Tôhoku Math. Journ.
35 (1983), 387-392.

TOTALLY GEODESIC FOLIATIONS AND KILLING FIELDS

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(Received July 6, 1982)

1. Introduction. In [3], Johnson and Whitt studied Killing fields on complete Riemannian manifolds admitting codimension-one totally geodesic foliations by compact leaves. They observed that any Killing field preserves the foliation. In general, the conclusion does not hold if we remove the assumption that the foliation consists of compact leaves (see [3, Remark following Theorem (3.1)]). However, if we assume the compactness of M, we have the same conclusion. That is, we prove the following.

THEOREM. *Let (M, g) be a closed connected Riemannian manifold and $\mathscr{F}$ be a codimension-one totally geodesic foliation of (M, g). Then any Killing field Z on (M, g) preserves $\mathscr{F}$, that is, the flow of Z maps each leaf of $\mathscr{F}$ to a leaf of $\mathscr{F}$.*

As a corollary, we have the following (cf. [3, Theorem (3.1)]).

COROLLARY. *Let $(M, \mathscr{F}, g)$ and Z be as in Theorem. If $\mathscr{F}$ has a compact leaf L_0 and Z is transverse to L_0 at some point, then all leaves of $\mathscr{F}$ are isometric to $(L_0, g|L_0)$ and Z is transverse to $\mathscr{F}$ everywhere on M.*

The proofs are given in § 3. As applications, in § 4, we study some properties of codimension-one totally geodesic foliations of closed Riemannian manifolds admitting Killing fields.

2. Preliminaries. Let (M, g) be a complete connected Riemannian manifold and $\mathscr{F}$ be a codimension-one totally geodesic foliation of (M, g). Let $\tilde{M}$ be the universal covering of M and $p: \tilde{M} \to M$ be the covering projection. We denote by $(\tilde{M}, \tilde{\mathscr{F}}, \tilde{g})$ the canonical lifting $(\tilde{M}, p^*\mathscr{F}, p^*g)$ of $(M, \mathscr{F}, g)$ to the universal covering $\tilde{M}$ of M. Then the following theorem is known.

THEOREM (Kashiwabara [4], Rummler [7]). *The foliated Riemannian manifold $(\tilde{M}, \tilde{\mathscr{F}}, \tilde{g})$ is isometric to a trivially foliated Riemannian manifold $(\tilde{L} \times \boldsymbol{R}^1, \{\tilde{L} \times (t)\}_{t \in \boldsymbol{R}^1}, \bar{g})$, where $\tilde{L}$ is a leaf of $\tilde{\mathscr{F}}$ and the metric $\bar{g}$ is of the form $ds^2 = ds_{\tilde{L}}^2 + f^2dt^2$. Here, f is a smooth positive function on $\tilde{M}$ and $ds_{\tilde{L}}^2$ is the metric of $\tilde{L}$ induced by the inclusion $\tilde{L} \to \tilde{M}$, and*

dt^2 is the canonical metric of $\boldsymbol{R}^1$.

Let $\pi\colon \tilde{M} = \tilde{L} \times \boldsymbol{R}^1 \to \tilde{L}$ and $\eta\colon \tilde{M} \to \boldsymbol{R}^1$ be the natural projections. We identify a vector field X on $\tilde{L}$ with the one $\tilde{X}$ on $\tilde{M}$ that is tangent to $\tilde{\mathscr{F}}$ and is π-related to X. We identify a vector field V on $\boldsymbol{R}^1$ with the one $\tilde{V}$ on $\tilde{M}$ that is orthogonal to $\tilde{\mathscr{F}}$ and is η-related to V. We also call $\tilde{X}$ (resp. $\tilde{V}$) the canonical lifting of X (resp. V). Let N be a unit vector field on M perpendicular to $\mathscr{F}$. We also denote by N the canonical lifting of N to $\tilde{M}$.

LEMMA 1. *Set $G = \operatorname{grad} f$. Then $\nabla_N N = -\mathscr{H}(G)/f$, where $\mathscr{H}_x$ is the orthogonal projection of $T_x\tilde{M}$ onto $T_x\tilde{\mathscr{F}}$. Equivalently, we have $f\theta + df = 0$ on $T\tilde{\mathscr{F}}$, where θ is the dual one-form of $\nabla_N N$ and d is the exterior differential of $\tilde{M}$.*

PROOF. Let V and W be the canonical liftings of vector fields on $\boldsymbol{R}^1$. Then we have $\mathscr{H}(\nabla_V W) = -\langle V, W\rangle \mathscr{H}(G)/f$ by the same computation as in [1, Lemma 7.3]. This formula is tensorial and $\mathscr{H}(\nabla_N N) = \nabla_N N$. Hence we have $\nabla_N N = -\mathscr{H}(G)/f$.

