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ON HYPERELLIPTIC POLARIZED VARIETIES

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Introduction. The purpose of this paper is to study polarized varieties which are double coverings of projective varieties of Δ-genus zero. Such varieties are called hyperelliptic polarized varieties (see (1.1) for a precise definition), because the Δ-genus is a higher-dimensional analogue of the genus of curves (cf. [F1], [F6] etc.). As examples of these varieties, we have double coverings of $\boldsymbol{P}^n$ (cf. [W]), $K3$-surfaces polarized by hyperelliptic curves on them (cf. [Sa 2]), Fano-threefolds whose anti-canonical linear systems are not very ample but have no base point (cf. [Is]), canonically polarized surfaces with $c_1^2 = 2p_g - 4$ (cf. [Ho 1]), etc. The present article is an outcome of the efforts to find a unified systematic method for the study of them. In particular, the works of Horikawa and Iskovskih were very stimulating for the author.

Compared with [Ho 1], our theory is still incomplete because of the ampleness assumption. This is almost equivalent to assuming that the branch loci are non-singular, which is not the case in many interesting examples such as Hilbert modular surfaces. Moreover, since the ampleness is not preserved under specialization, our result is not powerful enough to study deformations of hyperelliptic polarized varieties (cf. (8.33)). The author hopes to improve these points by systematically developing a theory of semipolarized varieties in future.

In §1 we give a characterization of hyperelliptic polarized varieties. In §2, assuming $\mathrm{char}(\mathfrak{k}) \neq 2$ from that time on, we review a general theory on the structure of double coverings. Then, according to the structures of the image varieties of Δ-genus zero, we classify hyperelliptic polarized manifolds into five types (I), (II), (IV), (Σ) and (*). Their structures are studied in more detail in §3, §4 and §5, where we classify them further according to the nature of the branch loci. The results are summarized in tables in §6 for the convenience in later use. §7 and §8 are devoted to the study of their deformations. In the Appendix we give generalized versions of classical results on curves.

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Notation, convention and terminology. Basically we employ the notation as in [F1] ~ [F6], which is similar to that of [EGA] and [Ha 2]. We work in the category of $\mathfrak{K}$-schemes of finite type, where $\mathfrak{K}$ is a fixed algebraically closed field of any characteristic (however, from §2 on, we assume $\mathrm{char}(\mathfrak{K}) \neq 2$). A point means a $\mathfrak{K}$-rational point. A variety means an irreducible reduced $\mathfrak{K}$-scheme, which is assumed to be proper over $\mathfrak{K}$ unless specifically stated to the contrary. A manifold means a non-singular variety. Line bundles are identified with the invertible sheaves of their sections. Tensor products of line bundles are denoted additively, while the intersection numbers of them are denoted multiplicatively.

Here are some of the symbols we use often.

$[\Lambda]$: The line bundle associated with a linear system Λ of Cartier divisors.

$\mathrm{Bs}\,\Lambda$: The set-theoretic intersection of all the members of Λ.

ρ_Λ: The rational mapping defined by Λ.

$|L|$: The complete linear system associated with a line bundle L.

$h^q(\mathscr{F}[L]) := \dim H^q(\mathscr{F} \otimes_{\mathscr{O}} \mathscr{L})$ for a coherent sheaf $\mathscr{F}$, where $\mathscr{L}$ is the invertible sheaf of sections of L.

L_T: The pull-back of L to a space T by a given morphism. However, when there is no danger of confusion, we write very often simply L instead of L_T. Similar convention is used also for Cartier divisors, vector bundles, etc.

ω_V: The dualizing sheaf of a locally Macaulay variety V.

K^M: The canonical bundle of a manifold M. So $\omega_M = \mathscr{O}_M[K^M]$.

$c_i(M)$: The i-th Chern class of M.

$p_g(M) := h^n(M, \mathscr{O}_M) = h^0(M, K^M)$, the geometric genus of M.

$b_i(M)$: The i-th Betti number of M.

T^M: The tangent bundle of M.

$\Theta_M := \mathscr{O}_M[T^M]$, the sheaf of vector fields on M.

$\boldsymbol{P}(E)$: The $\boldsymbol{P}^{r-1}$-bundle $E^\vee - \{\text{zero section}\}/\mathfrak{K}^\times$ associated with a vector bundle E of rank r, where $E^\vee$ is the dual bundle.

H^E: The tautological line bundle on $\boldsymbol{P}(E)$, corresponding to the invertible sheaf $\mathscr{O}(1)$ in the notation of [EGA].

$H_\alpha, H_\beta, \cdots$: The line bundle defined by hyperplane sections on projective spaces $\boldsymbol{P}_\alpha, \boldsymbol{P}_\beta, \cdots$ indicated by the same Greek letters.

$R_B(W)$: The double covering of W with branch locus B (cf. (2.1)).

$d(M, L) := L^n$, where (M, L) is a polarized variety and $n = \dim M$.

$g(M, L)$: The sectional genus of (M, L).
$\chi_j(M, L)$: The j-th sectional Euler-Poincaré characteristic of (M, L) (see (8.4; 1) for a precise definition).
$\Delta(M, L) := n + d(M, L) - h^0(M, L)$, the Δ-genus of (M, L).

1. Characterizations of hyperelliptic polarized varieties.

DEFINITION (1.1). A polarized variety (V, L) is said to by *hyperelliptic* if $\mathrm{Bs}\,|L| = \varnothing$, the morphism $\rho_{|L|}: V \to \boldsymbol{P}^N (N = \dim |L|)$ is of degree two onto its image W and if $\Delta(W, H) = 0$ for the hyperplane section H on W.

REMARK. The morphism $\rho: V \to W$ is finite since $L = \rho^* H$ is ample on V. If $\dim V = 1$, then V is a hyperelliptic curve, because W is a Veronese curve $\cong \boldsymbol{P}^1$.

LEMMA (1.2). *Let L be a line bundle on a variety V such that $\mathrm{Bs}\,|L| = \varnothing$. Let W be the image of the rational mapping $\rho_{|L|}: V \to \boldsymbol{P}^N$ $(N = \dim |L|)$ and let H be the hyperplane section bundle. Then the natural mapping $\rho^*: H^0(W, H) \to H^0(V, L)$ is bijective.*

PROOF. $H^0(\boldsymbol{P}^N, \mathcal{O}(1)) \to H^0(V, L)$ is bijective by the definition of $\rho_{|L|}$. This factors through $H^0(W, H)$. So ρ^* is surjective. On the other hand, ρ^* is injective since ρ is surjective. Hence ρ^* is bijective.

PROPOSITION (1.3). *$d(V, L) = 2\Delta(V, L)$ for any hyperelliptic polarized variety (V, L).*

PROOF. Let W and H be as in (1.1) and set $n = \dim V$, $w = \deg W$. Then $d(V, L) = 2w$. $\Delta(W, H) = 0$ means $h^0(W, H) = n + w$. So $\Delta(V, L) = n + d(V, L) - h^0(W, H) = w$ by (1.2). Thus we obtain $d(V, L) = 2\Delta(V, L)$.

(1.4) In the rest of this section we will consider the converse of the above fact. In particular, we will prove the following:

THEOREM. *Let (V, L) be a polarized variety such that $\mathrm{Bs}\,|L| = \varnothing$, $d = d(V, L) = 2\Delta(V, L) = 2\Delta$ and $g = g(V, L) > \Delta$. Then (V, L) is hyperelliptic unless L is simply generated and (V, L) is a Fano-$K3$ variety.*

The meaning of "Fano-$K3$ variety" is defined below.

DEFINITION (1.5). A polarized variety (V, L) is said to be *globally Macaulay* if $H^q(V, tL) = 0$ for any integers q, t with $0 < q < n = \dim V$. In this case V is locally Macaulay. (For a proof, see, e.g., [F6; (5.8)].)

(V, L) is said to be *globally Gorenstein* if it is globally Macaulay and if the dualizing sheaf ω_V is isomorphic to $\mathcal{O}_V[rL]$ for some integer

r. r is called the *index* and $r+n-1$ is called the *sectional index* of (V, L). Of course, V is locally Gorenstein if (V, L) is globally Gorenstein.

(V, L) is called a *Fano-K3 variety* if it is globally Gorenstein and if the sectional index is one.

REMARK (1.6). Let V be a projectively normal subvariety of $\boldsymbol{P}^N$ with hyperplane section H. Then the local ring at the vertex of the affine cone of V is Macaulay (resp. Gorenstein) if and only if (V, H) is globally Macaulay (resp. Gorenstein). But we do not need this fact in this paper and proof is omitted.

EXAMPLES (1.7). (1) $(\boldsymbol{P}^n, H)$ is globally Gorenstein with index $-n-1$. Conversely, any globally Gorenstein polarized variety with sectional index $\leqq -2$ is isomorphic to $(\boldsymbol{P}^n, H)$. For a proof, use the arguments in [F1]. Compare also (1.11) below.

(2) Any globally Gorenstein polarized variety with sectional index -1 is isomorphic to a hyperquadric. This is proved similarly as (1).

(3) Globally Gorenstein polarized varieties with sectional index 0 were called *Del Pezzo varieties* in [F6]. We have $\Delta(V, L)=1$ in this case and a classification theorem of Del Pezzo manifolds (=non-singular varieties) are obtained.

(4) Any polarized $K3$-surface is globally Gorenstein with sectional index one, and hence a Fano-$K3$ variety. Any canonical curve of genus $\geqq 2$ is a Fano-$K3$ variety. If M is a complex threefold with $L=-K_M$ being ample, then (M, L) is a Fano-$K3$ variety.

(5) If the canonical bundle K of a non-singular surface S is ample and if $H^1(S, \mathcal{O}_S)=0$, then (S, K) is a globally Gorenstein manifold with sectional index two.

(6) For any complete intersection V of type $(d_1, \cdots, d_r)$ in $\boldsymbol{P}^{n+r}$, (V, H) is globally Gorenstein with index $d_1+\cdots+d_r-n-r-1$.

PROPOSITION (1.8). *Let (V, L) be a polarized variety and let D be an irreducible reduced member of $|L|$ such that $H^0(V, tL) \to H^0(D, tL_D)$ is surjective for every t. Then*

(1) *(V, L) is globally Macaulay if so is (D, L_D).*

(2) *If (D, L_D) is globally Gorenstein, then (V, L) is also globally Gorenstein and the sectional indices are the same.*

PROOF. For (1), apply [F3; (2.1)]. To prove (2), set $\omega_D=\mathcal{O}_D[rL_D]$. V is locally Macaulay by (1) and we have $\omega_D=\omega_V[L]_D$ by the adjunction formula. Applying [F3; (2.2)], we infer $H^0(V, \omega_V[tL])=0$ for any $t \leqq -r$. So, from the exact sequence $0=H^0(V, \omega_V[-rL]) \to$

$H^0(V, \omega_V[(1-r)L]) \to H^0(D, \omega_D[-rL]) \to H^1(V, \omega_V[-rL]) = 0$, we infer that there is a homomorphism $\varphi: \mathcal{O}_V \to \omega_V[(1-r)L]$ such that φ_D is an isomorphism. Then the supports of both $\mathscr{K} = \mathrm{Ker}(\varphi)$ and $\mathscr{C} = \mathrm{Coker}(\varphi)$ are at most finite sets, because they do not meet the ample divisor D. Hence $\mathscr{K} = 0$, since it is a subsheaf of the torsion free sheaf $\mathcal{O}_V$. Using the exact sequence $0 \to \mathcal{O}_V \to \omega_V[(1-r)L] \to \mathscr{C} \to 0$ and $H^1(V, \mathcal{O}_V) = 0$, we obtain $h^0(\mathscr{C}) = h^0(V, \omega_V[(1-r)L]) - 1 = 0$. So $\mathscr{C} = 0$ and φ is an isomorphism. Thus we prove (2).

(1.9) PROOF OF THE THEOREM (1.4). Let (W, H) be as in (1.2) and set $w = \deg W$, $n = \dim V$. Let δ be the mapping degree of $\rho = \rho_{|L|}: V \to W$. Then $2\Delta = d = \delta w$ and $0 \leqq \Delta(W, H) = n + w - h^0(V, L) = w - \Delta$ by (1.2). Hence, if $\delta \geqq 2$, the equality must hold and $\delta = 2$. This means that (V, L) is hyperelliptic. So we consider the case $\delta = 1$.

Suppose first that $n = 1$. We claim $g = \Delta + 1$. Indeed, otherwise, we would have $h^0(\omega_V[-L-p]) = h^1(V, L) - 1 = g - \Delta - 1 > 0$ for a general point p on V. Moreover $\mathrm{Bs}\,|L-p| = \varnothing$ since ρ is birational. So (A4) in the Appendix would imply $h^0(L) + h^0(\omega[-L]) \leqq h^0(p) + h^0(\omega[-p]) = g$. But the left hand side equals $1 + g$. This contradiction proves our claim $g = \Delta + 1$. Therefore $h^0(\omega[-L]) = h^1(L) = g - \Delta = 1$ and we have a non-zero homomorphism $\varphi: \mathcal{O}[L] \to \omega$. By [F2; Lemma 1.4], φ is an isomorphism since $\deg L = d = 2\Delta = 2g - 2 = \deg(\omega)$. Applying (A1) we further see that L is simply generated. Thus the assertion is true in case $n = 1$.

For the case $n \geqq 2$, we use the induction on n. A general member D of $|L|$ is a regular rung of (V, L) by [F6; (3.7)]. So $d(D, L) = d$, $g(D, L) = g$ and $\Delta(D, L) = \Delta$. Hence L_D is simply generated and (D, L) is globally Gorenstein with sectional index one by the induction hypothesis. Then, by [F2; Cor. 2.3] (or [F3; (3.1)], [F1; Prop. 1.7]), we infer that L is simply generated on V and that $H^0(V, tL) \to H^0(D, tL)$ is surjective for every t. Applying (1.8) we complete the proof.

COROLLARY (1.10). *Let (V, L) be a polarized variety such that $\mathrm{Bs}\,|L| = \varnothing$, $d(V, L) = 2\Delta(V, L)$ and $g(V, L) > \Delta(V, L) + 1$. Then (V, L) is hyperelliptic.*

PROPOSITION (1.11). *Let (V, L) be a Fano-K3 variety with $n = \dim V \geqq 3$. Then $\chi(V, tL) = (2t + n - 2)(t + 1) \cdots (t + n - 3)(2^{-1}dt(t + n - 2) + n(n-1))/n!$ where $d = d(V, L)$. Moreover $d = 2\Delta(V, L)$ and $g(V, L) = \Delta(V, L) + 1$.*

PROOF. We use the same argument as in [F6; (5.9)]. We have

$H^n(V, tL) = 0$ for $t > 2 - n$ since $\omega_V = \mathcal{O}_V[(2-n)L]$. So $\chi(V, tL) = 0$ for $2 - n < t < 0$ and hence we may set $\chi(V, tL) = (t+1)\cdots(t+n-3)(dt^3 + at^2 + bt + c)/n!$ for some constants a, b and c. Moreover $c = n(n-1)(n-2)$ because $\chi(V, \mathcal{O}_V) = 1$. We infer $d = 2g(V, L) - 2$ from $\omega_V = (2-n)L$. Hence $a = 3d(n-2)/2$. Using $\chi(V, (2-n)L) = (-1)^n h^n(V, \omega_V) = (-1)^n$ in addition, we obtain $b = d(n-2)^2/2 + 2n(n-1)$. These calculations give the above formula for $\chi(V, tL)$. Now we have $h^0(V, L) = \chi(V, L) = n + 2^{-1}d$. This implies $d = 2\Delta(V, L)$ and $g(V, L) = \Delta(V, L) + 1$.

REMARK. If $n = 2$, we have $\chi(V, tL) = 2^{-1}dt^2 + 2$ and $d = 2\Delta$, $g = \Delta + 1$.

COROLLARY (1.12). *Let (V, L) be a Fano-K3 variety such that* $\mathrm{Bs}\,|L| = \emptyset$. *Then L is simply generated unless (V, L) is hyperelliptic.*

2. Structure of double coverings. From now on, throughout this paper, we assume $\mathrm{char}\,(\mathfrak{K}) \neq 2$.

(2.1) Let W be a variety and let F be a line bundle on W. Suppose that we have a member B of $|2F|$. Then, as is well known, there is a natural way to construct a divisor D on the A^1-bundle F over W, in such a way that the restriction π to D of the projection $F \to W$ makes D a branched double covering of W with branch locus B (cf., e.g., [F4; (2.3)]). D is denoted by $R_{B,F}(W)$ since it is determined uniquely by the triple (W, F, B). If there is no other line bundle F' with $B \in |2F'|$, we write $R_B(W)$ too. Note that:

(1) $R_{B,F}(W)$ is irreducible unless $B = 2B'$ for some $B' \in |F|$.

(2) $R_{B,F}(W)$ is non-singular if so are both W and B.

THEOREM (2.2). *Let W, F and B be as in* (2.1), *let $V = R_{B,F}(W)$ and let $\pi: V \to W$ be the natural morphism. Suppose that W and B are non-singular. Then*

(1) *there is a natural exact sequence* $0 \to \mathcal{O}_W \to \pi_*\mathcal{O}_V \to \mathcal{O}_W[-F] \to 0$ *and this sequence splits.*

(2) $\pi^*B = 2R$ *for some* $R \in |\pi^*F|$.

(3) $K^V = \pi^*K^W + R$, *where K^X denotes the canonical bundle of a manifold X.*

These are easy consequences of the construction of V. We will call B (resp. R) the *branch locus* (resp. *ramification locus*) of π. Equipped with the reduced structure, they are isomorphic to each other by π.

(2.3) Now we consider general double coverings. First we prove the following:

LEMMA. *Let f: $V \to W$ be a proper finite separable morphism between normal varieties V, W which may not be complete. Suppose that $f^{-1}(x)$ consists of two points for any general point x on W. Then there is a non-trivial involution i of V such that $f = f \circ i$. Moreover, W is isomorphic to the quotient V/i.*

PROOF. Let S_V and S_W be the singular loci of V and W respectively and let $W_0 = W - (S_W \cup f(S_V))$. Note that $\operatorname{codim}(W - W_0) \geqq 2$ since V and W are normal. f_0: $V_0 = f^{-1}(W_0) \to W_0$ is a finite flat morphism. Therefore $\mathscr{F} = f_*\mathscr{O}_V$ is locally free of rank two on W_0. Since $\mathscr{F}$ is an $\mathscr{O}_W$-algebra, we have the trace mapping τ: $\mathscr{F} \to \mathscr{O}_W$ on W_0. Let j: $\mathscr{O} = \mathscr{O}_W \to \mathscr{F}$ be the natural homomorphism and let $\mathscr{C}$ be the cokernel of it. Then τ gives a splitting $\mathscr{F} \cong \mathscr{O} \oplus \mathscr{C}$ on W_0 because $\operatorname{char}(\mathfrak{R}) \neq 2$. Furthermore, on W_0, we have $\mathscr{C} \cong \operatorname{Ker}(\tau)$ and this is an invertible sheaf. Let φ be a local base of $\mathscr{C}$ on some open set U in W_0. Calculating the trace with respect to the basis $(1, \varphi)$ of $\mathscr{F}$, we infer $\varphi^2 \in \operatorname{Im}(j)$ since $\varphi \in \operatorname{Ker}(\tau)$. Now, for any $\psi \in H^0(U, \mathscr{F})$, we can write $\psi = a + b\varphi$ for some $a, b \in H^0(U, \mathscr{O})$, and define $\iota_U(\psi) = a - b\varphi$. Then ι_U is an $\mathscr{O}$-algebra involution of $\mathscr{F}_U$, since $\varphi^2 \in \operatorname{Im}(j)$. Moreover, we can easily verify that this definition is independent of the choice of the local base φ of $\mathscr{C}$. Hence we can patch them together to obtain a global involution ι of $\mathscr{F}$ which is defined on W_0. Let g: $V_0 \to V$ and h: $W_0 \to W$ be the open embeddings respectively. Since $\operatorname{codim}(V - V_0) \geqq 2$ and V is normal, we have $g_*\mathscr{O}_{V_0} = \mathscr{O}_V$. Hence $h_*(\mathscr{F}_0) = \mathscr{F}$. So the above involution ι of $\mathscr{F}_0$ can be extended to an involution of $\mathscr{F}$ defined on the whole W. Since $V = \mathscr{S}pec(\mathscr{F})$ by the finiteness of f, this induces an involution i of V with the desired property. The natural morphism $V/i \to W$ is an isomorphism by Zariski's main theorem.

