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FURTHER REMARKS ON THE EXPONENT OF CONVERGENCE OF POINCARÉ SERIES

S. J. PATTERSON

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1. Introduction. Let G be a discrete subgroup of the automorphism group $\mathrm{Con}(n)$ of $(n+1)$-dimensional hyperbolic space $\boldsymbol{H}^{n+1}$. We shall associate in § 3 a certain number $\delta(G)$, $0 \leqq \delta(G) \leqq n$, to G called the *exponent of convergence of the Poincaré series attached to* G. It is related to several of the geometric properties of G; these properties have been the subject of many investigations but in [5] Sullivan has discussed these exhaustively and completed them in several important points.

The question with which this paper is concerned is that of estimating $\delta(G)$ for a given group G. In this it was prompted by a recent paper [1] in this journal in which the authors find upper bounds for $\delta(G_1 * G_2)$ where G_1, G_2 are considered known and $G_1 * G_2$ is the free product of G_1 and G_2 in $\mathrm{Con}(n)$ formed by the Klein Combination Theorem, when this is applicable. The technique is similar to the one used in [2] to bound $\delta(G_1 * G_2)$ from below. In [1] however the authors use a particular model of G_1 and G_2 and it seemed desirable to free the argument of this constraint. This will be done in § 3.

In § 5 we shall apply this estimate to show that, given n and $\varepsilon > 0$ there exists a discrete $G \subset \mathrm{Con}(n)$ with

(a) $\delta(G) < \varepsilon$

(b) G is of the first kind, which means that G does not operate discontinuously on any non-empty open subset of the boundary of $\boldsymbol{H}^{n+1}$. The author had given a proof of this in [3] in the case $n = 1$, $\varepsilon = 1/2$ which used uniformization theory and the perturbation theory of elliptic differential operators. This proof would not extend to the case $n > 1$. Both J. Elstrodt and D. Sullivan indicated to the author that it would be very desirable to give a geometric proof of this theorem.

The techniques used here do not depend on the dimension of the hyperbolic space. Usually one is interested only in the cases $n = 1$ (Fuchsian groups) and $n = 2$ (Kleinian groups) but to cover these uniformly it is convenient to work with hyperbolic space of arbitrary dimension. Although it is in principle well-known I have included a brief summary, in § 2, of the three basic models of $\boldsymbol{H}^{n+1}$, along with the relations between them and those formulae which will be of use to us here.

To return to the original question of estimating $\delta(G)$ for a given group, let us remark that it is necessary to make rather more precise what one means by "given". If one can 'list' the elements of G, as happens with Schottky groups and some other free products, then one has

$$\delta(G) = \lim_{X\to\infty} \log \operatorname{Card}\{g \in G;\, L(x_1, gx_2) \leqq X\}/\log X$$

(see [5, Cor. 10]) where x_1, x_2 are two fixed points of H^{n+1} and L is defined in § 2. This method can be carried out on computers to compute $\delta(G)$ as long as G is not too pathological. It would be interesting to know effective bounds for

$$\delta(G) - \log \operatorname{Card}\{g \in G;\, L(x_1, gx_2) \leqq X\}/\log X\,,$$

in terms, say, of a given fundamental domain of G.

There are other senses in which G may be "given"—for example, when $n = 1, 2$, by uniformization theory, or by group-theoretic constructions applied to another group—, but here very little is known.

2. Models of hyperbolic space. Although it is less frequently used than the other models of hyperbolic space the easiest to introduce is the Klein model. Let $n \geqq 1$.

(i) *The Klein model.* Let J be the $(n + 2) \times (n + 2)$ diagonal matrix $\operatorname{diag}(-1, 1, \cdots, 1)$; the associated quadratic form is

$$J(y) = -y_0^2 + y_1^2 + \cdots + y_{n+1}^2\,.$$

Let

$$GO(1, n + 1) = \{A \in GL_{n+2}(\boldsymbol{R});\ \text{there exists } \lambda \in \boldsymbol{R}_+^\times \text{ with } AJ{}^tA = \lambda J\}\,,$$

where ${}^tA$ denotes the transpose of A. This group has four connected components. Let $GO^\circ(1, n + 1)$ be the connected component of the identity; then $GO(1, n + 1)/GO^\circ(1, n + 1) \cong C_2 \times C_2$, where C_2 is the cyclic group with two elements. Let

$$L^+ = \{y \in \boldsymbol{R}^{n+2};\, J(y) < 0,\ y_0 > 0\}$$

be the positive light-cone. An element of $GO(1, n + 1)$ either preserves L^+ or maps it into $L^- = -L^+$. The subgroup of $GO(1, n + 1)$ preserving L^+ has then two components, determined by whether the determinant is positive or negative. Of course $GO^\circ(1, n + 1)$ preserves L^+ and we shall think of it as the "orientation-preserving" group with this property.

Next let

$$PGO^\circ(1, n+1) = GO^\circ(1, n+1)/\{\lambda I; \lambda \in \boldsymbol{R}_+^\times\} .$$

This will be, by definition, Con (n). It would make no difference to the discussion had we taken the full group preserving L^+ as our starting point rather than $GO^\circ(1, n+1)$.