LEMMA 2. *Let E be a unit vector of $T_x\tilde{\mathscr{F}}$. Then*

$$K(E, N) = -(\nabla^2 f)(E, E)/f\,,$$

where $\nabla^2 f$ is the Hessian of f defined by $(\nabla^2 f)(X, Y) = X(Y(f)) - \nabla_X Y(f)$ and $K(E, N)$ is the sectional curvature of the plane spanned by E and N.

PROOF. Extend E to a section of $T\tilde{\mathscr{F}}$. Then, by definition and Lemma 1, we have

$$\begin{aligned} K(E, N) &= \langle \nabla_E \nabla_N N, E\rangle - \langle \nabla_N \nabla_E N, E\rangle - \langle \nabla_{[E,N]} N, E\rangle \\ &= \langle \nabla_E(-\mathscr{H}(G)/f), E\rangle + \langle \nabla_N E, N\rangle\langle \nabla_N N, E\rangle \\ &= -E(1/f)E(f) - \langle \nabla_E \mathscr{H}(G), E\rangle/f - \langle \nabla_N N, E\rangle^2 \\ &= E(f)^2/f^2 - E\langle \mathscr{H}(G), E\rangle/f + \langle \mathscr{H}(G), \nabla_E E\rangle/f - E(f)^2/f^2 \\ &= -E(E(f))/f + \nabla_E E(f)/f = -(\nabla^2 f)(E, E)/f\,. \end{aligned}$$

LEMMA 3. *A vector field Z on $\tilde{M}$ is a Killing field if and only if*
(1) $\mathscr{H}Z(\cdot, t)$ is a Killing field on $\tilde{L}$ for each $t \in \boldsymbol{R}$,
(2) $N\langle Z, N\rangle = \theta(Z)$,
(3) $N\langle Z, E\rangle = -fE(\langle Z, N\rangle/f)$ for all vector fields E on $\tilde{L}$.

We omit the proof, because we can prove this lemma by the same computation as in [1, Lemma 7.11].

3. Proof of Theorem. We may assume that the foliation is transversely oriented. We use the same notations as in § 2. In the following, we assume that M is compact. Let Z be a Killing field on (M, g). We also denote by Z the canonical lifting of Z to $\tilde{M}$. Let $\tilde{L}$ be a leaf of $\tilde{\mathscr{F}}$ and $\gamma: \boldsymbol{R} \to \tilde{L}$ be a geodesic on $\tilde{L}$ with $\gamma(0) = p$ and $|\gamma'| = 1$. Then Z is a Jacobi field along γ. Set $\phi = \langle Z, N \rangle$. We also denote the restriction of ϕ (resp. f) to γ by ϕ (resp. f). By the Jacobi differential equation and the fact that $\mathscr{F}$ is totally geodesic, we have the following differential equation

$$\phi'' + K(T, N)\phi = 0 ,$$

where $T = \gamma'$, the differentiation of γ with respect to the parameter t. By Lemma 2, we also have

$$f'' + K(T, N)f = 0 .$$

Thus we get $f''\phi - f\phi'' = 0$. Hence we have the following linear differential equation

$$(1) \qquad \phi' = (\log f)'\phi + C/f \quad \text{for a constant } C .$$

We may assume $f(0) \equiv f(\gamma(0)) = f(p) = 1$. Then the solution of (1) with the initial condition $\gamma(0) = A$ is given by

$$(2) \qquad \phi(t) = Af(t) + Cf(t)\int_0^t f^{-2}(s)ds .$$

LEMMA 4. *For the function f, we have*

$$\limsup_{t\to+\infty} f(t)\int_0^t f^{-2}(s)ds = +\infty .$$

PROOF. By Lemma 1, we have $f' = -f\theta(T)$. As M is compact, the function $|\theta|$ is bounded. Thus we have

