Tôhoku Math. Journ.
34 (1982), 559-574.

ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

SATORU MURAKAMI

(Received February 13, 1982, revised May 18, 1982)

1. Introduction. The main purpose of this paper is to investigate the asymptotic behavior of solutions of the nonlinear differential equation on $[0, \infty) \times Q$,

$$(1) \qquad \dot{x} = f(t, x) ,$$

where Q is an open subset of R^n and $f(t, x)$ is continuous on $[0, \infty) \times Q$. Consider the following assumptions with respect to the equation (1): There exist nonnegative continuous functions $V(t, x)$ and $W(x)$ such that $V(t, x)$ is locally Lipschitzian with respect to x and $\dot{V}_{(1)}(t, x) \leqq -W(x)$ for $t \geqq 0$, $x \in Q$. Then it is well known that each bounded solution of (1) approaches the set $E = \{x \in Q: W(x) = 0\}$ as $t \to \infty$ under the assumption that $f(t, x)$ is bounded when x is bounded [7], [8], [9], [11]. Recently, LaSalle [4] obtained the same result under the weaker assumption that $f(t, x)$ satisfies Condition (B) (see Remark 1 below). In this paper, we analyze the problem posed above under a further relaxed assumption, Condition (C) below.

As an application, we shall investigate the asymptotic behavior of solutions of the second order scalar nonlinear differential equation

$$\ddot{x} + h(t, x, \dot{x})|\dot{x}|^{\alpha}\dot{x} + f(x) + g(t, x, \dot{x}) + p(t, x, \dot{x}) = 0 ,$$

where $\alpha \geqq 0$. In the case $\alpha = 0$, Onuchic [7], [8], [9] obtained sufficient conditions under which every solution, together with its derivative, tends to zero as $t \to \infty$. Since he applied the invariance principle, one of his most essential assumptions is the following: $h(t, x, y)$ is bounded when $x^2 + y^2$ is bounded. Many authors discussed the problem of relaxing the boundedness condition on h. One of these conditions is the growth condition on h. Thurston and Wong [10], Artstein and Infante [1] and others discussed this problem under the growth condition.

The author wishes to thank the referees for many useful suggestions and carefully reading the manuscript.

2. Notation, definition and preparatory lemmas. We denote by

R^n the n-dimensional real Euclidean space and by $|x|$ the Euclidean norm of $x \in R^n$. If a point x is in R^n and E is a subset of R^n, then we denote $\operatorname{dist}(x, E) = \inf\{|x - y|: y \in E\}$. We say that the function $V(t, x)$ is locally Lipschitzian with respect to x if for each (t_0, x_0) in $[0, \infty) \times Q$, there exist a neighborhood U of (t_0, x_0) and a constant $L(U) > 0$ such that for $(t, x), (t, y) \in U$ we have $|V(t, x) - V(t, y)| \leqq L(U)|x - y|$. Furthermore, we define the derivative of $V(t, x)$ along the solution of the equation (1) as follows:

$$\dot{V}_{(1)}(t, x) = \limsup_{h \to 0+} \{V(t + h, x + hf(t, x)) - V(t, x)\}/h .$$

Let $x(t)$ be a solution of (1) on $[t_0, \infty)$. We say $x(t)$ is bounded in the future when $x(t) \in K$ on $[t_0, \infty)$ for some compact set K in Q. In the case $Q = R^n$, it is well known that every solution of (1) is bounded in the future if

$$\dot{V}_{(1)}(t, x) \leqq 0 \quad \text{and} \quad a(|x|) \leqq V(t, x) ,$$

where $a(r)$ is a nonnegative continuous function on $[0, \infty)$ such that $a(r) \to \infty$ as $r \to \infty$.

Let $y(t)$ be a continuous function on an interval $[T, \infty)$ with values in R^n. A point $p \in R^n$ is said to be a positive limit point of $y(t)$ if there exists a sequence $\{t_m\}$, $t_m \to \infty$ as $m \to \infty$, such that $y(t_m) \to p$ as $m \to \infty$. The set of all positive limit points of $y(t)$ is denoted by Ω and is called the positive limit set of $y(t)$. It is well known that when $y(t)$ is bounded and continuous on $[T, \infty)$, the positive limit set Ω of $y(t)$ is a nonempty, compact and connected set and $y(t) \to \Omega$ as $t \to \infty$, that is, $\operatorname{dist}(y(t), \Omega) \to 0$ as $t \to \infty$.

