Tôhoku Math. Journ.
35 (1983), 321-339.

STIEFEL-WHITNEY HOMOLOGY CLASSES OF Z_2-POINCARÉ-EULER SPACES

AKINORI MATSUI

(Received December 14, 1981, revised June 14, 1982)

1. Introduction and the statement of results. Let K be a simplicial complex. It is said to be totally n-dimensional if for each $\sigma \in K$ there exists an n-dimensional simplex $\tau \in K$ such that $\sigma \prec \tau$ or $\sigma = \tau$. A polyhedron X is totally n-dimensional if so is a triangulation K of X. (See Akin [1].) A totally n-dimensional locally finite simplicial complex K is an n-dimensional Z_2-Euler complex if there exists a totally $(n-1)$-dimensional subcomplex L such that

1) The cardinality of $\{\tau \in L \mid \sigma \prec \tau\}$ is even for each $\sigma \in L$.
2) The cardinality of $\{\tau \in K \mid \sigma \prec \tau\}$ is odd for each $\sigma \in L$.
3) The cardinality of $\{\tau \in K \mid \sigma \prec \tau\}$ is even for each $\sigma \in K - L$.

We usually denote ∂K instead of L. A polyhedron X is Z_2-Euler if so is a triangulation K of X. Let $\partial X = |\partial K|$. A compact n-dimensional Z_2-Euler space X is said to be closed if ∂X is empty. Examples of Z_2-Euler spaces are PL-manifolds, Z_2-homology manifolds, complex analytic spaces and so on. (See Sullivan [16].)

Let K be a triangulation of a Z_2-Euler space X. Then the k-th Stiefel-Whitney homology class $s_k(X)$ is defined as the k-skelton $\bar{K}^k$ of the first barycentric subdivision $\bar{K}$ of K. (See Akin [1], Halperin and Toledo [7], Sullivan [16].) Since a differentiable manifold M has a triangulation, the k-th Stiefel-Whitney homology class $s_k(M)$ can be defined as above. Whitney [19] announced that the k-th Stiefel-Whitney homology class $s_k(M)$ of an n-dimensional differentiable manifold M is the Poincaré dual of the $(n-k)$-th Stiefel-Whitney class $w^{n-k}(M)$. Its proof was outlined by Cheeger [5] and given by Halperin and Toledo [7]. Taylor [18] generalized it to the case of Z_2-homology manifolds. This paper will give another proof of this result.

We will study the case of Z_2-Poincaré-Euler spaces. An n-dimensional Z_2-Euler space X is called an n-dimensional Z_2-Poincaré-Euler space if the cap products $[X]_\cap : H^*(X; Z_2) \to H_*^{\inf}(X, \partial X; Z_2)$ and $[X]_\cap : H^*(X, \partial X; Z_2) \to H_*^{\inf}(X; Z_2)$ are isomorphisms. Here $H_*^{\inf}$ is the homology theory of infinite chains.

Let X be an n-dimensional Z_2-Poincaré-Euler space. Define a cohomology class U_X in $H^*(X \times X, \partial X \times X; Z_2)$ as the Poincaré dual of $\Delta_*[X]$, where Δ is the diagonal map. Then $[X \times X] \cap U_X = \Delta_*[X]$. Define the Stiefel-Whitney class $w^*(X)$ by $w^*(X) = (\mathrm{Sq}\, U_X)/[X]$. There exists a proper PL-embedding $\varphi: (X, \partial X) \to (\boldsymbol{R}^\alpha_+, \partial \boldsymbol{R}^\alpha_+)$ for α sufficiently large, where $\boldsymbol{R}^\alpha_+ = \{(x_1, x_2, \cdots, x_\alpha) \mid x_\alpha \geqq 0\}$. (See Hudson [10].) Suppose that R is a regular neighborhood of X in $\boldsymbol{R}^\alpha_+$. Put $\tilde{R} = R \cap \partial \boldsymbol{R}^\alpha_+$ and $\bar{R} = \mathrm{cl}\,(\partial R - \tilde{R})$. Regard φ as a proper embedding from $(X, \partial X)$ to $(R, \tilde{R})$. We also call $(R; \tilde{R}, \bar{R}; \varphi)$ a regular neighborhood of X in $\boldsymbol{R}^\alpha_+$. We will define homomorphisms

$$e_\varphi: \mathfrak{N}_*(R, \bar{R}) \to Z_2 \quad \text{and} \quad \tilde{e}_\varphi: \mathfrak{N}_*(R, \bar{R}) \to Z_2\,, \quad \text{where} \quad \mathfrak{N}_*(R, \bar{R})$$

is the unoriented differentiable bordism group. We need the following:

TRANSVERSALITY THEOREM (Buoncristiano, Rourke and Sanderson [2] and Rourke and Sanderson [14]). *Let M and N be* PL-*manifolds. Suppose that $f: (M, \partial M) \to (N, \partial N)$ is a locally flat proper embedding and that X is a closed subpolyhedron in N. If $f(\partial M) \cap X = \varnothing$ or if $(\partial N, \partial N \cap X)$ is collared in (N, X) and $\partial N \cap X$ is block transverse to $f \mid \partial M: \partial M \to \partial N$, then there exists an embedding $g: M \to N$ ambient isotopic to f relative to ∂N such that X is block transverse to g.*

Let $f: (M, \partial M) \to (R, \bar{R})$ be in $\mathfrak{N}_*(R, \bar{R})$. There exists an embedding $g: (M, \partial M) \to (R \times D^\beta, \bar{R} \times D^\beta)$ for β sufficiently large, such that $g \simeq f \times \{0\}$ and that $(\varphi \times \mathrm{id})(X \times D^\beta)$ is block transverse to g by Transversality Theorem. Let $Y = (\varphi \times \mathrm{id})^{-1} \circ g(M)$. Then Y is a closed Z_2-Euler space with a normal block bundle ν in $X \times D^\beta$. Define $e_\varphi(f, M)$ as the modulo 2 Euler number $e(Y)$ of Y. Let $\psi: Y \to X \times D^\beta$ be the inclusion. Define $\tilde{e}_\varphi(f, M) = \langle \psi^* w^*(X \times D^\beta) \cup \bar{w}(\nu), [Y] \rangle$, where $\bar{w}(\nu)$ is the cohomology class determined by $w^*(\nu) \cup \bar{w}(\nu) = 1$. Now define a homomorphism $o_\varphi: \mathfrak{N}_*(R, \bar{R}) \to Z_2$ by $o_\varphi = \tilde{e}_\varphi - e_\varphi$. We can state the main theorem of this paper as follows:

THEOREM. *Let X be an n-dimensional Z_2-Poincaré-Euler space. Take a regular neighborhood $(R; \tilde{R}, \bar{R}; \varphi)$ of X in $\boldsymbol{R}^\alpha_+$. Then $[X] \cap w^*(X) = s_*(X)$ if and only if $o_\varphi = 0$.*

A totally n-dimentional polyhedron X is an n-dimensional Z_2-homology manifold if there exist a locally finite triangulation K of X and a totally $(n-1)$-dimensional subcomplex L such that

1) $H_*(Lk(\sigma; L); Z_2) = H_*(S^{n-i-2}; Z_2)$ for each i-simplex $\sigma \in L$.
2) $H_*(Lk(\sigma; K); Z_2) = H_*(pt; Z_2)$ for each i-simplex $\sigma \in L$.
3) $H_*(Lk(\sigma; K); Z_2) = H_*(S^{n-i-1}; Z_2)$ for each i-simplex $\sigma \in K - L$.

Theorem is applied to prove the following generalization of Whitney-

Cheeger-Halperin and Toledo theorem.

COROLLARY. *Let X be an n-dimensional Z_2-homology manifold with or without boundary. Then* $[X] \cap w^*(X) = s_*(X)$.

We remark that Taylor [18] proved the corollary for Z_2-homology manifolds without boundaries.

In Section 2, we study Stiefel-Whitney homology classes and the graded bordism theory of compact Z_2-Euler spaces. The structure of the graded bordism group of compact Z_2-Euler spaces is given in Proposition 2.3. The ungraded bordism theory was studied by Akin [1]. In Section 3, we study the Stiefel-Whitney classes of block bundles via the bordism group of compact Z_2-Euler spaces. The result will be used in Section 6. In Section 4, we study regular neighborhoods and the Stiefel-Whitney classes. These are necessary for calculation in Sections 5 and 6. In order to prove the above corollary, we need Propositions 4.6 and 4.7. In Section 5, we give a characterization of Stiefel-Whitney classes via the unoriented differentiable bordism group. In Section 6, we give a characterization of Stiefel-Whitney homology classes via the unoriented differentiable bordism group.

Our Theorem follows from Lemmas 5.1 and 6.1.

For completeness we add an appendix, where we give a detailed proof of Transversality Theorem by following the outline given in Buoncristiano, Rourke and Sanderson [2].

2. Stiefel-Whitney homology classes and bordism groups of Z_2-Euler spaces.

Let K be a simplicial complex. The barycentric subdivision $\bar{K}$ of K is defined by

$$\bar{K} = \{(\sigma_0, \cdots, \sigma_p) \mid \sigma_0 \prec \cdots \prec \sigma_p, \sigma_i \in K\} .$$

We denote the k-skelton of $\bar{K}$ by $\bar{K}^k$. Then we have the following:

PROPOSITION 2.1. *Let K be a Z_2-Euler complex. Then $\bar{K}^k$ is a Z_2-Euler complex such that* $\partial\bar{K}^k = \overline{\partial K}^{k-1}$.

