \in Z^1(\mathscr{U}^*, O_S(L))$$

where (i, j) runs all pairs such that $1 \leqq i < j \leqq 6$.

Then since $h_{ij|\nu} = 0$ for $(i, j) = (1, 2), (2, 3), (4, 5), (5, 6)$ because of the construction of the local coordinates, by Lemma 2.11, we obtain:

$$(2.31)\qquad |\{h_{ij|\nu}(z_i)\}|_m \leqq \left[\frac{B(A^\nu(u))^2}{1 - BA^\nu(u)}\right]_{\nu+1}$$

where $B = 2CP$ and $[\quad]_{\nu+1}$ denotes the coefficient of $u^{\nu+1}$ in the power series.

Step 4. In this step, we shall estimate the operator norm of the coboundary mapping $\delta: C^0(\mathscr{U}^*, O_S(N^{-\nu})) \to Z^1(\mathscr{U}^*, O_S(N^{-\nu}))$, $\nu = 1, 2, \cdots$ by modifying the method in [5] appendix. Let L be a line bundle over S. We introduce a norm on $C^0(\mathscr{U}^*, O_S(L))$ by

$$(2.32)\qquad |\{\eta_i\}| = \max_i \sup \{|\eta_i|: U_i\}\ ,\quad \{\eta_i\} \in C^0(\mathscr{U}^*, O_S(L))\ .$$

LEMMA 2.33. *Let E, L be line bundles on S. Suppose that $|L| \neq 1$ and $\delta: C^0(\mathscr{U}^*, O_S(E\otimes L^{-\nu})) \to Z^1(\mathscr{U}^*, O_S(E\otimes L^{-\nu}))$ is an isomorphism for $\nu \geqq 1$. Then there exists a positive constant K independent of ν such that the operator norm $|\delta|$ of the coboundary mapping $\delta: C^0(\mathscr{U}^*, O_S(E\otimes L^{-\nu})) \to Z^1(\mathscr{U}^*, O_S(E\otimes L^{-\nu}))$ is equal or lower than K, i.e. for every $\{\eta_i^\nu\} \in C^0(\mathscr{U}^*, O(E\otimes L^{-\nu}))$ the inequality*

$$(2.34)\qquad |\{\eta_i\}| \leqq K|\delta\{\eta_i\}|_m$$

holds, (where we take the transition functions $\{e_{ij}\}$, $\{l_{ij}\}$ of E, L to be constant and $e_{ij} = l_{ij} = 1$ for $(i, j) = (1, 2), (2, 3), (4, 5), (5, 6)$).

PROOF. Let Z be a holomorphic vector field on the Hopf surface S. We define a linear endmorphism of $C^q(\mathscr{U}^*, O_S(E\otimes L^{-\nu}))$ by

$$(2.35)\qquad Z: \eta = \{\eta_{i_0,\cdots,i_q}\} \to Z\eta = \{Z\eta_{i_0,\cdots,i_q}\}\ .$$

Since E and L are flat line bundle, this mapping is a cochain mapping, i.e. $\delta(Z\eta) = Z(\delta\eta)$, since the transition functions e_{ij}, l_{ij} are constants for $1 \leqq i, j \leqq 6$ $(i \neq j)$. We note that the Silov boundary of U_i $(1 \leqq i \leqq 6)$ is its edges. Then by Lemma 2.7, we can take two holomorphic vector fields Z_1, Z_2 such that the zero loci of $Z_1\wedge Z_2$ do not intersect the Silov boundary of U_i for every i. We introduce another norm on $Z^1(\mathscr{U}^*, O_S(E\otimes L^{-\nu}))$ by

$$(2.36)\qquad |\{\gamma_{ij}^\nu\}| = \max_{(i,j)} \sup \{\min\{|\gamma_{ij}^\nu|, |\gamma_{ji}^\nu|\}: U_i \cap U_j\}$$
$$\text{for}\quad \{\gamma_{ij}^\nu\} \in Z^1(\mathscr{U}^*, O_S(E\otimes L^{-\nu}))\ .$$

Then Cauchy inequality implies that there exists a positive constant d such that

$$(2.37)\qquad |Z_1^p Z_2^q \gamma^\nu| \leqq d^{p+q}|\gamma^\nu|_m \quad \text{for}\quad \gamma^\nu \in Z^1(\mathscr{U}^*, O_S(E\otimes L^{-\nu}))\ ,$$
$$\nu \geqq 1\ ,\quad p, q \geqq 0\ .$$

Since $\delta: C^0(\mathscr{U}^*, O(E\otimes L^{-\nu})) \to Z^1(\mathscr{U}^*, O_S(E\otimes L^{-\nu}))$ is a linear isomorphism, Kodaira-Spencer's augument (c.f. [12]) shows that there exists a positive constant K_ν which satisfies the following property:

$$(2.38)\qquad |\eta^\nu| \leqq K_\nu|\delta\eta^\nu|_m \quad \text{for every}\quad \eta_\nu \in C^0(\mathscr{U}^*, O_S(E\otimes L^{-\nu}))\ .$$

But K_ν may depend on ν.

We set

$$(2.39)\qquad E(\nu) = \{(\gamma^\nu, r) \in Z^1(\mathscr{U}, O_S(E\otimes L^{-\nu})) \times \boldsymbol{R}\ ;$$
$$|Z_1^p Z_2^q \gamma^\nu| \leqq d^{p+q} r \quad \text{for}\quad p, q \geqq 0\}\ .$$

Then (2.37) implies that $(\gamma^\nu, |\gamma^\nu|_m) \in E(\nu)$ for $\gamma^\nu \in Z^1(\mathscr{U}^*, O_S(E \otimes L^{-\nu}))$. We set $K'_\nu = \sup\{|\eta^\nu|/r; \delta\eta^\nu = \gamma^\nu, r \neq 0, (\gamma^\nu, r) \in E(\nu)\}$. It is clear that $K'_\nu \leqq K_\nu$. By (2.39), for our purpose, it is sufficient to show that K'_ν are bounded with respect to ν. Suppose not. Taking a subsequence, if necessary, we may assume that $K'_\mu \to \infty$ as μ goes to infinity. By the definition of K'_μ, we can find $\eta^\mu \in C^0(\mathscr{U}, O_S(E \otimes L^{-\mu}))$ and $(\gamma^\mu, r_\mu) \in E(\mu)$ satisfying:

$$(2.40) \qquad \delta\eta^\mu = \gamma^\mu , \qquad |\eta^\mu|/r_\mu \leqq K'_\mu \leqq 2|\eta^\mu|/r_\mu , \qquad |\eta^\mu| = 1 .$$