LEMMA 1. *Let $f(t)$ be a C^1-function on $[t_0, \infty)$ satisfying*

$$\dot{f}(t) \leqq h(t) \quad for \quad t \geqq t_0 , \tag{2}$$

where $\int_{t_0}^{t} h(s)ds$ is uniformly continuous on $[t_0, \infty)$, and suppose that there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, such that $f(t_n) \to b$ as $n \to \infty$. Then for any number c $(c < b)$, there exist $d > 0$ and a positive number n_0 such that $f(t) > c$ for all $t \in [t_n - d, t_n]$, $n \geqq n_0$.

PROOF. Since $H(t) = \int_{t_0}^{t} h(s)ds$ is uniformly continuous on $[t_0, \infty)$, there exists a positive number d such that $|H(t) - H(t')| < (b - c)/2$ for all $t, t' \in [t_0, \infty)$, $|t - t'| \leqq d$. Moreover, let n_0 be a positive integer such that $t_n \geqq t_0 + d$ and $|f(t_n) - b| < (b - c)/2$ for $n \geqq n_0$. Integrating both sides of (2) over $[t, t_n]$, $t_n - d \leqq t \leqq t_n$, we have $f(t_n) - f(t) \leqq H(t_n) -$

$H(t) < (b-c)/2$ for $n \geqq n_0$. Thus $f(t) > f(t_n) - (b-c)/2 > (b-(b-c)/2) - (b-c)/2 = c$ for $t \in [t_n - d, t_n]$, $n \geqq n_0$. q.e.d.

3. Condition (C) **and the main theorem.** We consider the following Codition (C) with respect to the function $f(t, x) = (f_1(t, x), \cdots, f_n(t, x))$.

Condition (C). The set of the indices $\{1, \cdots, n\}$ is decomposed into a disjoint union $I \cup J$ with the following properties: (i) For any $i \in I$, any continuous $u: [0, \infty) \to Q$ with a compact range and any $\gamma > 0$, there exist $T_i = T(\gamma, u, i) > 0$ and $\beta_i = \beta(\gamma, u, i) > 0$ such that for any $a > T_i$ and $t > 0$, $\left|\int_a^{a+t} f_i(s, u(s))ds\right| > \gamma$ implies $t > \beta_i$, and (ii) for any $j \in J$, any compact set K in Q and any continuous $u: [0, \infty) \to K$, we have $f_j(t, u(t))u_j(t) \leqq h_j(t)$, where $\int_0^t h_j(s)ds$ is uniformly continuous on $[0, \infty)$ and $u_j(t)$ represents the j-th component of $u(t)$.

We say that the equation (1) satisfies Condition (C) whenever $f(t, x)$ satisfies Condition (C).

REMARK 1. LaSalle [4] imposed the following Condition (B) on $f(t, x)$.

Condition (B). Given any compact set K in Q, any continuous $u: [0, \infty) \to K$, and any $\gamma > 0$, there exist $T = T(\gamma, K, u) > 0$ and $\beta = \beta(\gamma, K, u) > 0$ such that for $a > T$ and $t > 0$, $\left|\int_a^{a+t} f(s, u(s))ds\right| > \gamma$ implies $t > \beta$.

Clearly, Condition (B) implies Condition (C) with $J = \varnothing$. For example, the equation

$$\dot{x}_1 = x_2 , \qquad \dot{x}_2 = -h(t, x_1, x_2)x_2 - f(x_1) ,$$

where $h(t, x_1, x_2)$ is nonnegative on $[0, \infty) \times R^2$, satisfies Condition (C). In this case, we may set $I = \{1\}$, $J = \{2\}$.

Next, suppose $V(t, x)$ is a nonnegative continuous function on $[0, \infty) \times Q$ and locally Lipschitzian with respect to x and that

$$\dot{V}_{(1)}(t, x) \leqq -W(x) \qquad x \in Q , \quad t \geqq 0 ,$$

for some nonnegative continuous function $W(x)$. In this case, we call $V(t, x)$ a Liapunov function of (1) on $[0, \infty) \times Q$ with $W(x)$. For the function $W(x)$, we define the set $E = \{x \in Q: W(x) = 0\}$. Furthermore, when $f(t, x)$ satisfies Condition (C), we denote by $S(E)$ the set of all points $z = (z_1, \cdots, z_n)$ such that $z_i = x_i$ if $i \in I$ and $0 \leqq z_j \leqq x_j$ or $x_j \leqq z_j \leqq 0$ if $j \in J$, for some $x = (x_1, \cdots, x_n)$ in E.