$$(3) \qquad |f'| \leqq Lf \quad \text{for a constant } L .$$

Assume that there is a constant α with

$$(4) \qquad f(t)\int_0^t f^{-2}(s)ds \leqq \alpha < +\infty .$$

By the inequality (3), we have $-L/f^2 \leqq f'/f^3 \leqq L/f^2$. It follows that

$$-L\int_0^t f^{-2}(s)ds \leqq -[1/2f^2]_0^t \leqq L\int_0^t f^{-2}(s)ds .$$

By the inequality (4), we have

$$-2L\alpha f^{-1}(t) \leqq 1 - f^{-2}(t) \leqq 2L\alpha f^{-1}(t) .$$

It follows that $0 < a \leqq f(t) \leqq b < +\infty$ for some constants a and b. Then we have

$$\alpha \geqq f(t) \int_0^t f^{-2}(s)ds \geqq ab^{-2} \int_0^t ds \,,$$

which is a contradiction.

LEMMA 5. *On each* $\tilde{L} \in \tilde{\mathscr{F}}$, *we have*

$$\phi = \langle Z, N \rangle = af \quad \textit{for a constant } a \,.$$

PROOF. In the formula (2), if $C \neq 0$, then $\phi(t)$ is not bounded on $\boldsymbol{R}$ by Lemma 4. In fact, if $A \cdot C \geqq 0$, then

$$\limsup_{t \to +\infty} \phi(t) = \pm \infty \,,$$

while if $A \cdot C < 0$, then

$$\limsup_{t \to -\infty} \phi(t) = \pm \infty \,.$$

As M is compact, the function ϕ is bounded. Thus we have $C = 0$. Hence $\phi(t) = Af(t)$ on each geodesic γ, and we have $\phi = af$ on $\tilde{L}$ for a constant a.

PROOF OF THEOREM. Set $\phi \equiv \langle N, Z \rangle$. By Lemma 5, on each $\tilde{L} \in \tilde{\mathscr{F}}$ we have $\phi = af$ for a constant a. Thus, we have

$$d\phi + \phi\theta = 0 \quad \text{on} \quad T\tilde{\mathscr{F}} \,,$$

by Lemma 1. For $E \in \Gamma(T\tilde{\mathscr{F}})$, we have

$$\begin{aligned} \langle [Z, N], E \rangle &= \langle \nabla_Z N, E \rangle - \langle \nabla_N Z, E \rangle = \langle Z, N \rangle \theta(E) + E\langle Z, N \rangle \\ &= (\phi\theta + d\phi)(E) = 0 \,. \end{aligned}$$

Hence, we have $[Z, E] \in \Gamma(T\tilde{\mathscr{F}})$ for $E \in \Gamma(T\tilde{\mathscr{F}})$.

PROOF OF COROLLARY. Let A be the union of all compact leaves of $\mathscr{F}$. Then A is a closed set in M (cf. [6]). By [5, Theorem 3], the Killing field Z is transverse to $\mathscr{F}$ everywhere on A. As Z preserves $\mathscr{F}$, the set A is also open in M. Hence we have $A = M$ by the connectedness of M. The rest of the statement follows from [3, Theorem 3.1].

4. Applications. First note that the assumption on the compactness of M cannot be removed. In fact, let $(\boldsymbol{R}^n, \mathscr{F}, g_0)$ be a totally geodesic foliation by hyperplanes on the n-dimensional Euclidean space $(\boldsymbol{R}^n, g_0)$. Then a Killing field generating a rotation does not preserve $\mathscr{F}$.

Let $(M, \mathscr{F}, g)$ and Z be as in Theorem. In this section, we always assume that $\mathscr{F}$ is transversely orientable. We also use the same notations as in § 2. Thus N is a unit vector field on M perpendicular to $\mathscr{F}$.

LEMMA 6. *The horizontal part* $\mathscr{H}(Z)$ *of* Z *on* $\tilde{M}$ *is the canonical lifting of a Killing field on* $\tilde{L}$ *to* $\tilde{M}$.

PROOF. Let E be a vector field on $\tilde{L}$. Then, by Lemma 3, (3), we have $N\langle Z, E\rangle = -fE(\langle Z, N\rangle/f)$. By Lemma 5, the function $\langle Z, N\rangle/f$ is constant on each leaf of $\tilde{\mathscr{F}}$. Thus we have $E(\langle Z, N\rangle/f) = 0$. This shows that $\mathscr{H}(Z)$ is the canonical lifting of a Killing field on $\tilde{L}$ by Lemma 3, (1).