Then $r \to 0$ as $\mu \to \infty$. Note that Z_i $(i = 1, 2)$ are cochain mapping and $(\gamma^\mu, r_\mu) \in E(\mu)$ implies that $(Z_i\gamma^\mu, dr_\mu) \in E(\mu)$ $(i = 1, 2)$. Hence we have:

$$(2.41) \qquad |Z_i\eta^\mu| \leqq K'_\mu dr_\mu \leqq \frac{2|\eta^\mu|}{r_\mu} dr_\mu = 2d \quad \text{for} \quad i = 1, 2 .$$

Let us denote η^μ by $\{\eta_i^\mu\}$. Since the zero loci of $Z_1 \wedge Z_2$ do not intersect with the Silov boundary of U_i $(1 \leqq i \leqq 6)$, (2.41) implies that η_i^μ and its first derivatives $\partial\eta_i^\mu/\partial z_1$, $\partial\eta_i^\mu/\partial z_2$ are uniformly bounded with respect to μ, where $\partial/\partial z_1$ and $\partial/\partial z_2$ are vector fields on U_i naturally defined from the identification of U_i with a domain in C^2. Then taking a subsequence if necessary, we may assume that each η_i^μ converges to a holomorphic function h_i uniformly on U_i.

Let us denote γ^μ by $\{\gamma_{ij}^\mu\}$. Then the equality $\delta\eta^\mu = \gamma^\mu$ means that

$$(2.42) \qquad \eta_i^\mu - l_{ij}^{-\mu} e_{ij} \eta_j^\mu = \gamma_{ij}^\mu \quad \text{on} \quad U_i \cap U_j .$$

We note that $|l_{31}|$, $|l_{64}| < 1$ or $|l_{13}|$, $|l_{46}| < 1$ holds by the assumption. Suppose $|l_{31}|$, $|l_{64}| < 1$ holds. In this case, we conclude that $h_3 \equiv 0$ on U_3 and $h_6 \equiv 0$ on U_6 by the fact $r_\mu \to 0$. Then by the fact $l_{ij} = e_{ij} = 1$ for $(i, j) = (1, 2), (2, 3), (4, 5), (5, 6)$ and the fact $r_\mu \to 0$, (2.42) implies that all h_i vanish. Thus η_i^μ converges to 0 *uniformly* on U_i respctively. This contradicts the fact that $|\eta^\mu| = 1$ for all μ. In the case $|l_{13}|$, $|l_{46}| < 1$, by the same argument we obtain the proof of the lemma. q.e.d.

Now we return to the proof of Theorem 2.8. We assume that (2.23) holds. Then since $|N| < 1$, applying Lemma 2.33 to $E = O_S$ (trivial bundle), $L = N$, by (2.31), we obtain:

$$(2.43) \qquad |f_{i|\nu+1}(z_i(p))| \leqq \left[\frac{KB(A^\nu(u))^2}{1 - BA^\nu(u)} \right]_{\nu+1} \quad \text{for} \quad p \in U_i , \quad 1 \leqq i \leqq 6 .$$

We note that the relation:

$$(2.44) \qquad f_{i|\nu+1}(z_i) - t_{ij}^{-\nu} f_{j|\nu+1}(z_j) = h_{ij|\nu}(z_i) \quad \text{on} \quad U_i^* \cap U_j^* .$$

Since U_4 (resp. U_1) contains the Silov bondaries of U_1^*, U_2^*, U_3^* (resp.

U_4^*, U_5^*, U_6^*), by (2.43) and the assumption $|N| < 1$, we have:

$$|f_{i|\nu+1}(z_i(p))| \leqq \left[\frac{(K+1)B(A^\nu(u))^2}{1 - BA^\nu(u)}\right]_{\nu+1} \quad \text{for} \quad p \in U_i^*\,, \quad 1 \leqq i \leqq 6\,. \tag{2.45}$$

Now we define a power series $A(u) = u + \sum_{\nu=2}^{\infty} A_\nu u^\nu$ to be the solution of the functional equation:

$$A(u) - u = \frac{(K+1)B(A(u))^2}{1 - BA(u)}\,. \tag{2.46}$$

Clearly $A(u)$ exists and has a positive radius of convergence. Then we obtain:

$$f_i(z_i(p), u_i) \ll A(u_i) \quad p \in U_i^*\,, \quad 1 \leqq i \leqq 6\,. \tag{2.47}$$

by the induction on by using (2.45). This completes the proof of Theorem 2.8. q.e.d.

Next we prove the following theorem.

THEOREM 2.48. *Let S be a primary Hopf surface imbedded in a complex manifold M of dimension* 3. *Suppose that the following conditions are satisfied*:

(1) *The normal bundle N of S is of tangentially infinite type with* $|N| \neq 1$.

(2) $[S]$ *is a flat line bundle on some neighbourhood of S.*

Then there exists a tubular neighbourhood of S which is biholomorphic to a tubular neighbourhood of the 0*-section of N.*

PROOF. We divide the proof of this theorem into several steps. First we remark the condition (2) is equivalent to the condition:

(2)′ There exists a multiplicative holomorphic function on a neighbourhood of S with divisor S.

Step 1. Let w be the multiplicative holomorphic function in the assumption (2)′. We choose Stein coverings $\mathscr{U} = \{U_i\}$, $\mathscr{U}^* = \{U_i^*\}$ of S and a Stein coordinate covering $\mathscr{V}^* = \{V_i^*, (z_i, w_i)\}$ as (2.9), (2.10). We may assume that $w = w|V_i^*$ for $1 \leqq i \leqq 6$, where we choose the branch of w such that $t_{ij} = w_i/w_j = 1$ for the pairs $(i, j) = (1, 2), (2, 3), (4, 5), (5, 6)$. We identify S with 0-section of N as usual. We choose a Stein coordinate covering $\mathscr{W}^* = \{W_i^*, (y_i, s_i)\}$ of N as follows:

(2.49)
(1) $W_i^* = p^{-1}(U_i^*)$, where $p\colon N \to S$ is the bundle projection.
(2) $y_i = p^*(z_i | U_i^*)$.
(3) s_i is a restriction of a branch of a multiplicative holomorphic function with divisor S defined on whole N. And s_i satisfies the relation $s_i/s_j = t_{ij}$ on $W_i^* \cap W_j^*$.