(2.4) Let f, V, W and i be as above. Suppose in addition that V is non-singular. We will study the local structure of i around the fixed point set X of i.

For each point $x \in X$, the action of i on the tangent space of V at x is semisimple, and its eigenvalues are 1 and -1. Choose an (étale) local coordinate system $(y_1, \cdots y_k, y_{k+1}, \cdots, y_n)$ of V at x such that $i^*dy_\alpha = dy_\alpha$ at x for each $\alpha \leqq k$ and $i^*dy_\beta = -dy_\beta$ at x for each $\beta > k$. Put $z_\alpha = (y_\alpha + i^*y_\alpha)/2$ for $\alpha \leqq k$ and $z_\beta = (y_\beta - i^*y_\beta)/2$ for $\beta > k$. Then, thanks to the Jacobian criterion, $(z_1, \cdots, z_n)$ is a local coordinate system of V at x. Since $i^*z_\alpha = z_\alpha$ and $i^*z_\beta = -z_\beta$ for $\alpha \leqq k$ and $\beta > k$, X is the submanifold defined by $z_{k+1} = \cdots = z_n = 0$ in a neighborhood of x. Thus we conclude that X is a disjoint union $\bigcup_{k=0}^{n-1} X_k$, where each X_k is

a non-singular subspace (which is not necessarily connected) of V of pure dimension k.

(2.5) Now we can describe the possible singularities of $W \cong V/i$ as follows.

Let y be a singular point of W. Then y must be the image of a point x on X. If $x \in X_{n-1}$, then y is a simple point. If $x \in X_0$, then y looks like the vertex of the affine cone of the Veronese manifold $(\boldsymbol{P}^{n-1}, 2H)$. In general, if $x \in X_k$, then y looks like a vertex of a generalized cone over the Veronese manifold $(\boldsymbol{P}^{n-k-1}, 2H)$ whose set of vertices is a linear submanifold of dimension k in some $\boldsymbol{P}^N$.

THEOREM (2.6). *Let $f: V \to W$ be a finite morphism of mapping degree two between manifolds V and W. Then $V \cong R_{B,F}(W)$ for some $F \in \mathrm{Pic}(W)$ and $B \in |2F|$.*

PROOF. By (2.3), we have an involution i of V such that $W \cong V/i$. By (2.5), the fixed point set R of i is a non-singular divisor on V. R is mapped isomorphically onto $B = f(R)$, which is a non-singular divisor on W. Moreover, as we saw in (2.3), $f_*\mathcal{O}_V \cong \mathcal{O}_W \oplus \mathscr{E}$ for some invertible sheaf $\mathscr{E}$ on W. Take F so that $\mathcal{O}_W[-F] \cong \mathscr{E}$. For any section φ of $\mathscr{E}$ on some open set U of W, we have $\varphi^2 \in H^0(U, \mathcal{O}_W) \subset H^0(U, f_*\mathcal{O}_V)$ as in (2.3). This process gives rise to a homomorphism $\mathscr{E}^{\otimes 2} \to \mathcal{O}_W$. Letting β be the corresponding element of $H^0(W, 2F)$, we easily see that $B = \{\beta = 0\}$ and $V \cong R_{B,F}(W)$.

(2.7) We return to the situation (2.4) where V is non-singular but W may be singular. Let X be as in (2.4) and let V' be the blowing-up of V with center X. Then i induces an involution i' on V', whose fixed point set R is the exceptional divisor lying over X. R is non-singular since so is X. Hence $W' \to V'/i'$ is non-singular. We have a natural morphism $W' \to W$, which makes W' the blowing-up of W with center $Y = f(X)$. Applying (2.6) to $V' \to W'$, we can relate the structures of V and W via V' and W'.

(2.8) We will apply the preceding theory to hyperelliptic polarized manifolds. First we classify them roughly.

Let (M, L) be a hyperelliptic polarized manifold and let W and H be as in (2.1). Since $\Delta(W, H) = 0$, the polarized variety (W, H) is of one of the following types (cf. [F1] and [F6; §4]).

(a) $(W, H) \cong (\boldsymbol{P}^n, \mathcal{O}(1))$.

(b) W is a non-singular hyperquadric in $\boldsymbol{P}^{n+1}$ and $H = \mathcal{O}_W(1)$.

(c) $(W, H) \cong (\boldsymbol{P}(E), H^E)$ for an ample vector bundle E on $\boldsymbol{P}^1$.

(d) $(W, H) \cong (\boldsymbol{P}^2, \mathcal{O}(2))$.

(e) W is a generalized cone over a base manifold of one of the above types (a), (b), (c) and (d).

REMARK. In any case $\mathrm{Pic}(W)$ is free of 2-torsion.

DEFINITION (2.9). A hyperelliptic polarized manifold (M, L) is said to be of type (I) (resp. (II), (Σ), (IV) or $(*)$) if (W, H) is of the above type (a) (resp. (b), (c), (d) or (e)).

REMARK. When $(W, H) \cong (\boldsymbol{P}^1_\alpha \times \boldsymbol{P}^1_\beta, H_\alpha + H_\beta)$, this definition is a little ambiguous. However, usually, it is more convenient to consider (M, L) to be of type (Σ) rather than of type (II). See (3.3).

3. Type (I), (II) and (IV). (3.1) Let (M, L) be a hyperelliptic polarized manifold of type (I). Then, by (2.6), we have $M \cong R_B(\boldsymbol{P}^n)$ for some hypersurface B of even degree. We say (M, L) to be of type (I^n_a) if $\deg(B) = 2a + 2$. Clearly B is connected unless $n = 1$. Since $F = \mathcal{O}(a + 1)$ in this case, (2.2) implies

(1) $H^q(M, tL) \cong H^q(\boldsymbol{P}^n, \mathcal{O}(t)) \oplus H^q(\boldsymbol{P}^n, \mathcal{O}(t - a - 1))$ *for any integers* q, t. *In particular,* (1.2) *implies* $H^0(\boldsymbol{P}^n, \mathcal{O}(-a)) = 0$, *and hence we have*

(2) $a \geqq 1$.

Moreover, by easy calculations and (1), we obtain

(3) $\chi(M, tL) = \{(t + 1) \cdots (t + n) + (t - a) \cdots (t - a + n - 1)\}/n!$.

Combining (2.2; 3) and the above (1), we see:

(4) (M, L) *is globally Gorenstein with sectional index* $a - 1$.

Furthermore

(5) M *is a hypersurface of degree* $2a + 2$ *in the weighted projective space* $\boldsymbol{P}(a + 1, 1, \cdots, 1)$ (cf. [M]).

For a proof of (5), use the induction on n (like [M] and [F3; (3.2)]) and the following:

LEMMA (3.2). *Suppose that* $n \geqq 2$ *and let* D *be a general member of* $|L|$. *Then* (D, L_D) *is a hyperelliptic polarized manifold of type* (I^{n-1}_a).

Indeed, the image H of D is a general hyperplane on $\boldsymbol{P}^n$. Hence $Y = B \cap H$ is non-singular. So $D = R_Y(H)$ is non-singular, too. Then the assertion is obvious.

(3.3) Let (M, L) be a hyperelliptic polarized manifold of type (II). Then $M \cong R_B(\boldsymbol{Q})$, where $\boldsymbol{Q}$ is the non-singular hyperquadric with $\dim \boldsymbol{Q} = n$ and $B \in |2F|$ for some $F \in \mathrm{Pic}(\boldsymbol{Q})$. (M, L) is said to be of type (II^n_a) if $F = (a + 1)H$. When $n = 2$, F is not necessarily an integral multiple of $H = \mathcal{O}_Q(1)$. In such a case (M, L) will be considered to be of type (Σ).

(3.4) Similarly as in (3.1), we have the following results for any

hyperelliptic polarized manifold (M, L) of type (II_a^n).

(1) $H^q(M, tL) \cong H^q(\boldsymbol{Q}, tH) \oplus H^q(\boldsymbol{Q}, (t-a-1)H)$ *for any* q, t.

(2) $a \geqq 1$.

(3) $\chi(M, tL) = \{(t+1)\cdots(t+n-1)(2t+n) + (t-a)\cdots(t-a+n-2)(2t-2a+n-2)\}/n!$.

(4) (M, L) *is globally Gorenstein with sectional index* a.

(5) (M, L) *is a weighted complete intersection of type* $(2, 2a+2)$ *in* $\boldsymbol{P}(a+1, 1, \cdots, 1)$. (cf. [M] *and* [F3; (3.3)]).

(6) *For any general member* D *of* $|L|$, (D, L_D) *is a hyperelliptic polarized manifold of type* (II_a^{n-1}).

(3.5) Let (M, L) be a hyperelliptic polarized manifold of type (IV). Then $M \cong R_B(\boldsymbol{P}_a^2)$ for some $B \in |(2a+2)H_a|$ and $L = f^*[2H_a]$, where f is the natural morphism $M \to \boldsymbol{P}_a^2$. In this case (M, L) is said to be of type (IV_a^2) or (IV_a). Similarly as before, we have:

(1) $H^q(M, tL) \cong H^q(\boldsymbol{P}^2, \mathcal{O}(2t)) \oplus H^q(\boldsymbol{P}^2, \mathcal{O}(2t-a-1))$ *for any* q, t.

(2) $a \geqq 2$ (*compare* (3.1; 2)).

(3) $\chi(M, tL) = 4t^2 - (2a-4)t + (a^2 - a + 2)/2$.

(4) (M, L) *is globally Macaulay and* $K^M = (a-2)H_a$. *So* (M, L) *is globally Gorenstein if and only if* a *is even, and the sectional index is* $a/2$ *in that case.*

REMARK. (M, L) is not a weighted complete intersection even if a is even. Needless to say, the manifold M itself is the same as that of type (I_a^2).

(3.6) We will further calculate several invariants of (M, L) such as Betti numbers, Picard groups and so on. Our main tool is the Lefschetz theorem. So we make the following:

DEFINITION. A covariant (or contravariant) functor F from the category of algebraic varieties to the category of groups is called a *Lefschetz functor of degree* d if it has the following property:

For any non-singular ample divisor A *on any manifold* M *with* $\dim M = n > d+1$, *the homomorphism* $F(\iota)$ *is bijective for the inclusion* $\iota: A \to M$.

If the above assertion is valid in case $H^q(M, -tA) = 0$ for any $q < n$, $t > 0$, then F will be called a weakly Lefschetz functor of degree d.

EXAMPLES (3.7). (a) In case $\mathfrak{K} = \boldsymbol{C}$, the topological homotopy group π_i is a Lefschetz functor of degree i. The homology group $H_q(\cdot; \boldsymbol{Z})$ is a Lefschetz functor of degree q. The cohomology group $H^q(\cdot; \boldsymbol{Z})$ is a contravariant Lefschetz functor of degree q. The Picard group $\mathrm{Pic}(\cdot)$ is a contravariant Lefschetz functor of degree 2.

(b) In case $p = \mathrm{char}(\mathfrak{K}) > 0$, the l-adic cohomology group $H^q(\cdot; \boldsymbol{Q}_l)$ is a contravariant Lefschetz functor of degree q. The tame fundamental group $\pi_1^{(p)}$, which is defined similarly as the usual algebraic fundamental group by considering only étale coverings of degree prime to p, is a Lefschetz functor of degree 1. The Picard group is a weakly Lefschetz functor of degree two.

LEMMA (3.8). *Let A and B be hypersurfaces in $\boldsymbol{P}^n$ of degrees a and b respectively. Suppose that they intersect normally along $C = A \cap B$ and that $a > b$. Then there exists a non-singular hypersurface D of degree a such that $D \cap B = C$.*

PROOF. Let E be the exceptional divisor on the blowing-up of $\boldsymbol{P}^n$ with center C. The proper transforms A' and B' of A and B belong to $|aH - E|$ and $|bH - E|$ respectively, where H is the pull-back of $\mathcal{O}_{P^n}(1)$. Since $A' \cap B' = \varnothing$ and $[A'] = [B'] + (a - b)H$, we infer that $\mathrm{Bs}\,|aH - E| = \varnothing$ and that this linear system is very ample outside B'. Hence a general member D' of $|aH - E|$ is non-singular. Moreover the image D of D' in $\boldsymbol{P}^n$ has the desired property.

THEOREM (3.9). *Let (M, L) be a hyperelliptic polarized manifold of type (I_a^n) (resp. (II_a^n)) and let f be the morphism $M \to W \cong \boldsymbol{P}^n$ (resp. $\boldsymbol{Q}^n$). Then $F(f)$ is bijective for any weakly Lefschetz functor of degree $< n$.*

PROOF. Here we consider the case of type (II) only, because the same method works in case of type (I). By (3.8), there exists a non-singular hypersurface D in $\boldsymbol{P}^{n+1}$ such that $D \cap \boldsymbol{Q}^n = B$, the branch locus of f. Then $N = R_D(\boldsymbol{P}^{n+1})$ is hyperelliptic of type (I_a^{n+1}), and M is an ample divisor on N. The ramification locus R of $N \to \boldsymbol{P}^{n+1}$ is an ample divisor on N and is isomorphic to D. Therefore, by (3.1; 4) and (3.4; 4), we obtain: $F(M) \cong F(N) \cong F(R) \cong F(D) \cong F(\boldsymbol{P}^{n+1}) \cong F(\boldsymbol{Q}^n)$. So $F(f)$ is bijective.

COROLLARY (3.10). *Let (M, L) be as above. Then* $\mathrm{Pic}\,(M)$ *is generated by L if $n \geqq 3$.*

COROLLARY (3.11). *Let (M, L) be as above. Then, for any integer i with $i \neq n$, $b_i(M) = 0$ if i is odd and $b_i(M) = 1$ if i is even.*

For a proof, use also the Poincaré duality. Moreover, one can show that $H^{2i}(M; \boldsymbol{Z})$ is generated by $c_1(L)^i$ for $i < n/2$ in case $\mathfrak{K} = \boldsymbol{C}$.

REMARK. As for the Euler number, we have $e(M) = 2e(W) - e(B)$. So we can calculate $b_n(M)$ too, using (3.11).

COROLLARY (3.12). *Let (M, L) be as in* (3.9) *and suppose that $n \geqq 2$. Then $\pi_1(M) = \{1\}$ in case $\mathfrak{K} = \boldsymbol{C}$, and $\pi_1^{(p)} = \{1\}$ in case $p = \mathrm{char}(\mathfrak{K}) > 0$.*

COROLLARY (3.13). *Let* (M, L) *be as in* (3.12). *Then* $\mathrm{Pic}(M)$ *has no torsion prime to* $\mathrm{char}(\mathfrak{K})$. *In particular, it is free of* 2*-torsion.*

4. **Type** (*). (4.1) For a while until (4.4), let (M, L) be a hyperelliptic polarized manifold of type (*). Namely, the image W of $\rho = \rho_{|L|}$ is singular.

(4.2) The results in [F1; §4] and [F6; (4.11)] describe the structure of $W \subset \boldsymbol{P}^N$ as follows:

Let S be the set of singular points of W. Then S is a linear submanifold in $\boldsymbol{P}^N$. Let Y be a linear submanifold of $\boldsymbol{P}^N$ such that $Y \cap S = \varnothing$ and $\dim S + \dim Y = N - 1$. Then $T = Y \cap W$ is non-singular and $\Delta(T, H) = 0$. So T is one of the types (a), (b), (c) and (d) in (2.8). Moreover, W is the generalized cone $S * T$, that is, the closure of the union of all the lines passing a point on S and another point on T.

(4.3) Combining (4.2) and (2.5), we infer that $(T, H) \cong (\boldsymbol{P}^1_\beta, 2H_\beta)$ or $(\boldsymbol{P}^2_\beta, 2H_\beta)$. In view of (2.7), we let P' (resp. M') be the blowing-up of $\boldsymbol{P}^N$ (resp. M) with center S (resp. $X = \rho^{-1}(S)$), and let W' be the proper transform of W on P'. Then there is a natural double covering ρ': $M' \to W'$.

Since $W = S * T$, we infer that $W' = \boldsymbol{P}_T(2H_\beta \oplus V)$, where V is the direct sum of $(1 + \dim S)$ trivial line bundles on T, and the tautological line bundle H_α on W' is the pull-back of $\mathcal{O}_W(1) = H$. Moreover, the exceptional divisor E lying over S is the unique member of $|H_\alpha - 2H_\beta|$.

E is a component of the branch locus B of ρ'. So we write $B = E + A$. Then $E \cap A = \varnothing$, since B is non-singular. We may set $[A] = xH_\alpha + yH_\beta$, because $\mathrm{Pic}(W')$ is generated by (the pull-backs of) H_α and H_β. Then x is odd and y is even, since $[B]$ is divisible by two in $\mathrm{Pic}(W')$. On the other hand, $E \cong T \times \boldsymbol{P}^{\dim S}$ and $[H_\alpha]_E$ is the pull-back of $\mathcal{O}(1)$ of the second factor. Therefore, $0 = [A]_E$ implies that $y = 0$ and $\dim S = 0$. Now we make the following:

DEFINITION (4.4). (M, L) is said to be of type ($^*\mathrm{II}_a$) (resp. ($^*\mathrm{IV}_a$)) if $T \cong \boldsymbol{P}^1$ (resp. $\boldsymbol{P}^2$) and $x = 2a + 1$.

In any case, S is a point and W is the cone over the Veronese manifold T with vertex S. In particular $\dim M = 2$ (resp. 3) if (M, L) is of type ($^*\mathrm{II}_a$) (resp. ($^*\mathrm{IV}_a$)).

(4.5) Conversely, given a Veronese manifold $(T, H) \cong (\boldsymbol{P}^{n-1}_\beta, 2H_\beta)$ and any non-singular member D of $|(2a + 1)H|$ on the cone W over T, we can construct a polarized manifold (M, L) in the following way.

Let P' be the blowing-up of $\boldsymbol{P}^N$ at the vertex of W and let W', E, H_α be as in (4.3). Furthermore set $B = E + A$, where A is the lift of

D to W'. Let $F = (a+1)H_\alpha - H_\beta$ and $M' = R_{B,F}(W')$ and let C be the component of the ramification locus R of $M' \to W'$ lying over E. Clearly $C \cong E \cong \boldsymbol{P}^{n-1}$ and $[C]_C = \mathcal{O}(-1)$, since $[E]_E = \mathcal{O}(-2)$ and $[E]_{M'} = [2C]$. So C can be contracted to a non-singular point. Let M be the manifold obtained by this contraction. Then $M' \to W'$ induces a finite morphism $f: M \to W$. Setting $L = f^*\mathcal{O}_W(1)$ we get a polarized manifold (M, L), which is hyperelliptic of type ($^*\mathrm{II}_a$) (resp. ($^*\mathrm{IV}_a$)) if $n = 2$ (resp. $= 3$) and $a \geqq 1$. We can carry out this process even if $a = 0$, but then $H^0(W, H) \to H^0(M, L)$ is not surjective and one easily sees that $(M, L) \cong (\boldsymbol{P}^n, \mathcal{O}(2))$.