In order to prove Proposition 2.1, we need the following:

LEMMA 2.1. *Let K be a totally n-dimensional locally finite simplicial complex. If $b \in \bar{K}^{p-1}$, then the cardinality of $\{a \in \bar{K} - \bar{K}^p \mid a \succ b\}$ is even.*

PROOF. If $p = n$, then $\bar{K} - \bar{K}^p$ is empty. Thus we may assume that $p < n$. Let $a = \langle\sigma_0, \cdots, \sigma_s\rangle \in \bar{K} - \bar{K}^p$ and let $b = \langle\tau_0, \cdots, \tau_t\rangle \in \bar{K}^{p-1}$. Then $s > t + 1$. Since the cardinality of $\{\sigma \in K \mid \sigma_0 \prec \sigma \prec \sigma_1\}$ is even for each $\langle\sigma_0, \sigma_1\rangle \in \bar{K}$, we have that the cardinality of $\{a \in \bar{K} - \bar{K}^p | a \succ b\}$ is even for $b \in \bar{K}^{p-1}$. q.e.d.

PROOF OF PROPOSITION 2.1. Note that the cardinality of $\{a \in \bar{K} | a \succ b\}$ equals the sum of the cardinalities of $\{a \in \bar{K}^p | a \succ b\}$ and $\{a \in \bar{K} - \bar{K}^p | a \succ b\}$ for $b \in \bar{K}$. By Lemma 2.1, it follows that the cardinalities $\{a \in \bar{K} | a \succ b\}$ and $\{a \in \bar{K}^p | a \succ b\}$ are congruent modulo 2 for $b \in \bar{K}^{p-1}$. Then $\bar{K}^p$ is a Z_2-Euler complex such that $\partial\bar{K}^p = \overline{\partial\bar{K}^{p-1}}$. q.e.d.

We need the following proposition to prove Corollary 2.2 as well as Lemmas 3.2 and 3.3 and 6.1.

PROPOSITION 2.2. (Halperin and Toledo [8]). *Let X and Y be Z_2-Euler spaces. Then $s_k(X \times Y) = \sum_{p=0}^{k} s_p(X) \times s_{k-p}(Y)$.*

In [8], Z_2-Euler spaces without boundaries are studied but we can prove Proposition 2.2, using the same method as in [8].

Let $\{\mathfrak{B}_n, \partial\}$ be the bordism theory of compact Z_2-Euler spaces. Then $\{\mathfrak{B}_n, \partial\}$ is a homology theory. (See Akin [1].) Let (A, B) be a pair of polyhedra. Define a homomorphism $s: \mathfrak{B}_n(A, B) \to H_0(A, B; Z_2) + H_1(A, B; Z_2) + \cdots + H_n(A, B; Z_2)$ by $s(\varphi, X) = \sum_{i=0}^{n} \varphi_* s_i(X)$. Then s is well defined by Proposition 2.1. The following holds:

PROPOSITION 2.3. *The homomorphism $s: \mathfrak{B}_n(A, B) \to H_0(A, B; Z_2) + H_1(A, B; Z_2) + \cdots + H_n(A, B; Z_2)$ is an isomorphism.*

PROOF. Put $h_n(A, B) = H_0(A, B; Z_2) + H_1(A, B; Z_2) + \cdots + H_n(A, B; Z_2)$. Define the boundary operator $\partial_h: h_n(A, B) \to h_{n-1}(B)$ by that of the ordinary homology theory. Note that $\{h_n, \partial_h\}$ and $\{\mathfrak{B}_n, \partial\}$ are homology theories with compact supports and that s is a homomorphism from $\mathfrak{B}_n(A, B)$ to $h_n(A, B)$ such that $\partial_h \circ s = s \circ \partial$. Since $h_n(pt) = Z_2$ and $\mathfrak{B}_n(pt) = Z_2$, where pt is the space of one point, the homomorphism $s: \mathfrak{B}_n(A, B) \to h_n(A, B)$ is an isomorphism. (cf. See Spanier [15].) q.e.d.

This proposition implies directly the following:

COROLLARY 2.1. *Let (φ_1, X_1) and (φ_2, X_2) be in $\mathfrak{B}_n(A, B)$. Then (φ_1, X_1) is cobordant to (φ_2, X_2) in $\mathfrak{B}_n(A, B)$ if and only if $(\varphi_1)_* s_i(X_1) = (\varphi_2)_* s_i(X_2)$ in $H_i(A, B; Z_2)$ for all i.*

REMARK. Akin [1] showed this in the case of ungraded bordism groups.

Let $S^1 \vee S^1$ be the one point union of two circles. Then $S^1 \vee S^1$ is a 1-dimensional Z_2-Euler space such that the modulo 2 Euler number $e(S^1 \vee S^1) = 1$. The following holds:

COROLLARY 2.2. *Let* (φ, X) *be in* $\mathfrak{B}_n(A, B)$. *If* $\varphi_*[X] = 0$ *in* $H_n(A, B; Z_2)$, *then there exists* (φ, Y) *in* $\mathfrak{B}_{n-1}(A, B)$ *such that* (φ, X) *is cobordant to* $(\psi \circ \pi, Y \times (S^1 \vee S^1))$, *where* $\pi: Y \times (S^1 \vee S^1) \to Y$ *is the projection.*

PROOF. Let $\bar{K}^{n-1}$ be the $(n-1)$-skelton of the barycentric subdivision $\bar{K}$ of a triangulation K of X. Put $|\bar{K}^{n-1}| = X^{n-1}$. Then $\varphi_* s_{n-1}(X) = (\varphi | X^{n-1})_*[X^{n-1}]$. Let $p: X^{n-1} \times (S^1 \vee S^1) \to X^{n-1}$ be the projection. Then $\varphi_* s_{n-1}(X) = (\varphi | X^{n-1})_* \circ p_* s_{n-1}(X^{n-1} \times (S^1 \vee S^1))$ by Proposition 2.2. By induction, there exists (ψ, Y) in $\mathfrak{B}_{n-1}(A, B)$ such that $\varphi_* s_i(X) = \psi_* \circ \pi_* s_i(Y \times (S^1 \vee S^1))$ for $0 \leqq i \leqq n$, where $\pi: Y \times (S^1 \vee S^1) \to Y$ is the projection. By Corollary 2.1, we have (φ, X) is cobordant to $(\psi \circ \pi, Y \times (S^1 \vee S^1))$. q.e.d.

We need the following to prove Lemma 3.3.

PROPOSITION 2.4. (Blanton and McCrory [4]). *The k-th Stiefel-Whitney homology class* $s_k(\boldsymbol{P}^n)$ *of the n-dimensional real projective space* $\boldsymbol{P}^n$ *is equal to* ${}_{n+1}C_{k+1} j_*[\boldsymbol{P}^k]$, *where* $j: \boldsymbol{P}^k \to \boldsymbol{P}^n$ *is the canonical inclusion.*

3. Characterization of Stiefel-Whitney classes of block bundles via the bordism group of Z_2-Euler spaces. Let $\xi = (E(\xi), K, \iota)$ be a k-block bundle over a simplicial complex K. Then there exist PL-homeomorphisms $\varphi_\sigma: \sigma \times D^k \to E(\sigma)$, called the charts, for all σ in K. (See Rourke and Sanderson [14].) Put $\dot{E}(\xi) = \cup \varphi_\sigma(\sigma \times \partial D^k)$. Then $\dot{\xi} = (\dot{E}(\xi), K)$ is called the sphere bundle associated with ξ.

Let $\xi = (E(\xi), K, \iota_K)$ and $\eta = (E(\eta), L, \iota_L)$ be k-block bundles over simplicial complexes K and L. A map $(\bar{h}, h): (E(\xi), K) \to (E(\eta), L)$ is a bundle map if

1) $h: K \to L$ is a simplicial map,

2) $\iota_L \circ h = \bar{h} \circ \iota_K$, and

3) for each σ in K, there exist charts $\varphi_1: \sigma \times D^k \to E(\sigma)$ and $\varphi_2: h(\sigma) \times D^k \to E(h(\sigma))$ such that $\bar{h} \circ \varphi_1 = \varphi_2 \circ (h | \sigma \times \mathrm{id})$, where id is the identity of D^k.

Let $\xi = (E(\xi), X, \iota_X)$ and $\eta = (E(\eta), Y, \iota_Y)$ be k-block bundles over polyhedra X and Y. A map $(\bar{h}, h): (E(\xi), X) \to (E(\eta), Y)$ is a bundle map if there exist simplicial complexes K and L such that $|K| = X$, $|L| = Y$ and that $(\bar{h}, h): (E(\xi), K) \to (E(\eta), L)$ is a bundle map.

REMARK. If a map $(\bar{h}, h): (E(\xi), X) \to (E(\eta), Y)$ is a bundle map, then $\xi = h^*\eta$. Conversely, if $\xi = h^*\eta$, then there exists a bundle map $(\bar{h}, h): (E(\xi), X) \to (E(\eta), Y)$. (See [14].)