We project L^+ through the origin to $y_0 = 1$; the image is the unit ball

$$B^{n+1} = \{y \in \boldsymbol{R}^{n+1}; y_1^2 + y_2^2 + \cdots + y_{n+1}^2 < 1\} .$$

$PGO^\circ(1, n+1)$ acts on B^{n+1} through the projection. We can describe the action explicitly as follows. Let $g \in GO^\circ(1, n+1)$ be given by the matrix (g_{ij}) $(0 \leqq i, j \leqq n+1)$. Then the j^{th} component of $g(y)$ is given by

$$g(y)_j = (g_{j0} + \sum_{1\leqq i\leqq n+1} g_{ji}y_i)/(g_{00} + \sum_{1\leqq i\leqq n+1} g_{0i}y_i) \quad (1 \leqq j \leqq n+1) .$$

The subgroup of $PGO^\circ(1, n+1)$ which preserves 0 is $SO(n+1)$. It is easy to see that $PGO^\circ(1, n+1)$ acts transitively on B^{n+1}. This is our first model of hyperbolic space.

Now define, for $y, y' \in B^{n+1}$,

$$L_B(y, y') = \frac{1 - (y, y')}{\sqrt{1-(y, y)}\sqrt{1-(y', y')}} ,$$

where (,) denotes the usual Euclidean inner product. Then, by construction one has

$$L_B(gy, gy') = L_B(y, y') \quad (g \in PGO^\circ(1, n+1)) .$$

Define also

$$j_B(g, y) = \sqrt{1-(gy, gy)}/\sqrt{1-(y, y)}$$

and then one has

$$1 - (gy, gy') = j_B(g, y)j_B(g, y')(1 - (y, y'))$$

and

$$j_B(g_1g_2, y) = j_B(g_1, g_2y)j_B(g_2, y) .$$

Moreover one has explicitly,

$$j_B(g, y) = (g_{00} + \sum_j g_{0j}y_j)^{-1} .$$

The function $L_B(y, y')$ is closely related to the hyperbolic distance between y and y'. This is however easier to establish in those models of hyperbolic geometry where the hyperbolic metric and the underlying Euclidean metric are conformally equivalent.

(ii) *The Poincaré model.* The underlying set in Euclidean space of this model is again B^{n+1}, but we shall denote it now by D^{n+1} to indicate the different context. Let

$$\varphi: B^{n+1} \to D^{n+1} ; \quad y \mapsto y/(1+\sqrt{1-(y, y)})$$
$$\varphi^{-1}: D^{n+1} \to B^{n+1} ; \quad x \mapsto 2x/(1+(x, x))$$

which are inverse to one another. One can regard D^{n+1} as the image of the projection of L^+ onto the hyperplane $y_0 = 0$ through $(-1, 0, \cdots, 0)$. This is a 'stereographic' projection. One has then an action of $PGO^\circ(1, n+1)$ on D^{n+1} defined by

$$g\varphi(y) = \varphi(gy) .$$

It appears that in general there are no simple formulae describing this action.

There is another construction of φ, which I learnt from J. Elstrodt. We consider D^{n+1} as the subset $\{(0, y_1, \cdots, y_{n+1}); y_1^2 + y_2^2 + \cdots + y_{n+1}^2 < 1\}$ of $\boldsymbol{R}^{n+2}$. Let S^{n+1} be the subset $\{(y_0, y_1, \cdots, y_{n+1}); y_0^2 + y_1^2 + \cdots + y_{n+1}^2 = 1\}$ of $\boldsymbol{R}^{n+2}$. Let now φ_1 be the vertical projection upwards from B^{n+1} to S^{n+1} and let φ_2 be the projection from the upper hemisphere of S^{n+1} onto D^{n+1} through $(-1, 0, \cdots, 0)$. Then $\varphi = \varphi_2 \circ \varphi_1$.

Observe that φ has a continuous extension to the boundary of B^{n+1} where it acts as the identity.

If one now defines $L_D(x, x') = L_B(\varphi^{-1}(x), \varphi^{-1}(x'))$ then one finds that

$$L_D(x, x') = 1 + 2\frac{\|x - x'\|^2}{(1-\|x\|^2)(1-\|x'\|^2)} ,$$

where $\|x\|^2 = (x, x)$. We define

$$j_D(g, x) = (1 - \|gx\|^2)/(1 - \|x\|^2) \tag{2.1}$$

for which we have

$$\|gx - gx'\|^2 = j_D(g, x) j_D(g, x') \|x - x'\|^2 \tag{2.2}$$

and

$$j_D(g_1 g_2, x) = j_D(g_1, g_2 x) j_D(g_2, x) . \tag{2.3}$$

In this model the infinitesimal hyperbolic distance element is given by

$$ds^2 = |dx|^2/(1 - \|x\|^2)^2 ,$$

form which one sees that if the hyperbolic distance between x and x' is denoted by $d(x, x')$ then

$$L(x, x') = (1/2) \cosh 2d(x, x') .$$

Moreover, an invariant volume element is given by

$$d\sigma_D(x) = (1 - \|x\|^2)^{-n-1} dm(x)$$

where m is the standard Lebesgue measure.