To prove Theorem 2.48, it is sufficient to construct a system of vector valued holomorphic functions $\{g_i\}$ defined respectively on a neighbourhood W_i of $U_i(W_i \subset W_i^*)$ satisfying:

$$(2.50)\qquad \begin{aligned} &g_i(y_i, 0) = y_i ,\\ &g_i(\theta_{ji}(y_i), t_{ji}s_i) = \psi_{ji}(g_i(y_i, s_i), s_i) \quad \text{on} \quad W_i \cap W_j \end{aligned}$$

where $\theta_{ji}(y_i) = y_j$ on $W_i^* \cap W_j^*$ and $\psi_{ji}(z_i, w_i) = z_j$ on $V_i^* \cap V_j^*$. In fact, using g_i's we can define a biholomorphic mapping g from a sufficiently small neighbourbood W of the 0-section of N into M by

$$(2.51)\qquad g\colon (y_i, s_i) \in W_i^* \cap W \to (g_i(y_i, s_i), s_i) \in V_i^* .$$

Then it is clear that g is a well-defined holomorphic mapping. Now we construct $\{g_i\}$ as formal power series. We set

$$(2.52)\qquad g_j(y_i, s_i) = \sum_{\nu=0}^{\infty} g_{i|\nu}(y_i)s_i^\nu , \qquad g_{i|0}(y_i) = y_i .$$

We write $g_i(y_i, s_i) = \sum_{\mu=0}^{\nu} g_{i|\mu}(y_i)s_i^\mu$. For two power series $P(u)$, $Q(u)$ in u, we indicate by writing $P(u) \equiv_\nu Q(u)$ that the power series expansion of $P(u) - Q(u)$ in u contains no terms of degree $\leqq \nu$. With these notations (2.50) is equivalent to:

$$(2.53)_\nu \qquad g_j^\nu(\theta_{ji}(y_i), t_{ji}s_i) \equiv_\nu \psi_{ji}(g_i^\nu(y_i, s_i), s_i) \quad \text{for} \quad \nu = 0, 1, 2, \cdots .$$

We construct $g_i^\nu(y_i, s_i)$ satisfying (2.53) by induction on ν. By the definition of y_i, $\{g_i^0(y_i, s_i) = y_i\}$ satisfy $(2.40)_0$. Suppose $\{g_i^{\nu-1}(y_i, s_i)\}$ satisying $(2.40)_{\nu-1}$ are already determined. We define a system of holomorphic functions $\{g_{ij|\nu}(z_i)\}$ defined on $U_i \cap U_j$ respectively by

$$(2.54)\qquad \begin{aligned} g_{ij|\nu}(y_i) &= [g_i^{\nu-1}(\theta_{ij}(y_j), t_{ij}s_j) - \psi_{ij}(g_j^{\nu-1}(y_j, s_j), s_j)]_\nu \\ &= [-\psi_{ij}(g_j^{\nu-1}(y_j, s_j), s_j)]_\nu \end{aligned}$$

where $[\ \ ]_\nu$ denotes the coefficient of $s_i^\nu = t_{ij}^\nu s_j^\nu$ in the series. The proof of the following lemma is standard and hence we omit it.

LEMMA 2.55 (c.f. [12]).

(1) $\{g_{ij|\nu}\}$ *is an element of* $Z^1(\mathscr{U}, O_S(T_S \otimes N^{-\nu}))$.

(2) $g_i^\nu(y_i, s_i) = g_i^{\nu-1}(y_i, s_i) + g_{i|\nu}(y_i)s_i^\nu$ *satisfies* (2.53), *if and only if* $\delta\{g_{i|\nu}\} = \{g_{ij|\nu}\}$.

Since N is of tangentially infinite type, Lemma 2.55 completes the inductive construction of the formal power series.

Step 2. We now proceed as in the proof of Theorem 2.8. We expand $\psi_{ij}(z_i + u, v)$ into a power series in three variables u_1, u_2, v and let $L_{ij}(z_j, u, v)$ be the linear part of the power series, where u denotes the vector (u_1, u_2). Since ψ_{ij} is a vector valued holomorphic function on the closure of $V_i^* \cap V_j^*$, we can find a large number P such that

$$\psi_{ij}(z_j + u, v) - \theta_{ij}(z_j) - L_{ij}(z_j, u, v) \ll \sum_{\nu=2}^{\infty} P^\nu(u_1 + u_2 + v)^\nu \quad \text{for} \quad z_j \in U_i^* \cap U_j^* \tag{2.56}$$

where $\ll$ means that every element of the left-hand side is dominated by the right-hand side. Suppose that for some $\nu \geqq 2$ and a polynomial $A^{\nu-1}(v) = cv + \sum_{\mu=2}^{\nu-1} a_\mu v^\mu$ in v with constant coefficients

$$g_i^{\nu-1}(y_i(p), s_i) - y_i(p) \ll A^{\nu-1}(s_i) \quad \text{for} \quad p \in U_i\,, \quad 1 \leqq i \leqq 6 \tag{2.57$_{\nu-1}$}$$

is obtained. If we take $c > 1$ sufficiently large, then $(2.57)_1$ is satisfied. Then we obtain from (2.54) and (2.56):

$$|g_{ij|\nu}(y_i(p))| \leqq \left[\frac{9P^2(A^{\nu-1}(s_j))}{1 - 3PA^{\nu-1}(s_j)}\right]_\nu \quad \text{for} \quad p \in U_i^* \cap U_j \tag{2.58}$$

where $[\ \ ]_\nu$ denotes the coefficient of $s_i^\nu = t_{ij}^\nu s_j^\nu$ in the power series (note that $g_{ij|\nu}$ extends to $U_i^* \cap U_j$ holomorphically). So we obtain:

$$|t_{ij}^\nu g_{ij|\nu}(y_i(p))| \leqq \left[\frac{9P^2(A^{\nu-1}(v))^2}{1 - 3PA^{\nu-1}(v)}\right]_\nu \quad \text{for} \quad p \in U_i^* \cap U_j \tag{2.59}$$

where $[\ \ ]_\nu$ denotes the coefficient of v^ν in the power series. By (2.59) and Lemma 2.55 (1), we can find a positive constant C such that

$$|g_{ij|\nu}(y_i(p))| \leqq \left[C\frac{9P^2(A^{\nu-1}(v))^2}{1 - 3PA^{\nu-1}(v)}\right]_\nu \quad \text{for} \quad p \in U_i \cap U_j^*\,. \tag{2.60}$$