Let $\xi=(E(\xi), A, \iota)$ be an n-block bundle over a locally compact polyhedron A. Define $\bar{E}(\xi)$ to be the total space of the sphere bundle associated with ξ. Then we will define a homomorphism $e_\xi: \mathfrak{B}_*(E(\xi), \bar{E}(\xi)) \to Z_2$, where $\mathfrak{B}_*(E(\xi), \bar{E}(\xi))$ is the bordism group of compact Z_2-Euler spaces. Let R be a regular neighborhood of A embedded properly in $\boldsymbol{R}^\alpha$ for α sufficiently large. Let $i: A \subset R$ be the inclusion and let $p: R \to A$ be the retraction. Suppose that $p^*\xi=(E(p^*\xi), R, \iota_R)$ is the induced bundle. Then there exist bundle maps $(\bar{i}, i): (E(\xi), A) \to (E(p^*\xi), R)$ and $(\bar{p}, p): (E(p^*\xi), R) \to (E(\xi), A)$. For each (φ, X) in $\mathfrak{B}_*(E(\xi), \bar{E}(\xi))$, there exists an embedding $\tilde{\varphi}: (X, \partial X) \to (E(p^*\xi), \bar{E}(p^*\xi))$ such that $\tilde{\varphi} \simeq \bar{i} \circ \varphi$. By Transversality Theorem, we may assume that $\tilde{\varphi}(X)$ is block transverse to $\iota_R: R \to E(p^*\xi)$. Then we define $e_\xi(\varphi, X)$ as the modulo 2 Euler number $e(\tilde{\varphi}^{-1}(\iota_R(R)))$ of $\tilde{\varphi}^{-1}(\iota_R(R))$. We need the following to prove Lemma 3.3:

LEMMA 3.1. *Let $(\bar{h}, h): (E(\xi_1), A_1) \to (E(\xi_2), A_2)$ be a bundle map. Then $e_{\xi_1}(\varphi, X)=e_{\xi_2}(\bar{h} \circ \varphi, X)$ for each (φ, X) in $\mathfrak{B}_*(E(\xi_1), \bar{E}(\xi_1))$.*

PROOF. Let $i_k: A_k \subset R_k$ be the inclusions to regular neighborhoods embedded properly in $\boldsymbol{R}^\alpha$, for α sufficiently large, such that there exists an inclusion $h_R: R_1 \subset R_2$ with $i_2 \circ h \simeq h_R \circ i_1$. Let $p_k: R_k \to A_k$ be the retractions for $k=1,2$. Suppose that $p_k^*\xi_k=(E(p_k^*\xi_k), R_k, \bar{\iota}_k)$ are the induced bundles for $k=1,2$. Then there exists the following bundle maps

$$(\bar{i}_k, i_k): (E(\xi_k), A_k) \to (E(p_k^*\xi_k), R_k) ,$$
$$(\bar{p}_k, p_k): (E(p_k^*\xi_k), R_k) \to (E(\xi_k), A_k) ,$$

for $k=1,2$, and

$$(\bar{h}_R, h_R): (E(p_1^*\xi_1), R_1) \to (E(p_2^*\xi_2), R_2) ,$$

such that $\bar{h}_R$ is an embedding. For each (φ, X) in $\mathfrak{B}_*(E(\xi_1), \bar{E}(\xi_1))$, there exists an embedding $\tilde{\varphi}: (X, \partial X) \to (E(p_1^*\xi_1), \bar{E}(p_1^*\xi_1))$ such that $\tilde{\varphi} \simeq \bar{i}_1 \circ \varphi$ and that $\tilde{\varphi}(X)$ is block transverse to $\bar{\iota}_1: R_1 \to E(p_1^*\xi_1)$. Then $\bar{h}_R \circ \tilde{\varphi}(X)$ is block transverse to $\bar{\iota}_2: R_2 \to E(p_2^*\xi_2)$. Noting that $\bar{h}_R \circ \tilde{\varphi} \simeq i_2 \circ (\bar{h} \circ \varphi)$, we have $e_{\xi_2}(\bar{h} \circ \varphi, X)=e((\bar{h}_R \circ \tilde{\varphi})^{-1}(\bar{\iota}_2(R_2)))$. Since $\bar{\iota}_1(R_1)=\bar{h}_R^{-1}(\bar{\iota}_2(R_2))$ and $e_{\xi_1}(\varphi, X)=e(\tilde{\varphi}^{-1}(\bar{\iota}_1(R_1)))$, it follows that $e_{\xi_1}(\varphi, X)=e_{\xi_2}(\bar{h} \circ \varphi, X)$. q.e.d.

LEMMA 3.2. *Let $\xi=(E, A, \iota)$ be an n-block bundle over a locally compact polyhedron A. Then there exists a unique cohomology class $\Phi(\xi)$ in $H^*(E, \bar{E}; Z_2)$ satisfying $\langle \Phi(\xi), \varphi_* s_*(X) \rangle = e_\xi(\varphi, X)$ for each (φ, X) in $\mathfrak{B}_*(E, \bar{E})$.*

PROOF. First we will prove the existence of $\Phi(\xi)$. Let $\Phi^i(\xi)=0$ in

$H^i(E, \bar{E}; Z_2)$ for $i = 0, 1, \cdots, n-1$. Define a homomorphism $\tilde{\Phi}^n$: $\mathfrak{B}_n(E, \bar{E}) \to Z_2$ by $\tilde{\Phi}^n(\varphi, X) = e_\xi(\varphi, X)$. If $\varphi_*[X] = 0$ in $H_n(E, \bar{E}; Z_2)$, then by Corollary 2.2 there exists (ψ, Y) in $\mathfrak{B}_{n-1}(E, \bar{E})$ such that (φ, X) is cobordant to $(\psi \circ \pi, Y \times (S^1 \vee S^1))$, where $\pi: (Y \times (S^1 \vee S^1)) \to Y$ is the projection. Hence $e_\xi(\varphi, X) = e_\xi(\psi \circ \pi, Y \times (S^1 \vee S^1)) = e_\xi(\psi, Y) \cdot e(S^1 \vee S^1) = 0$. Thus we can define $\Phi^n(\xi)$ as the cohomology class determined by $\tilde{\Phi}^n$. As an induction hypothesis, we may assume that $\Phi^n(\xi), \cdots, \Phi^{n+i}(\xi)$ are determined so that $\langle \Phi^{n+p}(\xi), \varphi_*[X] \rangle = \sum_{j=0}^{p-1} \langle \Phi^{n+j}(\xi), \varphi_* s_{n+j}(X) \rangle + e_\xi(\varphi, X)$ for $p \leqq i$. Define a homomorphism $\tilde{\Phi}^{n+i+1}$: $\mathfrak{B}_{n+i+1}(E, \bar{E}) \to Z_2$ by $\tilde{\Phi}^{n+i+1}(\varphi, X) = \sum_{j=0}^{i} \langle \Phi^{n+j}(\xi), \varphi_* s_{n+j}(X) \rangle + e_\xi(\varphi, X)$. Suppose that $\varphi_*[X] = 0$. By Corollary 2.2, there exists (ψ, Y) in $\mathfrak{B}_{n+i}(E, \bar{E})$ such that (φ, X) is cobordant to $(\psi \circ \pi, Y \times (S^1 \vee S^1))$, where $\pi: Y \times (S^1 \vee S^1) \to Y$ is the projection. Note that $\pi_*(s_{n+j}(Y \times (S^1 \vee S^1))) = s_{n+j}(Y)$ for $j = 0, \cdots, i$, by Proposition 2.2 and that $e_\xi(\psi \circ \pi, Y \times (S^1 \vee S^1)) = e_\xi(\psi, Y)$. Then $\tilde{\Phi}^{n+i+1}(\varphi, X) = \sum_{j=0}^{i} \langle \Phi^{n+j}(\xi), \psi_* s_{n+j}(Y) \rangle + e_\xi(\psi, Y)$. Since $\langle \Phi^{n+i}(\xi), \varphi_* s_{n+i}(Y) \rangle = \sum_{j=0}^{i-1} \langle \Phi^{n+j}(\xi), \psi_* s_{n+j}(Y) \rangle + e_\xi(\psi, Y)$, it follows that $\tilde{\Phi}^{n+i+1}(\varphi, X) = 0$. Hence we can define $\Phi^{n+i+1}(\xi)$ as the cohomology class determined by $\tilde{\Phi}^{n+i+1}$. By induction, cohomology classes $\Phi^k(\xi)$ can be defined as above for every k, so that the following is satisfied, $\langle \Phi^{n+k}(\xi), \varphi_*[X] \rangle = \sum_{j=0}^{k-1} \langle \Phi^{n+j}(\xi), \varphi_* s_{n+j}(X) \rangle + e_\xi(\varphi, X)$ for each (φ, X) in $\mathfrak{B}_{n+k}(E, \bar{E})$. Put $\Phi(\xi) = \sum \Phi^k(\xi)$. Then for each (φ, X) in $\mathfrak{B}_m(E, \bar{E})$, it follows that

$$\begin{aligned} \langle \Phi(\xi), \varphi_* s_*(X) \rangle &= \sum_{k=0}^{m} \langle \Phi^k(\xi), \varphi_* s_k(X) \rangle \\ &= \langle \Phi^m(\xi), \varphi_* s_m(X) \rangle + \sum_{k=0}^{m-1} \langle \Phi^k(\xi), \varphi_* s_k(X) \rangle \\ &= e_\xi(\varphi, X) . \end{aligned}$$

Hence there exists a cohomology class $\Phi(\xi)$ satisfying the assumption.

The uniqueness of $\Phi(\xi)$ can be proved as follows. Setting $\Phi = \Phi^0 + \Phi^1 + \cdots + \Phi^\alpha$ in $H^*(E, \bar{E}; Z_2)$, suppose that $\langle \Phi, \varphi_* s_*(X) \rangle = 0$ for each (φ, X) in $\mathfrak{B}_*(E, \bar{E})$. Clearly $\Phi^0 = 0$. Suppose that $\Phi^0 = 0, \Phi^1 = 0, \cdots, \Phi^k = 0$. Since $\langle \Phi, \varphi_* s_*(X) \rangle = 0$ for (φ, X) in $\mathfrak{B}_{k+1}(E, \bar{E})$, it follows that $\langle \Phi^{k+1}, \varphi_*[X] \rangle = 0$ and $\Phi^{k+1} = 0$. Hence $\Phi = 0$ if $\langle \Phi, \varphi_* s_*(X) \rangle = 0$ for each (φ, X) in $\mathfrak{B}_*(E, \bar{E})$. This means that the cohomology class $\Phi(\xi)$ satisfying the assumption is unique. q.e.d.

Let $\xi = (E, X, \iota)$ be a block bundle. Let $\Phi(\xi)$ be the cohomology class defined as above. Define $\tilde{w}(\xi)$ by $\tilde{w}(\xi) = \iota^*(U_\xi \cup)^{-1}\Phi(\xi)$, where $\iota^*(U_\xi \cup)^{-1}$: $H^*(E, \bar{E}; Z_2) \to H^*(X; Z_2)$ is the Thom isomorphism of ξ. Then the following holds:

LEMMA 3.3. *If ξ is the block bundle induced by a vector bundle* *over a locally compact polyhedron X, then the cohomology class $\tilde{w}(\xi)$ coincides with the dual Stiefel-Whitney class $\bar{w}(\xi)$ of $w^*(\xi)$.*

In order to prove Lemma 3.3, it is sufficient to prove the following (cf. [12]):

1) Given a block bundle $\xi=(E(\xi), A, \iota)$ and a map $h\colon B\to A$, where A and B are locally compact polyhedra, we have $\tilde{w}(h^*\xi)=h^*\tilde{w}(\xi)$.