REMARK. (M, L) is determined uniquely by the divisor D. Hence they are all deformations of each other.

(4.6) Let things be as above (n being an arbitrary positive integer) and set $Z' = C + H_\beta \in \mathrm{Pic}\,(M')$. Then $[Z']_C = 0$ and hence Z' is the pull-back of $Z \in \mathrm{Pic}\,(M)$. Moreover we have:

(1) $L = 2Z$ *and* $K^M = (2a - n - 1)Z$ *in* $\mathrm{Pic}\,(M)$.

(2) $H^q(M, tL) \cong H^q(W', tH_\alpha) \oplus H^q(W', (t - a - 1)H_\alpha + H_\beta)$.

(3) $d(M, L) = 2^n$, $d(M, Z) = 1$, $g(M, Z) = a$ *and* $g(M, L) = 1 + (2a + n - 3)2^{n-2}$.

(4) (M, L) *and* (M, Z) *are globally Macaulay.*

(5) *For any general member* Y *of* $|L|$, (Y, Z) *is a hyperelliptic polarized manifold of type* (I^{n-1}_{2a}).

(6) *For any general member* X *of* $|Z|$, (X, Z_X) *is a polarized manifold of the same type as in* (4.5).

(7) (M, Z) *is a hypersurface of degree* $2(2a+1)$ *in the weighted projective space* $\boldsymbol{P}(2a+1, 2, 1, \cdots, 1)$.

(8) $F(M) \cong F(\boldsymbol{P}^n)$ *for any weakly Lefschetz functor* F *of degree* $< n$.

PROOF. We have $L = H_\alpha = [E] + 2H_\beta = 2C + 2H_\beta = 2Z'$ in $\mathrm{Pic}\,(M')$. Hence $L = 2Z$ in $\mathrm{Pic}\,(M)$. We have $K^{W'} = -2H_\alpha - (n-2)H_\beta$ because $W' \cong \boldsymbol{P}_T(2H_\beta \oplus \mathcal{O}_T)$, and $K^{M'} = K^{W'} + F = (a-1)H_\alpha - (n-1)H_\beta$ by (2.2). On the other hand $K^{M'} = K^M + (n-1)C$. Combining them we see $K^M = (2a - n - 1)Z$ on M'. Thus we prove (1).

(2) follows from (2.2) and $H^q(M, tL) \cong H^q(M', tL)$. (3) is a consequence of (2). In fact, it is easy to calculate $\chi(M, tL)$.

Next we will prove (5). Since $H^0(W, H) \cong H^0(M, L)$, Y is the pull-back of a general hyperplane section U of W. Then $U \cong T \cong \boldsymbol{P}^{n-1}_\beta$ and $D \cap U$ is a non-singular hypersurface in $U \cong \boldsymbol{P}_\beta$ of degree $2(2a+1)$.

Clearly $Y \cong R_{D\cap U}(U)$ and L is the pull-back of $\mathcal{O}_U(1) = 2H_\beta$. Hence (Y, Z) is of type (I^{n-1}_{2a}).

Applying [F3; (2.1) and (2.2)], we infer from (5) and (3.1.4) that (M, Z) is globally Macaulay. Thus we prove (4).

In order to show (6) we use the following:

LEMMA (4.7). *Let Γ be a general member of $|H_\beta|$ on W'. Then $\Gamma \cap A$ is non-singular.*

PROOF. Recall that D is isomorphic to the divisor A on W', which is a $\boldsymbol{P}^1$-bundle over T. So we have a natural homomorphism $\varphi: \Theta_D \to (\Theta_T)_D$, where Θ's denote tangent bundles of the given manifolds. Let Σ be the set of points on D at which φ is not surjective. We claim that $\Sigma \neq D$.

Assume that $\Sigma = D$ and let R be the image of φ. In view of the restriction of the exact sequence $0 \to \Theta_{W'/T} \to \Theta_{W'} \to \Theta_T \to 0$ to A, we infer that $\mathrm{Ker}\,(\varphi) \cong (\Theta_{W'/T})_D \cong [2H_\alpha - 2H_\beta]_D$ and that R is a subbundle of $(\Theta_T)_D$ of corank one. Moreover we see $(\Theta_T)_D/R \cong (\Theta_{W'})_D/\Theta_D \cong [A] = (2a+1)H_\alpha$ by 9-lemma. Therefore $c_{n-1}((\Theta_T)_A[-A]) = 0$. On the other hand, using $[H_\alpha - 2H_\beta]_A = E_A = 0$ and the exact sequence $0 \to \mathcal{O} \to H_\beta \oplus \cdots \oplus H_\beta \to \Theta_T \to 0$, we obtain $c(\Theta_T[-A]) = (1 + c_1(H_\beta) - c_1([A]))^n \sum_{j=0} c_1([A])^j$. On A we have $[A] = (2a+1)H_\alpha = (4a+2)H_\beta$. So $c_{n-1}(\Theta_T[-A])_A = H_\beta^{n-1}\{A\} \cdot (1 - (-4a-1)^n)/(4a+2)$ by an easy calculation. Clearly this is not zero. This contradiction proves the claim.

Thus we see $\dim \Sigma < \dim D = n-1$. On the other hand, for any point x on Σ, there exists only one member of the linear system $\Lambda = |H_\beta|_A$ which is singular at x, because $\mathrm{rank}\,(\varphi_x) = n-2$. Clearly any member of Λ is non-singular outside Σ. Hence the dimension of the family of singular members of Λ is at most $n-2$. So a general member of Λ is non-singular, proving the lemma.

REMARK. If $\mathrm{char}\,(\mathfrak{K}) = 0$, this lemma is obvious by Bertini's theorem.

(4.8) PROOF OF (4.6), CONTINUED. Let Λ' be the linear system $C + |H_\beta|$ on M'. Since $h^0(M, Z) = n$, one easily sees that Λ' is the pull-back of $|Z|$. So a general member X of $|Z|$ determines a general member Γ of $|H_\beta|_{W'}$. The image of Γ on W is a cone over a Veronese manifold $(\boldsymbol{P}^{n-2}_\beta, 2H_\beta)$ and $\Gamma \cap A$ is non-singular by (4.7). This observation shows (6).

We prove (7) by induction on n. When $n = 1$, M is a hyperelliptic curve of genus a and $Z = [C]$, C being a Weierstrass point of M. Hence (7) is valid in this case. When $n \geqq 2$, we apply [F3; (3.2)] since

$H^0(M, tZ) \to H^0(X, tZ_X)$ is surjective for every integer t by virtue of (4).

To show (8) we embed $T \cong \boldsymbol{P}_\beta^{n-1}$ in $P \cong \boldsymbol{P}_\beta^n$ as a hyperplane and we consider W to be a subspace of the cone V over the Veronese manifold $(P, 2H_\beta)$. We claim that $|(2a+1)H_V|$ contains a non-singular member G such that $G_W = D$. Indeed, the linear subsystem of $|(2a+1)H_V|$ consisting of the member containing D is very ample outside W. Therefore, similarly as in (3.8), any general member G of this subsystem is non-singular and $G_W = D$, as claimed. Starting from V and G, we construct a polarized manifold (N, Z_N) as in (4.5) in such a way that M is a member of $|Z_N|$. Let Y be a general member of $|2Z_N|$. Then $F(Y) \cong F(\boldsymbol{P}^n)$ by (5) and (3.9). Moreover we have $F(Y) \cong F(N) \cong F(M)$. Thus we obtain (8).

REMARK (4.9). Of course, the analogues of (3.10) $\sim$ (3.13) are valid in this case, too. In particular, $\mathrm{Pic}\,(M)$ is generated by Z if $n \geqq 3$ (this follows also from (7) and [M]).

(4.10) Any polarized manifold (M, Z) with $d(M, Z) = \Delta(M, Z) = 1$ and $g(M, Z) = a$ can be constructed as in (4.5) if $a \leqq 2$ and $n > a + 1$. This will be proved in [F4-3].

5. Type (Σ). (5.1) In this section we consider the case in which (M, L) is of type (Σ) and $\dim M = n \geqq 2$ (cf. (2.9)). So $(W, H) \cong (\boldsymbol{P}(E), H^E)$ for an ample vector bundle E on $\boldsymbol{P}^1$, where (W, H) is as in (1.1). As is well-known, E is a direct sum of line bundles.

(5.2) Notations and Definitions. Given a sequence $\delta = (\delta_1, \cdots, \delta_n)$ of positive integers such that $\delta_1 \geqq \cdots \geqq \delta_n$, we denote δ_1, δ_n and $\delta_1 + \cdots + \delta_n$ by $\delta_{\max}$, $\delta_{\min}$ and $|\delta|$ respectively. By $E(\delta)$ or E_δ we denote the vector bundle $\bigoplus_{j=1}^n [\delta_j H_\beta]$ on $\boldsymbol{P}_\beta^1$. $W(\delta)$ or W_δ denotes $\boldsymbol{P}(E_\delta)$, and the tautological line bundle on W_δ is denoted by $H(\delta)$ or H_δ. (W_δ, H_δ) is called a *rational scroll* of type (δ). The following facts can be easily proved.

(5.3; 1) $H^q(W_\delta, tH_\delta + sH_\beta) \cong H^q(\boldsymbol{P}_\beta^1, \mathrm{S}^t(E_\delta) \otimes [sH_\beta])$ for any $t \geqq 0$, $q \in \boldsymbol{Z}$.

(5.3; 2) $\chi(W_\delta, tH_\delta + sH_\beta) = (t+1)\cdots(t+n-1)\{t|\delta| + n(s+1)\}/n!$ for any integers t, s.

(5.3; 3) $K^W = -nH_\delta + (|\delta| - 2)H_\beta$.

(5.4) If (W, H) is a rational scroll of type (δ), then (M, L) is said to be of type $(\Sigma(\delta_1, \cdots, \delta_n))$ or $(\Sigma^n|\delta|)$. For example, (M, L) is of type $(\Sigma^2 4)$ if it is either of type $(\Sigma(2, 2))$ or $(\Sigma(3, 1))$. Clearly we have $d(M, L) = 2d(W, H) = 2|\delta|$ and $\Delta(M, L) = |\delta|$.

By virtue of (2.6), $M \cong R_B(W)$ for the branch locus B of $\rho: M \to W$. We consider the following cases separately:

(a) B is connected.

(b) B consists of several fibers of $\pi: W \to \boldsymbol{P}^1_\beta$.

(c) All the other cases.

(5.5) For a while, until (5.20), we consider the case (a) above. In view of (2.6) we set $B = 2aH_\delta + 2bH_\beta$ in $\mathrm{Pic}(W)$, where a, b are integers. In this case (M, L) is said to be of type $(\Sigma^n(\delta)^+_{a,b})$, and also of type $(\Sigma^n|\delta|^+_{a,b})$.

REMARK. In the case of type $(\Sigma(1,1)^+_{a,b})$, we have $W \cong \boldsymbol{P}^1_\alpha \times \boldsymbol{P}^1_\beta$ and $H = H_\alpha + H_\beta$. So the choice of the two rulings of W is completely optional, and hence this can be viewed to be of type $(\Sigma(1,1)^+_{a+b,-b})$, too.

(5.6) By virtue of (2.2), we have:

(1) $H^p(M, tL) \cong H^p(W, tH) \oplus H^p(W, (t-a)H - bH_\beta)$ *for any integers* p, t.

(2) $K^M = (a-n)L + (b + |\delta| - 2)H_\beta$.

Furthermore, using (5.3), we obtain an explicit formula for $\chi(M, tL)$. In particular we have:

(3) $g(M, L) = a|\delta| + b - 1$, $\chi_{n-2}(M, L) = (a-1)(a|\delta| + 2b - 2)/2 + 1$ *and* $\chi(M, \mathcal{O}_M) = 1 + (n!)^{-1}(a-1)\cdots(a-n+1)(a|\delta| + bn - n)$.

(5.7) We easily see that a polarized manifold (M, L) of type $(\Sigma^n(\delta)^+_{a,b})$ exists if and only if b is greater than a constant which depends on (δ) and a. If (δ) and a are fixed, we can determine this constant without any essential trouble. However, at present, no explicit "formula" for this purpose is found. We have only the following partial results, which are enough for many purposes.

(1) We should have $H^0(W, H) \cong H^0(M, L)$, and hence $H^0(W, (1-a)H_\delta - bH_\beta) = 0$ by (5.6; 1). This implies $a = 1$ and $b > 0$, or $a \geqq 2$.

(2) $\delta_{\max} + (2a-1)\delta_{\min} + 2b \geqq 0$. This is a consequence of Reid's lemma (cf. [Is; (7.4)]). This inequality follows also from the observation below.

Let C be the section of $W \to \boldsymbol{P}^1_\beta$ corresponding to the quotient bundle $\delta_n H_\beta$. Then $HC = \delta_n$ and $BC = 2a\delta_n + 2b$. So the assertion is obvious if $C \not\subset B$. Suppose $C \subset B$. Then we have a surjection from the normal bundle N of C in W onto $[B]_C$. Since N is isomorphic to $E(\delta_n - \delta_1, \cdots, \delta_n - \delta_{n-1})$, we have $BC \geqq \mathrm{Min}(\delta_n - \delta_j) = \delta_{\min} - \delta_{\max}$. This implies the desired inequality.

(3) *Let j be the largest integer such that $a\delta_j + b \geqq 0$. Then $2j \geqq n$. Or equivalently, $a\delta_i + b \geqq 0$ for any i with $2i \leqq n+1$.*

To see this, let Y be the subspace of W corresponding to the quotient bundle $\delta_{j+1}H_\beta \oplus \cdots \oplus \delta_n H_\beta$ of E_δ. Then $Y \subset B$ by Reid's

lemma. Clearly $Y \cong W(\delta_{j+1}, \cdots, \delta_n)$ and the normal bundle N of Y in W is $\bigoplus_{i=1}^{j}[H_\delta - \delta_i H_\beta]$. Since we have a surjective homomorphism $N \to [B]_Y$, we have divisors $D_1, \cdots, D_j$ on Y such that $D_i \in |B - H_\delta + \delta_i H_\beta|$ for each i and that $D_1 \cap \cdots \cap D_j = \varnothing$. This is impossible unless $j > \dim Y - 1 = n - j - 1$. So $2j \geqq n$.

(4) When $n = 2$, we have $a\delta_{\min} + b \geqq 0$. Indeed, in this case, B is an irreducible curve, and hence cannot contain the curve C as in (2).

(5) On the other hand, for any (a, b) with $a \geqq 2$ and $a\delta_{\min} + b \geqq 0$, there exists a non-singular member B of $|2aH_\delta + 2bH_\beta|$, and hence we have a hyperelliptic polarized manifold (M, L) of type $(\Sigma^n(\delta)^+_{a,b})$.

Indeed, the assertion is clear if $a\delta_{\min} + b > 0$, because then B is very ample. So we consider the case in which $a\delta_n + b = 0$. Let j be the largest integer such that $a\delta_j + b > 0$. Then $\delta_{j+1} = \cdots = \delta_n = \delta_{\min}$. Let Y be the subspace of $W \cong W_\delta$ corresponding to the quotient bundle $E(\delta_{i+1}, \cdots, \delta_n)$ of $E(\delta)$. Clearly we have $\mathrm{Bs}\,|H_\sigma| = \varnothing$ on W, where $H_\sigma = H_\delta - \delta_{\min} H_\beta$. Let ρ be the induced rational mapping into $\boldsymbol{P}_\sigma$ and let $W' = \rho(W)$ and $S = \rho(Y)$. Then W' is a generalized cone over a base $\cong W(\delta_1 - \delta_n, \cdots, \delta_j - \delta_n)$ with the vertex set S, which is a linear subspace of dimension $n - j - 1$. From the converse viewpoint, W is the blowing-up of W' with center S. Now, let B' be a general member of $|2aH_\sigma|$ on W'. Since H_σ is very ample on W', the singular locus of B' must be contained in $\mathrm{Sing}\,(W') = S$. Moreover, we can take B' to be transverse to S. Then, we easily see that $B = \rho^* B'$ is a non-singular divisor on W, and $B \in |2aH_\delta + 2bH_\beta|$, as required.

Of course, (M, L) is obtained if we set $M = R_B(W)$.

PROPOSITION (5.8). *Suppose (M, L) is of type $(\Sigma^n(\delta)^+_{a,b})$. Then $H^p(M, \mathcal{O}_M) = 0$ for $0 < p < n$, except when $\delta_1 = \cdots = \delta_n$, $W \cong \boldsymbol{P}^1_\beta \times \boldsymbol{P}^{n-1}_\alpha$, $B \in |2aH_\alpha|$, $M \cong \boldsymbol{P}^1_\beta \times R_A(\boldsymbol{P}^{n-1}_\alpha)$ for a hypersurface A of $\boldsymbol{P}^{n-1}_\alpha$ of degree $2a$, $a \geqq n \geqq 3$ and $p = n - 1$.*

PROOF. Suppose $H^p(M, \mathcal{O}_M) \neq 0$. Then, using (5.3) and (5.6), we infer that $0 \neq h^p(W, -aH_\delta - bH_\beta) = h^{n-p}(W, (a - n)H_\delta + (b + |\delta| - 2)H_\beta)$. This implies $p = n - 1$, $a \geqq n$ and $(a - n)\delta_n + b + |\delta| \leqq 0$. Combining this with (5.7; 2), we obtain $0 \leqq (2n - 1)\delta_n + \delta_1 - 2|\delta| = (\delta_n - \delta_1) + \sum_{j=2}^{n} 2(\delta_n - \delta_j)$. Hence $\delta_1 = \cdots = \delta_n$. This implies $W \cong \boldsymbol{P}^1_\beta \times \boldsymbol{P}^{n-1}_\alpha$ and $H_\delta = H_\alpha + \delta_n H_\beta$. Moreover we have $a\delta_n + b = 0$, hence $B \in |2aH_\delta + 2bH_\beta| = 2aH_\alpha$. The rest of the assertion is now obvious.

COROLLARY (5.9). $p_g(M) = \{(b - 1)n + a|\delta|\}(a - 1) \cdots (a - n + 1)/n!$ *except when $h^{n-1}(M, \mathcal{O}_M) \neq 0$, as described in* (5.8).

For a proof, use (5.6; 3).

PROPOSITION (5.10). *A hyperelliptic polarized manifold of type* $(\Sigma^n(\delta)^+_{a,b})$ *is globally Macaulay if and only if* $1 - |\delta| \leqq b \leqq 1$.

PROOF. For $0 < p < n$, we have $h^p(M, tL) = h^p(W, (t-a)H - bH_\beta) = h^{n-p}(W, (a-n-t)H_\delta + (b + |\delta| - 2)H_\beta)$. If (M, L) is globally Macaulay, substituting $t = a$ and $t = a - n$ and using (5.3; 1), we obtain $b \leqq 1$ and $b + |\delta| > 0$. Conversely, suppose $1 - |\delta| \leqq b \leqq 1$. Then, by virtue of (5.3; 1), $h^p(M, tL) = 0$ follows from $b \leqq 1$ for $t \geqq a$, or from $b + |\delta| > 0$ for $t \leqq a - n$, or is true automatically for $a - n < t < a$. Thus (M, L) is globally Macaulay.