(2.59) and (2.60) assert that $g_{ij|\nu}(y_i)$ extends holomorphically to W_{ij}^*, where $W_{ij}^* = (U_i^* \cap U_j) \cup (U_j^* \cap U_i)$ except for $(i, j) = (1, 2), (2, 3), (4, 5), (5, 6)$ and $W_{ij}^* = U_i^* \cap U_j^*$ for $(i, j) = (1, 2), (2, 3), (4, 5), (5, 6)$ (we note that $g_{ij|\nu}(y_i) = 0$ for $(i, j) = (1, 2), (2, 3), (4, 5), (5, 6)$ by the construction of the coordinates. We introduce a subspace of $Z^1(\mathscr{U}, \Theta_S \otimes O_S(N^{-\nu}))$ as follows:

$$\begin{aligned} &Z_*^1(\mathscr{U}, \Theta_S \otimes O_S(N^{-\nu})) \\ &\quad = \{\gamma^\nu \in Z^1(\mathscr{U}, \Theta_S \otimes O_S(N^{-\nu}));\ \text{each } \gamma_{ij}^\nu \text{ is holomorphic on } \\ &\qquad W_{ij}^* \text{ and } \gamma_{ij}^\nu = 0 \text{ for } (i, j) = (1, 2), (2, 3), (4, 5), (5, 6)\}\,, \end{aligned} \tag{2.61}$$

And we introduce a norm on $Z_*^1(\mathscr{U}, \Theta_S \otimes O_S(N^{-\nu}))$ by

$$|\gamma^\nu|_m = \max_{(i,j)} \sup\{\min\{|\gamma_{ij}^\nu|, |\gamma_{ji}^\nu|\} : W_{ij}^*\} \quad \text{for} \quad \gamma^\nu \in Z_*^1(\mathscr{U}, \Theta_S \otimes O_S(N^{-\nu})) \tag{2.62}$$

where $\gamma_{ij}^\nu = \max\{|\gamma_{ij1}^\nu|, |\gamma_{ij2}^\nu|\}$ $(\gamma_{ij}^\nu = (\gamma_{ij1}^\nu, \gamma_{ij2}^\nu))$. Then (2.60) asserts that

$$(2.63)\qquad |\{g_{ij|\nu}(y_i(p))\}|_m \leqq \left[\frac{9CP(A^{\nu-1}(v))^2}{1-3PA^{\nu-1}(v)}\right]_\nu .$$

Step 3. Since T_S is not a flat vector bundle in general, we need the following lemma.

LEMMA 2.64. *$Z^1(\mathscr{U}^*, \Theta_S \otimes O_S(N^{-\nu}))$, $Z^1(\mathscr{U}^*, O_S(L_k \otimes N^{-\nu}))$ $(k=1, 2)$ $(L_k = L(\alpha_1, \alpha_2, \alpha_k)_t)$ are metric spaces with respect to the norms*:

$$|\gamma^\nu|_m = \max_{(i,j)} \sup \{\min \{|\gamma^\nu_{ij}|, |\gamma^\nu_{ji}|\} : W^*_{ij}\} \quad \text{for} \quad \gamma^\nu \in Z^1(\mathscr{U}^*, \Theta_S \otimes O_S(L^{-\nu}))$$

$$|\xi^\nu|_m = \max_{(i,j)} \sup \{\min \{|\xi^\nu_{ij}|, |\xi^\nu_{ji}|\} : W^*_{ij}\} \quad \text{for} \quad \xi^\nu \in Z^1(\mathscr{U}^*, O_S(L_k \otimes N^{-\nu}))$$

$$k = 1, 2$$

respectively. And $Z^1(\mathscr{U}^*, \Theta_S \otimes O_S(N^{-\nu}))$ is a direct sum of $Z^1(\mathscr{U}^*, O_S(L_k \otimes N^{-\nu}))$ as metric spaces.

PROOF. By (2.5), we have the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^0(O_S(L_1 \otimes N^{-\nu})) & \longrightarrow & C^0(\Theta_S \otimes O_S(L^{-\nu})) & \longrightarrow & C^0(O_S(L_2 \otimes N^{-\nu})) & \longrightarrow & 0 \\ & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\ 0 & \longrightarrow & Z^1(O_S(L_1 \otimes N^{-\nu})) & \longrightarrow & Z^1(S \otimes O_S(L^{-\nu})) & \longrightarrow & Z^1(O_S(L_2 \otimes N^{-\nu})) & \longrightarrow & 0 . \end{array}$$

The first line is exact since $\mathscr{U}^*$ is a Stein covering and it clearly splits. And by the assumption, every column gives a linear isomorpism. Hence the second line is exact and splits. It is clear that this splitting is a splitting as metric spaces by the definition of the norms. q.e.d.

By Lemma 2.64, the same argument as in Lemma 2.33 works, since $|N| \neq 1$. Then we have the following lemma.

LEMMA 2.65. *For every element γ^ν of $Z^1_*(\mathscr{U}, \Theta_S \otimes O_S(N^{-\nu}))$, there exists a unique element η^ν of $C^0(\mathscr{U}, \Theta_S \otimes O_S(N^{-\nu}))$ satisfying $\delta\eta^\nu = \gamma^\nu$. And the inequality $|\eta^\nu| \leqq K''|\gamma^\nu|_m$ holds, where K'' is a positive constant independent of ν and $|\eta^\nu|$ denotes* $\max_i \sup \{|\eta^\nu_i| : U_i\}$.

Step 4. We define a power series $A(v) = cv + \sum_{\mu=2}^\infty a_\mu v^\mu$ of positive radius of convergence by:

$$(2.66)\qquad A(v) - cv = \frac{9CK''P^2(A(v))^2}{1-3PA(v)} .$$

Then by using (2.63) and Lemma 2.65, we can prove

$$(2.67)\qquad g_i(y_i(p), s_i) \ll A(s_i) \quad \text{for} \quad p \in U_i ,\quad 1 \leqq i \leqq 6$$

as before. This completes the proof of Theorem 2.48. q.e.d.