(iii) *The upper half-space model.* This is obtained from D^{n+1} by means of an inversion about $(1, 0, \cdots, 0)$. Let

$$\boldsymbol{H}^{n+1} = \{(z_0, z_1, \cdots, z_n) \in \boldsymbol{R}^{n+1}; z_0 > 0\} .$$

We then define

$$\psi: \boldsymbol{D}^{n+1} \to \boldsymbol{H}^{n+1} ; \qquad x \mapsto (1 - 2x_1 + \|x\|^2)((1 - \|x\|^2)/2,\ x_2, \cdots, x_{n+1}) ,$$
$$\psi^{-1}: \boldsymbol{H}^{n+1} \to \boldsymbol{D}^{n+1} ; \qquad z \mapsto (1/4 + z_0 + \|z\|^2)^{-1}(\|z\|^2 - 1/4,\ z_1, \cdots, z_n) ,$$

which again are inverse to one another. This time one finds that if $L_H(z, z') = L_D(\psi^{-1}(z), \psi^{-1}(z'))$ then

$$L_H(z, z') = 1 + \|z - z'\|^2/2 \operatorname{Im}(z) \operatorname{Im}(z')$$

where

$$\operatorname{Im}((z_0, z_1, \cdots, z_n)) = z_0 .$$

Let

$$j_H(g, z) = \operatorname{Im}(gz)/\operatorname{Im}(z)$$

and then one has

$$\|gz - gz'\|^2 = j_H(g, z) j_H(g, z') \|z - z'\|^2 ,$$

and

$$j_H(g_1 g_2, z) = j_H(g_1, g_2 z) j_H(g_2, z) .$$

The infinitesimal distance element is

$$ds^2 = |dz|^2/4 \operatorname{Im}(z)^2$$

and the volume element is

$$d\sigma_H(z) = 2^{-(n+1)} \operatorname{Im}(z)^{-n-1} dm(z) .$$

These constructs are all that we need of hyperbolic geometry.

The group $PGO^\circ(1, n + 1)$ also acts on the boundary S^n of B^{n+1} or D^{n+1}, and, as we have already remarked, the action is the same in both cases. The boundary of $\boldsymbol{H}^{n+1}$ we take to be $\boldsymbol{R}^n_\infty = \boldsymbol{R}^n \cup \{\infty\}$ where $\boldsymbol{R}^n$ is that subspace defined by $\operatorname{Im}(z) = 0$ and ∞ has the usual formal properties.

For us the most convenient model with generally be the Poincaré model. The usefulness of the upper half-space model lies in that it allows us to emphasize one point of the boundary. However for most of the time we do not have to commit ourselves to any particular model and we shall write H for a hyperbolic space, ∂H for its boundary, $\operatorname{Con}(n)$

for $PGO^\circ(1, n+1)$, and L, j, d and σ for the functions constructed above. Let $\bar{H} = H \cup \partial H$.

3. The Klein Combination Theorem and estimates for $\delta(G)$. Let G_1 and G_2 be two discrete subgroups of Con(n), and suppose that we are given two open sets F_1, F_2 in H, $\bar{H}$ or ∂H. Let F_j^c denote the complement of F_j in the relevant set, which is the same for F_1 and F_2. We shall suppose that

(1) $gF_j \cap F_j = \varnothing$ if $g \in G_j - \{I\}$, $j = 1, 2$,

(2) $F_1^c \cap F_2^c = \varnothing$.

Then it is easy to see that the group $G_1 * G_2$ generated by G_1 and G_2 is isomorphic to the free product of G_1 and G_2 and is also discrete since

$$g(F_1 \cap F_2) \cap (F_1 \cap F_2) = \varnothing \quad (g \in G_1 * G_2 - \{I\}) . \tag{3.1}$$

In fact, to prove the first assertion one writes down an arbitrary word of $G_1 * G_2$ and one verifies that, by induction and 1 and 2, the last assertion holds; we shall see a refined version of this argument below. This particularly simple version of the Klein Combination Theorem I learnt from A. F. Beardon.

Let now G be any discrete subgroup of Con(n). Then we define

$$\delta(G) = \operatorname{Inf}\{s > 0;\ \sum_{g \in G} L(x, gx')^{-s} < \infty\} .$$

This does not depend on the choice of x, $x' \in H$. One has

$$0 \leqq \delta(G) \leqq n .$$

The problem with which we shall be concerned here is that of estimating $\delta(G_1 * G_2)$ given $\delta(G_1)$, $\delta(G_2)$ and some sharper forms of Condition 2 above. (For estimates from below see [2, Theorem 1 ff.].)

For these purposes we have to establish some inequalities. We shall work in the Poincaré model and we rephase (2.2) as

$$\| g_1^{-1}x' - g_2 x \|^2 = j_D(g_1^{-1}, x')j_D(g_1^{-1}, g_1 g_2 x)\| x' - g_1 g_2 x \|^2 ,$$

which we write in the form

$$j_D(g_1 g_2, x) = j_D(g_1^{-1}, x')j_D(g_2, x)\| x' - g_1 g_2 x \|^2/\| g_1^{-1}x' - g_2 x \| ,$$

which we shall use to estimate $j_D(g_1 g_2, x)$.

Suppose now that $g_2 \in G_1 * G_2$ is of the form

$$\gamma_1^{(2)}\, \gamma_1^{(1)}\, \gamma_2^{(2)}\, \gamma_2^{(1)} \cdots$$

where $\gamma_b^{(a)} \in G_a - \{I\}$. Let us observe next that

$$g_2(x) \in F_2^c$$

if $x \in F_1 \cap F_2$. This follows by induction on the length of g_2, since, by the induction hypothesis $\gamma_1^{(1)}\gamma_2^{(2)} \cdots (x) \in F_1^c \subset F_2$ by Condition 2), and so the assertion follows since by Condition 1 $\gamma_1^{(2)}(F_2) \subset F_2^c$. Moreover, if $g_1 \in G_1 - \{I\}$ then $g_1 g_2(x) \in F_1^c$. Thus with these assumptions about g_1, g_2 we obtain

$$(3.2) \qquad j_D(g_1 g_2, x) \leqq j_D(g_2, x) \operatorname*{Inf}_{x' \in B^{n+1}} \frac{\operatorname*{Sup}_{w \in F_1^c} \|x' - w\|^2 j_D(g_1^{-1}, x')}{\operatorname*{Inf}_{w' \in F_2^c} \|g_1^{-1}x' - w'\|^2} .$$

For convenience we write $\Delta_{12}(g_1)$ for the second factor on the right-hand side of this inequality, so that the inequality now reads

$$(3.3) \qquad j_D(g_1 g_2, x) \leqq j_D(g_2, x)\Delta_{12}(g_1) .$$

If $g \in G_2 - \{I\}$ then we can analogously define $\Delta_{21}(g)$.