THEOREM 1. *Suppose that $f(t, x)$ satisfies Condition* (C) *and that*

$V(t, x)$ is a Liapunov function of (1) *on* $[0, \infty) \times Q$ *with* $W(x)$. *Then each solution of* (1) *which is bounded in the future approaches* $S(E)$ *as* $t \to \infty$.

PROOF. Let $x(t)$ be any solution of (1) satisfying $x(t) \in A$, $t \geqq t_0$, for some compact set A in Q. Clearly,

$$V(t, x(t)) - V(t_0, x(t_0)) \leqq -\int_{t_0}^{t} W(x(s))ds$$

for all $t \geqq t_0$, and hence we have

$$\int_{t_0}^{\infty} W(x(s))ds < \infty . \tag{3}$$

Fix $a = (a_1, \cdots, a_n) \notin S(E \cap A) \cap A =: \bar{S}$ and define the sets $J_0 = \{j \in J: a_j = 0\}$, $J_1 = \{j \in J: a_j > 0\}$ and $J_2 = \{j \in J: a_j < 0\}$. First of all, we shall prove that there exists a positive number ε such that

$$\begin{aligned} \{x = (x_1, \cdots, x_n) : |x_i - a_i| < \varepsilon \quad (i \in I) , \quad & x_j > a_j - \varepsilon \quad (j \in J_1) \\ \text{and} \quad x_k < a_k + \varepsilon \quad & (k \in J_2)\} \cap U(\bar{S}, \varepsilon) = \varnothing , \end{aligned} \tag{4}$$

where $U(\bar{S}, \varepsilon)$ is an ε-neighborhood of $\bar{S}$. Suppose this is not the case. Then there exist a sequence of positive numbers $\{\varepsilon_n\}$, $\varepsilon_n \to 0$ as $n \to \infty$, and two sequences $\{x_n\}$, $\{z_n\}$ in R^n such that

$$\begin{aligned} (*) \qquad & |x_{n,i} - a_i| < \varepsilon_n \ (i \in I) , \qquad x_{n,j} \geqq a_j - \varepsilon_n \ (j \in J_1) , \\ & x_{n,k} \leqq a_k + \varepsilon_n \ (k \in J_2) \ \text{ and } \ |x_n - z_n| < \varepsilon_n , \qquad z_n \in \bar{S} \ \text{ for all } \ n , \end{aligned}$$

where $x_{n,i}$ represents the i-th component of x_n. Since $\varepsilon_n \to 0$ as $n \to \infty$, we may assume $x_{n,j} > 0$ $(j \in J_1)$, $x_{n,k} < 0$ $(k \in J_2)$, $z_{n,j} > 0$ $(j \in J_1)$ and $z_{n,k} < 0$ $(k \in J_2)$ for a sufficient large number n. On the other hand, by the definition of $\bar{S}$, we obtain another sequence $\{\bar{z}_n\}$ in $E \cap A$ such that

$$\begin{aligned} (**) \qquad \bar{z}_{n,i} = z_{n,i} \ (i \in I) , \qquad & \bar{z}_{n,j} \geqq z_{n,j} > 0 \ (j \in J_1) \\ \text{and} \quad & \bar{z}_{n,k} \leqq z_{n,k} < 0 \ (k \in J_2) . \end{aligned}$$

Since the sequences $\{x_n\}$, $\{z_n\}$ and $\{\bar{z}_n\}$ are bounded, we may suppose that $x_n \to x_0$, $z_n \to z_0$ and $\bar{z}_n \to \bar{z}_0$ as $n \to \infty$, taking a subsequence if necessary. Letting $n \to \infty$ in $(*)$ and $(**)$, we have $x_0 = z_0$, $x_{0,i} = a_i$ $(i \in I)$, $x_{0,j} \geqq a_j$ $(j \in J_1)$, $x_{0,k} \leqq a_k$ $(k \in J_2)$, $\bar{z}_{0,i} = z_{0,i}$ $(i \in I)$, $\bar{z}_{0,j} \geqq z_{0,j} \geqq 0$ $(j \in J_1)$ and $\bar{z}_{0,k} \leqq z_{0,k} \leqq 0$ $(k \in J_2)$. Consequently, $a_i = \bar{z}_{0,i}$ $(i \in I)$, $0 \leqq a_j \leqq \bar{z}_{0,j}$ or $\bar{z}_{0,j} \leqq a_j \leqq 0$ $(j \in J)$. Since $\bar{z}_0 \in E \cap A$, this means $a \in S(E \cap A)$. This contradicts $a \notin \bar{S}$. Therefore we can choose a positive number ε which satisfies (4).