3. Local Cohomology. In this section, we compute some local cohomologies.

THEOREM 3.1. *Let S be a primary Hopf surface and let L be a line bundle over S. Suppose that $|L| \neq 1$. Then the followings are true (we identify S with the* 0*-section).*

(1) *If L^{-1} is of infinite type, then we have that*

$$H_S^1(O_L) = 0\ , \qquad H_S^2(O_L) = 0\ .$$

(2) *If L^{-1} is of both tangentially infinite type and infinite type, then we have that $H_S^1(\Theta_L) = 0$, $H_S^2(\Theta_L) = 0$.*

(3) *If L^{-1} is of both smooth type and infinite type and L is of infinite type, then $H_S^1(\Omega_L^1 \otimes \Omega_L^3) = 0$, $H^1(L, \Omega_L^1 \otimes \Omega_L^3) = 0$.*

PROOF. Let us identify S with the 0-section of L. Let F be the natural compactification of L as a $\boldsymbol{P}^1$-bundle and let S_∞ be the infinity section of F. Let $f: F \to S$ be the bundle projection. Let $\mathscr{S} = O_F$ (resp. Θ_F, $\Omega_F^1 \otimes \Omega_F^3$). Let us consider the exact sequence:

$$\begin{aligned} (3.1) \qquad & 0 \to H^0(F, \mathscr{S}) \to H^0(F - S, \mathscr{S}) \to H_S^1(\mathscr{S}) \to H^1(F, \mathscr{S}) \\ & \to H^1(F - S, \mathscr{S}) \to H_S^2(\mathscr{S}) \to H^2(F, \mathscr{S}) \to \cdots . \end{aligned}$$

We study $H^i(F - S, \mathscr{S})$ for $i = 1, 2$. Clearly $F - S$ is biholomorphic to L^{-1}. Let $f^\#: L^{-1} \to S_\infty$ be the bundle projection. We identify $F - S$ with L^{-1} hereafter. By identifying S_∞ with S, we consider L^{-1} as a line bundle over S. We choose a Stein covering $\mathscr{U}^* = \{U_i^*\}$ of S and a Stein coordinate covering $\mathscr{W}^* = \{W_i^*, (y_i, s_i)\}$ of L^{-1} as (2.9), (2.10), (2.49). First we study $H^0(F - S, \mathscr{S})$. Let h be an element of $H^0(L^{-1}, \mathscr{S})$. We expand $h_i = h \mid W_i^*$ into a power series:

$$(3.2) \qquad h_i(y_i, s_i) = \sum_{\nu=0}^{\infty} h_{i|\nu}(y_i) s_i^\nu\ .$$

Then $h_{i|\nu} \in H^0(S, \mathscr{F} \otimes O_S(L^\nu))$, where $\mathscr{F} = O_S$ (resp. $\Theta_S \oplus O_S(L^{-1})$, $(\Omega_S^1 \oplus O_S(L)) \otimes \Omega_S^2 \otimes O_S(L)$). Then by the assumption we have that

$$\begin{aligned} & H^0(F - S, O_F) \cong H^0(S, O_S) \\ (3.3) \qquad & H^0(F - S, \Theta_F) \cong H^0(S, \Theta_S) + H^0(S, O_S(L^{-1})) + H^0(S, O_S) \\ & H^0(F - S, \Omega_F^1 \otimes \Omega_F^3) = 0\ . \end{aligned}$$

Secondly we study $H^1(F - S, \mathscr{S})$. Let $\{h_{ij}\}$ be an element of $Z^1(\mathscr{W}^*, \mathscr{S})$. Then each h_{ij} has a power series expansion on $W_i^* \cap W_j^*$:

$$(3.4) \qquad h_{ij}(y_i, s_i) = \sum_{\nu=0}^{\infty} h_{ij|\nu}(y_i) s_i^\nu\ .$$

Let $\mathscr{U}^{\delta} = \{U_i^{\delta}\}$ $(0 < \delta < \varepsilon)$ be a Stein covering of S satisfying

$$\text{(3.5)}\quad \begin{aligned} &\text{(i)}\quad U_i^{\delta} \Subset U_i^* \text{ for } 1 \leqq i \leqq 6,\ 0 < \delta < \varepsilon . \\ &\text{(ii)}\quad \partial U_i^{\delta} \to \partial U_i^* \text{ as } \delta \to 0 \text{ uniformly with respect to some} \\ &\qquad \text{complete Riemannian metric on } S \text{ for } 1 \leqq i \leqq 6 . \end{aligned}$$

We note that $\{h_{ij|\nu}\} \in Z^1(\mathscr{U}^*, \mathscr{F} \otimes O_S(L^{\nu}))$. Suppose that $\{h_{ij}\}$ is cohomologous to 0 formally, i.e. every $\{h_{ij|\nu}\}$ is cohomologous to 0. Then we can find a system of formal power series $\{h_i\}$, $h_i(y_i, s_i) = \sum_{\nu=1}^{\infty} h_{i|\nu}(y_i)s_i^{\nu}$ satisfying:

$$\text{(3.6)}\quad \begin{aligned} &\text{(i)}\quad h_{ij}(y_j, s_j) = h_i(y_i, s_i) - h_j(y_j, s_j) \text{ on } W_i^* \cap W_j^* \text{ as formal} \\ &\qquad \text{power series .} \\ &\text{(ii)}\quad h_{i|\nu}(y_i) \text{ is holomorphic on } U_i^* \text{ for } 1 \leqq i \leqq 6,\ \nu \geqq 0 . \end{aligned}$$

for every $0 < \delta < \varepsilon$ and $R > 0$, we can find a power series $A(\delta, R)(s) = \sum_{\nu=1}^{\infty} A_{\nu}(\delta, R)s^{\nu}$ in s with constant coefficients $A_{\nu}(\delta, R)$ satisfying the condition:

$$\text{(3.7)}\quad \sum_{\nu=1}^{\infty} h_{ij|\nu}(y_i(p)) \ll A(\delta, R)(s_i) \quad \text{for} \quad p \in U_i \cap U_j \quad \text{for} \quad 1 \leqq i \neq j \leqq 6$$

and $A(\delta, R)(s)$ has a radius of convergence greater than R. By Lemma 2.33 and the bundle exact sequence (2.5), there exists a positive constant K_{δ} such that $\sum_{\nu=1}^{\infty} h_{i|\nu}(y_i)s_i^{\nu} \ll K_{\delta}A(\delta, R)(s_i)$ holds for all i. Letting $\delta \to 0$ and $R \to \infty$, we conclude that every h_i is holomorphic on W_i^*. This implies that we can compute $H^1(F - S, \mathscr{S})$ formally. Then by the assumption we obtain:

$$\text{(3.8)}\quad \begin{aligned} &H^1(F - S, O_F) \cong H^1(S, O_S) \quad (\text{induced by } (f^{\#})^*) \\ &H^1(F - S, \Theta_F) \cong H^1(S, \Theta_S) + H^1(S, O_S(L^{-1})) + H^1(S, O_S) \\ &H^1(F - S, \Omega_F^1 \otimes \Omega_F^3) = 0 . \end{aligned}$$

Next we study $H^i(F, \mathscr{S})$. Let us consider the Leray spectral sequence: $E_2^{p,q} = H^p(S, R^q f_* \mathscr{S}) \Rightarrow H^{p+q}(F, \mathscr{S})$.