We recall the following elementary lemma:

LEMMA 1. *If $x \in F_1 \cap F_2$ then*

$$\delta(G) = \operatorname{Inf}\{s > 0; \sum_{g \in G_1 * G_2} j_D(g, x)^s < \infty\} .$$

The proof of this will be left as an exercise for the reader.

LEMMA 2. *This series $\sum_{g \in G_1 * G_2} j_D(g, x)^s$ is dominated by*

$$\begin{aligned} 1 + (\sum_{k \geq 0} (\sum_{g_1 \in G_1 - \{I\}} \Delta_{12}(g_1)^s)^k (\sum_{g_2 \in G_2 - \{I\}} \Delta_{21}(g_2)^s)^k) \\ \times (((\sum_{g_1 \in G_1 - \{I\}} j_D(g_1, x)^s)(\sum_{g_2 \in G_2 - \{I\}} \Delta_{21}(g_2)^s + 1) \\ + (\sum_{g_2 \in G_2 - \{I\}} j_D(g_2, x)^s)(\sum_{g_1 \in G_1 - \{I\}} \Delta_{12}(g_1)^s + 1)) .\end{aligned}$$

PROOF. If $g \in G_1 * G_2$ can be written in the form

$$\gamma_1^{(1)}\gamma_2^{(2)} \cdots \gamma_k^{(2)}$$

where $\gamma_b^{(a)} \in G_a - \{I\}$ then we shall refer to g as a (1, 2) word; i.e., it begins with an element of $G_1 - \{I\}$ and ends with one of $G_2 - \{I\}$. Analogously one can define (2, 1), (1, 1) and (2, 2) words.

We consider first the partial sum

$$\sum j_D(g, x)^s$$

where g runs through the set of (1, 2) words of $G_1 * G_2$. If g has the form $\gamma_1^{(1)}\gamma_1^{(2)} \cdots \gamma_k^{(2)}$ then by successive applications of (3.3) we have

$$j_D(g, x) \leqq \Delta_{12}(\gamma_1^{(1)})\,\Delta_{21}(\gamma_1^{(2)}) \cdots \Delta_{12}(\gamma_k^{(1)}) j_D(\gamma_k^{(2)}, x) .$$

Raising this to the s^{th} power, summing first over all possible $\gamma_1^{(1)}$, $\gamma_1^{(2)}$, $\cdots$,

$\gamma_k^{(2)}$, and then over k we obtain the following majorant for the partial sum under consideration

$$\sum_{k\geq 0}\Big(\sum_{g_1\in G_1-\{I\}}\Delta_{12}(g_1)^s\Big)^{k+1}\Big(\sum_{g_2\in G_2-\{I\}}\Delta_{21}(g_2)^s\Big)^k\sum_{g\in G_2-\{I\}}j_D(g,x)^s\ .$$

The partial sum over the (2, 1) words is bounded by an analogous expression. Similarly the partial sum of (1, 1) words is dominated by

$$\sum_{k\geq 0}\Big(\sum_{g_1\in G_1-\{I\}}\Delta_{12}(g_1)^s\Big)^{k}\Big(\sum_{g_2\in G_2-\{I\}}\Delta_{21}(g_2)^s\Big)^k\sum_{g\in G_1-\{I\}}j_D(g,x)^s$$

and the partial sum of (2, 2) words likewise. Adding these and the term corresponding to $g = I$ we obtain the assertion of the lemma.

COROLLARY. *If* $s > \operatorname{Max}(\delta(G_1), \delta(G_2))$ *is such that*

$$\Big(\sum_{g_1\in G_1-\{I\}}\Delta_{12}(g_1)^s\Big)\Big(\sum_{g_2\in G_2-\{I\}}\Delta_{21}(g_2)^s\Big) < 1$$

then

$$\delta(G_1 * G_2) \leqq s\ .$$

REMARK 1. The arguments used here are those of [1, § 4], freed of unnatural restrictions.

REMARK 2. Note that, from (3.3),

$$\Delta_{12}(g) \geqq j_D(g, g_2x) \quad (g \in G_1 - \{I\})$$

for any fixed g_2 in G_2. Thus in order that

$$\sum_{g\in G_1-\{I\}}\Delta_{12}(g)^s$$

should converge, it is necessary that $s \geqq \delta(G_1)$. Likewise, if there exists $c > 0$ such that $\|w_1 - w_2\|^2 \geqq c$ for $w_1 \in F_1^c$, $w_2 \in F_2^c$ then, taking x' in $F_1 \cap F_2$

$$\Delta_{12}(g) \leqq (4/c^2)j_D(g^{-1}, x')$$

and so, under these circumstances $s > \delta(G_1)$ implies that the series above converges.

REMARK 3. The corollary is too general to be of much significance as it stands. It is usually not all that easy to estimate Δ_{12} and Δ_{21}.

REMARK 4. One can analogously find lower bounds which sharpen those of [2]. Since we have no application for these in mind we shall not discuss them here.