2) For block bundles ξ_1 and ξ_2 over locally compact polyhedra, we have $\tilde{w}(\xi_1)\times\tilde{w}(\xi_2)=\tilde{w}(\xi_1\times\xi_2)$.

3) For the canonical 1-disk bundle η^1 over the projective space $\boldsymbol{P}^n$, we have $\tilde{w}(\eta^1)=1+\alpha+\cdots+\alpha^n$, for the generator α of $H^1(\boldsymbol{P}^n; Z_2)$.

PROOF. 1) Let $h^*\xi=(E(h^*\xi), B, \iota_B)$ be the induced bundle. There exists a bundle map $(\bar{h}, h)\colon (E(h^*\xi), B)\to(E(\xi), A)$. Since $(\bar{h}\circ\varphi, X)$ is in $\mathfrak{B}_*(E(\xi), \bar{E}(\xi))$ for (φ, X) in $\mathfrak{B}_*(E(h^*\xi), \bar{E}(h^*\xi))$ and $e_\xi(\bar{h}\circ\varphi, X)=e_{h^*\xi}(\varphi, X)$ by Lemma 3.1, it follows that $\langle\Phi(\xi), (\bar{h}\circ\varphi)_*s_*(X)\rangle=e_{h^*\xi}(\varphi, X)$. Note that $\Phi(h^*\xi)=\bar{h}^*\Phi(\xi)$ by Lemma 3.2. Since $\tilde{w}(h^*\xi)=\iota_B^*(U_{h^*\xi}\cup)^{-1}\bar{h}^*\Phi(\xi)$ and $\bar{h}\circ\iota_B\simeq\iota\circ h$, it follows that $\tilde{w}(h^*\xi)=h^*\circ\iota^*(U_\xi\cup)^{-1}\Phi(\xi)$, hence $\tilde{w}(h^*\xi)=h^*\tilde{w}(\xi)$.

2) Let $\xi_i=(E_i, B_i, \iota_i)$ be block bundles over locally compact polyhedra B_i for $i=1, 2$. Let $\bar{E}_i$ be the total space of the sphere bundle associated with ξ_i. Since $(\varphi_1\times\varphi_2)_*s_*(X_1\times X_2)=(\varphi_1)_*s_*(X_1)\times(\varphi_2)_*s_*(X_2)$ for (φ_i, X_i) in $\mathfrak{B}_*(E_i, \bar{E}_i)$, by Proposition 2.2, it follows that

$$\begin{aligned}&\langle\Phi(\xi_1)\times\Phi(\xi_2), (\varphi_1\times\varphi_2)_*s_*(X_1\times X_2)\rangle\\&=\langle\Phi(\xi_1), (\varphi_1)_*s_*(X_1)\rangle\langle\Phi(\xi_2), (\varphi_2)_*s_*(X_2)\rangle\\&=e_{\xi_1}(\varphi_1, X_1)\cdot e_{\xi_2}(\varphi_2, X_2)\\&=e_{\xi_1\times\xi_2}(\varphi_1\times\varphi_2, X_1\times X_2)\,.\end{aligned}$$

By the uniqueness of $\Phi(\xi_1\times\xi_2)$, we have $\Phi(\xi_1)\times\Phi(\xi_2)=\Phi(\xi_1\times\xi_2)$, hence $\tilde{w}(\xi_1)\times\tilde{w}(\xi_2)=\tilde{w}(\xi_1\times\xi_2)$.

3) Let $\eta^1=(E^{n+1}, \boldsymbol{P}^n, \iota)$ be the canonical 1-disk bundle over the real projective space. Define $h\colon (E^{n+1}, \partial E^{n+1})\to(\boldsymbol{P}^n, pt)$ by the canonical identification $E^{n+1}/\partial E^{n+1}=\boldsymbol{P}^n$. Then $h_*\colon H_*(E^{n+1}, \partial E^{n+1}; Z_2)\to H_*(\boldsymbol{P}^{n+1}, pt; Z_2)$ is an isomorphism and $h\circ\iota=j_n$, where $j_k\colon \boldsymbol{P}^k\to\boldsymbol{P}^{n+1}$ are the canonical inclusions. Let $\bar{j}_k\colon (E^k, \boldsymbol{P}^{k-1})\to(E^{n+1}, \boldsymbol{P}^n)$ be the canonical inclusions. Then $h_*(\bar{j}_k, E^k)=(j_k, \boldsymbol{P}^k)$. Note that $h_*\colon \mathfrak{B}_k(E^{n+1}, \partial E^{n+1})\to\mathfrak{B}_k(\boldsymbol{P}^{n+1}, pt)$ is an isomorphism by Proposition 2.3. Since $\mathfrak{B}_*(\boldsymbol{P}^{n+1}, pt)$ is generated by $\{(j_k, \boldsymbol{P}^k)\}$, we see that $\mathfrak{B}_*(E^{n+1}, \partial E^{n+1})$ is generated by $\{(\bar{j}_k, E^k)\}$. In order to prove the assertion 3), it is sufficient to prove

$$\langle U_{\eta^1}\cup(\iota^*)^{-1}(1+\alpha+\cdots+\alpha^n), (\bar{j}_k)_*s_*(E^k)\rangle=e_{\eta^1}(\bar{j}_k, E^k)\,.$$

Let β be the generator of $H^1(\boldsymbol{P}^{n+1}; Z_2)$. Then $U_{\eta^1} = h^*\beta$ and $(\iota^*)^{-1}\alpha^i = h^*\beta^i$. Since $h_* \circ (\bar{j}_k)_* s_*(E^k) = (j_k)_* s_*(\boldsymbol{P}^k)$, we have

$$\langle U_{\eta^1} \cup (\iota^*)^{-1}(1 + \alpha + \cdots + \alpha^n), (\bar{j}_k)_* s_*(E^k)\rangle$$
$$= \langle \beta + \beta^2 + \cdots + \beta^{n+1}, (j_k)_* s_*(\boldsymbol{P}^k)\rangle .$$

By Proposition 2.4, it follows that

$$(j_k)_* s_*(\boldsymbol{P}^k) = \sum_{p=0}^{k} {}_{k+1}C_{p+1}(j_p)_*[\boldsymbol{P}^p] .$$

Then

$$\langle U_{\eta^1} \cup (\iota^*)^{-1}(1 + \alpha + \cdots + \alpha^n), (\bar{j}_k)_* s_*(E^k)\rangle$$
$$= \sum_{p=1}^{k} {}_{k+1}C_{p+1} = k .$$

Note that $e_{\eta^1}(\bar{j}_k, E^k) = e(\boldsymbol{P}^{k-1}) = k$. Hence

$$\langle U_{\eta^1} \cup (\iota^*)^{-1}(1 + \alpha + \cdots + \alpha^n), (\bar{j}_k)_* s_*(E^k)\rangle = e_{\eta^1}(\bar{j}_k, E^k) .$$

By the above, we have $\tilde{w}(\eta^1) = 1 + \alpha + \cdots + \alpha^n$. q.e.d.

COROLLARY 3.1. *Let $\nu = (E, M, \iota)$ be the normal block bundle of a proper embedding from a compact triangulated differentiable manifold M into $\boldsymbol{R}_+^\alpha$. Then*

$$\langle U_\nu \cup (\iota^*)^{-1}w^*(M), \varphi_* s_*(X)\rangle = e_\nu(\varphi, X) \quad \textit{for each} \quad (\varphi, X)$$

in the bordism group $\mathfrak{B}_*(E, \bar{E})$ of compact Z_2-Euler spaces, where $\bar{E}$ is the total space of the sphere bundle associated with ν.

PROOF. Since ν is induced by a vector bundle, it follows that $\langle U_\nu \cup (\iota^*)^{-1}\bar{w}(\nu), \varphi_* s_*(X)\rangle = e_\nu(\varphi, X)$ by Lemma 3.3. Since $w^*(M) = \bar{w}(\nu)$, we have

$$\langle U_\nu \cup (\iota^*)^{-1}w^*(M), \varphi_* s_*(X)\rangle = e_\nu(\varphi, X) . \qquad \text{q.e.d.}$$

4. Regular neighborhoods and Stiefel-Whitney classes. Let $(R; \tilde{R}, \bar{R}; \varphi)$ be a regular neighborhood of an n-dimensional Z_2-Poincaré-Euler space X in $\boldsymbol{R}_+^\alpha$. Define a cohomology class $U(\varphi)$ in $H^k(R, \bar{R}; Z_2)$ as the Poincaré dual of $\varphi_*[X]$ in $H_n^{\inf}(R, \tilde{R}; Z_2)$. Then $[R] \cap \varphi^* U(\varphi) = \varphi_*[X]$. The following holds:

PROPOSITION 4.1. *Let $(R; \tilde{R}, \bar{R}; \varphi)$ be a regular neighborhood of an n-dimensional Z_2-Poincaré-Euler space X in $\boldsymbol{R}_+^{n+k}$. Then there exist the following isomorphisms:*

1) $t_1\colon H^i(X; Z_2) \to H^{i+k}(R, \bar{R}; Z_2)$ *defined by* $t_1(\alpha) = U(\varphi) \cup (\varphi^*)^{-1}\alpha$.

2) $t_2\colon H^i(X, \partial X; Z_2) \to H^{i+k}(R, \partial R; Z_2)$ *defined by* $t_2(\alpha) = U(\varphi) \cup (\varphi^*)^{-1}\alpha$.