(1) Case $\mathscr{S} = O_F$. In this case we have the following:

$$\text{(3.9)}\quad \begin{aligned} &E_2^{1,0} = H^1(S, O_S)\ , \qquad E_2^{0,1} = 0\ , \qquad E_2^{2,0} = H^2(S, O_S) = 0\ , \\ &E_2^{1,1} = 0\ , \qquad E_2^{0,2} = 0\ . \end{aligned}$$

Hence we have that

$$\text{(3.10)}\quad H^1(F, O_F) \cong H^1(S, O_S) \quad (\text{induced by } f^*)\ , \qquad H^2(F, O_F) = 0\ .$$

(2) Case $\mathscr{S} = \Theta_F$. In this case we have that $R^0 f_* \Theta_F \cong \Theta_S \oplus O_S(L^{-1}) \oplus O_S \oplus O_S(L)$, $R^1 f_* \Theta_F = 0$, $R^2 f_* \Theta_F = 0$. So we obtain:

(3.11) $$E_2^{1,0} = H^1(S, O_S(L^{-1})) + H^1(S, O_S) + H^1(S, \Theta_S)\,, \qquad E_2^{0,1} = 0\,,$$
$$E_2^{2,0} = H^2(S, O_S(L^{-1}))\,, \qquad E_2^{1,1} = 0\,, \qquad E_2^{0,2} = 0\,.$$

Hence we have:

$$H^0(F, \Theta_F) = H^0(S, \Theta_S) + H^0(S, O_S(L^{-1})) \oplus H^0(S, O_S)$$

(3.12) $$H^1(F, \Theta_F) = H^1(S, \Theta_F) + H^1(S, O_S) + H^1(S, O_S(L^{-1}))$$

$$H^2(F, \Theta_F) = H^2(S, O_S(L^{-1}))\,.$$

We claim that:

(3.13) $$\operatorname{Ker}(H^2(F, \Theta_F) \to H^2(F - S, \Theta_F)) = 0\,.$$

Let $\{h_{ijk}(y_i)\} \in Z^2(\mathscr{U}^*, O_S(L^{-1}))$ be a 2-cycle not cohomologous to 0. Then $\{h_{ijk}(y_i)s_i^2\,\partial/\partial s_i\} \in Z^2(\mathscr{W}^*, \Theta_{F-S})$ is a 2-cycle not cohomologous to 0 even formally. This proves the claim.

(3) Case $\mathscr{S} = \Omega_F^1 \otimes \Omega_F^3$. By (3.12) and the assumption, we have that $H^2(F, \Theta_F) = 0$. Then by Serre duality, we have

(3.14) $$H^1(F, \Omega_F^1 \otimes \Omega_F^3) = 0.$$

By (3.1), (3.3), (3.8), (3.10), we see that $H_S^1(O_L) = 0$, $H_S^2(O_L) = 0$, in the Case (1).

By (3.1), (3.3), (3.8), (3.12), (3.13), we see that $H_S^1(\Theta_L) = 0$, $H_S^2(\Theta_L) = 0$ in the Case (2).

By (3.1), (3.3), (3.8), (3.14), we see that $H_S^1(\Omega_L^1 \otimes \Omega_L^3) = 0$, in the Case (3). q.e.d.

COROLLARY 3.15. *Let $M = M^{\pm}(\alpha, A, m)$. If M is diffeomorphic to $S^3 \times S^3$. Then M has the following properties*:

(1) *M contains a cycle of primary Hopf surfaces of length* 4, i.e. *there exists a divisor $D = S_0 + S_1 + S_2 + S_3$ such that*:

(i) *Each S_i is a nonsingular primary Hopf surface.*

(ii) *$S_i \cap S_j \neq \emptyset$, iff $(i, j) = (0, 1), (1, 2), (2, 3), (3, 0)$ and $\{S_i\}$ intersect transversally.*

(2) *For suitable indexing the normal bundles N_0, N_2 of S_0, S_2 satisfy $|N_0| < 1$, $|N_2| > 1$.*

Moreover if N_0 and N_2 are torsion free, then $q(M) = \dim H^1(M, O_M) = 1$.

PROOF. By (1.12), M is of the form:

(3.16) $$M = (H(\alpha) - C) \cup U(S_\infty)$$

where $C = \{[z_1, z_2, z_3] \in H(\alpha); z_1 = z_2 = 0\}$ and S_∞ is a primary Hopf surface and $U(S_\infty)$ is a tubular neighbourhood of S_∞ in a $\boldsymbol{P}^1$-bundle $P(\beta)$ over a primary Hopf surface in which S_∞ is the ∞-section. Let S_2 be S_∞. Then

$|N_2| > 1$, because $|N_2| = |\beta_3^{-1}| > 1$. Let S_0 be the primary Hopf surface $\{[z_1, z_2, z_3] \in H(\alpha); z_3 = 0\}$. Then $|N_0| < 1$, because $|N_0| = |\alpha_3| < 1$. Let S_1^*, S_3^* be the surfaces in $H(\alpha) - C$ defined respectively by: $S_1^* = \{[z_1, z_2, z_3] \in H(\alpha) - C; z_1 = 0\}$, $S_3^* = \{[z_1, z_2, z_3] \in H(\alpha) - C; z_2 = 0\}$. Then the closures of S_1^*, S_3^* in M are nonsingular primary Hopf surfaces intersect with S_2 transversally because of the patching of $H(\alpha) - C$ and $U(S_\infty)$ (c.f. Section 1). Let S_1, S_3 be the closures of S_1^*, S_3^* respectively. Then $D = S_0 + S_1 + S_2 + S_3$ is the desired cycle of primary Hopf surfaces.