REMARK 5. One has a natural homomorphism $\theta\colon G_1 * G_2 \to G_1$. Let G_{12} be the kernel of this map. Then G_{12} is the normal subgroup generated by G_2 in $G_1 * G_2$. The methods used above can be applied to find upper and lower bounds for $\delta(G_{12})$; these are the same as those for $\delta(G_1 * G_2)$. Akaza and Furusawa suggest in [1] that one should have

$$\delta(G_{12}) = \delta(G_1 * G_2) ,$$

at least when G_2 is 'elementary'. This seems to be an over-hasty inference. The question as to the relationship in general between $\delta(N)$ and $\delta(G)$ where N is a normal subgroup in G is a difficult and only partially answered one. The overall situation is unclear, but see [4] for some very interesting results. At any rate one can have in some cases $\delta(N) = \delta(G)$ (see, for example, [2, Theorem 3 ff]), whereas some examples are known with $\delta(N) < \delta(G)$ ([3, Theorem 4.4]).

4. **Examples.** As an illustration of these techniques we shall first consider the case of the Hecke groups $G(\lambda)$ which operate on D^2. We shall continue to use the Poincaré model. In this case elements of Con (1) can be conveniently written since

$$PSU(1, 1) = \left\{\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}; |\alpha|^2 - |\beta|^2 = 1\right\} \Big/ \{\pm I\}$$

operates on D^2 by

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}(z) = (\alpha z + \beta)/(\bar{\beta} z + \bar{\alpha})$$

where the arithmetical operations are carried out in the complex plane. This identifies $PSU(1, 1)$ with Con (1).

The group $G(\lambda)$ is generated by $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $\begin{pmatrix} 1 + i\lambda & -i\lambda \\ i\lambda & 1 - i\lambda \end{pmatrix}$. We take

$$G_1 = \left\{I, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right\}$$

$$G_2 = \left\{\begin{pmatrix} 1 + im\lambda & -im\lambda \\ im\lambda & 1 - im\lambda \end{pmatrix}; m \in \boldsymbol{Z}\right\} ,$$

and

$$F_1 = \{x \in S^1; x_0 > 0\}$$

$$F_2 = \{x \in S^1; x_0 < (\lambda^2 - 1)/(\lambda^2 + 1)\} .$$

Then Condition 1 of § 3 is clearly satisfied since F_1 is clearly a 'fundamental domain' for G_1 and F_2 is the one for G_2 constructed by the method of isometric circles. Moreover, if $\lambda > 1$ then $F_1^c \cap F_2^c = \varnothing$ and hence the Klein Combination Theorem applies. The group $G_1 * G_2$ will be denoted by $G(\lambda)$. We shall estimate $\delta(G(\lambda))$ from above.

We have to estimate $\Delta_{12}\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right)$ and $\Delta_{21}\left(\begin{pmatrix} 1+im\lambda & -im\lambda \\ im\lambda & 1-im\lambda \end{pmatrix}\right)$. In the first case

$$\Delta_{12}\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right) = \operatorname*{Inf}_{x'} \frac{\operatorname*{Sup}_{w \in F_1^c} \| x' - w \|^2 j(g_1^{-1}, x')}{\operatorname*{Inf}_{w' \in F_2^c} \| g_1^{-1}(x') - w' \|^2}$$

where $g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. We take $x' = 0$ in the inner expression on the right-hand side, which yields a value greater than the infimum. This value is 1, so that

$$\Delta_{12}\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right) \leqq 1 .$$

Likewise if $g \in G_2 - \{I\}$ one has

$$\Delta_{21}(g) \leqq \frac{\operatorname*{Sup}_{w \in F_2^c} \| 0 - w \|^2 j_D(g^{-1}, 0)}{\operatorname*{Inf}_{w' \in F_1^c} \| g^{-1}(0) - w' \|^2} .$$

The denominator here exceeds $((\sqrt{5} - 1)/2)^2$ since $g^{-1}(0)$ lies on the circle $(x_0 - 1/2)^2 + x_2^2 = 1/4$. Thus

$$\Delta_{21}\left(\begin{pmatrix} 1+im\lambda & -im\lambda \\ im\lambda & 1-im\lambda \end{pmatrix}\right) \leqq \left(\frac{\sqrt{5}+1}{2}\right)^2 (1 + m^2\lambda^2)^{-1} .$$

Hence if s satisfies

$$\left(\frac{\sqrt{5}+1}{2}\right)^{2s} \sum_{m \neq 0} (1 + m^2\lambda^2)^{-s} < 1$$

then $\delta(G(\lambda)) \leqq s$. By Cauchy's inequality

$$\sum_{m \neq 0} (1 + m^2\lambda^2)^{-s} \leqq \int_{-\infty}^{\infty} (1 + \lambda^2\xi^2)^{-s} d\xi = \lambda^{-1}\pi^{1/2}\Gamma(s - 1/2)/\Gamma(s) .$$

Hence if s satisfies

$$\left(\frac{\sqrt{5}+1}{2}\right)^{2s} \pi^{1/2}\Gamma(s - 1/2)/\Gamma(s) < \lambda$$

then $\delta(G(\lambda)) \leqq s$. In particular one sees that

$$\delta(G(\lambda)) = 1/2 + O(\lambda^{-1})$$

as $\lambda \to \infty$. This result was proved by Beardon by a similar method.

REMARK. The same method can be used to give lower bounds for

$\delta(G(\lambda))$. One finds, for example, that if s is such that

$$4^{-s}\sum_{m\neq 0}(1+m^2\lambda^2)^{-s}>1$$

then $\delta(G(\lambda))\geqq s$.