Now, we shall prove that $\Omega \subset \bar{S}$, where Ω is the positive limit set of $x(t)$. Suppose that this is not the case. Then there exist a point $a =$

$(a_1, \cdots, a_n)$ and a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, such that $a \notin \bar{S}$ and $x(t_n) \to a$ as $n \to \infty$. For this point a, choose a positive number ε so that (4) holds, where we may assume that ε is sufficiently small. Moreover choose two numbers T and β such that $T > T_i$, $\beta < \beta_i$ for all $i \in I$ and $0 < \beta < 1$, where T_i and β_i are the numbers corresponding to $x(t)$, $\gamma = \varepsilon/2$ and f_i in the definition of Condition (C). Furthermore, choose a positive integer n_0 so that if $n \geqq n_0$, then $|x_i(t_n) - a_i| < \varepsilon/2$ for all $i \in I$ and $t_n > T + 1$. Then we have

$$|x_i(t_n) - x_i(t)| = \left|\int_t^{t_n} f_i(s, x(s))ds\right| < \varepsilon/2$$

for $t \in [t_n - \beta, t_n + \beta]$, $n \geqq n_0$, and therefore

$$(5) \qquad |x_i(t) - a_i| < \varepsilon \quad \text{for} \quad t \in [t_n - \beta, t_n + \beta]\,, \quad n \geqq n_0\,.$$

On the other hand, for $j \in J$, there exists a function $h_j(t)$ such that $\int_0^t h_j(s)ds$ is uniformly continuous on $[0, \infty)$ and $(d/dt)x_j(t)^2 = 2f_j(t, x(t))x_j(t) \leqq h_j(t)$. Since $x_j(t_n) \to a_j$ as $n \to \infty$, Lemma 1 implies that there exist positive numbers d and n_1 such that

$$(6) \qquad x_j(t) > a_j - \varepsilon \ (j \in J_1) \quad \text{and} \quad x_k(t) < a_k + \varepsilon \ (k \in J_2)$$
$$\text{for} \quad t \in [t_n - d, t_n]\,, \quad n \geqq n_1\,.$$

Put $\delta_0 = \min(\beta, d)$ and $n_2 = \max(n_0, n_1)$. Taking a subsequence of $\{t_n\}$ if necessary, we may assume that the intervals $[t_n - \delta_0, t_n]$, $n \geqq n_2$, are mutually disjoint. From (4), (5) and (6), it follows that $x(t) \notin U(\bar{S}, \varepsilon)$ and hence $\operatorname{dist}(x(t), E) \geqq \varepsilon$ for all $t \in [t_n - \delta_0, t_n]$, $n \geqq n_2$. Therefore there exists a $c > 0$ such that $W(x(t)) > c$ for $t \in [t_n - \delta_0, t_n]$, $n \geqq n_2$. Consequently,

$$\int_{t_0}^{\infty} W(x(s))ds \geqq \sum_{n=n_2}^{\infty} \int_{t_n - \delta_0}^{t_n} W(x(s))ds = \infty\,.$$

This contradicts (3). Thus we have $\Omega \subset \bar{S} \subset S(E)$. Since $x(t) \to \Omega$ as $t \to \infty$, we conclude that $x(t) \to S(E)$ as $t \to \infty$. q.e.d.

The following corollary is immediate. This is a result given in [4].

COROLLARY 1. *Suppose $f(t, x)$ satisfies Condition* (B), *that is, $J = \emptyset$ in Condition* (C). *Moreover suppose $V(t, x)$ is a Liapunov function of* (1) *on $[0, \infty) \times Q$ with $W(x)$. Then each solution of* (1) *which is bounded in the future approaches the set E as $t \to \infty$.*

COROLLARY 2. *Suppose $f(t, x)$ satisfies Condition* (C) *and $f(t, 0) = 0$ for $t \geqq 0$. Moreover suppose that there exists a continuous function $V(t, x)$ on $[0, \infty) \times Q$ such that*

(a) *$V(t, x)$ is locally Lipschitzian with respect to x, and $V(t, 0) = 0$, $t \geqq 0$,*

(b) *$a(|x|) \leqq V(t, x)$ for $t \geqq 0$, $x \in Q$, where $a(r)$ is a continuous and positive definite function, and*

(c) *$\dot{V}_{(1)}(t, x) \leqq -c(|x|)$, $t \geqq 0$, $x \in Q$, where $c(r)$ is continuous and positive definite.*

Then the zero solution of (1) *is asymptotically stable.*

PROOF. Define $W(x) = c(|x|)$. Then $V(t, x)$ is a Liapunov function of (1) with $W(x)$ and $S(E) = E = \{\text{the origin}\}$. Thus we have Corollary 2 by the standard argument and Theorem 1. q.e.d.