3) t_3: $H^{\inf}_{i+k}(R, \bar{R}; Z_2) \to H^{\inf}_i(X; Z_2)$ *defined by* $t_3(a) = (\varphi_*)^{-1}(a \cap U(\varphi))$.

4) t_4: $H^{\inf}_{i+k}(R, \partial R; Z_2) \to H^{\inf}_i(X, \partial X; Z_2)$ *defined by* $t_4(a) = (\varphi_*)^{-1}(a \cap U(\varphi))$.

PROOF. Note that the diagram

$$\begin{array}{ccc} H^i(X; Z_2) & \xrightarrow{t_1} & H^{i+k}(R, \bar{R}; Z_2) \\ {\scriptstyle [X]\cap}\downarrow & & \downarrow{\scriptstyle [R]\cap} \\ H^{\inf}_{n-i}(X, \partial X; Z_2) & \xrightarrow{\varphi_*} & H^{\inf}_{n-i}(R, \tilde{R}; Z_2) \end{array}$$

is commutative and that homomorphisms $[X]_\cap$, $[R]_\cap$ and φ_* are isomorphisms. Thus t_1 is an isomorphism.

We can prove 2), 3) and 4) similarly. q.e.d.

Let $(R; \tilde{R}, \bar{R}; \varphi)$ be a regular neighborhood of a Z_2-Poincaré-Euler space X in $\boldsymbol{R}^\alpha_+$. The k-th Stiefel-Whitney class $w^k(\varphi)$ of φ is defined by $w^k(\varphi) = \varphi^* \circ (U(\varphi) \cup)^{-1} \mathrm{Sq}^k U(\varphi)$. The total Stiefel-Whitney class is $w^*(\varphi) = 1 + w^1(\varphi) + \cdots = \varphi^* \circ (U(\varphi) \cup)^{-1} \mathrm{Sq}\, U(\varphi)$. If φ has a normal block bundle ν, then $w^*(\varphi) = w^*(\nu)$. The following gives an alternative definition for $w^*(X)$.

PROPOSITION 4.2. *Let $(R; \tilde{R}, \bar{R}; \varphi)$ be a regular neighborhood of a Z_2-Poincaré-Euler space X in $\boldsymbol{R}^\alpha_+$. Then $w^*(X) \cup w^*(\varphi) = 1$.*

PROOF. Let $r: (R, \tilde{R}) \to (X, \partial X)$ be a deformation retraction. Let $U_X \in H^*(X \times X, \partial X \times X; Z_2)$ and $U_R \in H^*(R \times R, \partial R \times R; Z_2)$ be the diagonal classes of X and R respectively. Note that the cap product $\cap (U(\varphi) \times 1_R)$: $H^{\inf}_*(R \times R, \bar{R} \times R \cup R \times \tilde{R}; Z_2) \to H^{\inf}_*(R \times R, R \times \tilde{R}; Z_2)$ is an isomorphism. Since $[R \times R] \cap ((r \times r)^* U_X \cup (U(\varphi) \times U(\varphi))) = (\varDelta_R)_* \circ \varphi_*[X]$ and $(\varDelta_R)_*[R] \cap (U(\varphi) \times 1_R) = (\varDelta_R)_* \circ \varphi_*[X]$, we have $U_R = (r \times r)^* U_X \cup (1_R \times U(\varphi))$. Since $w^*(R) = 1_R$, we have $\mathrm{Sq}\, U_R = U_R$. Noting $\mathrm{Sq}\, U(\varphi) = r^* w^*(\varphi) \cup U(\varphi)$, we see that $(r \times r)^* \mathrm{Sq}\, U_X \cup (1_R \times r^* w^*(\varphi)) \cup (1_R \times U(\varphi)) = (r \times r)^* U_X \cup (1_R \times U(\varphi))$. Note that the cup product $(1_R \times U(\varphi)) \cup : H^*(R \times R; Z_2) \to H^*(R \times R, R \times \bar{R}; Z_2)$ and $r^*: H^*(R; Z_2) \to H^*(X; Z_2)$ are isomorphisms. Then $\mathrm{Sq}\, U_X \cup (1_X \times w^*(\varphi)) = U_X$. Since $[\mathrm{Sq}\, U_X \cup (1_X \times w^*(\varphi))]/[X] = \mathrm{Sq}\, U_X/[X] \cup w^*(\varphi) = w^*(X) \cup w^*(\varphi)$, we have only to prove $U_X/[X] = 1_X$. Note that $[X] \cap U_X/[X] = (p_2)_*([X \times X] \cap U_X)$, where p_2 is the projection of $X \times X$ to the second factor and that $p_2 \circ \varDelta$: $X \to X$ is the identity. Then we have $[X] \cap U_X/[X] = 1_X$, hence $U_X/[X] = 1_X$. q.e.d.

We need the following for the calculation in Section 5.

PROPOSITION 4.3. *Let X and Y be Z_2-Poincaré-Euler space. Then*

$w^*(X \times Y) = w^*(X) \times w^*(Y)$.

PROOF. Let U_X, U_Y and $U_{X\times Y}$ be the diagonal classes of X, Y and $X\times Y$ respectively. Then $w^*(X\times Y) = (\mathrm{Sq}\, U_{X\times Y})/[X\times Y] = (\mathrm{Sq}\, U_X)/[X] \times (\mathrm{Sq}\, U_Y)/[Y] = w^*(X) \times w^*(Y)$. q.e.d.

In order to apply our main Theorem to Z_2-homology manifolds, we need Propositions 4.4 and 4.5.

PROPOSITION 4.4. *Given Z_2-homology manifolds X and Y, let ψ: $(Y, \partial Y) \to (X, \partial X)$ be an embedding with a normal block bundle ν. Then $\psi^* w^*(X) = w^*(Y) \cup w^*(\nu)$.*

PROOF. Let E be the total space of a normal block bundle ν of ψ and let $\bar{E}$ be the total space of the sphere bundle induced by ν. First we will prove that $w^*(E) = i^* w^*(X)$, where $i: E \to X$ is the inclusion. Put $\tilde{E} = \mathrm{cl}\,(\partial E - \bar{E})$. Let $P = \{(x_1, \cdots, x_\alpha) \mid x_\alpha \geqq 0, x_{\alpha-1} \leqq 0\}$ and $Q = \{(x_1, \cdots, x_\alpha) \mid x_\alpha \geqq 0, x_{\alpha-1} \geqq 0\}$. Then $\boldsymbol{R}^\alpha_+ = P \cup Q$. Let $\tilde{P} = \{(x_1, \cdots, x_\alpha) \mid x_\alpha = 0, x_{\alpha-1} \leqq 0\}$, $\bar{P} = \bar{Q} = P \cap Q$ and $\tilde{Q} = \{(x_1, \cdots, x_\alpha) \mid x_\alpha = 0, x_{\alpha-1} \geqq 0\}$. Note that there exists a proper embedding φ: $X \to \boldsymbol{R}^\alpha_+$ such that $\varphi | E$: $(E; \tilde{E}, \bar{E}) \to (P; \tilde{P}, \bar{P})$ and $\varphi | \mathrm{cl}\,(X - E)$: $(\mathrm{cl}\,(X - E), \mathrm{cl}\,(\partial X - \tilde{E}), \bar{E}) \to (Q; \tilde{Q}, \bar{Q})$ are proper. (See Hudson [10].) Let $(R_P; \tilde{R}_P, \bar{R}_P; \varphi | E)$, $(R_Q; \tilde{R}_Q, \bar{R}_Q; \varphi | \mathrm{cl}\,(X - E))$ and $(R; \tilde{R}, \bar{R}; \varphi)$ be regular neighborhoods of E in P, of $\mathrm{cl}\,(X - E)$ in Q and of X in $\boldsymbol{R}^\alpha_+$, respectively, such that $R = R_P \cup R_Q$ and $\bar{R} = \bar{R}_P \cup \bar{R}_Q$. Define $U(\varphi | E) \in H^*(R_P, \bar{R}_P; Z_2)$ as the Poincaré dual of $(\varphi | E)_*[E]$. Then $U(\varphi | E) = j^* U(\varphi)$, where $j: P \to \boldsymbol{R}^\alpha_+$ is the inclusion, hence $w^*(\varphi | E) = i^* w^*(\varphi)$. Thus $w^*(E) = i^* w^*(X)$. Note that $U(\psi_Y) = U(\varphi | E) \cup [(\varphi | E)^*]^{-1} U_\nu$, where $(R_P; \tilde{R}_P \cap \tilde{P}, \bar{R}_P \cup (\tilde{R}_P \cap \bar{P}); \psi_Y)$ is a regular neighborhood of Y in $\boldsymbol{R}^\alpha_+$. Let $\psi_\nu: Y \to E$ be the canonical inclusion. Then $w^*(\psi_Y) = \psi_\nu^* w^*(\varphi | E) \cup w^*(\nu)$. By Proposition 4.2, we have $\psi_\nu^* w^*(E) = w^*(Y) \cup w^*(\nu)$. Since $i \circ \psi_\nu = \psi$, we have $\psi^* w^*(X) = w^*(Y) \cup w^*(\nu)$. q.e.d.

PROPOSITION 4.5. *Let X be a closed Z_2-Poincaré-Euler space. Then $\langle w^*(X), [X] \rangle = e(X)$, where $e(X)$ is the modulo 2 Euler number of X.*

The proof in the case of smooth manifolds given in Milnor [12] can be applied to this proposition without any changes.

We need the following to prove Lemmas 5.2 and 6.2 in subsequent sections.