Now we shall turn to a question which generalizes that which we have just considered and which we shall need in § 5. We fix first two groups G_1 and G_2 which we shall "decouple" by constructing G_1 and hG_2h^{-1} where h shall, in a certain sense, tend to infinity. Then G_1 and hG_2h^{-1} will have less and less interaction with each other so that one would expect that $\delta(G_1 * hG_2h^{-1})$ would approach $\operatorname{Max}(\delta(G_1), \delta(G_2))$. Our purpose now is to give conditions under which this can occur.

Let $H\subset \operatorname{Con}(n)$ be a countable ordered set, so that we can speak of $h\to\infty$ $(h\in H)$. We shall assume that we are given H, G_1, G_2, F_1, F_2 so that

(1) $g(F_j)\cap F_j=\varnothing \quad (g\in G_j-\{I\})$

(2) $F_1^c\cap h(F_2^c)=\varnothing \quad (h\in H)$.

Let $d(S_1, S_2)$ denote the Euclidean distance between the two sets S_1 and S_2. Then we define

$$E_1(h)=d(F_1^c, hF_2^c)$$

and

$$E_2(h)=\operatorname*{Sup}_{w\in F_2^c} j_D(h, w)\ .$$

THEOREM 1. *Suppose that as $h\to\infty$ in H*

$$E_1(h)^{-1}E_2(h)\to 0\ .$$

Then

$$\delta(G_1 * (hG_2h^{-1}))\to \operatorname{Max}(\delta(G_1), \delta(G_2))\ .$$

REMARK. This is a formalized version of Theorems 2, 3 and 4 in [1].

PROOF. We estimate $\Delta_{12}(g)$ $(g\in G_1-\{I\})$ and $\Delta_{21}(hgh^{-1})$ $(g\in G_2-\{I\})$.

The set hF_2 satisfies Condition 1 of § 3 for hG_2h^{-1} and hence we can apply the results found there.

The quantity $\Delta_{12}(g)$ is easily estimated. We fix $x'\in F_1\cap F_2$. Then

$$\Delta_{12}(g)\leqq \frac{\operatorname*{Sup}_{w\in F_1^c}\|x'-w\|^2}{\operatorname*{Inf}_{w'\in hF_2^c}\|g^{-1}x'-w'\|^2}\, j_D(g^{-1}, x')\ .$$

As we have already noted $g^{-1}(x')\in F_1^c$ so that

$$\Delta_{12}(g)\leqq 4E_1(h)^{-1}j_D(g^{-1}, x')\ .$$

Now we turn to $\Delta_{21}(hgh^{-1})$; here we estimate the outer infimum by choosing x' to be of the form hx'' where $x'' \in F_1 \cap F_2$. Then

$$\Delta_{21}(hgh^{-1}) \leqq \frac{\operatorname{Sup}_{w \in hF_2^c} \| hx'' - w \|^2}{\operatorname{Inf}_{w' \in F_1^c} \| hg^{-1}x'' - w' \|^2} j_D(hg^{-1}h^{-1}, hx'') .$$

In this expression we first replace w by $h(w)$ with w now in F_2^c. Then we apply (2.2) to the numerator. We also apply (2.3) to the j-factor. Finally, note that $g^{-1}x'' \in F_2^c$. We thus obtain

$$\Delta_{21}(hgh^{-1}) \leqq E_1(h)^{-1} \operatorname{Sup}_{w \in F_2^c} \| x'' - w \|^2 j_D(h, w) j_D(h, g^{-1}x'') j_D(g^{-1}, x'') .$$

But now

$$j_D(h, w) \leqq E_2(h) \quad \text{as} \quad w \in F_2^c$$

and

$$j_D(h, g^{-1}x'') \leqq E_2(h) \quad \text{as} \quad g^{-1}(x'') \in F_2^c .$$

Thus it follows that

$$\Delta_{21}(hgh^{-1}) \leqq 4E_1(h)^{-1}E_2(h)^2 j_D(g^{-1}, x'') .$$

Thus if $s > \operatorname{Max}(\delta(G_1), \delta(G_2))$ one has that

$$\Big(\sum_{g_1 \in G_1 - \{I\}} \Delta_{12}(g_1)^s\Big) \cdot \Big(\sum_{g_2 \in G_2 - \{I\}} \Delta_{21}(hg_2h^{-1})^s\Big)$$

is less than

$$2^{4s}(E_1(h)^{-1}E_2(h))^{2s}\Big(\sum_{g_1 \in G_1 - \{I\}} j_D(g_1, x')^s\Big) \cdot \Big(\sum_{g_2 \in G_2 - \{I\}} j_D(g_2, x'')^s\Big) .$$

From this and the corollary of § 3 the theorem follows at once.

REMARK. It seems that one obtains generally good estimates if one chooses an x' (as in the definition of Δ_{12}) to lie in $F_1 \cap F_2$. One can use this to prove further results of the same type. For example, suppose that $d(F_1, F_2) > 0$ and that, if $\delta(G_1) \geqq \delta(G_2)$,

$$\sum_{g \in G_1 - \{I\}} j_D(g, x)^{\delta(G_1)}$$

diverges. Then it follows from the corollary to Lemma 2 that

$$\delta(G_1 * G_2) > \delta(G_1) .$$

The condition on G_1 can be verified for some classes of groups, see, for example [2, § 2] and [5, Theorem 8]. Such inequalities were first proved by Beardon who used them to show that if G is not elementary then $\delta(G) > 0$.