REMARK 2. Corollary 2 is a generalization of a theorem given by Marachkov [11, Theorem 7.10], where it was assumed that $f(t, x)$ is bounded when x is bounded.

4. Applications. Consider the following second order scalar differential equation

$$\ddot{x} + h(t, x, \dot{x})|\dot{x}|^{\alpha}\dot{x} + f(x) + g(t, x, \dot{x}) + p(t, x, \dot{x}) = 0 , \tag{7}$$

where $\alpha \geqq 0$, and the system equivalent to (7),

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -h(t, x, y)|y|^{\alpha}y - f(x) - g(t, x, y) - p(t, x, y) . \end{aligned} \tag{8}$$

An equation of this type was discussed also by Ballieu and Peiffer [3]. Throughout this section, we suppose that the following hypotheses are satisfied:

(H1) $h(t, x, y)$, $g(t, x, y)$ and $p(t, x, y)$ are continuous on $[0, \infty) \times R^2$ and $h(t, x, y) \geqq k(x, y) \geqq 0$, $y \cdot g(t, x, y) \geqq 0$ and $|p(t, x, y)| \leqq \beta(t)$ for $x, y \in R$, $t \geqq 0$, where $k(x, y)$ and $\beta(t)$ are continuous and $\int_0^\infty \beta(s)ds < \infty$.

(H2) $f(x)$ is continuous on R and there exists a $\rho > 0$ such that $x \cdot f(x) > 0$ for $0 < |x| \leqq \rho$ and

$$F(x) = \int_0^x f(u)du \to \infty \quad \text{as} \quad |x| \to \infty .$$

Define

$$V(t, x, y) = (y^2 + 2F(x) + M)^{1/2} + \int_t^\infty \beta(s)ds , \tag{9}$$

where M is chosen so that $2F(x) + M > 0$. An easy computation shows that

$$\dot{V}_{(8)}(t, x, y) \leqq -h(t, x, y)|y|^{\alpha+2}(y^2 + 2F(x) + M)^{-1/2} \leqq -W(x, y) \leqq 0 \,, \tag{10}$$

where $W(x, y) = k(x, y)|y|^{\alpha+2}(y^2 + 2F(x) + M)^{-1/2}$.

From (H2) and (9), $V(t, x, y) \to \infty$ as $|x| + |y| \to \infty$ uniformly for t. Therefore, it follows from (10) that every solution of (8) is bounded in the future. Furthermore, from (H1), it is obvious that the system (8) satisfies Condition (C) with $I = \{1\}$. For the above $W(x, y)$, we obtain $E = \{(x, y): y \cdot k(x, y) = 0\}$ and $S(E) = \{(x, z): 0 \leqq z \leqq y$ or $y \leqq z \leqq 0$ for some $(x, y) \in E\}$. Then by Theorem 1, we have:

THEOREM 2. *Every solution of* (8) *is bounded in the future and approaches* $S(E)$ *as* $t \to \infty$.

Furthermore, by Corollary 1, we have:

COROLLARY 3. *Suppose that* $h(t, x, y)$ *satisfies Condition* (B) *and that*

(H3) $\int_s^{s+t} g(u, x(u), y(u))du \to 0$ *as* $s \to \infty$ *uniformly on* $[0, 1]$ *for any bounded continuous function* $(x(t), y(t))$.

Then every solution of (8) *approaches* E *as* $t \to \infty$.

Now we give sufficient conditions under which every solution of (8) approaches the x-axis, i.e., $R_x = \{(x, 0): -\infty < x < \infty\}$.

COROLLARY 4. *In addition to all the assumptions of Corollary* 3, *suppose that*

(H4) $R_x^+ = \{(x, 0): x > 0\}$ *and* $R_x^- = \{(x, 0): x < 0\}$ *are connected components of* $E - \{(0, 0)\}$.

Then every solution of (8) *approaches* R_x *as* $t \to \infty$.

PROOF. Let $(x(t), y(t))$ be any solution of (8). Then Corollary 3 implies $\Omega \subset E$, where Ω is the positive limit set of $(x(t), y(t))$. In order to prove $y(t) \to 0$ as $t \to \infty$, we shall employ the argument used in [7]. We must have $\Omega \cap R_x \neq \varnothing$. Indeed, if this is not the case, it would follow that $|x(t)| \to \infty$ as $t \to \infty$. This contradicts Theorem 2.