LEMMA 4.1. *Let $(R; \tilde{R}, \bar{R}; \varphi)$ be a regular neighborhood of an n-dimensional Z_2-Poincaré-Euler space X in $\boldsymbol{R}^\alpha_+$. Suppose that a* PL-*embedding f: $(M, \partial M) \to (R, \bar{R})$ is given with a normal block bundle $\xi = (E, M, f_E)$, such that $\varphi(X)$ is transverse to ξ, where M is a compact* PL-

manifold. Let U_ξ be the Thom class of ξ. Let $j_E: E \to R$ be the inclusion. Define $Y = \varphi^{-1} \circ f(M)$ and $X_E = \varphi^{-1} \circ j_E(E)$. Let $\varphi_E: X_E \to E$ and ψ_M: $Y \to M$ be embeddings defined by $\varphi_E = j_E^{-1} \circ \varphi$ and $\psi_M = f^{-1} \circ (\varphi | Y)$. Then the following hold:

1) $(f_E)_*([M] \cap f^* U(\varphi)) = (\varphi_E)_*[X_E] \cap U_\xi$.
2) $[M] \cap f^* U(\varphi) = (\psi_M)_*[Y]$.

PROOF. 1) Note that $j_E \circ f_E = f$ and $[E] \cap U_\xi = (f_E)_*[M]$. Hence $(f_E)_*([M] \cap f^* U(\varphi)) = ([E] \cap j_E^* U(\varphi)) \cap U_\xi$. Thus it suffices to prove $[E] \cap j_E^* U(\varphi) = (\varphi_E)_*[X_E]$. Let $\tilde{\tilde{R}} = \mathrm{cl}\,(R - j_E(E))$ and let $j_R: (R; \tilde{R}, \bar{R}) \to (R; \tilde{\tilde{R}}, \bar{R})$ be defined as the identity. Regard j_E as a map $j_E: (E; \tilde{E}, \bar{E}) \to (R; \bar{R}, \tilde{\tilde{R}})$, where $\tilde{E} = \mathrm{cl}\,(\partial E - \bar{E})$. Note that $(j_E)_*[E] = (j_R)_*[R]$ and $[R] \cap U(\varphi) = \varphi_*[X]$. Hence $(j_E)_*([E] \cap (j_E)^* U(\varphi)) = (j_R)_* \circ \varphi_*[X] = (j_E)_* \circ (\varphi_E)_*[X_E]$. Since $(j_E)_*: H_*^{\mathrm{inf}}(E, \bar{E}; Z_2) \to H_*^{\mathrm{inf}}(R, \tilde{\tilde{R}}; Z_2)$ is an isomorphism, we have $[E] \cap (j_E)^* U(\varphi) = (\varphi_E)_*[X_E]$.

2) Note that $[X_E] \cap (\varphi_E)_* U_\xi = (\psi_E)_*[Y]$, where $\psi_E: Y \to X_E$ is the inclusion. By 1), we have $(f_E)_*([M] \cap f^* U(\varphi)) = (\varphi_E)_* \circ (\psi_E)_*[Y]$. Since $\varphi_E \circ \psi_E = f_E \circ \psi_M$ and since $(f_E)_*: H_*^{\mathrm{inf}}(M, \partial M; Z_2) \to H_*^{\mathrm{inf}}(E, \tilde{E}; Z_2)$ is an isomorphism, we have $[M] \cap f^* U(\varphi) = (\psi_M)_*[Y]$. q.e.d.

5. Characterization of Stiefel-Whitney classes via unoriented differentiable bordism groups. Let $(R; \tilde{R}, \bar{R}; \varphi)$ be a regular neighborhood of an n-dimensional Z_2-Poincaré-Euler space X in $\boldsymbol{R}_+^\alpha$. Suppose that $\tilde{e}_\varphi$: $\mathfrak{N}_*(R, \bar{R}) \to Z_2$ is the homomorphism defined in Section 1. Then the following holds:

LEMMA 5.1. *For each $(f, M) \in \mathfrak{N}_*(R, \bar{R})$, it follows that*

$$\langle U(\varphi) \cup (\varphi^*)^{-1} w^*(X), f_*([M] \cap w^*(M)) \rangle = \tilde{e}_\varphi(f, M) .$$

In order to prove this lemma, we need the following:

LEMMA 5.2. *Let $f: (M, \partial M) \to (R, \bar{R})$ be a* PL-*embedding with the normal block bundle ξ, where M is a compact triangulated differentiable manifold. If $\varphi(X)$ is transverse to ξ, then*

$$\langle U(\varphi) \cup (\varphi^*)^{-1} w^*(X), f_*([M] \cap w^*(M)) \rangle = \tilde{e}_\varphi(f, M) .$$

PROOF. We use the notations in Lemma 4.1. By 2) of Lemma 4.1, we have $\langle U(\varphi) \cup (\varphi^*)^{-1} w^*(X), f_*([M] \cap w^*(M)) \rangle = \langle f^* \circ (\varphi^*)^{-1} w^*(X) \cup w^*(M), (\psi_M)_*[Y] \rangle$. Let $\psi_X: Y \to X$ be the inclusion. Note that $f \circ \psi_M = \varphi \circ \psi_X$. Hence $\langle U(\varphi) \cup (\varphi^*)^{-1} w^*(X), f_*([M] \cap w^*(M)) \rangle = \langle \psi_X^* w^*(X) \cup \psi_M^* w^*(M), [Y] \rangle = \langle \psi_X^* w^*(X) \cup \psi_M^* \bar{w}(\xi), [Y] \rangle = \langle \psi_X^* w^*(X) \cup \bar{w}(\psi_M^* \xi), [Y] \rangle$. Thus $\langle U(\varphi) \cup (\varphi^*)^{-1} w^*(X), f_*([M] \cap w^*(M)) \rangle = \tilde{e}_\varphi(f, M)$ by the definition of $\tilde{e}_\varphi$. q.e.d.

PROOF OF LEMMA 5.1. Let (f, M) be in $\mathfrak{N}_*(R, \bar{R})$. Then there exists an embedding $g: (M, \partial M) \to (R \times D^\beta, \bar{R} \times D^\beta)$ such that $g \simeq f \times \{0\}$ and $(\varphi \times \mathrm{id})(X \times D^\beta)$ is block transverse to g by Transversality Theorem. By Lemma 5.2, it follows that $\langle (U(\varphi) \times 1) \cup [(\varphi \times \mathrm{id})^*]^{-1} w^*(X \times D^\beta), g_*([M] \cap w^*(M)) \rangle = \tilde{e}_\varphi(f, M)$. Since $\langle (U(\varphi) \cup (\varphi^*)^{-1} w^*(X), f_*([M] \cap w^*(M)) \rangle = \langle (U(\varphi) \times 1) \cup [(\varphi \times \mathrm{id})^*]^{-1} w^*(X \times D^\beta), g_*([M] \cap w^*(M)) \rangle$ by Proposition 4.3, we have

$$\langle U(\varphi) \cup (\varphi^*)^{-1} w^*(X), f_*([M] \cap w^*(M)) \rangle = \tilde{e}_\varphi(f, M) . \qquad \text{q.e.d.}$$

The following and Lemma 5.1 give a characterization of Stiefel-Weitney classes.

LEMMA 5.3. *Let (A, B) be a pair of polyhedra. Given $\Phi \in H^*(A, B; Z_2)$, if $\langle \Phi, f_*([M] \cap w^*(M)) \rangle = 0$ for every $(f, M) \in \mathfrak{N}_*(A, B)$, then $\Phi = 0$.*

PROOF. Let $\Phi = \Phi^0 + \Phi^1 + \cdots + \Phi^n$ for $\Phi^i \in H^i(A, B; Z_2)$. Since $\langle \Phi, f_*([M] \cap w^*(M)) \rangle = \langle \Phi^0, f_*[M] \rangle$ for $(f, M) \in \mathfrak{N}_0(A, B)$, $\langle \Phi, f_*([M] \cap w^*(M)) \rangle = 0$ for every (f, M) implies that $\Phi^0 = 0$. Suppose that $\Phi^0 = 0$, $\Phi^1 = 0, \cdots, \Phi^k = 0$. Then $\langle \Phi, f_*([M] \cap w^*(M)) \rangle = \langle \Phi^{k+1}, f_*[M] \rangle$ for $(f, M) \in \mathfrak{N}_{k+1}(A, B)$. Hence, if $\langle \Phi, f_*([M] \cap w^*(M)) \rangle = 0$ for every (f, M), it follows that $\Phi^{k+1} = 0$. By induction on k, we have $\Phi = 0$. q.e.d.

6. Characterization of Stiefel-Whitney homology classes via unoriented differentiable bordism groups. Let $(R; \tilde{R}, \bar{R}; \varphi)$ be a regular neighborhood of an n-dimensional Z_2-Poincaré-Euler space X in $\boldsymbol{R}^\alpha_+$. Suppose that $e_\varphi: \mathfrak{N}_*(R, \bar{R}) \to Z_2$ is the homomorphism defined in Section 1. Then the following holds:

LEMMA 6.1. *For each $(f, M) \in \mathfrak{N}_*(R, \bar{R})$, it follows that*

$$\langle U(\varphi) \cup (\varphi^*)^{-1} \circ ([X] \cap)^{-1} s_*(X), f_*([M] \cap w^*(M)) \rangle = e_\varphi(f, M) .$$

In order to prove this lemma, we need the following:

LEMMA 6.2. *Let $f: (M, \partial M) \to (R, \bar{R})$ be a PL-embedding with a normal block bundle ξ, where M is a compact triangulated differentiable manifold. If $\varphi(X)$ is transverse to ξ, then*

$$\langle U(\varphi) \cup (\varphi^*)^{-1} \circ ([X] \cap)^{-1} s_*(X), f_*([M] \cap w^*(M)) \rangle = e_\varphi(f, M) .$$