Next, suppose that N_0 and N_2 are torsion free. Then by Theorem 3.1 (1), $H^1_{S_2}(O_M) = 0$ and $H^2_{S_2}(O_M) = 0$. So we have that $H^1(M, O_M) \cong H^1(M - S_2, O_M)$. Since $M - S_2 = H(\alpha) - C$, $M - S_2$ is identified with N_0. We compactify N_0 naturally to a $\boldsymbol{P}^1$-bundle F_0 over S_0. Since the normal bundle of the infinity section of F_0 is isomorphic to N_0^{-1}, by Theorem 3.1 we have $H^1(F_0, O_{F_0}) \cong H^1(M - S_2, O_M) \cong H^1(M, O_M)$ (note that if a line bundle L over a primary Hopf surface is torsion free, then L^{-1} is also torsion free). Let $\pi: F_0 \to S_0$ be the bundle projection. Then by using the Leray spectral sequence $E_2^{p,q} = H^p(S_0, \pi_* O_{F_0}) \to H^{p+q}(F_0, O_{F_0})$, we see that $H^1(F_0, O_{F_0}) \cong H^1(S, O_S)$ by π^*. Since $\dim H^1(S, O_S) = 1$, $\dim H^1(M, O_M) = \dim H^1(F_0, O_{F_0}) = \dim H^1(S, O_S) = 1$. q.e.d.

4. Characterization of $M^{\pm}(\alpha, A, m)$.

In this section, we study the converse of Corollary 3.15.

THEOREM 4.1. *Let V be a compact simply connected complex manifold of dimension* 3. *Suppose the following conditions are satisfied.*

(1) $q(V) = \dim H^1(V, O_V) = 1$, $b_2(V) = 0$.

(2) *V contains a cycle of primary Hopf surfaces of length* 4, i.e. *there exists a divisor $D = S_0 + S_1 + S_2 + S_3$ such that:*

(i) *Each S_i is a nonsingular primary Hopf surface.*

(ii) $S_i \cap S_j \neq \emptyset$, *iff* $(i, j) = (0, 1), (1, 2), (2, 3), (3, 0)$ *and* $\{S_i\}$ *intersect transversally.*

(3) *S_0, S_2 are nonelliptic and their normal bundles N_0, N_2 are torsion free with $|N_0| < 1$, $|N_2| > 1$.*

Then V is biholomorphic to $M^{\pm}(\alpha, A, m)$ for some α, A, m and V is diffeomorphic to $S^3 \times S^3$.

PROOF. By Lefschetz duality theorem, we have:

$$H^1(V - S_0, \boldsymbol{C}) \cong H_5(V, S_0, \boldsymbol{C}) . \tag{4.1}$$

We consider the exact sequence:

$$\to H_5(V, \boldsymbol{C}) \to H_5(V, S_0, \boldsymbol{C}) \to H_4(S_0, \boldsymbol{C}) \to H_4(V, \boldsymbol{C}) \to \cdots . \tag{4.1}$$

Then by the assumption and (4.1), (4.2), we have:

(4.3) $$H^1(V - S_0, \boldsymbol{C}) \cong \boldsymbol{C}\,.$$

Next we consider the exact sequence:

(4.4) $$0 \to H^0(V - S_0, dO_V) \to H^1(V - S_0, \boldsymbol{C}) \xrightarrow{j} H^1(V - S_0, O_V) \to \cdots\,.$$

LEMMA 4.5. *j is an isomorphism.*

PROOF. First we compute $\dim H^1(V - S_0, O_V)$. By Theorem 2.8 and Theorem 2.48, S_0 has a tubular neighbourhood which is biholomorphic to a tubular neighbourhood of the 0-section of N_0. Then by Theorem 3.1, we have $H^1_{S_0}(O_V) = 0$ and $H^2_{S_0}(O_V) = 0$. By the exact sequence:

(4.6) $$\to H^1_{S_0}(O_V) \to H^1(V, O_V) \to H^1(V - S_0, O_V) \to H^2_{S_0}(O_V) \to \cdots$$

and the assumption $q(V) = 1$, we have that $\dim H^1(V - S_0, O_V) = 1$.

Next we prove that $\dim H^0(V - S_0, dO_V) = 0$. The proof of the following sublemma is easy. Hence we omit it.

SUBLEMMA. $H^0(S_2, \Omega^1_{V|S_2} \otimes N_2^{-\nu}) = 0$ *for all* $\nu \geqq 0$.

This sublemma means that every holomorphic 1-form on $V - S_0$ has zero of infinity order along S_2. Hence we have that $\dim H^0(V - S_0, dO_V) = 0$. By using (4.3) and the above calculations, we see that j is an isomorphism. q.e.d.

LEMMA 4.7. *For every line bundle L over V, $L|V - S_0$ is a flat line bundle.*

PROOF. Let us consider the commutative diagram:

(4.8)
$$\begin{array}{ccccccccccc}
\longrightarrow & H^1(\boldsymbol{Z}) & \longrightarrow & H^1(\boldsymbol{C}) & \longrightarrow & H^1(\boldsymbol{C}^*) & \longrightarrow & H^2(\boldsymbol{Z}) & \longrightarrow & H^2(\boldsymbol{C}) & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & H^1(\boldsymbol{Z}) & \longrightarrow & H^1(O_V) & \longrightarrow & H^1(O_V^*) & \xrightarrow{\delta^*} & H^2(\boldsymbol{Z}) & \longrightarrow & H^2(O_V) & \longrightarrow \cdots
\end{array}$$

where $H^i(-)$ means $H^i(V - S_0, -)$. We note that $\delta^*(L|V - S_0) = 0$ in $H^2(V - S_0, \boldsymbol{C})$ because of $b_2(V) = 0$. Then by using Lemma 4.5, one can prove this lemma only by chasing the diagram (4.8). q.e.d.

By Lemma 4.7, $[S_2]_{V-S_0}$ is a flat line bundle. Then the assumption and Theorem 2.48 imply that there exists a tubular neighbourhood of S_2 which is biholomorphic to a tubular neighbourhood of the 0-section of N_2. Then by the inverse of the surgery in Section 1, we can replace S_2 by an elliptic curve C, since $|N_2| > 1$. We denote the result of this surgery by V^*.

We claim that V^* is a primary Hopf manifold of dimension 3.

LEMMA 4.9. *For every line bundle L over V, $L|V - S_2$ is a flat line bundle.*

PROOF. Since S_2 has a tubular neighbourhood which is biholomorphic to a tubular neighbourhood of the 0-section of N_2, by the same argument as in Lemma 4.5, we obtain that the natural homomorphism $H^1(V - S_2, C) \to H^1(V - S_2, O_V)$ is an isomorphism. Then by using the same diagram as (4.8), one can prove this lemma. q.e.d.

By Lemma 4.9, there exist multiplicative holomorphic functions f_1, f_2, f_3 with divisors $S_1 \cap (V - S_2)$, S_0, $S_3 \cap (V - S_2)$ respectively on $V - S_2$.

LEMMA 4.10. $H_1(V^*, \boldsymbol{Z}) \cong \boldsymbol{Z}$.