Consider the two possibilities:

(a) $(0, 0) \notin \Omega$ and hence $\Omega \subset E - \{(0, 0)\}$, and

(b) $(0, 0) \in \Omega$.

Case (a) implies $\Omega \subset R_x$, since we have (H4) and $\Omega \cap R_x \neq \varnothing$, and hence $y(t) \to 0$ as $t \to \infty$. Consider the case (b). Since $(0, 0) \in \Omega$, there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, such that $(x(t_n), y(t_n)) \to (0, 0)$ as $n \to \infty$.

Consider the function

$$U(t, x, y) = (y^2 + 2F(x))^{1/2} + \int_t^\infty \beta(s)ds$$

defined for $t \geqq 0$ and $|x| + |y| \leqq \rho$. Then by an easy computation, we have $\dot{U}_{(8)}(t, x, y) \leqq 0$ and $U_n = U(t_n, x(t_n), y(t_n)) \to 0$ as $n \to \infty$. Therefore as long as $|x(t)| + |y(t)| \leqq \rho$, $t \geqq t_n$, we have

$$|y(t)| \leqq (y(t)^2 + 2F(x(t)))^{1/2} + \int_t^\infty \beta(s)ds \leqq U_n \,. \tag{11}$$

Since $U_n \to 0$ as $n \to \infty$, if we choose a sufficient large number n_0, then we conclude that (11) holds for all $t \geqq t_n$, $n \geqq n_0$. Hence $y(t) \to 0$ as $t \to \infty$. q.e.d.

COROLLARY 5. *Suppose that*

(H5) $h(t, x, y) \geqq k(x, y) > 0$ $(y \neq 0)$.

Then every solution of (8) *approaches* R_x *as* $t \to \infty$.

PROOF. Since $E = R_x$, it follows that $S(E) = R_x$. Therefore, Theorem 2 implies Corollary 5. q.e.d.

COROLLARY 6. *Suppose that* (H4) *holds and*

(H6) $y \cdot k_y(x, y) \geqq 0$, $x, y \in R$,

where $k_y(x, y)$ *denotes the partial derivative of* $k(x, y)$ *with respect to* y. *Then every solution of* (8) *approaches* R_x *as* $t \to \infty$.

PROOF. Let $(x(t), y(t))$ be any solution of (8). Since we have $S(E) = E$ by (H6), Theorem 2 implies that $(x(t), y(t)) \to E$ as $t \to \infty$. The remainder can be proved by the same argument as in the proof of Corollary 4. q.e.d.

To obtain more precise information on the asymptotic behavior of solution of (8) as $t \to \infty$, we need the following lemma which is a generalization of the result obtained by Thurston and Wong [10] in the case $p = q = 2$.

LEMMA 2. *Let* $H(s)$ *and* $u(s)$ *be nonnegative real-valued functions on* $[0, \infty)$ *such that for positive constants* p *and* q *which satisfy* $1/p + 1/q = 1$,

(i) $\int_0^\infty H(s)u(s)^q ds < \infty$, *and*

(ii) *there exist a sequence of positive numbers* $\{s_n\}$ *and a positive constant* d *such that* $s_{n+1} - s_n \geqq d$ *and that*

(a) $H(s) = 0$ *on* $[s_n, s_n + d]$ *for all* n, *or*

(b) $\sum_{n=1}^\infty \left[\int_{s_n}^{s_n+d} H(s)ds\right]^{-q/p} = \infty$.

Then there exists a subsequence $\{t_n\}$ *of* $\{s_n\}$ *such that*

$$\int_{t_n}^{t_n+d} H(s)u(s)ds \to 0 \quad as \quad n \to \infty .$$

PROOF. If (a) holds, then we can set $\{t_n\} = \{s_n\}$. We assume that (b) holds. Suppose that the assertion in the lemma does not hold. Then there exist $\delta > 0$ and a positive integer n_0 such that

$$\delta \leqq \int_{s_n}^{s_n+d} H(s)u(s)ds , \qquad n \geqq n_0 .$$

By Hölder's inequality, we have

$$\delta \leqq \left[\int_{s_n}^{s_n+d} H(s)ds\right]^{1/p}\left[\int_{s_n}^{s_n+d} H(s)u(s)^q ds\right]^{1/q} .$$

Thus

$$\begin{aligned}\sum_{n=n_0}^{\infty}\left[\int_{s_n}^{s_n+d} H(s)ds\right]^{-q/p} &\leqq \delta^{-q}\sum_{n=n_0}^{\infty}\left[\int_{s_n}^{s_n+d} H(s)u(s)^q ds\right] \\ &\leqq \delta^{-q}\int_{s_{n_0}}^{\infty} H(s)u(s)^q ds < \infty .\end{aligned}$$

This contradicts (b). Hence the assertion in the lemma holds. q.e.d.