COROLLARY (5.11). *(M, L) is globally Gorenstein if and only if* $b = 2 - |\delta|$.

For a proof, use (5.10) and (5.6; 2).

REMARK. This can happen only when $a \geqq 2$, provided $n \geqq 2$. Indeed, if $a = 1$, then $b \geqq 1$ by (5.7; 1). So $|\delta| \leqq 1$, which is possible only when $n = 1$.

PROPOSITION (5.12). *$-K^M$ is ample only in the following cases*:
(1) $\delta_1 = \delta_n + 1$, $\delta_2 = \cdots = \delta_n$, $a\delta_n + b = 0$ *and* $a < n$.
(2) $\delta_1 = \cdots = \delta_n$, $a\delta_n + b = 0$ *and* $a < n$.
(3) $\delta_1 = \cdots = \delta_n$, $a\delta_n + b = 1$ *and* $a < n$.

PROOF. By (5.6; 2), $(a-n)H_\delta + (b + |\delta| - 2)H_\beta$ is a negative line bundle on W. So we have $a < n$ and (♯): $(a-n)\delta_n + b + |\delta| \leqq 1$. Subtracting 2(♯) from (5.7; 2), we get $(2n-1)\delta_n + \delta_1 - 2|\delta| \geqq -2$. Hence $2 \geqq \delta_1 - \delta_n + 2\sum_{j=2}^n (\delta_j - \delta_n)$. So we infer that $\delta_2 = \cdots = \delta_n$ and $\delta_1 - \delta_n \leqq 2$. Suppose that $\delta_1 = \delta_n + 2$. Then $a\delta_2 + b \leqq -1$ by (♯), which contradicts (5.7; 3 or 4). Therefore $\delta_1 - \delta_n = 0$ or 1.

Suppose $\delta_1 = \delta_n + 1$. Then $2(a\delta_n + b) + 1 \geqq 0$ by (5.7; 2). So $a\delta_n + b \geqq 0$. Together with (♯), this implies $a\delta_n + b = 0$. This is the case (1).

Suppose $\delta_1 = \delta_n$. Then $a\delta_n + b = 0$ or 1 by (5.7; 2) and (♯). So we are in case (2) or (3).

(5.13) We study further the above three cases.

In case (1), $W \cong W(1, 0, \cdots, 0)$, which is isomorphic to the blowing-up of $\boldsymbol{P}^n_\alpha$ with center C being a linear subspace of codimension two. Moreover, under this isomorphism, we have $H_\beta = H_\alpha - E_C$ and $H_\delta = H_\alpha + \delta_n H_\beta$. $B \in |2aH_\alpha|$, which means, B is the total transform of a hypersurface A on $\boldsymbol{P}^n_\alpha$. Since B is non-singular, A must be non-singular and intersect C transversally. From this we infer that $M \cong R_B(W)$ is isomorphic to

the blowing-up of $N = R_A(\boldsymbol{P}^n_\alpha)$ with center being the full inverse image of C on N. By (3.10), Pic (N) is generated by H_α if $n \geqq 3$. Consequently, Pic (M) is generated by L and H_β.

In case (2), $W \cong \boldsymbol{P}^1_\beta \times \boldsymbol{P}^{n-1}_\alpha$, $H = H_\alpha + \delta_n H_\beta$ and $B \in |2aH_\alpha|$, that means, B is of the form $\boldsymbol{P}^1_\beta \times A$ for a hypersurface A on $\boldsymbol{P}^{n-1}_\alpha$. $n \geqq 3$ since B is connected. M is isomorphic to $\boldsymbol{P}^1_\beta \times R_A(\boldsymbol{P}^{n-1}_\alpha)$, and Pic $(M) \cong$ Pic (W). Furthermore, an analogue of (3.9) holds for the morphism $M \to W$.

In case (3), $W \cong \boldsymbol{P}^1_\beta \times \boldsymbol{P}^{n-1}_\alpha$, $H = H_\alpha + \delta_n H_\beta$ and $B \in |2aH_\alpha + 2H_\beta|$. In particular, B is ample on W. If $n \geqq 4$, we have Pic $(M) \cong$ Pic $(B) \cong$ Pic (W).

COROLLARY (5.14). *Suppose that $K^M = -mL$ for some positive integer m. Then one of the following conditions is satisfied:*

(1) $m = 1$, $\delta_1 = 2$, $\delta_2 = \cdots = \delta_n = 1$, $a = n - 1$, $b = 1 - n$ *and* $n \geqq 3$.

(2) $m = 1$, $\delta_1 = \cdots = \delta_n = 1$, $a = n - 1$, $b = 2 - n$ *and* $n \geqq 3$.

(3) $m = 2$, $\delta_1 = \cdots = \delta_n = 1$, $a = n - 2$, $b = 2 - n$ *and* $n \geqq 4$.

(4) $m = 1$, $\delta_1 = \cdots = \delta_n = 2$, $a = n - 1$, $b = 2 - 2n$ *and* $n \geqq 3$.

PROOF. By (5.6; 2), we have $a = n - m$ and $b = 2 - |\delta|$. In case (5.12; 1), $a\delta_n + b = 0$ implies $m\delta_n = 1$. So $m = \delta_n = 1$ and we verify the condition (1). Similarly, in case (5.12; 3), we obtain the condition (2). Here $n \geqq 3$ follows from (5.7; 1). In case (5.12; 2), we have $m\delta_n = 2$ and obtain the condition (3) or (4).

REMARK. This argument is a generalization of that in [Is; §7].

(5.15) For the sake of comparison, we consider what happens when $K^M = 0$. By virture of (5.6: 2), we have $a = n$ and $b = 2 - |\delta|$. But there are infinitely many possible δ's even if we impose the stronger condition $a\delta_n + b \geqq 0$ in (5.7; 5), which guarantees the existence of (M, L) of type $(\Sigma^n(\delta)^+_{a,b})$. In particular, there is no bound for $|\delta|$. However, there are only finitely many possible isomorphism types for the manifold W itself.

Similarly, for any fixed positive integer m, there are infinitely many types of (M, L) such that $K^M = mL$. Moreover, there are infinitely many possible isomorphism classes for W.

PROPOSITION (5.16). *Let (M, L) be of type $(\Sigma^n(\delta)^+_{a,b})$. Then, for a general member D of L, (D, L_D) is of type $(\Sigma^{n-1}(\delta')^+_{a,b})$ and there is an exact sequence $0 \to [0] \to E(\delta) \to E(\delta') \to 0$ of vector bundles on $\boldsymbol{P}^1$. In particular, $|\delta'| = |\delta|$, $\delta'_{\min} \geqq \delta_{\min}$ and $\delta'_{\max} \geqq \delta_{\max}$.*

PROOF. We have $M \cong R_B(W)$. Let H be the hyperplane section of

W corresponding to D. Since D is general, H is non-singular and meets B transversally. Therefore $D \cong R_{B\cap H}(H)$ is non-singular. $B \cap H$ is connected because it is ample on B. Moreover, letting $E(\delta') = \pi_* \mathcal{O}_H(1)$ where π is the morphism $H \to \boldsymbol{P}^1$, we see $H \cong W(\delta')$. So (D, L) is of type $(\Sigma^{n-1}(\delta')^+_{\alpha,b})$. Furthermore, by the definition of (δ'), we have an exact sequence $0 \to [0] \to E(\delta) \to E(\delta') \to 0$. It is easy to see the rest of the assertion.

COROLLARY (5.17). *Let (M, L) be of type $(\Sigma^n(\delta)^+)$. Then $\pi_1^{(p)}(M) = \{1\}$ for $p = \mathrm{char}\,(\mathfrak{K})$. Moreover, M is topologically simply connected in case $\mathfrak{K} = \boldsymbol{C}$.*

A proof will be given in §8 by means of a deformation theory of (M, L) (cf. (8.15)).

PROPOSITION (5.18). *Let (M, L) be of type $(\Sigma^n(\delta)^+_{a,b})$. Then*

(1) *$g(M, L) \geqq \Delta(M, L)$ in general,*

(2) *$g(M, L) = \Delta(M, L)$ if and only if (M, L) is of type $(\Sigma^n(\delta)^+_{1,1})$, $(\Sigma^3(1, 1, 1)^+_{2,-2})$, $(\Sigma^2(\mu + 1, \mu)^+_{2,-2\mu})$ or $(\Sigma^2(\mu, \mu)^+_{2,1-2\mu})$.*

(3) *$g(M, L) = \Delta(M, L) + 1$ if and only if (M, L) is of type $(\Sigma^n(\delta)^+_{1,2})$, $(\Sigma^4(1, 1, 1, 1)^+_{2,-2})$, $(\Sigma^3(2, 2, 2)^+_{2,-4})$, $(\Sigma^3(2, 1, 1)^+_{2,-2})$, $(\Sigma^3(1, 1, 1)^+_{2,-1})$, $(\Sigma^2(\mu + \varepsilon, \mu)^+_{2,2-\varepsilon-2\mu})$ with $\mu \geqq 1$, $\varepsilon = 0, 1, 2$, or $(\Sigma^2(1, 1)^+_{3,-2})$.*

PROOF. We first prove (2). By (5.6; 3), $g(M, L) = \Delta(M, L)$ if and only if $(a - 1)|\delta| + b = 1$. If $a = 1$, then $b = 1$. Suppose $a = 2$. Then $|\delta| = 1 - b$ and $3\delta_n + 2b + \delta_1 \geqq 0$ by (5.7; 2). Hence $2 \geqq \delta_1 + 2\delta_2 + \cdots + 2\delta_{n-1} - \delta_n$. This is impossible unless $n \leqq 3$. If $n = 3$, we have $2 \geqq \delta_1 + 2\delta_2 - \delta_3 \geqq \delta_1 + \delta_2$, so $1 = \delta_1 = \delta_2 = \delta_3$ and $b = -2$. Thus (M, L) is of type $(\Sigma^3(1, 1, 1)^+_{2,-2})$. If $n = 2$, we have $2\delta_2 + b \geqq 0$ by (5.7; 4). Since $b = 1 - \delta_1 - \delta_2$, we infer $\varepsilon = \delta_1 - \delta_2 = 0$ or 1. Thus (M, L) is of type $(\Sigma^2(\mu + 1, \mu)^+_{2,-2\mu})$ or $(\Sigma^2(\mu, \mu)^+_{2,1-2\mu})$. Finally we consider the case $a \geqq 3$. Then $(2a - 1)\delta_n + 2 - 2(a - 1)|\delta| + \delta_1 \geqq 0$. So $2 \geqq 2(a - 1)|\delta| - \delta_1 - (2a - 1)\delta_n \geqq (2a - 3)\delta_1 + \{2(a - 1)(n - 1) - (2a - 1)\}\delta_n$. From this we obtain $n = 2$. Then, by (5.7; 4), we have $0 \leqq a\delta_2 + b = a\delta_2 + 1 - (a - 1)|\delta|$. So $1 \geqq (a - 1)\delta_1 - \delta_2 \geqq (a - 2)\delta_1$. Hence $a = 3$ and $\delta_1 = \delta_2 = 1$. So $b = -3$. But then $W \cong \boldsymbol{P}^1_\beta \times \boldsymbol{P}^1_\alpha$ and $B \in |6H_\alpha|$ cannot be connected. Thus we exclude this possibility.

The assertion (3) is proved by a similar elementary argument.

(1) is proved by a similar method, or by the following observation: By virtue of (5.16), we can find a ladder $M = V_n \supset V_{n-1} \supset \cdots \supset V_1$ of (M, L) such that (V_j, L) is of type $(\Sigma^j|\delta|^+_{a,b})$ for each $j \geqq 2$, and V_1 is a non-singular hyperelliptic curve. $H^1(V_j, \mathcal{O}_{V_j}) = 0$ by (5.8). Therefore,

this ladder is regular (cf. [F6; (1.5)]). So we have $g(M, L) = g(V_1, L) \geqq \Delta(V_1, L) = \Delta(M, L)$.

PROPOSITION (5.19). *Let (M, L) be of type $(\Sigma^n(\delta)^+_{a,b})$. Then $H^1(M, L) = 0$ except in the following cases*: (1) $a = 1$. (2) $n = 2$ *and* $(a-2)\delta_2 + \delta_1 + b \leqq 0$.

For a proof, use (5.6) and (5.3). These results (5.18) and (5.19) will be used in the study of deformations of (M, L).

(5.20) The Kodaira dimension $\kappa(M)$ is calculated as follows. In view of (5.6; 2), we set $D = (a-n)H_\delta + (b + |\delta| - 2)H_\beta$ and $b' = (a-n)\delta_{\max} + b + |\delta| - 2$. Of course, we have $\kappa(M) = \kappa(D, W)$ by (5.6; 2) and [F5; (3.17)].

(1) $\kappa(M) < 0$ if $a < n$.

(2) When $a \geqq n$, we have $\kappa(M) < 0$ if and only if $b' < 0$. We further analyse this case. Together with (5.7; 2), $b' < 0$ implies $2 \geqq (2(a-n)+1)(\delta_1 - \delta_n) + 2\sum_{j\geqq 2}(\delta_j - \delta_n)$. So $\delta_2 = \delta_n$. Hence $a\delta_n + b \geqq 0$ by (5.7; 3 or 4). From this we obtain $1 \geqq (a - n + 1)(\delta_1 - \delta_n)$. So $\delta_1 = \delta_n$ or $a - n = \delta_1 - \delta_n - 1 = 0$. Thus there are following three possibilities:

(2a) $\delta_1 = \delta_n + 1$, $\delta_2 = \delta_n$, $a = n$ and $b = -n\delta_n$. In this case (M, L) has a structure similar to those in (5.13; 1).

(2b) $\delta_1 = \delta_n$, $a\delta_n + b = 0$. In this case (M, L) is similar to those in (5.13; 2). In particular, $M \cong \boldsymbol{P}^1 \times R_A(\boldsymbol{P}^{n-1}_a)$ for a hypersurface A in $\boldsymbol{P}^{n-1}_a$ of degree $2a$.

(2c) $\delta_1 = \delta_n$, $a\delta_n + b = 1$. In this case (M, L) is similar to those in (5.13; 3).

(3) $\kappa(M) = 0$ if $a = n$ and $b' = 0$. In this case $K^M = 0$. By a method similar to above, one can show $\delta_3 = \delta_n$ in this case, and $(\delta_1 - \delta_n, \delta_2 - \delta_n) = (0, 0), (1, 0), (2, 0), (1, 1)$ or $(2, 1)$.

(4) $\kappa(M) = 1$ if $a = n$ and $b' > 0$.

(5) $\kappa(M) = n$ if $a > n$ and $b' > 0$. Indeed, since $|H - \delta_1 H_\beta| \neq \varnothing$, we have $\kappa(D, W) = \kappa((t+1)D, W) \geqq \kappa(D + tb'H_\beta, W) = n$ for $t \gg 0$.

(6) It remains to consider the case $a > n$ and $b' = 0$. By (5.7; 2) we obtain $4 \geqq (2(a-n)+1)(\delta_1 - \delta_n) + 2\sum_{j\geqq 2}(\delta_j - \delta_n)$. From this we infer $\delta_2 = \delta_n$. So $a\delta_n + b \geqq 0$ by (5.7; 3 or 4). Hence $2 \geqq (a - n + 1)(\delta_1 - \delta_n)$, which implies $\delta_1 = \delta_n$ or $a - n = \delta_1 - \delta_n = 1$. Thus there are the following two possibilities:

(6a) $\delta_1 = \delta_n + 1$, $\delta_2 = \delta_n$, $a = n + 1$, $b = -(n+1)\delta_n$. In this case, as in (5.13; 1), W is a blowing-up of $\boldsymbol{P}^n$ and D turns out to be the exceptional divisor on it. So $\kappa(M) = 0$.

(6b) $\delta_1 = \delta_n$, $b = 2 - a\delta_n$. In this case, as in (5.13; 2), $W \cong \boldsymbol{P}^1_\beta \times$

P_α^{n-1} and $D = (a - n)H_\alpha$. Hence $\kappa(M) = n - 1$. Note that $n \geqq 3$, because otherwise B is not connected.

REMARK. When $n = 2$ and $\kappa(M) < 0$, M is a rational surface since we have $q(M) = 0$ by (5.8).

(5.21) Now we consider the case (5.4; b). Clearly $B \in |2bH_\beta|$ for some positive integer b. In this case (M, L) is said to be of type $(\Sigma^n(\delta)_b^0)$.

The image Y of B on P_β^1 is a divisor of degree $2b$. Clearly $C = R_Y(P_\beta^1)$ is a hyperelliptic curve of genus $b - 1$ and M is isomorphic to the P^{n-1}-bundle $P(E_C)$ over C. In particular, $q(M) = b - 1$ and $h^p(M, \mathcal{O}_M) = 0$ for $p > 1$.

THEOREM (5.22). *Let (M, L) be of type $(\Sigma^n(\delta)_b^0)$. Then*

(1) $H^p(M, tL) \cong H^p(W, tH) \oplus H^p(W, tH - bH_\beta)$ *for every t, p.*

(2) $b > \delta_{\max}$.

(3) $\chi_j(M, L) = 2 - b$ *for* $j = 0, 1, \cdots, n - 1$. *In particular* $g(M, L) = b - 1$.

(4) $K^M = -nL + (b + |\delta| - 2)H_\beta$. *So, this cannot be a multiple of L.*

PROOF. (1) follows from (2.2). (2) is a consequence of (1) and $H^0(W, H) \cong H^0(M, L)$. By calculation using (5.3), we get a formula for $\chi(M, tL)$, which yields (3). (4) follows from (2.2; 3). $b = 2 - |\delta|$ is impossible by (2).

(5.23) Now we consider the remaining case (5.4; c). Because (b) is not the case, we must have a component B_1 of B such that $\pi(B_1) = P_\beta^1$. Moreover, we claim that $\pi(B_j) = P_\beta^1$ for any component B_j of B. Indeed, otherwise, B_j would be a fiber of π, and hence $B_1 \cap B_j \neq \emptyset$, contradicting the smoothness of B.

Furthermore, we claim $n = \dim M = 2$. To see this, let F be a general fiber of π. Then $F \cong P^{n-1}$. $F \cap B_1$ and $F \cap B_j$ are hypersurfaces in F not intersecting each other. This is impossible if $n \geqq 3$.

Thus, $W \cong W(\delta_1, \delta_2)$, which is isomorphic to the Hirzebruch surface Σ_k with $k = \delta_1 - \delta_2$.

(5.24) Suppose that $k = 0$. Then $W \cong P_\beta^1 \times P_\alpha^1$ and $H = H_\alpha + \delta_1 H_\beta$. We have $B \in |2aH_\alpha + 2b'H_\beta|$ for some integers a, b'. If $b' > 0$, then B is ample and hence connected. So $b' = 0$ and B consists of $2a$ horizontal components. In this case (M, L) is said to be of type $(\Sigma^2(\mu, \mu)_a^=)$, where $\mu = \delta_1 = \delta_2$.

REMARK. The type $(\Sigma(1, 1)_a^=)$ is the same as $(\Sigma(1, 1)_a^0)$. Compare the Remark to (5.5).

(5.25) If $k > 0$, then Σ_k has a unique section X such that $X^2 < 0$. Moreover, setting $H_\alpha = H - \delta_2 H_\beta$, we have $H_\alpha^2 = k$, $H_\alpha X = 0$, $[X] = H_\alpha - kH_\beta = H - \delta_1 H_\beta$ and $X^2 = -k$. For each component B_j of B, we set $[B_j] = x_j H_\alpha + y_j H_\beta$. Since $\pi(B_j) = \boldsymbol{P}^1$, we have $x_j > 0$. Moreover, $y_j = XB_j \geqq 0$ unless $B_j = X$. Suppose that there are two components B_1, B_2 different from X. Then $x_1, x_2 > 0$ and $y_1, y_2 \geqq 0$, hence $B_1 B_2 = kx_1x_2 + x_1y_2 + x_2y_1 > 0$. This contradicts $B_1 \cap B_2 = \varnothing$. Thus we conclude: There is only one component of B other than X.