5. Groups of the first kind. The objective of this section is the proof of Theorem 2 below. The history of this result has already been referred to in § 1 and further details can be found in [3]. One should note that, in [3], it was suggested that if G_j $(j \geqq 1)$, $G_1 \subset G_2 \subset G_3 \subset \cdots$ are subgroups of a discrete subgroup G of Con(n) and $\bigcup_{j \geq 1} G_j = G$ then one should have

$$\lim \delta(G_j) = \delta(G) .$$

This has since been proved by Sullivan in [5, Cor. 6]. It forms an essential part of the proof of Theorem 2.

Let G be a discrete subgroup of Con(n) which we take to operate on D^{n+1}. Then the limit set $L(G)$ of G is the intersection of all nonempty, closed, G-invariant subsets of D^{n+1} (or S^n). A group G is said to be of the *first kind* if $L(G) = S^n$, and of the *second kind* otherwise.

THEOREM 2. *Let $\varepsilon > 0$ be given. Then there exists a discrete subgroup G of* Con(n) *of the first kind with $\delta(G) \leqq \varepsilon$.*

The condition that $\delta(G) \leqq \varepsilon$ expresses that G should be "small"; that G be of the first kind means that G should be "large". The point of the theorem is that the two notions are in general independent of one another. Cf. however, [5, Theorem 22 ff]. If one considers instead of *closed* subsets of S^n *measurable* subsets then one can show that whenever $\delta(G) < n/2$ then there is a measurable subset U of S^n such that

(a) $U \cup gU = \varnothing$ $(g \neq I)$

(b) meas $(S^n - \bigcup_{g \in G} gU) = 0$,

where meas denotes the n-dimensional Lebesgue measure on S^n. For a discussion of this property see [3].

PROOF. We choose a countable dense ordered subset $\zeta_1, \zeta_2, \zeta_3, \cdots$ of S^n. Choose $\varepsilon_1, \varepsilon_2, \cdots$ a strictly increasing sequence of positive numbers, such that $\varepsilon_j \to \varepsilon$. Let U be an open neighborhood of 0 with $\bar{U} \subset D^{n+1}$. Then we shall construct a sequence of discrete subgroups

$$G_1 \subset G_2 \subset G_3 \subset \cdots$$

such that

(a) $\delta(G_j) \leqq \varepsilon_j$,

(b) $g(U) \cap U = \varnothing$ $(g \in G_j - \{I\})$,

(c) $\zeta_1, \zeta_2, \zeta_3, \cdots, \zeta_j \in L(G_j)$.

This will clearly suffice, since, if $G = \bigcup_j G_j$ then G is discrete by (b), of the first kind by (c) and that $\delta(G) \leqq \varepsilon$ holds follows from the result of Sullivan quoted above.

The construction is thus inductive; we suppose that G_j has already been constructed and we shall now show how to construct G_{j+1}. There exists a set $F_j \supset U$ such that $gF_j \cap F_j = \varnothing$ when $g \in G_j - \{I\}$; indeed we can take $F_j = U$.

If $\zeta_{j+1} \in L(G_j)$ then we take $G_{j+1} = G_j$. Suppose now that $\zeta_{j+1} \notin L(G_j)$. We shall also assume that G_j has no elements of finite order apart from the identity. As it happens G_j will be a free product of infinite cyclic groups, and this condition will also be preserved in the inductive step. It therefore follows that ζ_{j+1} is not a fixed point of any element of $G_j - \{I\}$; for if it were, that element would be of infinite order and ζ_{j+1} would then belong to the limit set of this cyclic group. To verify this one shows, for example by computing the stabilizer of $(1, 0, 0, \cdots, 0)$ in the Klein model, that the stabilizer of ∞ in the upper half-space model is the set of maps of the form

$$(z_0, z) \mapsto (\lambda z_0, \lambda A(z) + b)$$

where $\lambda > 0$, $A \in SO_n$, $z, b \in \boldsymbol{R}^n$. It is clear that any discrete infinite cyclic subgroup of this group has ∞ as a limit point.

Thus as $\zeta_{j+1} \notin L(G_j)$ we can find an open neighborhood V of ζ_{j+1} in $\bar{D}^{n+1}$ such that

$$V \cap g(V) = \varnothing \quad (g \in G_j - \{I\}) .$$

We can, and shall choose V such that

$$U \cap (\bigcup_{g \in G_j} gV) = \varnothing .$$

Thus we can take $F_j = U \cup V$ which therefore satisfies the condition above and moreover $\zeta_{j+1} \in F_j$.

Now we shall construct a cyclic group Γ_j and an open set Φ_j such that

$$\gamma\Phi_j \cap \Phi_j = \varnothing \quad (\gamma \in \Gamma_j - \{I\}) .$$

We shall arrange that ζ_{j+1} shall be in the limit set of Γ_j. We shall also construct an ordered set H_j of elements of $\mathrm{Con}(n)$ such that ζ_{j+1} is a fixed point of each element of H_j and moreover if V is an open neighborhood of ζ_{j+1} in $\bar{D}^{n+1}$ then $h(\Phi_j^c) \subset V$ for all sufficiently large h (in the sense of the order on H_j). In particular, for h large enough $(h\Phi_j^c) \subset F_j$. We shall also arrange that $\delta(\Gamma_j) = 0$. With these assumptions we need only verify that if

$$E_1(h) = d(F_j^c, h(\Phi_j^c))$$

and

$$E_2(h) = \operatorname*{Sup}_{w \in \Phi_j^c} j(h, w) ,$$

then

$$E_1(h)^{-1}E_2(h) \to 0 ,$$

since then, by Theorem 1, $\delta(G_j * (h\Gamma_j h^{-1})) \to \delta(G_j)$ as $h \to \infty$, and so for suitable h we have

(a) $\delta(G_j * h\Gamma_j h^{-1}) \leqq \varepsilon_{j+1}$

(b) $F_j \cap h\Phi_j \supset U$,

and, since by the Klein Combination Theorem the translates of $F_j \cap h\Phi_j$ under $G_j * h\Gamma_j h^{-1}$ are pairwise disjoint, we see that $G_{j+1} = G_j * h\Gamma_j h^{-1}$ satisfies the conditions of the inductive step.