THEOREM 3. *Suppose that in addition to* (H1) *and* (H3), *the condition*

(H2)′ $x \cdot f(x) > 0$ $(x \neq 0)$ *and* $F(x) = \int_0^x f(s)ds \to \infty$ *as* $|x| \to \infty$

is satisfied. Moreover suppose that the following condition (H7) *holds*:

(H7) *For any pair* $(x(t), y(t))$ *of bounded continuous functions on* $[0, \infty)$, *there exist a sequence of positive numbers* $\{s_n\}$ *and a positive constant* d *such that* $s_{n+1} - s_n \geqq d$ *and that* $h(s, x(s), y(s)) = 0$ *on* $[s_n, s_n + d]$ *for all* n *or*

$$\sum_{n=1}^{\infty}\left[\int_{s_n}^{s_n+d} h(s, x(s), y(s))ds\right]^{-1/(\alpha+1)} = \infty .$$

Furthermore let $(x(t), y(t))$ *be any solution of* (8) *which approaches* R_x *as* $t \to \infty$. *Then* $(x(t), y(t)) \to (0, 0)$ *as* $t \to \infty$.

PROOF. By Theorem 2, the solution $(x(t), y(t))$ is bounded on a half interval $[t_0, \infty)$, that is, there exists a compact set A in R^2 such that $(x(t), y(t)) \in A$ for $t \geqq t_0$. Consider the Liapunov function defined by (9). Then for $t \geqq t_0$,

$$\begin{aligned}&(d/dt)\,V(t, x(t), y(t)) \\ &\quad \leqq -h(t, x(t), y(t))\,|y(t)|^{\alpha+2}[y(t)^2 + 2F(x(t)) + M]^{-1/2} \\ &\quad \leqq -h(t, x(t), y(t))\,|y(t)|^{\alpha+2}/L ,\end{aligned}$$

where $L = \sup_{(x,y)\in A}[y^2 + 2F(x) + M]^{1/2}$. Since $V(t, x(t), y(t))$ is nonnegative and nonincreasing, we conclude that

$$\int_{t_0}^{\infty} h(s, x(s), y(s))\,|y(s)|^{\alpha+2}ds < \infty \,, \tag{12}$$

and that $V(t, x(t), y(t)) \to k$ as $t \to \infty$ for some k. Since the positive limit set of $(x(t), y(t))$ is connected and $y(t) \to 0$ as $t \to \infty$, it follows from (H2)$'$ that $x(t) \to c$ as $t \to \infty$ for some constant c. Applying Lemma 2 to $u(s) = |y(s)|^{\alpha+1}$, $H(s) = h(s, x(s), y(s))$, $p = \alpha + 2$ and $q = (\alpha+2)/(\alpha+1)$, it follows from (12) that there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, such that

$$\int_{t_n}^{t_n+d} h(s, x(s), y(s))\,|y(s)|^{\alpha+1}ds \to 0 \quad \text{as} \quad n \to \infty$$

and hence

$$\int_{t_n}^{t_n+t} h(s, x(s), y(s))\,|y(s)|^{\alpha}y(s)ds \to 0 \quad \text{as} \quad n \to \infty$$

uniformly for $t \in [0, d]$. Integrating the second equation of (8) over $[t_n, t_n + d]$, we obtain for $t \in [0, d]$,

$$\begin{aligned} y(t_n + t) - y(t_n) = &-\int_{t_n}^{t_n+t} h(s, x(s), y(s))\,|y(s)|^{\alpha}y(s)ds - \int_0^t f(x(t_n + s))ds \\ &- \int_{t_n}^{t_n+t} g(s, x(s), y(s))ds - \int_{t_n}^{t_n+t} p(s, x(s), y(s))ds \,. \end{aligned}$$

Letting $n \to \infty$, we have $\int_0^t f(c)ds = 0$ for $t \in [0, d]$. Therefore, (H2)$'$ implies $c = 0$. q.e.d.