So X must be a component of B, since B is not connected. Let B' be the component other than X and set $[B'] = xH_\alpha + yH_\beta$. Then $y = 0$ because $B'X = 0$. Thus we have $[B] = (x+1)H_\alpha - kH_\beta$, which is $2F$ for some $F \in \mathrm{Pic}\,(W)$. Therefore $x = 2a - 1$ and $k = 2\gamma$ for some integer a, γ. Thus, in this case, (M, L) is said to be of type $(\Sigma(\delta_1, \delta_2)_a^-)$.

THEOREM (5.26). *Suppose (M, L) is of type $(\Sigma(\mu, \mu)_a^=)$. Then*

(1) $W \cong \boldsymbol{P}_\alpha^1 \times \boldsymbol{P}_\beta^1$, $H = H_\alpha + \mu H_\beta$ *and* $B \in |2aH_\alpha|$.

(2) $H^p(M, tL) \cong H^p(W, tH) \oplus H^p(W, tH - aH_\alpha)$.

(3) $a \geqq 2$.

(4) $g(M, L) = a\mu - 1$, $q(M) = a - 1$ *and* $p_g(M) = 0$.

(5) $K^M = (a-2)H_\alpha - 2H_\beta = (a-2)L - ((a-2)\mu + 2)H_\beta$. *So* $c_1(M)^2 = -8a + 16$.

For a proof, use (2.2) and (5.3).

THEOREM (5.27). *Suppose that (M, L) is of type $(\Sigma(\mu + 2\gamma, \mu)_a^-)$, where $\gamma > 0$. Then*

(1) $W \cong \Sigma_{2\gamma}$ *and* $B = B_1 + B_2$, $B_1 \in |H_\alpha - 2\gamma H_\beta|$, $B_2 \in |(2a-1)H_\alpha|$, *where $H_\alpha = H - \mu H_\beta$. B_1 is the unique curve on W with negative self intesection number.*

(2) $H^p(M, tL) \cong H^p(W, tH) \oplus H^p(W, tH - 2H_a + \gamma H_\beta)$. *This implies* $a \geqq 2$.

(3) $g(M, L) = a\mu + 2a\gamma - \gamma - 1$, $q(M) = 0$ *and* $p_g(M) = (a-1)(a\gamma - \gamma - 1)$.

(4) $K^M = (a-2)H_\alpha + (\gamma - 2)H_\beta = (a-2)L - ((a-2)\mu - \gamma + 2)H_\beta$ *and* $c_1(M)^2 = 4(a-2)(a\gamma - \gamma - 2)$.

(5) (M, L) *is globally Macaulay if and only if* $(a-2)\mu \leqq \gamma - 1$.

(6) (M, L) *is globally Gorenstein if and only if* $(a-2)\mu = \gamma - 2$.

(7) $g(M, L) \geqq \Delta(M, L)$ *in general.* $g = \Delta$ *if and only if* $\gamma = 1$ *and* $a = 2$. $g = \Delta + 1$ *if and only if* $\gamma = a = 2$.

(8) $H^1(M, L) \neq 0$ *if and only if* $a > 2$ *and* $\mu \geqq \gamma$.

These are proved similarly as the preceding results.

6. Summary. (6.1) Now we can give a classification table of hyperelliptic polarized manifolds. See Tables I and II.

TABLE I Hyperelliptic polarized surfaces.

type	$d(M, L)$	$g(M, L)$	$q(M)$	$p_g(M)$	$c_1(M)^2$
(I_a)	2	a	0	$a(a-1)/2$	$2(a-2)^2$
(IV_a)	8	$2a+1$	0	$a(a-1)/2$	$2(a-2)^2$
$(^*\mathrm{II}_a)$	4	$2a$	0	$a(a-1)$	$(2a-3)^2$
$(\Sigma(\delta_1, \delta_2)^+_{a,b})$	$2\lvert\delta\rvert$	$a\lvert\delta\rvert+b-1$	0	$(a-1)(a\lvert\delta\rvert+2b-2)/2$	$2(a-2)(a\lvert\delta\rvert+2b-4)$
$(\Sigma(\delta_1, \delta_2)^0_b)$	$2\lvert\delta\rvert$	$b-1$	$b-1$	0	$-8b+16$
$(\Sigma(\mu, \mu)^=_a)$	4μ	$a\mu-1$	$a-1$	0	$-8a+16$
$(\Sigma(\mu+2\gamma, \mu)^-_a)$	$4(\mu+\gamma)$	$a\mu+2a\gamma-\gamma-1$	0	$(a-1)(a\gamma-\gamma-1)$	$4(a-2)(a\gamma-\gamma-2)$

TABLE II Hyperelliptic polarized manifolds with $\dim M \geqq 3$.

type	$d(M, L)$	$g(M, L)$	$q(M)$	$b_2(M)$	$\chi_{n-2}(M, L)$
(I^n_a)	2	a	0	1	>0
(II^n_a)	4	$2a+1$	0	1	>0
$(^*\mathrm{IV}_a)$	8	$4a+1$	0	1	>0
$(\Sigma^n(\delta)^+_{a,b})$	$2\lvert\delta\rvert$	$a\lvert\delta\rvert+b-1$	0	$\geqq 2$	>0
$(\Sigma^n(\delta)^0_b)$	$2\lvert\delta\rvert$	$b-1$	$b-1$	2	$\leqq 0$

REMARK. (II^2_a) is the same type as $(\Sigma(1, 1)^+_{a+1,0})$ and hence is omitted in Table I. Note also that $(\Sigma(1, 1)^+_{a,,b}) = (\Sigma(1, 1)^+_{a+b,-b})$ (cf. (5.5)) and that $(\Sigma(1, 1)^0_a) = (\Sigma(1, 1)^=_a)$ (cf. (5.24)).

(6.2) $q(M) = 0$ unless (M, L) is of type $(\Sigma^n(\delta)^0)$ or $(\Sigma^2(\delta)^=)$. These two types are characterized by the property $\chi_{n-2}(M, L) \leqq 0$.

(6.3) $H^1(M, L) = 0$ unless (M, L) is of type $(\Sigma^n(\delta)^0)$, $(\Sigma(\mu, \mu)^=)$, $(\Sigma^n(\delta)^+_{1,b})$ with $b \geqq 2$, $(\Sigma^2(\delta)^+_{a,b})$ with $(a-2)\delta_{\min} + b + \delta_{\max} \leqq 0$, or $(\Sigma^2(\mu + 2\gamma, \mu)^-_a)$ with $a \geqq 3$, $\mu \geqq \gamma$.

(6.4) (M, L) is globally Macaulay if and only if it is of type (I), (II), (IV), $(^*)$, $(\Sigma(\delta)^+_{a,b})$ with $1 - \lvert\delta\rvert \leqq b \leqq 1$ or $(\Sigma^2(\mu + 2\gamma, \mu)^-_a)$ with $(a-2)\mu \leqq \gamma - 1$.

(6.5) (M, L) is globally Gorenstein in all the cases where K^M is a multiple of L, namely, the cases $(\mathrm{I}_{\sigma+1})$, (II_σ), $(\mathrm{IV}_{2\sigma})$, $(^*\mathrm{IV}_\sigma)$, $(\Sigma^n(\delta)_{\sigma+1,2-\lvert\delta\rvert})$ and $(\Sigma^2(\mu + 2(\sigma-1)\mu + 4, \mu)^-_{\sigma+1})$, where σ is the sectional index.

(6.6) If (M, L) is globally Gorenstein with negative index, then (M, L) is either of type (I^n_a) with $1 \leqq a < n$, (II^n_a) with $1 \leqq a \leqq n-2$, $(^*\mathrm{IV}_1)$, or the types described in (5.14). In particular, $d(M, L) = 2$, 4, $2n$, $2n+2$ or $4n$.

(6.7) $g(M, L) \geqq \Delta(M, L)$ except when $q(M) > 0$ (cf. (6.2)). When $q(M) = 0$, we have $g \geqq \Delta + 2$ except in the following cases: $g = \Delta$: (I^n_1), $(^*\mathrm{II}_1)$, $(\Sigma(\mu+2, \mu)^-_2)$ and those in (5.18; 2). $g = \Delta + 1$: (I^n_2), (II^n_1), (IV_2),

(*IV_1), ($\Sigma(\mu+4, \mu)_2^-$) and those in (5.18; 3).

(6.8) The canonical bundle K^M of M cannot be very ample.

In fact, if (M, L) is not of type (*), we have $M \cong R_{B,F}(W)$. So, by (2.2), $H^0(M, K^M) \cong H^0(W, K^W + F) \oplus H^0(W, K^W)$. On the other hand, we have $p_g(W) = 0$ since $\Delta(W, H) = 0$ (recall (2.8)). Hence $H^0(W, K^W + F) \cong H^0(M, K^M)$. This implies that the rational mapping defined by K^M factors through W, and hence cannot be birational. If in addition $K^W + F$ is very ample on W, the canonical mapping of M is nothing but the morphism $M \to W$.

When (M, L) is of type (*), let M' and W' be as in (4.3). Then, similarly as above, the rational mapping defined by the canonical bundle of M' must factor through W'. Hence it is not birational, and so $\rho_{|K^M|}$ is not birational.

(6.9) The calculation of the Kodaira dimension of M is easy except possibly in the case of type $(\Sigma^n(\delta)^+_{a,b})$, which was treated in (5.20). The results are summarized in Table III.

REMARK. In all the cases where $\kappa(M) = 0$, M is birationally equivalent to a manifold with trivial canonical bundle, and $q(M) = 0$. Therefore, if $n = 2$, M is (birationally) a $K3$-surface.

TABLE III Kodaira dimension of M.

value of $\kappa(M)$	n	$n-1$	1	0	$-\infty$
(I^n_a)	$a>n$	—	—	$a=n$	$a<n$
(II^n_a)	$a>n-1$	—	—	$a=n-1$	$a<n-1$
(IV_a)	$a>2$	—	—	$a=2$	—
$(*\mathrm{II}_a)$	$a>1$	—	—	—	$a=1$
$(*\mathrm{IV}_a)$	$a>2$	—	—	$a=2$	$a=1$
$(\Sigma^n(\delta)^+_{a,b})$ (cf. (5.20))	case (5)	case (6b)	case (4)	case (3) & (6a)	case (1) & (2)
$(\Sigma^n(\delta)^0_b)$	—	—	—	—	any b
$(\Sigma(\mu, \mu)^=_a)$	—	—	—	—	any a
$(\Sigma(\mu+2\gamma, \mu)^-_a)$	$a>2$	—	$a=2$ & $\gamma>2$	$a=\gamma=2$	$a=2$ & $\gamma=1$

7. Small deformations.

DEFINITION (7.1). A deformation family of prepolarized manifolds consists of a quadruple $(\mathscr{M}, X, \pi, \mathscr{L})$ of manifolds $\mathscr{M}$ and X both of which are connected but may not be complete, a proper smooth morphism $\pi: \mathscr{M} \to X$ and a line bundle $\mathscr{L}$ on $\mathscr{M}$.

Given a point $x \in X$, we set $M_x = \pi^{-1}(x)$ and let L_x be the restriction of $\mathscr{L}$ to M_x. This prepolarized manifold (M_x, L_x) is said to be a member of the above deformation family.

We say that any small deformation of a given prepolarized manifold

(M, L) has a certain property (♯) if, for any deformation family $(\mathscr{M}, X, \pi, \mathscr{L})$ as above and any $o \in X$ such that $(M_o, L_o) \cong (M, L)$, there exists a neighborhood U of o in X such that (M_x, L_x) has the property (♯) for every $x \in U$. We use the terminology "small deformation" for various objects in a similar sense.

(7.2) The purpose of this section is to consider the following:

PROBLEM. *Let (M, L) be a hyperelliptic polarized manifold of certain type. Then, is any small deformation of (M, L) hyperelliptic?*

REMARK (7.3). If $H^2(M, \mathcal{O}_M) = 0$, then any small deformation $\{M_x\}_{x \in X}$ of M carries a family of line bundles $\{L_x\}$, and thus can be made a family of prepolarized manifolds.

When $n = \dim M \geqq 3$, this condition is satisfied except when (M, L) is of type $(\Sigma(u, u, u)^+_{a,-au})$. For a proof, use (5.8) and (5.22; 1) (compare also Table II in §6).

REMARK (7.4). If $H^1(M, L) = 0$, then any small deformation of (M, L) satisfies $d(M_x, L_x) = 2\Delta(M_x, L_x)$.

Indeed, we have $h^1(M_x, L_x) = 0$ by the semicontinuity theorem. Applying the theory of Grothendieck [EGA; Chap. III] (or see [Ha 2; Corollary 12.6]), we infer that $\pi_* \mathscr{L}$ is locally free of rank $h^0(M, L)$. So $h^0(M_x, L_x) = h^0(M, L)$, which implies the assertion.

When $n \geqq 2$, we have $H^1(M, L) = 0$ unless (M, L) is of type $(\Sigma^n(\delta)^0)$, $(\Sigma(u, u)^=)$, $(\Sigma^n(\delta)^+_{1,b})$ with $b \geqq 2$, $(\Sigma^2(\delta)^+_{a,b})$ with $(a-2)\delta_{\min} + b + \delta_{\max} \leqq 0$, or $(\Sigma^2(u + 2\gamma, u)^-_a)$ with $a \geqq 3$, $u \geqq \gamma$. For a proof, see (6.4), (5.19) and (5.27; 8).

(7.5) Assume $d(M_x, L_x) = 2\Delta(M_x, L_x)$ in the question (7.2). Then any small deformation of (M, L) is hyperelliptic if $g(M, L) > \Delta(M, L)$ and if (M, L) is not a Fano-$K3$ variety.

Indeed, taking a smaller neighborhood of o if necessary, we may assume that L_x is ample and $\mathrm{Bs}|L_x| = \varnothing$ for any x in U. Moreover, we have $h^0(M, F) = 0$ or $h^0(M, -F) = 0$ for $F = K^M + (n-2)L$ since (M, L) is not Fano-$K3$. So we may assume $h^0(M_x, F_x) = 0$ or $h^0(M_x - F_x) = 0$ for any $x \in U$. Hence (M_x, L_x) is not Fano-$K3$. So (1.4) applies.

REMARK (7.6). We have $g(M, L) \geqq \Delta(M, L) + 2$ except when (M, L) is of the types given in (6.7). Clearly this condition implies that (M, L) is not Fano-$K3$.

(7.7) Combining the above observations we obtain the following:

THEOREM. *Let (M, L) be a hyperelliptic polarized manifold with*

$n = \dim M \geqq 2$. *Then any small deformation of* (M, L) *is a hyperelliptic polarized manifold if* (M, L) *is of one of the following types*: (I_a^n) *with* $a \geqq 3$, (II_a^n) *with* $a \geqq 2$, (IV_a) *with* $a \geqq 3$, $(^*\mathrm{II}_a)$ *with* $a \geqq 2$, $(^*\mathrm{IV}_a)$ *with* $a \geqq 2$, $(\Sigma^n(\delta)_{a,b}^+)$ *with* $a \geqq 2$ *and* $(a-2)\delta_{\min} + \delta_{\max} + b > 0$, $(\Sigma(u+2\gamma, u)_2^-)$ *with* $\gamma \geqq 3$, $(\Sigma(u+2\gamma, u)_a^-)$ *with* $a \geqq 3$ *and* $u < \gamma$.

REMARK. The same is true for the type (I_2^n), too. In this case we have $g(M, L) = \Delta(M, L) + 1$, but L_x cannot be simply generated because $h^0(M_x, L_x) = n + 1$. Hence (1.4) applies.

COROLLARY (7.8). *Let* (M, L) *be a hyperelliptic polarized manifold of dimension* $n \geqq 3$. *Then any small deformation of* M *carries a structure of a hyperelliptic polarized manifold if* (M, L) *is of one of the following types*: (I_a^n) *with* $a \geqq 2$, (II_a^n) *with* $a \geqq 2$, $(^*\mathrm{IV}_a)$ *with* $a \geqq 2$, $(\Sigma^n(\delta)_{a,b}^+)$ *with* $a \geqq 2$ *and* $(a-2)\delta_{\min} + \delta_{\max} + b > 0$.

(7.9) Now we consider the same problem (7.2) from an entirely different viewpoint by the aid of the theory of Kodaira-Spencer-Horikawa.

Let things be as in (2.1) and suppose that W and B (and hence $V = R_{B,F}(W)$ also) are non-singular. Then we have a natural homomorphism $\Theta_V \to \pi^*\Theta_W$ on V, where Θ denotes the sheaf of vector fields. Taking direct images we obtain $\theta: \pi_*\Theta_V \to \Theta_W \otimes \pi_*\mathcal{O}_V$. The involution i of V over W acts on these sheaves equivariantly. Let $\pi_*\Theta_V = \Theta_V^+ \oplus \Theta_V^-$ be the decomposition such that the action of i is the identity on Θ_V^+ and is (-1)-times the identity on Θ_V^-. Then θ induces an isomorphism $\theta: \Theta_V^- \cong \Theta_W \otimes \pi_*\mathcal{O}_V^- \cong \Theta_W[-F]$ and an injective homomorphism $\theta^+: \Theta_V^+ \to \Theta_W$ such that $\mathrm{Coker}(\theta^+) \cong \mathcal{O}_B[B]$, the normal sheaf of B in W. Taking cohomologies and setting $\tau_V^+ = H^1(W, \Theta_V^+)$ and $\tau_V^- = H^1(W, \Theta_V^-)$, we obtain $\tau_V = \tau_V^+ \oplus \tau_V^-$ and an exact sequence $H^0(W, \Theta_W) \to H^0(B, [B]) \to \tau_V^+ \to \tau_W \to H^1(B, [B]) \to H^2(\Theta_V^+) \to H^2(\Theta_W)$ where τ_M denotes $H^1(M, \Theta_M)$ for any manifold M. The meanings of these mappings may be interpreted roughly as follows.

In general, τ_M is the Zariski tangent space of the formal moduli space for the deformation functor of M. τ_V^+ corresponds to the subspace of τ_V along which the structure of a double covering is preserved. Similarly, $H^0(B, [B])$ parametrizes infinitesimal displacements of B in W. A family $\{B_x\}$ gives rise to a family $\{V_x = R_{B_x, F_x}(W)\}$ of double coverings of W. This assignment corresponds to the mapping $H^0(B, [B]) \to \tau_V^+$. Moreover we can show the following:

THEOREM (7.10). *Let things be as above and suppose that* $\tau_V^- = 0$. *Then any formal deformation of* V *turns out to be a double covering of a formal deformation of* W. *This means that, for any proper smooth morphism* $f: \mathfrak{V} \to S$ *of formal schemes such that* S *has only one closed*

point o and that the fiber V_o over o is isomorphic to V, there exists a formal scheme $\mathfrak{W}$ over S together with a morphism $\Pi: \mathfrak{V} \to \mathfrak{W}$ which makes $\mathfrak{V}$ a double covering of $\mathfrak{W}$ such that $\Pi_o: V_o \to W_o$ is isomorphic to $\pi: V \to W$.