PROOF. By 1) of Lemma 4.1, we have $\langle U(\varphi) \cup (\varphi^*)^{-1} \circ ([X] \cap)^{-1} s_*(X), f_*([M] \cap w^*(M)) \rangle = \langle w^*(M) \cup f^* \circ (\varphi^*)^{-1} \circ ([X] \cap)^{-1} s_*(X), (f_E)^{-1}((\varphi_E)_*[X_E] \cap U_\xi) \rangle$. Note that $j_E \circ f_E = f$. Then $\langle U(\varphi) \cup (\varphi^*)^{-1} \circ ([X] \cap)^{-1} s_*(X), f_*([M] \cap w^*(M)) \rangle = \langle U_\xi \cup (f_E^*)^{-1} w^*(M), ((\varphi_E)_*[X_E]) \cap j_E^* \circ (\varphi^*)^{-1} \circ ([X] \cap)^{-1} s_*(X) \rangle$. Since there exists the commutative diagram

$$\begin{array}{ccccc} H^*(X; Z_2) & \xleftarrow{\varphi^*} & H^*(R; Z_2) & \xrightarrow{j_E^*} & H^*(E; Z_2) \\ \Big\downarrow [X]\cap & & & & \Big\downarrow ((\varphi_E)_*[X_E])\cap \\ H_*(X, \partial X; Z_2) & \xrightarrow[\varphi_*]{} & H_*(R, \mathrm{cl}\,(R-E); Z_2) & \xleftarrow[(j_E)_*]{} & H_*(E, \bar{E}; Z_2) \end{array}$$

and since $[X]\cap$, φ^* and $(j_E)_*$ are isomorphisms, we have

$$((\varphi_E)_*[X_E]) \cap j_E^* \circ (\varphi^*)^{-1} \circ ([X]\cap)^{-1} s_*(X) = [(j_E)_*]^{-1} \circ \varphi_* s_*(X) = (\varphi_E)_* s_*(X_E) .$$

Note that $\langle U_\xi \cup (f_E^*)^{-1} w^*(M), (\varphi_E)_* s_*(X_E)\rangle = e(Y)$ by Corollary 3.1. Thus $\langle U(\varphi) \cup (\varphi^*)^{-1} \circ ([X]\cap)^{-1} s_*(X), f_*([M] \cap w^*(M))\rangle = e_\varphi(f, M)$. q.e.d.

PROOF OF LEMMA 6.1. Let (f, M) be in $\mathfrak{N}_*(R, \bar{R})$. Then there exists an embedding $g\colon (M, \partial M) \to (R \times D^\beta, \bar{R} \times D^\beta)$ such that $g \simeq f \times \{0\}$ and that $(\varphi \times \mathrm{id})(X \times D^\beta)$ is block transverse to g by Transversality Theorem. By Lemma 6.2, it follows that

$$\langle (U(\varphi) \times 1) \cup [(\varphi \times \mathrm{id})^*]^{-1} \circ ([X \times D^\beta]\cap)^{-1} s_*(X \times D^\beta), g_*([M] \cap w^*(M))\rangle = e_\varphi(f, M) .$$

Since $\langle U(\varphi) \cup (\varphi^*)^{-1} \circ ([X]\cap)^{-1} s_*(X), f_*([M] \cap w^*(M))\rangle = \langle (U(\varphi) \times 1) \cup [(\varphi \times \mathrm{id})^*]^{-1} \circ ([X \times D^\beta]\cap)^{-1} s_*(X \times D^\beta), g_*([M] \cap w^*(M))\rangle$ by Proposition 2.2, we have $\langle U(\varphi) \cup (\varphi^*)^{-1} \circ ([X]\cap)^{-1} s_*(X), f_*([M] \cap w^*(M))\rangle = e_\varphi(f, M)$. q.e.d.

Now we are in a position to prove the following theorem announced in Section 1.

THEOREM. *Let X be an n-dimensional Z_2-Poincaré-Euler space. Take a regular neighborhood $(R; \tilde{R}, \bar{R}; \varphi)$ of X in $\boldsymbol{R}_+^\alpha$. Then $[X] \cap w^*(X) = s_*(X)$ if and only if $o_\varphi = 0$.*

PROOF. If $[X] \cap w^*(X) = s_*(X)$, then $\tilde{e}_\varphi(f, M) = e_\varphi(f, M)$. This means $o_\varphi = 0$. Conversely suppose that $o_\varphi = 0$. By Lemmas 5.1, 5.3 and 6.1, we have $U(\varphi) \cup (\varphi^*)^{-1} w^*(X) = U(\varphi) \cup (\varphi^*)^{-1} \circ ([X]\cap)^{-1} s_*(X)$. Hence $[X] \cap w^*(X) = s_*(X)$ by Proposition 4.1. q.e.d.

This Theorem can be applied to Z_2-homology manifolds.

COROLLARY. *Let X be an n-dimensional Z_2-homology manifold with or without boundary. Then $[X] \cap w^*(X) = s_*(X)$.*

PROOF. Let $\psi\colon Y \to X \times D^\beta$ be the embedding used to define e_φ and $\tilde{e}_\varphi$. Note that ψ has a normal block bundle ν in $X \times D^\beta$. Then Y is a Z_2-homology manifold. Therefore $\psi^* w^*(X \times D^\beta) = w^*(Y) \cup w^*(\nu)$ by Proposition 4.4. In view of the definition of e_φ and $\tilde{e}_\varphi$, we have $o_\varphi = 0$

by Proposition 4.5. Thus $[X] \cap w^*(X) = s_*(X)$ by Theorem. q.e.d.

EXAMPLE 1. We construct a simple example of Z_2-Poincaré-Euler space X which is not a Z_2-homology manifold. Let $X_1 = D^2/\{a, b, c\}$ where $D^2 = [-1, 1]^2$ and a, b, c are distinct points in ∂D^2. Then X_1 is a Z_2-Euler space. Let $X_2 = \text{cone } \partial X_1$. Then there exists a canonical PL-homeomorphism $\iota: \partial X_1 \to \partial X_2$. Put $X = X_1 \underset{\iota}{\cup} X_2$. Then X is homotopy equivalent to S^2 and is not a Z_2-homology manifold.

EXAMPLE 2. We construct a little more complicated example of Z_2-Poincaré-Euler space X which does not satisfy $[X] \cap w^*(X) = s_*(X)$. In particular, X is not a Z_2-homology manifold. Let X_1 be the quotient space of $[-1, 1] \times [0, 1]$ by the identification $(-1, t) = (0, t)$ and $(1, t) = (0, 1 - t)$ for each t in $[0, 1]$. Then X_1 is a Z_2-Euler space. Put $Y = \partial X_1/([0, 1] \times \{0\})$. Let $\varphi: \partial X_1 \to Y$ be the quotient map. Let X_2 be the mapping cylinder of φ. Then X_2 is a Z_2-Euler space such that $\partial X_2 = \partial X_1 \cup Y$. Let $X_3 = ([0, 1]^2 \cup [-1, 0]^2)/\{(0, 0), (1, 1)\}$. Then X_3 is a Z_2-Euler space such that ∂X_3 is PL-homeomorphic to Y. Define $X = X_1 \cup X_2 \cup X_3$. Then X is a Z_2-Euler space and is homotopy equivalent to $\boldsymbol{P}^2$. Hence $w^1(X) \neq 0$. Since $s_1(X) = 0$, it follows that X is a Z_2-Poincaré-Euler space which does not satisfy $[X] \cap w^*(X) = s_*(X)$.

Appendix. Proof of Transversality Theorem.

A.1. BLOCK TRANSVERSALITY AND MOCK TRANSVERSALITY. Let M and N be PL-manifolds. Suppose that $f: M \to N$ is a locally flat PL-embedding and that X is a subpolyhedron of N. Then X is block transverse to f in N, if there exists a normal block bundle $\nu = (E(\nu), M, f_E)$ of f such that $X \cap E(\nu) = E(\nu | X \cap f(M))$. (See [2] and [14].)

Let $f: (M, \partial M) \to (N, \partial N)$ be a PL-embedding. The collars $c_1: \partial M \times I \to M$ and $c_2: \partial N \times I \to N$ are said to be compatible with f, if $f \circ c_1(x, t) = c_2(f(x), t)$ for every (x, t) in $\partial M \times I$. (See [10].)

Let X and Y be polyhedra and let K be a ball complex (cf. [2]) such that $X = |K|$. A proper PL-embedding $f: Y \to X$ is transverse to K, if $f^{-1}(\sigma)$ is a compact PL-manifold with boundary $f^{-1}(\partial\sigma)$ and if the PL-embedding $f | f^{-1}(\sigma): f^{-1}(\sigma) \to \sigma$ has compatible collars for every σ in K.

In order to prove Transversality Theorem, we need the following. The next section is devoted to its proof.

PROPOSITION A.1. (cf. Buoncristiano, Rourke and Sanderson [2]). *Let X and Y be polyhedra. Let K be a ball complex such that $X = |K|$. Suppose that a subdivision K' of K does not subdivide a subcomplex*

L of K and that a proper PL-*embedding f: $Y \to X$ is transverse to K. Then there exists a proper* PL-*embedding g: $Y \to X$ which is transverse to K' and ambient isotopic to f relative to $|L|$.*

Let M and N be PL-manifolds. Suppose that $f: M \to N$ is a locally flat proper PL-embedding and that X is a subpolyhedron of N. We say that f is mock transverse to X in N, if there exists a ball complex K which contains a subcomplex L such that $|K| = N$ and $|L| = X$ and if f is transverse to K.

We also need the following to prove Transversality Theorem. We do not repeat the proof here since an adequate proof is given in [2].

PROPOSITION A.2. (Buoncristiano, Rourke and Sanderson [2, II, Theorem 4.4]). *Let M and N be* PL-*manifolds. Suppose that $f: M \to N$ is a locally flat proper* PL-*embedding and X is a closed subpolyhedron of N. The* PL-*embedding f is mock transverse to X in N if and only if X is block transverse to f in N.*

PROOF OF TRANSVERSALITY THEOREM. Noting the assumption of Transversality Theorem, there exists a normal block bundle $\nu = (E(\nu), M, f_E)$ of f to which a regular neighborhood R of $\partial N \cap X$ in X is transverse in N. Let K be a ball complex such that blocks $E(\sigma)$ of ν are balls of K, that $|K| = N$ and that $K|R$ is contained in K as a subcomplex. Then f is transverse to K. Choose a subdivision K' of K which does not subdivide $K|\partial N$ and which contains a subcomplex K_X of K' where $|K_X| = X$. Put $L = K|\partial N$. Then by Proposition A.1, there exists an PL-embedding $g: M \to N$ which is transverse to K' and ambient isotopic to f relative to $|L| = \partial N$. Thus g is mock transverse to X, and X is block transverse to g by Proposition A.2. q.e.d.