The following proposition imposes a strong restriction on the horizontal part $\mathscr{H}(Z)$ of a Killing field Z.

PROPOSITION 1. *If there is a point* p *with* $\mathscr{H}_p(Z)=0$, *then* $Z \in \Gamma(T\mathscr{F})$ *or* $\mathscr{F}$ *is without holonomy.*

PROOF. We denote the flow of Z (resp. N) on M by z_t (resp. n_s). Note that $[N, Z] = 0$. Thus we have $n_s(z_t(p)) = z_t(n_s(p))$. Suppose $Z_p = 0$. Then, for all s and $t \in \boldsymbol{R}$, we have $n_s(p) = z_t(n_s(p))$. This implies that $Z_{n_s(p)} = 0$. As the set $\{n_s(p) \mid s \in \boldsymbol{R}\}$ intersects all leaves of $\mathscr{F}$ (see [3, Lemma 1.9]), the Killing field Z has zeros on each leaf of $\mathscr{F}$. By Theorem, the Killing field Z preserves $\mathscr{F}$. Thus Z must be tangent to $\mathscr{F}$ everywhere on M. Now suppose $Z_p \neq 0$. Then Z is perpendicular to $\mathscr{F}$ at p. By Lemma 6, the vector field $\mathscr{H}(Z)$ is zero on the set $\{n_s(p) \mid s \in \boldsymbol{R}\}$. If $Z = 0$ at $n_s(p)$ for some s, then, by the above argument, the Killing field Z is tangent to $\mathscr{F}$ everywhere on M. This contradicts the fact that Z is perpendicular to $\mathscr{F}$ at p. Thus $Z \neq 0$ on $\{n_s(p) \mid s \in \boldsymbol{R}\}$. Hence Z is perpendicular to $\mathscr{F}$ on $\{n_s(p) \mid s \in \boldsymbol{R}\}$ by Lemma 6. As $\{n_s(p) \mid s \in \boldsymbol{R}\} \cap L \neq \emptyset$ for all $L \in \mathscr{F}$, we have $z_t(L) \neq L$ for some $t \in \boldsymbol{R}$. This shows that $\mathscr{F}$ is without holonomy.

Now we consider the relation between $\mathscr{F}$ and the identity component $I(M, g)_0$ of the isometry group of the closed Riemannian manifold (M, g). The following is a direct consequence of Corollary (cf. [5, Proposition]).

PROPOSITION 2. *Let* $(M, \mathscr{F}, g)$ *and* L_0 *be as in Corollary. If* $\dim I(M, g)_0 = \dim I(L_0, g|L_0)_0 + 1$, *then all leaves of* $\mathscr{F}$ *are isometric to* $(L_0, g|L_0)$, *hence, in particular, all leaves of* $\mathscr{F}$ *are compact.*

In fact, if we choose a Killing field Z on M corresponding to the term "+1", then Z satisfies the assumptions of Corollary.

Finally we prove the following (cf. [3, Proposition 3.2]).

PROPOSITION 3. *Let $(M, \mathscr{F}, g, N)$ be a codimension-one totally geodesic foliation of a closed manifold. Suppose that there is a Killing field Z with $\langle N, Z\rangle > 0$ on M. Then, for any Killing field Y on M, there is a constant C with $\langle N, Y\rangle = C\langle N, Z\rangle$.*

PROOF. By Theorem and the assumption on Z, the foliation $\mathscr{F}$ is defined by a non-vanishing smooth closed one-form. Thus, by [2], the following two cases occur:

(i) All leaves of $\mathscr{F}$ are compact.

(ii) All leaves of $\mathscr{F}$ are dense in M.

Case (i). Proposition 3 follows from [3, Proposition 3.2].

Case (ii). Fix a point p of M. Then there is a constant c with $\langle N, Y\rangle_p = c\langle N, Z\rangle_p$. Denote by L_p the leaf of $\mathscr{F}$ through p. Then $Y - cZ$ is tangent to $\mathscr{F}$ on L_p, because $Y - cZ$ preserves $\mathscr{F}$ by Theorem. Thus $\langle Y - cZ, N\rangle = 0$ on L_p. As L_p is dense in M, the smooth function $\langle Y - cZ, N\rangle = 0$ on M.

REMARK. The foliations appearing in Proposition 3 are Riemannian foliations (cf. Kamber and Tondeur [8]).

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