For a proof, it is enough to show the following:

LEMMA (7.11). *Let R be an Artinian $\mathfrak{R}$-algebra with the maximal ideal $\mathfrak{m}$ and let I be an ideal of R such that $I \cdot \mathfrak{m} = 0$. Let $f: V \to \mathrm{Spec}\,(R)$ and $g_I: W_I \to \mathrm{Spec}\,(R/I)$ be proper smooth morphisms and suppose that there is a morphism $\Pi_I: V_I \to W_I$ which makes $V_I = V \times_{\mathrm{Spec}\,(R)} \mathrm{Spec}\,(R/I)$ a double covering of W_I. Suppose in addition that the restriction of Π_I over the closed point of $\mathrm{Spec}\,(R)$ is isomorphic to $\pi: V \to W$ (where we have $\tau_V^- = 0$ by assumption). Then there exists an R-scheme $g: W \to \mathrm{Spec}\,(R)$ and a morphism $\Pi: V \to W$ such that their restrictions over $\mathrm{Spec}\,(R/I)$ is isomorphic to W_I and Π_I.*

This is proved by the same argument as that in [Ho 2; Lemma in p. 276]. Indeed, the injectivity of $H^2(W, \Theta_W) \to H^2(V, \pi^*\Theta_W) \cong H^2(W, \Theta_W \otimes \pi_*\mathcal{O}_V)$ is clear and the surjectivity of $H^1(W, \Theta_W) \to H^1(V, \pi^*\Theta_W) \cong H^1(W, \Theta_W \otimes \pi_*\mathcal{O}_V)$ follows from $\tau_V^- = H^1(W, \Theta_W[-F]) = 0$.

COROLLARY (7.12). *Let (M, L) be a hyperelliptic polarized manifold and let (W, H) be as in (1.1). Suppose that W is non-singular and that $\tau_M^- = 0$. Then any small deformation M_x of M carries a line bundle L_x such that (M_x, L_x) is a hyperelliptic polarized manifold.*

PROOF. Given any deformation family $\mathscr{M} \to X$ with $M_o \cong M$ for some point o on X, by (7.10) we can find a formal deformation $\mathfrak{W} \to \mathfrak{X}$ of W over the formal completion of X at o together with a double covering $\Pi: \mathfrak{M} \to \mathfrak{W}$ whose restriction over o is $\pi: M \to W$, where $\mathfrak{M}$ is the formal completion of $\mathscr{M}$ along M_o. Since $H^2(W, \mathcal{O}_W) = 0$, there is a line bundle $\mathfrak{H}$ on $\mathfrak{W}$ whose restriction to $W_o \cong W$ is H. $\mathfrak{H}$ is ample on $\mathfrak{W}$ since so is H on W. Hence its pull-back to $\mathfrak{M}$ is ample. From this we infer that the morphism $\mathfrak{M} \to \mathfrak{W}$ is algebraisable. So, there is a neighborhood U of o in X (with respect to the étale topology) over which we have a morphism $\mathscr{M}_U \to \mathscr{W}$ of families as an extension of $\mathfrak{M} \to \mathfrak{W}$. Shrinking U further if necessary, we see that $\pi_x: M_x \to W_x$ is a finite double covering and $\Delta(W_x, H_x) = 0$ for any $x \in U$. Thus we obtain the conclusion.

PROPOSITION (7.13). *Let (M, L) be a hyperelliptic polarized manifold with $n = \dim M \geqq 2$. Then $\tau_M^- = 0$ if (M, L) is of one of the following types:*

(1) (I_a^n) *with* $n \geqq 3$ *or* $a \geqq 3$.

(1′) (IV_a) *with* $a \geqq 3$.

(2) (II_a^n) *with* $a \geqq 2$.

(3) $(\Sigma^n(\delta)_{a,b}^+)$ *with* $a \geqq 2$ *and* $n \geqq 3$, *except when* $n = a = 3$, $\delta_1 = \delta_2 = \delta_3$, $b = -\delta_3$ *and* $M \cong \boldsymbol{P}^1 \times R_A(\boldsymbol{P}^2)$ *for a hypersurface* A *of degree* 6 *on* $\boldsymbol{P}^2$.

(4) $(\Sigma(\delta_1, \delta_2)_{a,b}^+)$ *with* $a \geqq 3$ *and* $a\delta_2 + b + \delta_1 - \delta_2 \geqq 3$.

(5) $(\Sigma(u + 2\gamma, u)_a^-)$ *with* $a \geqq 3$ *and* $\gamma \geqq 3$.

PROOF. In case (1) we have $W \cong \boldsymbol{P}^n$ and $F = (a+1)H$. There is an exact sequence $0 \to \mathcal{O}_W \to \Lambda \otimes [H] \to \Theta_W \to 0$ on W where Λ is the dual space of $H^0(W, H)$ (cf., e.g., [Ha 2; p. 182]). This yields an exact sequence $0 \to H^1(W, \Theta_W[-F]) \to H^2(W, -(a+1)H) \to \Lambda \otimes H^2(W, -aH)$. So $\tau_M^- = 0$ unless $n=2$. When $n=2$, the last mapping is the dual of the natural mapping $H^0(W,H) \otimes H^0(W, K^W + aH) \to H^0(W, K^W + (a+1)H)$, which is surjective unless $a=2$. So $\tau_M^- = 0$ if $a \geqq 3$. Case (1′) follows from (1).

In case (2), W is a hyperquadric in $P = \boldsymbol{P}^{n+1}$ and $F = (a+1)H$. Using the exact sequence $0 \to \mathcal{O}_W \to H^0(W, H)^\vee \otimes [H] \to (\Theta_P)_W \to 0$, we infer that $H^1(W, \Theta_P[-F]_W) = 0$ by the same argument as above. Using the exact sequence $0 \to \Theta_W \to (\Theta_P)_W \to 2H \to 0$, we obtain $\tau_M^- = H^1(W, \Theta_W[-F]) = 0$.

In case (3), let $T^{W/P}$ denote the relative tangent bundle of $W \to \boldsymbol{P}_\beta^1$. Then we have two natural exact sequences: (♯) $0 \to T^{W/P} \to \Theta_W \to 2H_\beta \to 0$ and (♯♯) $0 \to \mathcal{O}_W \to H_\delta \otimes E_\delta^\vee \to T^{W/P} \to 0$, where the notation is as in (5.2). Since $n \geqq 3$, $H^1(W, -tH_\delta + uH_\beta) = 0$ for any $t > 0$ and any integer u by (5.3; 1) and the Serre duality. In particular $H^1(W, 2H_\beta - F) = 0$. Therefore, by virtue of (♯), it suffices to show $H^1(W, T^{W/P}[-F]) = 0$. We have $H^1(W, H_\delta - \delta_j H_\beta - F) = 0$ because $a \geqq 2$. In view of (♯♯), we infer that $\tau_M^- = 0$ if $H^2(W, -F) = 0$. If this is not the case, then $0 < h^{n-2}(W, K^W + F) \leqq h^{n-2}(M, K^M) = h^2(M)$. By (5.8), this is possible only when $n = 3$, $\delta_1 = \delta_2 = \delta_3$, $a\delta_3 + b = 0$, $W \cong \boldsymbol{P}_\beta^1 \times \boldsymbol{P}_\alpha^2$ and $M \cong \boldsymbol{P}_\beta^1 \times R_A(\boldsymbol{P}_\alpha^2)$ for a hypersurface A on $\boldsymbol{P}_\alpha^2$ of degree $2a$ with $a \geqq 3$. Moreover, if $a \geqq 4$, the mapping $H^2(W, -F) \to H^2(W, H_\delta \otimes E_\delta^\vee[-F])$ is injective, because it is the dual of the surjective mapping $H^0(\boldsymbol{P}_\alpha^2, (a-4)H_\alpha) \otimes H^0(\boldsymbol{P}_\alpha^2, H_\alpha) \to H^0(\boldsymbol{P}_\alpha^2, (a-3)H_\alpha)$. So $a = 3$ if $\tau_M^- \neq 0$.

In case (4), W is isomorphic to the Hirzebruch surface Σ_k with $k = \delta_1 - \delta_2$. We use the exact sequence $0 \to [2H_\delta - \Delta H_\beta] \to \Theta_W \to 2H_\beta \to 0$, where $\Delta = \delta_1 + \delta_2$. Using the Serre duality and (5.3), we obtain $H^1(W, 2H_\delta - \Delta H_\beta - F) = 0$ and $H^1(W, 2H_\beta - F) = 0$ by assumption. Hence $H^1(W, \Theta_W[-F]) = 0$.

In case (5), we have $W \cong \Sigma_{2\gamma}$ and $F = aH_\alpha - \gamma H_\beta$ where $H_\alpha = H_\delta - uH_\beta$. So we get $H^1(W, \Theta_W[-F]) = 0$ by the same argument as above.

COROLLARY (7.14). *Let (M, L) be a globally Gorenstein hyperelliptic polarized manifold with $n = \dim M \geqq 2$. If (M, L) is not of type $(*)$, then any small deformation of M admits a structure of a hyperelliptic polarized manifold unless M is a $K3$-surface or (M, L) is of type (II_1^n).*

PROOF. In view of (6.5), we infer $\tau_M^- = 0$ by (7.13). So (7.12) applies.

REMARK. As a matter of fact, the conclusion is true even if (M, L) is of type $(*)$. Compare (7.8).

EXAMPLE (7.15). Let (M, L) be a hyperelliptic polarized manifold of type (II_1^n). Then (M, L) is a weighted complete intersection of type $(2, 4)$ in $\boldsymbol{P}(2, 1, \cdots, 1)$ (see (3.4; 5)). But a general complete intersection of this type is a hypersurface of degree four in $\boldsymbol{P}^{n+1}$, and hence not hyperelliptic.

In this case the conditions in (7.3) and (7.4) are satisfied when $n \geqq 3$, but (7.5) does not apply because (M, L) is Fano-$K3$. We see also $\tau_M^- \neq 0$. In fact, $\dim(\tau_M^-) = 1$ when $n \geqq 3$ and $\dim(\tau_M^-) = 2$ when $n = 2$ (in this case M is a $K3$-surface).

REMARK (7.16). In case M is a $K3$-surface, we must have $\tau_M^- \neq 0$ because any general small deformation of M is non-algebraic. In fact, (7.3) is not true. Moreover, usually, a small deformation (M_x, L_x) of (M, L) is not hyperelliptic, because L_x turns out to be very ample.

REMARK (7.17). We have $\tau_M^- \neq 0$ if (M, L) is of type $(\Sigma^n(\delta)_{1,b}^+)$. Note that $b \geqq 1$ by (5.7; 1). When $b \geqq 2$, the condition (7.4) does not hold. When $b = 1$, (7.5) does not apply because $g(M, L) = \Delta(M, L)$ (see (5.18; 2)).

We have $\tau_M^- \neq 0$ in case (M, L) is of the exceptional type described in (7.13; 3). In this case (7.3) is not true. Indeed, the product factor $R_A(\boldsymbol{P}^2)$ of M is a $K3$-surface.

Thus, the assertions (7.13; 1, 2 and 3) are the best possible. Perhaps, however, (4) and (5) may not be so. Compare (7.7).

8. Deformation equivalence.

DEFINITION (8.1). Prepolarized manifolds (M, L) and (M', L') are said to be *deformation equivalent* if there exist prepolarized manifolds $(M_0, L_0) = (M, L), (M_1, L_1), \cdots, (M_r, L_r) = (M', L')$ such that (M_j, L_j) and (M_{j-1}, L_{j-1}) are members of one and the same deformation family for each $j = 1, 2, \cdots, r$. In this case we write $(M, L) \sim (M', L')$.

In this section we will study deformation equivalences among hyperelliptic polarized manifolds. When (M, L) is of a certain type $(\sharp)$,

we write $(M, L) \in (\sharp)$. If in addition $(M, L) \sim (M', L')$, we write $(M', L') \sim (\sharp)$. If $(M', L') \in (\sharp')$, then we write $(\sharp) \sim (\sharp')$.

(8.2) For any fixed integers n and a, all the hyperelliptic polarized manifolds of type (I_a^n) form a single deformation family. Indeed, they correspond to non-singular members of $|(2a+2)H|$ on $\boldsymbol{P}^n$, which are parametrized by a Zariski open (hence connected) subset of the projective space $|(2a+2)H|$.

Similarly, if $(\sharp) = (\mathrm{II}_a^n)$, (IV_a), $(^*\mathrm{II}_a)$, $(^*\mathrm{IV}_a)$, $(\Sigma^n(\delta)_{a,b}^+)$, $(\Sigma^n(\delta)_b^0)$, $(\Sigma(u, u)_a^-)$ or $(\Sigma(u+2\gamma, u)_a^-)$ with indices such as a, n, (δ), γ, b being fixed, all the hyperelliptic polarized manifolds of type $(\sharp)$ form a single deformation family. Therefore, for any types $(\sharp)$ and $(\sharp')$ as above, $(\sharp) \sim (\sharp')$ implies $(M, L) \sim (M', L')$ for any $(M, L) \in (\sharp)$ and $(M', L') \in (\sharp')$.

(8.3) An invariant i of a prepolarized manifold (M, L) is called a *deformation invariant* if $(M, L) \sim (M', L')$ implies $i(M, L) = i(M', L')$. In this case $i(\sharp)$ is well-defined for any types $(\sharp)$ as in (8.2).

(8.4) Examples of deformation invariants. (1) The Hilbert polynomial $\chi(M, tL)$ is a deformation invariant. By the formula $\chi(M, tL) = \sum_{j=0}^n \chi_j(M, L) t^{[j]}/j!$ where $t^{[j]} = t(t+1)\cdots(t+j-1)$, we infer that $\chi_j(M, L)$ is also a deformation invariant for every j. In particular, $d(M, L) = \chi_n(M, L)$ and $g(M, L) = 1 - \chi_{n-1}(M, L)$ are deformation invariants.

(2) In case $\mathfrak{K} = \boldsymbol{C}$, all the topological invariants of M are deformation invariants. In particular, so are $q(M)$, $p_g(M)$ and all the Hodge numbers $h^{p,q}(M)$. This is not always true when $\mathrm{char}(\mathfrak{K}) > 0$.

(3) If $\mathrm{char}(\mathfrak{K}) = p \geqq 0$, the tame fundamental group $\pi_1^{(p)}(M)$ is a deformation invariant.

(4) The l-adic cohomology ring $H^\cdot(M; \boldsymbol{Q}_l)$ is a deformation invariant. Moreover, if $(M, L) \sim (M', L')$, there exists a (non-canonical) isomorphism $H^\cdot(M; \boldsymbol{Q}_l) \to H^\cdot(M'; \boldsymbol{Q}_l)$ which maps $c_1(L)$ and the total Chern class of M to $c_1(L')$ and to total Chern class of M', respectively. So, in particular, all the Chern numbers are deformation invariants.

(8.5) In order to study deformation equivalences within the type (Σ), we need several preparations.

DEFINITION. Given $(\delta) = (\delta_1, \delta_2, \cdots, \delta_n)$ as in (5.2), we define $u(\delta) = \delta_{\max} - \delta_{\min}$ and $v(\delta) = \mathrm{Min}(\sharp\{\delta_j \mid \delta_j = \delta_{\max}\}, \sharp\{\delta_j \mid \delta_j = \delta_{\min}\})$. For example $v(3, 3, 2, 1, 1, 1) = 2$ and $v(4, 2, 1, 1) = 1$.

(δ) is said to be *stable* if $u(\delta) \leqq 1$. We see easily that any stable integral vector (δ) is determined uniquely by $|\delta|$.

DEFINITION (8.6). The vector bundle $E(\delta)$ is said to be a *specializa-*

tion of $E(\delta')$ if there exists a deformation family of vector bundles E_t parametrized by $t \in A^1$ such that $E_0 \cong E(\delta)$ and $E_t \cong E(\delta')$ for every $t \neq o$. Here notations are as in (5.2). In the above case, we say that $W(\delta)$ is a specialization of $W(\delta')$.

REMARK (8.7). If $E(\delta)$ is a specialization of $E(\delta')$, then $\delta_{\max} \geqq \delta'_{\max}$, $\delta_{\min} \leqq \delta'_{\min}$, $u(\delta) \geqq u(\delta')$, and $|\delta| = |\delta'|$ by the semicontinuity theorem. In particular, $(\delta') = (\delta)$ if (δ) is stable.

LEMMA (8.8). *If $a - b \geqq 2$, then $E(a, b)$ is a specialization of some $E(a', b')$ with $a > a' \geqq b' > b$.*

PROOF. Let T be a one dimensional subspace of $\mathrm{Ext}^1(\mathcal{O}(a), \mathcal{O}(b)) \cong H^1(\boldsymbol{P}^1, \mathcal{O}(b - a))$, which is not trivial by the assumption $a - b \geqq 2$. For each $t \in T$, let E_t be the vector bundle with the natural extension $0 \to \mathcal{O}(b) \to E_t \to \mathcal{O}(a) \to 0$. Clearly $E_0 \cong E(a, b)$ and $E_t \cong E(a', b')$ for every $t \neq 0$ for some fixed (a', b') with $a' \geqq b'$, because t's differ only up to scalar multiplication. So, it suffices to show $b' > b$.

The image of t under the isomorphism $H^1(\boldsymbol{P}^1, \mathcal{O}(b - a)) \cong \mathrm{Hom}(H^0(\boldsymbol{P}^1, \omega(a - b)), H^1(\boldsymbol{P}^1, \omega))$ gives the first mapping of the long exact sequence $H^0(\boldsymbol{P}^1, \mathcal{O}(a-b-2)) \to H^1(\boldsymbol{P}^1, \mathcal{O}(-2)) \to H^1(\boldsymbol{P}^1, E_t(-b-2)) \to H^1(\boldsymbol{P}^1, \mathcal{O}(a - b - 2)) = 0$. From this we infer $H^1(\boldsymbol{P}^1, E_t(-b - 2)) = 0$ for $t \neq 0$, which implies $b' > b$.

COROLLARY (8.9). *If (δ) is not stable, then $E(\delta)$ is a specialization of another vector bundle $E(\delta')$ such that $u(\delta') < u(\delta)$, or $u(\delta') = u(\delta)$ and $v(\delta') < v(\delta)$.*

PROOF. $E(\delta_1, \delta_n)$ is a specialization of some $E(a', b')$ with $\delta_1 > a' \geqq b' > \delta_n$ by (8.8). Then $E(\delta') = E(a', b') \oplus E(\delta_2, \cdots, \delta_{n-1})$ has the required property.