A.2. PROOF OF PROPOSITION A.1. In order to prove Proposition A.1, it suffices to prove the following:

LEMMA A.1. *Let X and Y be polyhedra. Let K be a ball complex such that $|K| = X$. Suppose that a subdivision K' of K does not subdivide a subcomplex L of K and that a proper* PL-*embedding f: $Y \to X$ is transverse to K. Then there exists a proper* PL-*embedding g: $Y \to X$ transverte to K' and an ambient isotopy F: $X \times I \to X \times I$ relative to $|L|$ between f and g such that $F(\sigma \times I) = \sigma \times I$ for each σ in K.*

We will prove this lemma by induction on the dimension of X. For the induction step, we need the following:

LEMMA A.2. *Let M be a compact* PL-*manifold. Let K be a ball* *complex such that* $|K| = D^n$. *Let* $f: M \to D^n$ *be a proper* PL-*embedding such that* $f|\partial M: \partial M \to \partial D^n$ *is transverse to* $K|\partial D^n$. *Then there exists an* PL-*embedding* $g: M \to D^n$ *transverse to* K *and ambient isotopic to* f *relative to* ∂D^n.

We need the following to prove Lemma A.2:

UNIQUENESS THEOREM OF COLLARS. (Hudson and Zeeman [9]). *If* c_0 *and* c_1 *are two collars of* M, *then there exists an ambient isotopy* F *of* M *fixed on* ∂M *such that* $c_1 = F_1 \circ c_0$ *and* F_0 *is the identity, where* $F(x, t) = (F_t(x), t)$.

LEMMA A.3. *Let* Δ *be a ball complex which contains only one* n-*ball such that* $|\Delta| = D^n$. *Let* Λ *be the subcomplex of* Δ *containing all balls except the* n-*ball and one* $(n-1)$-*ball. If* X *is a compact* PL-*manifold and if a* PL-*embedding* $f: X \to |\Lambda|$ *is transverse to* Λ, *then there exists a* PL-*embedding* $F: X \times I \to D^n$ *transverse to* Δ *such that* $F(x, 0) = f(x)$ *for every* x *in* X.

PROOF. Since there exists a PL-homeomorphism $h: |\Lambda| \times I \to |\Delta|$ such that $h(y, 0) = y$ for every y in $|\Lambda|$, an PL-embedding $F: X \times I \to |\Delta|$ can be defined by $F(x, t) = h(f(x), t)$. Clearly F is transverse to Δ and $F(x, 0) = f(x)$. q.e.d.

PROOF OF LEMMA A.2. Clearly there exists a subdivision K' of K which does not subdivide $K|\partial D^n$ such that $\partial D^n \times I = |K' - \sigma|$ for some n-ball σ in K'. Note that the ball complex $K' - \sigma$ collapses to $K|\partial D^n$. By if $\dim M = n$, there is nothing to prove. Otherwise by using Lemma A.3, we can construct a subpolyhedron X of $|K' - \sigma|$ such that X collapses to $f(\partial M)$ and that the inclusion $i: X \subset |K' - \sigma|$ is transverse to $K' - \sigma$. Since the inclusion i has a normal block bundle (see [2]), X is a PL-manifold. Therefore there exists a PL-homeomorphism $h: \partial M \times I \to X$. Define $\tilde{f}: \partial M \times I \to |K' - \sigma|$ by $\tilde{f} = i \circ h$. Then $\tilde{f}$ is transverse to $K' - \sigma$. By the uniqueness theorem of regular neighborhoods (see [10]), there exists a collar $c_1: \partial D^n \times I \to D^n$ such that $c_1(\partial D^n \times I) = |K' - \sigma|$ and $c_1(f(x), t) = j \circ \tilde{f}(x, t)$ for (x, t) in $\partial M \times I$, where $j: |K' - \sigma| \to D^n$ is the inclusion. Let $c: \partial M \times I \to M$ and $c_0: \partial D^n \times I \to D^n$ be compatible collars with f. By the uniqueness theorem of collars, there exists an ambient isotopy $F: D^n \times I \to D^n \times I$ relative to $\partial D^n \times I$ such that F_0 is the identity and $c_1 = F_1 \circ c_0$, where $F(x, t) = (F_t(x), t)$ for every (x, t) in $D^n \times I$. Define $g: M \to D^n$ by $g = F_1 \circ f$. Note that $\tilde{f}$ is transverse to $K' - \sigma$. Thus g is transverse to K', and hence g is transverse to K. q.e.d.

PROOF OF LEMMA A.1. We prove Lemma A.1 by induction on the dimension of X. The case $\dim X = 0$ is trivial. Suppose that Lemma A.1 holds whenever the dimension of X is smaller than $n+1$ and assume that $\dim X = n+1$. Suppose that a PL-embedding $f\colon Y \to X$ is transverse to a ball complex structure K of X. Then $f|f^{-1}(|K^n|)\colon f^{-1}(|K^n|) \to |K^n|$ is transverse to K^n, where K^n is the n-skelton of K. Put $(K^n)' = \{\sigma \in K' | \sigma \subset |K^n|\}$. By induction assumption, there exist a PL-embedding $g\colon f^{-1}(|K^n|) \to |K^n|$ transverse to $(K^n)'$ and an ambient isotopy $\tilde{G}\colon K^n \times I \to K^n \times I$ between $f|f^{-1}(|K^n|)$ and g relative to $|K^n| \cap |L|$ such that $\tilde{G}(\sigma \times I) = \sigma \times I$ for each σ in K^n. Clearly there exists an isotopy $G\colon X \times I \to X \times I$ relative to $|L|$ such that $G|K^n| \times I = \tilde{G}$ and $G(\sigma \times I) = \sigma \times I$ for every σ in K. Thus we may assume that f is transverse to $\bar{K}^n$, where $\bar{K}^n = (K^n)' \cup (K - K^n)$. Applying Lemma A.2 to PL-embeddings $f|f^{-1}(\sigma)\colon f^{-1}(\sigma) \to \sigma$ for all σ in $K - K^n$, there exists a PL-embedding $g\colon Y \to X$ transverse to K' an ambient isotopy $F\colon X \times I \to X \times I$ between f and g relative to $|K^n| \cup |L|$ such that $F(\sigma \times I) = \sigma \times I$ for every σ in K. q.e.d.

REFERENCES

[1] E. AKIN, Stiefel-Whitney homology classes and cobordism, Trans. Amer. Math. Soc. 205 (1975), 341–359.

[2] S. BUONCRISTIANO, C. R. ROURKE AND B. J. SANDERSON, A geometric approach to homology theory, London Math. Soc. Lecture Notes 18, 1976.

[3] J. D. BLANTON AND P. A. SCHWEITZER, Axiom for characteristic classes of manifolds, Proc. Symp. in Pure Math. 27 (1975), Amer. Math. Soc. 349–356.

[4] J. D. BLANTON AND C. MCCRORY, An axiomatic proof of Stiefel conjecture, Proc. Amer. Math. Soc. 77 (1979), 409–414.

[5] G. CHEEGER, A combinatorial formula for Stiefel-Whitney classes, in Topology of Manifolds, (Cantrel and Edward, eds.), Markham Publ., 1970, 470–471.

[6] P. CONNER AND E. FLOYD, Differentiable periodic maps, Ergebnisse der Mathematik und ihrer Grenzgebiete 33, Springer-Verlag, Berlin, Heidelberg, New York, 1964.

[7] S. HALPERIN AND D. TOLEDO, Stiefel-Whitney homology classes, Ann. of Math. 96 (1972), 511–525.

[8] S. HALPERIN AND D. TOLEDO, The product formula for Stiefel-Whitney homology classes, Proc. of Amer. Math. Soc. 48 (1975), 239–244.

[9] J. F. P. HUDSON AND E. C. ZEEMAN, On combinatrial isotopy, Publ. Math. Inst. Hautes Et. Sci. 19 (1964), 69–94.

[10] J. F. P. HUDSON, Piecewise Linear Topology, Benjamin, New York, 1969.

[11] C. MCCRORY, Cone complexes and PL transversality, Trans. Amer. Math. Soc. 207 (1975), 269–291.

[12] J. MILNOR AND J. STASHEFF, Characteristic classes, Ann. of Math. Studies 76, Princeton Univ. Press. 1974.

[13] C. P. ROURKE, Block structures in geometric and algebraic topology, Actes Congres intern. Math. Nice, 1970, 127–132.

[14] C. P. ROURKE AND B. J. SANDERSON, Block bundles, I and II, Ann. of Math. 87 (1968), 1–28 and 255–277.

[15] E. H. SPANIER, Algebraic Topology, McGraw-Hill, New York, 1966.
[16] D. SULLIVAN, Combinatorial invariant of analytic spaces, Proc. Liverpool Singularities I, Lecture Notes in Math. 192, Springer-Verlag, Berlin, Heidelberg, New York, 1971, 165–168.
[17] D. SULLIVAN, Singularities in spaces, Proc. Liverpool Singularities II, Lecture Notes in Math. 209, Springer-Verlag, Berlin, Heidelberg, New York, 1971, 196–206.
[18] L. TAYLOR, Stiefel-Whitney homology classes, Quart. J. Oxford 28 (1977), 381–387.
[19] H. WHITNEY, On the theory of sphere bundles, Proc. Nat. Acad. Sci. U.S.A. 26 (1940), 148–153.

ICHINOSEKI TECHNICAL COLLEGE
ICHINOSEKI, 021
JAPAN