It now remains to describe the construction of Γ_j, Φ_j and H_j, and to verify that the objects constructed have the properties claimed. To do this it is best to refer to the upper half-space model of hyperbolic space, and we shall make ζ_{j+1} correspond to ∞. We observe again that the group fixing ∞ consists of elements of the form

$$(z_0, z) \mapsto (\lambda z_0, \lambda Az + b)$$

with the same convensions as above. First of all we fix $\lambda_0 > 1$ and let

$$\Gamma_j = \{(z_0, z) \mapsto \lambda_0^m(z_0, z); m \in \boldsymbol{Z}\} .$$

Then ∞ is a limit point of this group and $\delta(\Gamma_j) = 0$.

Next let

$$\Phi_j = \{x; 1 < \|x\| < \lambda_0\} ,$$

which satisfies

$$\gamma\Phi_j \cap \Phi_j = \varnothing \quad (\gamma \in \Gamma_j - \{I\}) .$$

Now choose $b_0 \in \boldsymbol{R}^n$, $\|b_0\| = \lambda_0^{1/2}$ and set

$$H_j = \{x \mapsto 2^m(x - b_0) + b_0; m \in \boldsymbol{N}\} .$$

This is a hyperbolic semigroup for which $b_0 \in \bar{\Phi}_j$ is the repulsive fixed point and ∞ the attractive fixed point. This means, in the Poincaré model, that for any neighborhood V' of ζ_{j+1} in $\bar{D}^{n+1}$ for all sufficiently large h in H_j one has

$$h(\Phi_j)^c \subset V' ,$$

as required. Moreover, since F_j is an open neighborhood of ζ_{j+1} this statement is also valid for $V' = F_j$. Further one has, again for large enough h,

$$d(h(\Phi_j^c), F_j) \geqq c_1 > 0 .$$

Thus G_j and $h\Gamma_j h^{-1}$ satisfy the conditions of the Klein Combination Theorem, and also

$$E_1(h) \geqq c_1 > 0 .$$

It therefore remains to check that

$$\operatorname*{Sup}_{w \in \Phi_j^c} j_D(h, w) \to 0$$

as $h \to \infty$.

This involves computing $j_D(h, w)$ and it is convenient to work in the upper half-space model of hyperbolic space. We choose, as in § 2, maps

$$\psi: D^{n+1} \to H^{n+1}$$
$$\psi^{-1}: H^{n+1} \to D^{n+1}$$

so that $\psi(\zeta_{j+1}) = \infty$. To be able to make use of our earlier formulae we suppose that $\zeta_{j+1} = (1, 0, \cdots, 0)$, an assumption that involves no loss of generality.

If $z \in H^{n+1}$ then

$$1 - \|\psi^{-1}(z)\|^2 = \operatorname{Im}(z)/(1/4 + \operatorname{Im}(z) + \|z\|^2) ,$$

where $\operatorname{Im}: H^{n+1} \to R_+^\times$ was defined in § 2. $h \in H_j$ then has the form

$$h(\psi^{-1}(z)) = \psi^{-1}(2^m(z - b_0) + b_0) ,$$

so that

$$\begin{aligned} j_D(h, \psi^{-1}(z)) &= (1 - \|h\psi^{-1}(z)\|^2)/(1 - \|\psi^{-1}(z)\|^2) \\ &= \frac{2^m(1/4 + \operatorname{Im}(z) + \|z\|^2)}{1/4 + 2^m \operatorname{Im}(z) + \|2^m(z - b_0) + b_0\|^2} . \end{aligned}$$

Since z lies in $\{z: \|z - b_0\|^2 \geqq c\}$ for a certain $c > 0$ we have to verify that this expression tends uniformly to zero on such a set as $m \to \infty$. Let $y = \operatorname{Im}(z)$, $R = (z_1^2 + \cdots + z_n^2)^{1/2}$; then we have to verify that in the region

$$y^2 + R^2 \geqq c , \quad y > 0,\ R > 0$$

the expression

$$2^m((y + 1/2)^2 + R^2)/((2^m y + 1/2)^2 + (2^m R)^2)$$

tends uniformly to 0 as $m \to \infty$. If we let $A = R^2 + y^2$ then the expression is

$$2^m(A + y + 1/4)/(2^{2m} A + 2^m y + 1/4) .$$

If $2^{m+1}c \geqq 1$ then one sees that this is, as a function of y, increasing. However since $y \leqq A^{1/2}$ this means that the expression is bounded by

$$2^m(A^{1/2} + 1/2)^2/(2^m A^{1/2} + 1/2)^2 ,$$

which is itself a decreasing function of A, and so is bounded by

$$2^m(c^{1/2} + 1/2)^2/(2^m c^{1/2} + 1/2)^2$$

which tends to zero as $m \to \infty$. This proves the assertion and with it the theorem.

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MATHEMATISCHES INSTITUT DER UNIVERSITÄT
BUNSENSTRASSE 3-5
D-34 GÖTTINGEN
WEST GERMANY