COROLLARY (8.10). *For any (δ), there exists a chain $(\delta)_0 = (\delta), (\delta)_1, \cdots, (\delta)_k$ of integral vectors such that $E(\delta)_{j-1}$ is a specialization of $E(\delta)_j$ for each $j = 1, \cdots, k$ and that $(\delta)_k$ is stable.*

LEMMA (8.11). *Suppose that $E(\delta)$ is a specialization of $E(\delta')$. Then $(\Sigma^n(\delta)^+_{a,b}) \sim (\Sigma^n(\delta')^+_{a,b})$ if $a\delta_{\min} + b \geqq 0$.*

PROOF. We have vector bundle E on $\boldsymbol{P}^1_\beta \times A^1$ such that $E_o \cong E(\delta)$ and $E_t \cong E(\delta')$ for $t \neq 0$, where the subscript t indicates the restriction over $t \in A^1$. Set $W = \boldsymbol{P}(E)$ and let H be the tautological line bundle on W. Set $\mathscr{B} = \mathcal{O}_W(2aH + 2bH_\beta)$ and let $f: W \to A^1$ be the natural morphism. $H^1(W_o, \mathscr{B}_o) = 0$ since $a\delta_{\min} + b \geqq 0$. By [Ha 2; Chap. III, §12], we infer that $\mathscr{D} = f_*\mathscr{B}$ is a locally free sheaf on A^1 of rank

$r = h^0(W_o, \mathscr{B}_o)$. Let X be the corresponding vector bundle, which we consider to be an A^r-bundle over A^1. Let $\tilde{W}$ be the fiber product of W and X over A^1. The natural homomorphism $\mathscr{O}_X \to \mathscr{D}_X$ induces a homomorphism $\mathscr{O}_{\tilde{W}} \to \mathscr{D}_{\tilde{W}}$, while we have $\mathscr{D}_{\tilde{W}} \to \mathscr{B}_{\tilde{W}}$ induced by $f^*\mathscr{D} \to \mathscr{B}$ on W. Combining them, we get $\mathscr{O}_{\tilde{W}} \to \mathscr{B}_{\tilde{W}}$, which defines a divisor B on $\tilde{W}$ such that $[B] = [2aH + 2bH_\beta]_{\tilde{W}}$. Set $M = R_B(\tilde{W})$. Then, over each point x on X, we have a double covering $M_x \to W_{p(x)}$ with branch locus B_x, where p is the projection $X \to A^1$.

Now, thanks to (5.7; 5), we find an open set U of X such that $p(U) = A^1$ and B_x is non-singular for every $x \in U$. Thus we have a family $\{M_x\}$ of hyperelliptic polarized manifolds over U. (M_x, H_x) is of type $(\Sigma^n(\delta)^+_{a,b})$ if $p(x) = o$ and of type $(\Sigma^n(\delta')^+_{a,b})$ if $p(x) \neq o$. Therefore $(\Sigma^n(\delta)^+_{a,b}) \sim (\Sigma^n(\delta')^+_{a,b})$, since U is connected.

THEOREM (8.12). *$(\Sigma^n(\delta)^+_{a,b}) \sim (\Sigma^n(\delta')^+_{a,b})$ if $|\delta| = |\delta'|$, $a\delta_{\min} + b \geqq 0$ and $a\delta'_{\min} + b \geqq 0$.*

For a proof, use (8.10) and (8.11).

QUESTION (8.13). *Is the above assertion true even if $a\delta_{\min} + b < 0$?*

According to Horikawa, there is an example where the answer is affirmative. But we do not know the answer in general.

THEOREM (8.14). *$(\Sigma^n(\delta)^0_b) \sim (\Sigma^n(\delta')^0_b)$ if $|\delta| = |\delta'|$.*

The proof is almost identical to that of (8.12).

(8.15) Now, as an application of (8.12), we will prove (5.17). However, since the same method works for $\pi_1^{(p)}$ in case $p = \text{char}(\mathfrak{K})$, we show only $\pi_1(\Sigma^n(\delta)^+_{a,b}) = \{1\}$ in case $\mathfrak{K} = \boldsymbol{C}$. Our proof consists of several steps.

Step 1, the case in which $n \geqq 3$ and $a\delta_{\min} + b > 0$. In this case B is ample on $W(\delta)$ and the ramification locus R of $M \to W$ is ample on M. Therefore, by the Lefschetz theorem, we infer $\pi_1(M) = \pi_1(R) = \pi_1(B) = \pi_1(W) = \{1\}$.

Step 2, $\pi_1(\Sigma(2, 2)^+_{a,b}) = \{1\}$. Since the assertion has nothing to do with L, it suffices to show $\pi_1(\Sigma(1, 1)^+_{a,a+b}) = \{1\}$. If $a + b = 0$, (3.12) applies. So, thanks to Remark to (5.5), we may assume that $a + b > 0$. Then there exists (M, L) of type $(\Sigma(2, 1, 1)^+_{a,b})$ by (5.7; 5). By (5.16), we have $(S, L) \in (\Sigma(2, 2)^+_{a,b})$ for a general member S of $|L|$. Then $\pi_1(S) = \pi_1(M) = \{1\}$ by the Lefschetz theorem and Step 1. This completes Step 2.

Step 3, $\pi_1(\Sigma(2, 1)^+_{a,b}) = \{1\}$. We have $a + b \geqq 0$ by (5.7; 4). So there exists (M, L) of type $(\Sigma(1, 1, 1)^+_{a,b})$. Then $(S, L_S) \in (\Sigma(2, 1)^+_{a,b})$ for a general

member S of $|L|$. So it suffices to show $\pi_1(M) = \{1\}$. If $a + b > 0$, Step 1 applies. If $a + b = 0$, then $M \cong \boldsymbol{P}^1 \times R_A(\boldsymbol{P}^2)$ similarly as in (5.13; 2). Hence $\pi_1(M) = \{1\}$ by (3.12) and the Künneth formula.

Step 4, the case $n = 2$. We have $a\delta_{\min} + b \geqq 0$ by (5.7; 4). So, by (8.12), we may assume (δ) to be stable. Then, replacing L if necessary, we reduce the problem to either Step 2 or Step 3.

Step 5, the general case. Using (5.16) and the Lefschetz theorem, we prove the assertion by induction on n.

(8.16) So far, we have seen that $(M, L) \sim (M', L')$ if they are of the (almost) same type. From now on, we consider the converse problem. In the following, (M, L) is always a hyperelliptic polarized manifold.

THEOREM (8.17). *$(M, L) \sim (\mathrm{I}_a^n)$ implies $(M, L) \in (\mathrm{I}_a^n)$.*

Indeed, $d(M, L) = 2$ and $g(M, L) = a$ imply $(M, L) \in (\mathrm{I}_a^n)$. See Tables I and II.

COROLLARY (8 18). *Any small deformation of type (I_a^n) is a hyperelliptic polarized manifold of type (I_a^n).*

For a proof, use (7.7) and (8.17).

THEOREM (8.19). *$(M, L) \sim (\mathrm{II}_a^n)$ implies $(M, L) \in (\mathrm{II}_a^n)$ if $n \geqq 3$.*

THEOREM (8.20). *$(M, L) \sim ({}^*\mathrm{IV}_a)$ implies $(M, L) \in ({}^*\mathrm{IV}_a)$.*

THEOREM (8.21). *$(M, L) \sim (\Sigma^n(\delta)_b^0)$ implies $(M, L) \in (\Sigma^n(\delta')_b^0)$ for some (δ') with $|\delta'| = |\delta|$.*

PROOF. If $n \geqq 3$, consult Table II. If $n = 2$, from $\chi_0(M, L) \leqq 0$ we infer $(M, L) \in (\Sigma(\delta')_x^0)$ or $(\Sigma(u, u)_y^=)$. Comparing $c_1(M)^2$ and $g(M, L)$, we get $x = b$ in the former case, or $y = b$ and $u = 1$ in the latter case. In view of Remark to (5.24), we finish the proof.

LEMMA (8.22). *$(M, L) \sim (\Sigma^n(\delta)_{a,b}^+)$ and $(M, L) \in (\sum^n |\delta|^+)$ imply $(M, L) \in (\Sigma^n(\delta')_{a,b}^+)$ for some (δ') with $|\delta'| = |\delta|$.*

PROOF. Comparing d, g, χ_{n-2} and $(K^M + (n-2)L)^2 L^{n-2}$ (cf. (8.4; 4)), we obtain (1): $|\delta'| = |\delta|$, (2): $x|\delta| + y = a|\delta| + b$, (3): $(x-1)(x|\delta| + 2y - 2) = (a-1)(a|\delta| + 2b - 2)$ and (4): $(x-2)(x|\delta| + 2y - 4) = (a-2)(a|\delta| + 2b - 4)$, where we assume $(M, L) \in (\Sigma^n(\delta')_{x,y}^+)$. Using (2), we get from (3) − (4) the equality (5): $2x + y = 2a + b$. Together with (2), this implies $x = a$ and $y = b$ unless $|\delta| = 2$. If $|\delta| = 2$, we must have $n = 2$, and $(\delta) = (\delta') = (1, 1)$. We easily see that there are two possible solutions: $(x, y) = (a, b)$ or $(a + b, -b)$. Recalling the Remark to (5.5), we obtain the conclusion.

THEOREM (8.23). *$(M, L) \sim (\Sigma^n(\delta)_{a,b}^+)$ implies $(M, L) \in (\Sigma^n(\delta')_{a,b}^+)$ for some*

(δ') *with* $|\delta'| = |\delta|$, *if* $n \geqq 3$.

PROOF. In view of Table II, we see that (8.22) applies.

(8.24) The preceding results altogether determine the deformation equivalences among hyperelliptic polarized manifolds of dimension $\geqq 3$, except the question (8.13).

To study the case of surfaces, we need a couple of results.

PROPOSITION (8.25). *Let* $(\mathscr{M}, X, f, \mathscr{L})$ *be a deformation family of prepolarized manifolds. Suppose that there exists a point* o *on* X *such that* $\pi_1^{(p)}(M_o) = \{1\}$ *and* $L_o = mF_o$ *for some* $F_o \in \mathrm{Pic}(M_o)$, *where* m *is a positive integer prime to* $p = \mathrm{char}(\mathfrak{K})$. *Then for every* $x \in X$, *there is* $F_x \in \mathrm{Pic}(M_x)$ *such that* $L_x = mF_x$.

PROOF (due to A. Ogus). Consider the exact sequence $0 \to \mu_m \to \mathscr{O}_M^\times \to \mathscr{O}_M^\times \to 0$, where $\mathscr{O}_M^\times \to \mathscr{O}_M^\times$ is the m-power homomorphism and μ_m is the constant sheaf of m-th roots of unity. This gives rise to an exact sequence $\mathrm{Pic}(\mathscr{M}) \to \mathrm{Pic}(\mathscr{M}) \to H^2(\mathscr{M}; \mu_m)$, the second homomorphism of which will be denoted by $c^{(m)}$. Clearly the difinition of $c^{(m)}$ is functorial, and $c_x^{(m)}: \mathrm{Pic}(M_x) \to H^2(M_x; \mu_m)$ and $c^{(m)}$ are compatible with respect to restrictions. So, we have $c^{(m)}(\mathscr{L})_o = c_o^{(m)}(L_o) = 0$. What we should show is $c^{(m)}(\mathscr{L})_x = 0$ for every $x \in X$.

We have $R^1 f_* \mu_m = 0$ since $\pi_1^{(p)}(M_o) = \{1\}$. So, by the Leray spectral sequence, we get a natural exact sequence $0 \to H^2(X, f_*\mu_m) \to H^2(\mathscr{M}, \mu_m) \to H^0(X, R^2 f_* \mu_m) \to 0$. Since $R^2 f_* \mu_m$ is a locally constant sheaf (with respect to the étale topology), $c^{(m)}(\mathscr{L})_o = 0$ implies that the image of $c^{(m)}(\mathscr{L})$ in $H^0(X, R^2 f_* \mu_m)$ vanishes. Hence $c^{(m)}(\mathscr{L})$ comes from $H^2(X, f_*\mu_m)$, which implies $c^{(m)}(\mathscr{L})_x = 0$ for every $x \in X$.

PROPOSITION (8.26). *If* $(M, L) \in (\Sigma^2(\delta)^+)$, *then* L *is not divisible by* 2 *in* $\mathrm{Pic}(M)$.

PROOF. By virtue of (8.12), (8.15) and (8.25), we may assume that (δ) is stable. If L is divisible by 2, then $L^2 = 2|\delta|$ is divisible by 4. Hence we may assume $|\delta|$ is even and $\delta_1 = \delta_2$. So $W \cong \boldsymbol{P}_\beta^1 \times \boldsymbol{P}_\alpha^1$, $H = H_\alpha + \delta_2 H_\beta$, $M \cong R_B(W)$ and $B \in |2aH_\alpha + 2bH_\beta|$ for some positive integers a, b. Note that $K^M = (a-2)\alpha + (b-2)\beta$, where α and β denote the pull-backs of H_α and H_β on M respectively. Set $e = 0$ if δ_2 is even, and $e = 1$ if δ_2 is odd. Then, if L is divisible by two, we have $\alpha + e\beta = 2F$ for some $F \in \mathrm{Pic}(M)$. We will derive a contradiction from this.

CLAIM. *If* $|F + x\alpha + y\beta| \neq \emptyset$, *then* $2x + 1 > a$ *and* $2y + e > b$.

To prove this claim, we may assume (x, y) to be a minimal pair

among those such that $|F+x\alpha+y\beta| \neq \emptyset$. So, for any $D \in |F+x\alpha+y\beta|$, there is no non-trivial effective divisor D' such that $D-D'$ is effective and that $[D']$ comes from $\mathrm{Pic}(W)$. In particular, the ramification locus R of $M \to W$, which is member of $|a\alpha+b\beta|$, is not a component of D. Furthermore, if i is the involution of M covering W, D and i^*D have no common component C, because then $C+i^*C$ would be a part of D and come from $\mathrm{Pic}(W)$. Therefore $N=D\cap R$ is a 0-dimensional subscheme of M. $i^*N=N$ since the restriction of i to R is the identity. On the other hand we have $i^*N=i^*D\cap R$. So $N\subset D\cap i^*D$. From this we obtain $D\cdot i^*D \geqq DR$. We have $[2i^*D]=i^*[2D]=2D$ in $\mathrm{Pic}(M)$ because $2D$ comes from $\mathrm{Pic}(W)$. Now, calculating the intersection numbers on both sides, we get: $4xy+2y+2ex+e \geqq b(2x+1)+a(2y+e)$, which yields: $(2x+1-a)(2y+e-b) \geqq ab > 0$. Therefore, if our claim were not true, both factors of the left hand side would be negative. On the other hand we have $2x+1=\alpha D \geqq 0$ and $2y+e=\beta D \geqq 0$. Combining them we infer $2x+1=2y+e=0$, but this is impossible unless $D=0$. If $D=0$, then F is an integral combination of α and β, and we get a non-trivial relation for α and β in $\mathrm{Pic}(M)$. This is absurd. Thus we prove the claim.

Returning to the proof of the proposition, we note that $F^2=e$ and $FK^M=b-2+e(a-2)$. Since $F^2 \equiv FK^M \pmod 2$, there are four possible cases: (1) a, b and e are even. (2) a and e are odd, b is even. (3) b and e are even, a is odd. (4) b and e are odd, a is even.

Case (1), $a=2a'$, $b=2b'$ and $e=0$. Set $Z=F+(a'-2)\alpha+(b'-1)\beta$. Then $K^M-Z=F+(a'-1)\alpha+(b'-1)\beta$. So $h^0(M, Z)=0=h^0(M, K^M-Z)=h^2(M, Z)$ by the above claim. Hence $\chi(M, Z)=-h^1(M, Z) \leqq 0$. On the other hand, by the Riemann-Roch theorem, we have $\chi(M, Z)=(Z^2-K^MZ)/2+\chi(M, \mathcal{O}_M)=2a'b' > 0$. Thus we get a contradiction.

Case (2), $a=2a'+1$, $b=2b'$ and $e=1$. Set $Z=F+(a'-1)\alpha+(b'-2)\beta$. Then $K^M-Z=F+(a'-1)\alpha+(b'-1)\beta$. Similarly as above, we obtain $\chi(M, Z) \leqq 0$ using the claim. On the other hand, we have $\chi(M, Z)=(2a'+1)b' > 0$ by the Riemann-Roch theorem.

Case (3), $a=2a'+1$, $b=2b'$, $e=0$. Set $Z=F+(a'-1)\alpha+(b'-1)\beta$. Then $\chi(M, Z) \leqq 0$ by the claim, while we have $\chi(M, Z)=b'(2a'+1) > 0$ by the Riemann-Roch theorem.

Case (4). The situation is the same as in case (2), except that the role of the two rulings of W are interchanged.

Thus, in any case, we derive a contradiction, as desired.

(8.27) We come back to the problem of deformation equivalences among hyperelliptic polarized surfaces.

THEOREM. *$(M, L) \sim (\mathrm{IV}_a)$ implies $(M, L) \in (\mathrm{IV}_a)$.*

PROOF. Using (8.25), we infer that L is divisible by two. Hence (M, L) is not of type $(\Sigma(\delta)^+)$ by (8.26). (M, L) is neither of type $(\Sigma(\delta)^0)$ nor of type $(\Sigma(\delta)^=)$, because $\chi(M, \mathcal{O}_M) > 0$. If (M, L) is of type $(\Sigma(u + 2\gamma, u)_x^-)$, then $u = \gamma = 1$ because $d(M, L) = 8$. Comparing g, χ and c_1^2, we get (1): $3x - 2 = 2a + 1$, (2): $2(x - 1)(x - 2) = a(a - 1)$, (3): $2(x - 2)(x - 3) = (a - 2)^2$. (2) − (3) yields $4(x - 1) = 3a$. Together with (1), this implies $a = x - 1 = 0$, which is absurd. Now, in view of Table I, we conclude that (M, L) is of type (IV). Comparing $g(M, L)$ we see $(M, L) \in (\mathrm{IV}_a)$.

THEOREM (8.28). *$(M, L) \sim (^*\mathrm{II}_a)$ implies $(M, L) \in (^*\mathrm{II}_a)$.*

PROOF. $c_1(M)^2 = (2a - 3)^2$ is odd. So, in view of Table I, we infer $(M, L) \in (^*\mathrm{II}_a)$.

THEOREM (8.29). *$(M, L) \sim (\Sigma(u, u)_a^=)$ implies $(M, L) \in (\Sigma(u, u)_a^=)$.*

PROOF. Since $\chi(M) = 2 - a \leqq 0$, (M, L) is either of type $(\Sigma(\delta')_b^0)$ or of type $(\Sigma(v, v)_x^=)$. In the former case, as we saw in (8.21), we have $(M, L) \in (\Sigma(1, 1)_a^0) = (\Sigma(1, 1)_a^=)$. So we need not worry about this possibility. In the latter case, we obtain $v = u$ and $x = a$ by the comparison of $d(M, L)$ and $g(M, L)$.

THEOREM (8.30). *$(M, L) \sim (\Sigma^2 \Delta_{a,b}^+)$ implies either*

(1) *$(M, L) \in (\Sigma^2(\delta)_{a,b}^+)$ for some (δ) with $|\delta| = \Delta$, or*

(2) *$(M, L) \in (\Sigma(u + 2\gamma, u)_a^-)$, $\Delta = 2(u + \gamma)$, u is odd, $\gamma = (a - 2)u + 2$ and $b = 2 - \Delta$.*

PROOF. If (M, L) is of type $(\Sigma(\delta)^+)$ for some (δ), then (8.22) applies. Otherwise, in view of Table I and the preceding results altogether, we infer that $(M, L) \in (\Sigma(u + 2\gamma, u)_x^-)$ for some u, γ and x. Comparing d, g, χ and c_1^2, we get $2(u + \gamma) = \Delta$, $x = a$ and $b = -au - \gamma$. Furthermore, using (8.25) and (8.26), we infer that L is not divisible by two. This implies that u is odd. Indeed, by (5.27; 1), we have $M \cong R_B(W)$, $B = B_1 + B_2$ and $B_1 \in |L - (u + 2\gamma)H_\beta|$. So $L = 2R_1 + (u + 2\gamma)H_\beta$, where R_1 is the component of the ramification locus lying over B_1. Thus, if u were even, L would be divisible by two.