PROOF. By Lefschetz duality theorem, we have:

$$H_1(V - S_2, \boldsymbol{Z}) \cong H^5(V, S_2, \boldsymbol{Z}) . \tag{4.11}$$

We consider the exact sequence:

$$\to H^4(V, \boldsymbol{Z}) \to H^4(S_2, \boldsymbol{Z}) \to H^5(V, S_2, \boldsymbol{Z}) \to H^5(V, \boldsymbol{Z}) \to \cdots . \tag{4.12}$$

Combining the assumption and (4.11), (4.12), we have:

$$H_1(V - S_2, \boldsymbol{Z}) \cong \boldsymbol{Z} . \tag{4.13}$$

Since the elliptic curve C has codimension 2 in V^*, we have that $\pi_1(V^*) \simeq \pi_1(V - S_2)$. So we have $H_1(V^*, \boldsymbol{Z}) \simeq H_1(V - S_2, \boldsymbol{Z}) \simeq \boldsymbol{Z}$. q.e.d.

By considering the universal covering manifold of V^*, we can extend f_1, f_2, f_3 to multiplicative holomorphic functions defined on V^* by Hartogs' extension theorem and we denote them by the same notations. Let us take a generator γ of $H_1(V^*, \boldsymbol{Z}) \simeq \boldsymbol{Z}$ and let $\alpha_1, \alpha_2, \alpha_3$ be the monodromy multipliers of f_1, f_2, f_3 along γ respectively (c.f. [10] II, p. 701, Lemma 11). Clearly $|\alpha_i| \neq 1$ for $i = 1, 2, 3$. And by reversing the orientation of γ, if necessary, we may assume that $|\alpha_i| < 1$ for $i = 1, 2, 3$. In fact, for instance, $\rho = -\log|\alpha_1|$ and $\sigma = \log|\alpha_2|$ were both positive, then $|f_1^\sigma f_2^\rho|$ would be a single valued continuous function on V^*. This contradicts that $f_1^\sigma f_2^\rho$ is a nonconstant multivalued holomorphic function on some open set in V^*.

Then we can define a holomorphic mapping $\Phi: V^* \to H(\alpha)$ $(\alpha = (\alpha_1, \alpha_2, \alpha_3))$ by:

$$\Phi: p \in V^* \to [f_1(p), f_2(p), f_3(p)] \in H(\alpha) . \tag{4.14}$$

Let S_1', S_3' be the zero loci of f_1, f_3 respectively. Since S_0 is a nonelliptic primary Hopf surface, Φ is a biholomorphic mapping onto its image on some tubular neighbourhood of S_0. In fact $S_1' \cap S_0$ and $S_3' \cap S_0$ exhaust all the curves in S_0 and they are zero loci of $f_1|S_0$ and $f_3|S_0$ respectively.

Then it is well-known that the holomorphic mapping $\zeta: p \in S_0 \to [f_1(p), f_3(p)] \in S_{(\alpha_1, \alpha_3)}$ is biholomorphic (c.f. [10], II). Then the above assertion is clear. This implies that Φ is generally one to one. We note that $H(\alpha)$ contains only three surfaces and they are primary Hopf surfaces, because S_0 is nonelliptic and N_0 is torsion free. Then we have that Φ is biholomorphic onto its image on some neighbourhood of $S_1' \cup S_0 \cup S_3'$. Let us consider the divisor D^* of $df_1 \wedge df_2 \wedge df_3$. Since $H(\alpha)$ contains only three curves which are the intersections of the three surfaces, we have that $\Phi(D^*)$ consists of finite number of points in $H(\alpha)$. Then it is clear that V^* contains a global spherical shell (c.f. [9]). This fact implies that V^* is a small deformation of a compact complex manifold which is a modification of a primary Hopf manifold of dimension 3 at finitely many points (c.f. [9]). Since $b_2(V) = 0$, this implies that Φ is a biholomorphic mapping. Hence V is the result of a surgery of $H(\alpha)$ which replaces the elliptic curve C by S_2 by identifying a tubular neighbourhood T of C in $H(\alpha)$ minus C (we identify T with a tubular neighbourhood of the 0-section of the rank 2 vector bundle $L(\alpha_1, \alpha_3, \alpha_2)_0 \to \boldsymbol{C}^*/\langle\alpha_2\rangle$) with a tubular neighbourhood T' of the 0-section of $L(\beta_1, \beta_2, \beta_3)$ minus the 0-section for some $(\beta_1, \beta_2, \beta_3)$ which is determined by the relation of Lemma 1.3 for some A, m. Let $v: T^* \to T'^*$ be the identification where $T^* = T - C$ and $T'^* = T' - $ (0-section). We write v by using the coordinates as in Lemma 1.3 as follows:

$$(4.15) \qquad v([z_1, z_2, z_3]) = ([h_1(z_3)z_1, h_2(z_3)z_2, h_3(z_3)] + [h'(z_1, z_2, z_3)])$$
$$(\text{or} = ([h_1(z_3)z_2, h_2(z_3)z_1, h_3(z_3)] + [h'(z_1, z_2, z_3)]$$

where $h_i(z_3)$, $1 \leqq i \leqq 3$ are multiplicative holomorphic functions on the elliptic curve C and $h'(z_1, z_2, z_3)$ is a sum of higher order terms in z_1, z_2. Let V_t ($|t| \leqq 1$) be the manifold constructed from $H(\alpha)$ by the surgery which replaces C with S_2 by using the following identification v_t:

$$(4.16) \qquad v_t([z_1, z_2, z_3]) = ([h_1(z_3)z_1, h_2(z_3)z_2, h_3(z_3)] + [th'(z_1, z_2, z_3)])$$
$$(\text{resp.} = ([h_1(z_3)z_2, h_2(z_3)z_1, h_3(z_3)] + [th'(z_1, z_2, z_3)])$$

(take T^* and T'^* small enough).
Then since $H^1(V_t, \Theta_{V_t}) \simeq H^1(V_t - S_2, \Theta_{V_t})$ by Theorem 3.1, the complex analytic family $\{V_t\}_{|t| \leqq 1}$ is trivial. This implies that $V = V_1$ is biholomorphic to $V_0 = M((\alpha_1, \alpha_3, \alpha_2), A, m)$. Since V is simply connected by Corollary 1.15, V is diffeomorphic to $S^3 \times S^3$. q.e.d.

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DEPARTMENT OF MATHEMATICS
TOKYO METROPOLITAN UNIVERSITY
SETAGAYA, TOKYO, 158
JAPAN