We should further show that $b + \Delta - 2 = \gamma - (a - 2)u - 2 = 0$. Assuming $\neq 0$, we set $b + \Delta - 2 = p^e m$, where e is a non-negative integer and m is an integer prime to $p = \mathrm{char}\,(\mathfrak{K})$ (if $p = 0$, we let $p^e = 1$). Note that $K^M - (a - 2)L = p^e m H_\beta$ and that $K' - (a - 2)L' = p^e m \beta'$, where $(M, L) \sim (M', L') \in (\Sigma^2 \Delta_{a,b}^+)$, K' is the canonical bundle of M', and β' is the pull-back of H_β on M'. Therefore we have $K^M - (a - 2 - m)L = m(L + p^e H_\beta) = m(2R_1 + (u + p^e + 2\gamma)H_\beta)$, which is divisible by $2m$ because

p^e is odd. So, by virtue of (8.25), $K' - (a-2-m)L' = m(L' + p^e\beta')$ is divisible by $2m$ in Pic(M'). Hence $L' + p^e\beta' = L''$ is divisible by 2, because Pic(M') has no torsion prime to p (cf. (5.17) and (8.15)). This contradicts (8.26) since $(M', L'') \in (\Sigma^2(\delta'')^+)$ for some (δ''). Thus we complete the proof.

REMARK. The last condition of the above case (2) is equivalent to saying that $K^M = (a-2)L$.

THEOREM (8.31). $(M, L) \sim (\Sigma(u+2\gamma, u)_a^-)$ *implies* $(M, L) \in (\Sigma(u+2\gamma, u)_a^-)$ *except in the case* (8.30; 2), *where* u *is odd and* $\gamma = (a-2)u + 2$.

PROOF. In view of Table I and the preceding results, we infer $(M, L) \in (\Sigma(u' + 2\gamma', u')_x^-)$ except in the case (8.30; 2). In the former case we obtain $x = a$, $u' = u$ and $\gamma' = \gamma$ by comparison of d, g, χ and c_1^2.

(8.32) The results in this section may be summarized as follows.

THEOREM. *Aside from the problem* (8.13), *hyperelliptic polarized manifolds of the same type such as* (I_a^n), (II_a^n), (IV_a), $(^*\mathrm{II}_a)$, $(^*\mathrm{IV}_a)$, $(\Sigma^n\Delta_{a,b}^+)$, $(\Sigma^n\Delta_b^0)$, $(\Sigma(u, u)_a^=)$, $(\Sigma(u+2\gamma, u)_a^-)$ *are deformation equivalent to each other. Conversely, these classes are stable under deformation equivalence, but for the exceptional possibility* (8.30; 2).

(8.33) It is a delicate problem whether the case (8.30; 2) does really happen or not.

When $a = 2$, M is a $K3$-surface. We show that the surjectivity of the period mapping for polarized $K3$-surfaces implies $(\Sigma(u+4, u)_2^-) \sim (\Sigma^2|2u+4|_{2,-2u-2}^+)$ for any odd positive integer u. To see this, let $(M_1, L_1) \in (\Sigma(u+4, u)_2^-)$ and $(M_2, L_2) \in (\Sigma^2|2u+4|_{2,-2u-2}^+)$. We claim that both polarizations are primitive, that means, there is no ample line bundle F such that $L_i = mF$ for some $m > 1$. Indeed, for $i = 1$, this follows from $L_1R_1 = u$ and $L_1H_\alpha = 2(u+8)$, where the notations are as in (5.27) and R_1 is the ramification divisor lying over B_1. As for the case $i = 2$, we use (8.26) and $LH_\beta = 2$.

Now, by the results in the Appendix of [PS], it follows that there is a bijection $f: H^2(M_1; \boldsymbol{Z}) \to H^2(M_2; \boldsymbol{Z})$ such that $f(c_1(L_1)) = c_1(L_2)$ and f is compatible with the intersection pairings. So they define the same marked lattice Λ. Let $D(\Lambda)$ be the period domain for polarized $K3$-surfaces with lattice Λ. One easily sees that the connectedness of $D(\Lambda)$ and the surjectivity of the period mapping imply $(M_1, L_1) \sim (M_2, L_2)$.

On the other hand, in case $\mathfrak{R} = \boldsymbol{C}$ and $a = 3$ (this implies $K^M = L$), Horikawa [Ho 1] showed that $(\Sigma(3u+4, u)_3^-)$ and $(\Sigma^2|4u+4|_{3,-4u-2}^+)$ are not deformation equivalent to each other. His method involves the

study of all the possible deformations of such surfaces, where the canonical bundles may not be ample. He classified them into two species, and then showed that both are stable under small deformations (cf. [Ho 1; §7]).

His method seems to be generalized in case $a \geqq 3$. Indeed, for any deformation (M, L) of such polarized surfaces, M is a minimal surface of general type. So L is semiample, i.e., there is a positive integer m such that $\mathrm{Bs}\,|mL| = \varnothing$. If in addition $\mathrm{Bs}\,|L| = \varnothing$, we can show, by the techniques in [F6; §3], that M is a double covering of a polarized variety of Δ-genus zero. Although the covering may not be a finite morphism and the branch locus may have certain singularities, it is not very difficult to transplant the techniques of Horikawa. When a is odd, we can actually prove $\mathrm{Bs}\,|L| = \varnothing$ as Horikawa did in case $a = 3$. When a is even, the problem seems a little subtler.

The author hopes to carry out the above plan in detail in a future paper.

Appendix. The main purpose of this appendix is to prove the following:

THEOREM (A1). *Let K be the canonical line bundle of a locally Gorenstein curve C. Suppose that the rational mapping defined by K is birational. Then K is simply generated and hence very ample.*

This was proved by Max Noether when C is a non-singular curve defined over the complex number field. Saint-Donat [Sa 1] showed that this is valid in positive characteristic cases, too. There it was assumed that C is non-singular, but this assumption can be omitted. Indeed, the crucial lemma on p. 162 is proved by the same argument since the Jacobian variety of C is non-singular even if C is singular. Moreover, it is easy to generalize Clifford's theorem on singular curves (see (1.9)). Here we provide a different proof.

(A2) From now on, we fix an irreducible reduced curve C with $h^1(C, \mathcal{O}_C) = g$. C is locally Macaulay and the dualizing sheaf will be denoted by ω.

(A3) A shaf $\mathcal{F}$ on C is said to be *quasi-invertible* if it is torsion free and of rank one. If so, we define $d(\mathcal{F}) = \chi(\mathcal{F}) - 1 + g$ and $\Delta(\mathcal{F}) = 1 + d(\mathcal{F}) - h^0(\mathcal{F}) = g - h^1(\mathcal{F})$ as in [F2]. When $\mathcal{F}$ is invertible, we have $d(\mathcal{F}) = \deg(\mathcal{F})$ by the Riemann-Roch theorem.

PROPOSITION (A4). *Let D be an effective divisor, L be a line bundle and $\mathcal{F}$ be a quasi-invertible sheaf on C. Let $\alpha \in \mathrm{Hom}(\mathcal{O}, \mathcal{O}[L - D])$*

and $\beta \in \mathrm{Hom}(\mathcal{O}, \mathcal{F}[-D])$ *such that* $\mathrm{Supp}(\mathrm{Coker}(\alpha)) \cap \mathrm{Supp}(\mathrm{Coker}(\beta)) = \varnothing$. *Then* $h^0(\mathcal{F}) + h^0(L) \leqq h^0(\mathcal{F}[L-D]) + h^0(D)$.

PROOF. Consider the homomorphism $\mu: H^0(\mathcal{F}) \oplus H^0(L) \to H^0(\mathcal{F}[L-D])$ defined by $\mu(\varphi \oplus \psi) = \alpha_{\mathcal{F}}(\varphi) - \beta_L(\psi)$, where $\alpha_{\mathcal{F}} \in \mathrm{Hom}(\mathcal{F}, \mathcal{F}[L-D])$ and $\beta_L \in \mathrm{Hom}(L, \mathcal{F}[L-D])$ are induced by α and β respectively. By assumption we infer that $\alpha_{\mathcal{F}}(\varphi) = \beta_L(\psi)$ implies the existence of $\delta \in H^0(D)$ such that $\psi = \alpha_D(\delta)$ and $\varphi = \beta_D(\delta)$, where $\alpha_D \in \mathrm{Hom}(\mathcal{O}[D], \mathcal{O}[L])$ and $\beta_D \in \mathrm{Hom}(\mathcal{O}[D], \mathcal{F})$ are the induced homomorphisms. Therefore $h^0(\mathcal{F}[L-D]) \geqq \dim(\mathrm{Im}(\mu)) = h^0(\mathcal{F}) + h^0(L) - \dim(\mathrm{Ker}(\mu)) \geqq h^0(\mathcal{F}) + h^0(L) - h^0(D)$, which proves the assertion.

PROPOSITION (A5). *Let* L *be a line bundle with* $\mathrm{Bs}|L| = \varnothing$ *and let* $\mathcal{F}$ *be a quasi-invertible sheaf such that* $2h^0(\mathcal{F}) \geqq h^0(\mathcal{F}[L]) + h^0(\mathcal{F}[-L])$. *Then the natural homomorphism* $H^0(\mathcal{F}) \otimes H^0(L) \to H^0(\mathcal{F}[L])$ *is surjective.*

PROOF. Take $\alpha, \beta \in H^0(L) = \mathrm{Hom}(\mathcal{O}, \mathcal{O}[L])$ in such a way that the supports of their cokernels do not meet. Consider the homomorphism $\mu: H^0(\mathcal{F}) \oplus H^0(\mathcal{F}) \to H^0(\mathcal{F}[L])$ induced by α and β. Similarly as in (A4), we infer $\dim(\mathrm{Ker}(\mu)) \leqq h^0(\mathcal{F}[-L])$. Hence, by assumption, we get $\dim(\mathrm{Im}(\mu)) = 2h^0(\mathcal{F}) - \dim(\mathrm{Ker}(\mu)) \geqq h^0(\mathcal{F}[L])$. Thus μ is surjective, hence so is $H^0(\mathcal{F}) \otimes H^0(L) \to H^0(\mathcal{F}[L])$.

LEMMA (A6). *Let* L *be a line bundle and let* $\mathcal{F}$ *be a quasi-invertible sheaf. Let* p *be a simple point on* C *such that* $p \notin \mathrm{Bs}|L|$ *and* $\varphi(p) \neq 0$ *for some* $\varphi \in H^0(\mathcal{F})$. *Suppose that* $H^0(\mathcal{F}) \otimes H^0(L-p) \to H^0(\mathcal{F}[L-p])$ *is surjective. Then the natural mapping* $H^0(\mathcal{F}) \otimes H^0(L) \to H^0(\mathcal{F}[L])$ *is surjective.*

Proof is almost identical to that of [F2; Lemma 1.8, (b)].

(A7) Clearly (A1) follows from the result below.

THEOREM. *Let* L *be a line bundle such that* $\mathrm{Bs}|L| = \varnothing$. *If* $\dim|L| \geqq 2$, *assume in addition that* $\rho_{|L|}$ *is birational. Then the natural mapping* $\mu: H^0(\omega) \otimes H^0(L) \to H^0(\omega[L])$ *is surjective.*

PROOF. We use the induction on $h^0(L)$. If $h^0(L) = 2$, (A5) applies by virtue of the Riemann-Roch theorem. If $h^0(L) \geqq 3$, then $\mathrm{Bs}|L-x| = \varnothing$ for a general point x on C because $\rho_{|L|}$ is assumed to be birational. So (A6) applies if $h^0(L) = 3$. Thus it suffices to consider the case $h^0(L) \geqq 4$.

Let $C' \subset \boldsymbol{P}^h$ be the image of C via $\rho_{|L|}$, where $h = \dim|L| = h^0(L) - 1$. If a general secant of C' is not a multiscant, then $\rho_{|L-x|}$ is birational for any general point x on C. Hence our assertion follows from (A6) and the induction hypothesis. So we may assume that any secant of C'

is a multi-secant.

Let H be a general hyperplane in $\boldsymbol{P}^h$. Then the divisor $D = H \cap C'$ on C' is a non-singular scheme. Since $\dim(\mathrm{Im}(H^0(\boldsymbol{P}^h, H) \cong H^0(C, L) \to H^0(D, L_D))) = h^0(L) - 1 = h$, we infer that D is not contained in any hyperplane of $H \cong \boldsymbol{P}^{h-1}$. Hence we can find $h-1$ points of D which span $\boldsymbol{P}^{h-2}$. Let B be the divisor on C consisting of the points on this $\boldsymbol{P}^{h-2}$. Then $h^0(L-B) = 2$ and $\mathrm{Bs}\,|L-B| = \varnothing$. However, unlike the previous cases, $\deg B > h-1$ and hence $H^0(C, L) \to H^0(B, L_B)$ is not surjective. Instead we claim $h^0(C, B) = 1$.

To prove the claim it suffices to show that any hyperplane S of $\boldsymbol{P}^h$ containing $D - B$ must contain B, too. Take a point x of D not in $\mathrm{Span}(B) \cong \boldsymbol{P}^{h-2}$. For any point q of B we have a third point y of D on the line $x * q$, because any secant of C' is a multi-secant. $y \notin \mathrm{Span}(B)$ because otherwise $x \in y * q \subset \mathrm{Span}(B)$. Hence $x, y \in D - B \subset S$. So $q \in x * y \subset S$, as required.

Now we have $h^1(\omega[-B]) = 1$ by the Serre duality. So $H^1(\omega[-B]) \to H^1(\omega)$ is injective and $H^0(\omega) \to H^0(B, \omega_B)$ is surjective. Therefore $H^0(\omega[L])$ and $\mathrm{Im}(\mu)$ have the same image in $H^0(B, \omega[L]_B)$. So $H^0(\omega[L]) \subset \mathrm{Im}(\mu) + \mathrm{Im}(\beta)$, where β is the natural mapping $H^0(\omega[L-B]) \otimes H^0(B) \to H^0(\omega[L])$.

On the other hand, $H^0(\omega) \otimes H^0(L-B) \to H^0(\omega[L-B])$ is surjective by (A5) (or by the induction hypothesis in case $h^0(L) = 2$). So $\mathrm{Im}(\beta)$ comes from $H^0(\omega) \otimes H^0(L-B) \otimes H^0(B)$, hence $\mathrm{Im}(\beta) \subset \mathrm{Im}(\mu)$. Putting things together we obtain $H^0(\omega[L]) \subset \mathrm{Im}(\mu)$.

REMARK (A8). It is actually possible that any general secant of C is a multi-secant.

Let C_0 be the affine curve in A^3 given parametrically by $x = t$, $y = t^q$, $z = (t^q)^q$, where q is a power of $p = \mathrm{char}(\mathfrak{K})$. Let C be its closure in $\boldsymbol{P}^3$. Then C is a complete intersection of type (q, q) with one singular point at infinity. One can easily check that any general secant line of C is a q-secant, that means, passes exactly q points on C.

Of course, however, such a phenomenon is impossible if $\mathrm{char}(\mathfrak{K}) = 0$ (cf., e.g., [Ha 2; p. 312]). Correspondingly we can improve the result (A7) in the following way.

LEMMA (A9). (In the sequel we assume $\mathrm{char}(\mathfrak{K}) = 0$.) *Let L be a line bundle such that $\mathrm{Bs}\,|L| = \varnothing$ and that the morphism $\rho_{|L|}$ defined by $|L|$ is birational. Then, for any general point p on C, we have $\mathrm{Bs}\,|L-p| = \varnothing$. Moreover, the morphism $\rho_{|L-p|}$ is birational unless $\dim|L| \leqq 2$.*

This is clear because any general secant of $C' = \rho_{|L|}(C) \subset \boldsymbol{P}^h$ is not

a multi-secant.

THEOREM (A10). *Let L be a line bundle such that* $\mathrm{Bs}\,|L| = \emptyset$. *If* $\dim |L| \geqq 2$, *assume in addition that $\rho_{|L|}$ is birational. Let $\mathscr{F}$ be a quasi-invertible sheaf such that $h^0(\mathscr{F}) > 0$ and $2h^0(\mathscr{F}) + h^0(L) - 2 \geqq h^0(\mathscr{F}[L]) + h^0(\mathscr{F}[-L])$. Then the natural mapping $H^0(\mathscr{F}) \otimes H^0(L) \to H^0(\mathscr{F}[L])$ is surjective.*

PROOF. Similarly as in (A7), we use the induction on $h^0(L)$. By virtue of (A6) and (A9), it suffices to show the following inequality (♯) for any general point p on C.

$$(\sharp) \qquad 2h^0(\mathscr{F}) + h^0(L - p) - 2 \geqq h^0(\mathscr{F}[L - p]) + h^0(\mathscr{F}[-L + p])\,.$$

To show this, we first consider the case in which $h^1(\mathscr{F}[-L]) = 0$. Then $h^1(\mathscr{F}[-L + p]) = 0$ by [F2; Lemma 1.4], and similarly we have $h^1(\mathscr{F}) = h^1(\mathscr{F}[L - p]) = 0$ since $|L - p| \neq \emptyset$. Hence $2h^0(\mathscr{F}) = h^0(\mathscr{F}[L - p]) + h^0(\mathscr{F}[-L + p])$. So $h^0(L - p) \geqq 2$ implies (♯) unless $h^0(L) = 2$.

If $h^1(\mathscr{F}[-L]) > 0$, then $h^1(\mathscr{F}[-L + p]) = \dim \mathrm{Hom}\,(\mathscr{F}, \omega[L - p]) = \dim \mathrm{Hom}\,(\mathscr{F}, \omega[L]) - 1 = h^1(\mathscr{F}[-L]) - 1$ since p is general. So $h^0(\mathscr{F}[-L + p]) = h^0(\mathscr{F}[-L])$. On the other hand $h^0(\mathscr{F}[L - p]) = h^0(\mathscr{F}[L]) - 1$ and $h^0(L - p) = h^0(L) - 1$. Combining them with the assumed inequality, we obtain (♯).

REMARK (A11). The assumed inequality in (A10) is true in the following cases.

(1) $\mathscr{F} = \omega$.

(2) $\deg(\mathscr{F}) \geqq 2\Delta(\mathscr{F})$ and $h^0(\mathscr{F}[-L]) = 0$.

(3) $\deg(\mathscr{F}) \geqq 2\Delta(\mathscr{F})$ and $\deg(L) \geqq 2g + 1$.

PROOF. (1) The left hand side is $h^0(L) + 2g - 2$, while the right hand side is equal to $\chi(\omega[L]) + h^1(L) = h^1(L) - 1 + d(L) + g$. They are equal by the Riemann-Roch theorem.

(2) $d(\mathscr{F}) \geqq 2\Delta(\mathscr{F})$ implies $2h^0(\mathscr{F}) \geqq d(\mathscr{F}) + 2$. On the other hand $h^0(L) \geqq h^0(\mathscr{F}[L]) - d(\mathscr{F})$ because $h^1(\mathscr{F}[L]) \leqq h^1(L)$ by [F2; Lemma 1.4]. Combining them we get the inequality.

(3) We may assume $h^0(\mathscr{F}[-L]) > 0$ by (2). So $d(\mathscr{F}) \geqq d(L) \geqq 2g + 1$, and $h^0(\mathscr{F}) \geqq g + 2$. Take an effective divisor D such that $\deg(D) = g + 1$ and $h^0(\mathscr{F}[-D]) = h^0(\mathscr{F}) - g - 1$. Then $H^0(L - D) > 0$ since $h^0(L) = 1 + d(L) - g \geqq g + 2$. So $h^0(\mathscr{F}[-L]) \leqq h^0(\mathscr{F}[-D]) = h^0(\mathscr{F}) - g - 1$. Combining them we obtain the desired inequality.

QUESTION (A12). *Is* (A10) *true in case* $\mathrm{char}\,(\mathfrak{k}) > 0$?

Note that (A11) is valid in positive characteristic cases too.

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