REMARK 3. To obtain a result similar to Theorem 3, Artstein and Infante [1] considered the following condition in the case where $\alpha = 0$:

(H8) For any pair $(x(t), y(t))$ of bounded continuous functions on $[0, \infty)$, there exists a positive constant B such that for all $T > 1$,

$$T^{-(\alpha+2)} \int_0^T h(s, x(s), y(s))ds \leqq B \,.$$

This is somewhat easily checked in applications. However (H8) implies (H7). In fact, suppose that (H8) holds. Then we have $n^{-p}\sum_{k=1}^{n} a_k \leqq B$, where $a_n = \int_{n-1}^{n} h(s, x(s), y(s))ds$ and $p = \alpha + 2$. Here we may assume that $a_n \neq 0$ for all sufficiently large n. Put $q = (\alpha + 2)/(\alpha + 1)$. By Hölder's inequality, for $n > N$, we have

$$n - N \leqq \left[\sum_{k=N+1}^{n} a_k\right]^{1/p}\left[\sum_{k=N+1}^{n} a_k^{-q/p}\right]^{1/q} \leqq nB^{1/p}\left[\sum_{k=N+1}^{n} a_k^{-q/p}\right]^{1/q} .$$

Thus,

$$\sum_{k=1}^{\infty}(a_k^{-q/p}) \geqq \sum_{k=1}^{\infty}\sum_{n=2^k+1}^{2^{k+1}} a_n^{-q/p} \geqq B^{-q/p}\sum_{k=1}^{\infty}(1-2^k/2^{k+1})^q = \infty \ .$$

Put $s_n = n$ and $d = 1$. Then (H7) holds for such a choice of $\{s_n\}$ and d. Further, for the function $h(t) = (t+2)\log(t+2)$ and for $\alpha = 0$, we easily see that (H8) does not hold but (H7) does. Therefore in the case $\alpha = 0$, (H7) is strictly weaker than (H8). Furthermore, if we can certify (12), then (H8) implies the condition considered by Artstein and Infante [1], that is, $T^{-2}\int_0^T h(s, x(s), y(s))|y(s)|^\alpha ds \leqq B_1$ for some constant B_1. Indeed, suppose that (12) and (H8) hold. Then for some constants M and B, we have $\sum_{n=1}^N b_n \leqq M$ and $\sum_{n=1}^N c_n \leqq BN^{\alpha+2}$ for all N, where $b_n = \int_{n-1}^n h(s, x(s), y(s))|y(s)|^{\alpha+2}ds$ and $c_n = \int_{n-1}^n h(s, x(s), y(s))ds$. By Hölder's inequality, for all N, we have

$$\begin{aligned}\int_0^N h(s, x(s), y(s))|y(s)|^\alpha ds &\leqq \sum_{n=1}^N (b_n^{\alpha/(\alpha+2)}c_n^{2/(\alpha+2)}) \\ &\leqq \left[\sum_{n=1}^N c_n\right]^{2/(\alpha+2)}\left[\sum_{n=1}^N b_n\right]^{\alpha/(\alpha+2)} \leqq B^{2/(\alpha+2)}N^2M^{\alpha/(\alpha+2)} \ .\end{aligned}$$

Therefore we can put $B_1 = B^{2/(\alpha+2)}M^{\alpha/(\alpha+2)}$.

We immediately obtain the following corollary which is a generalization of a result given in [1] and [7].

COROLLARY 7. *Let* (H1), (H2)′, (H3) *and* (H7) *hold, and suppose that all the assumptions in Corollary* 5 *or* 6 *are satisfied. Then every solution of* (8) *tends to the origin as* $t \to \infty$.

If all the assumptions of Corollary 3 are satisfied, then (H8) obviously holds. Therefore we have:

COROLLARY 8. *Let* (H1), (H2)′, (H3) *and* (H4) *hold, and suppose that* $h(t, x, y)$ *satisfies Condition* (B). *Then every solution of* (8) *tends to the origin as* $t \to \infty$.

Ballieu and Peiffer [3] investigated the equation (8) in the case $k(x, y) = \psi(x)$ and $g = p = 0$, under (H1), (H2)′ and the following assumptions:

(i) $0 \leqq \psi(x) \leqq h(t, x, y) \leqq b(t)\phi(x, y)$, b, ϕ, ψ being continuous, $b(t) > 0$, $\phi(x, y) \geqq 0$,

(ii) $\int_{-\eta}^{\eta}\psi(x)dx = m(\eta) > 0$ for all $\eta > 0$,

(iii) $\int_0^\infty dt/b(t) = \infty$, $b(t)$ nondecreasing.

Now we show that (i) and (iii) imply (H7). Since $b(t)$ is nondecreasing,

(iii) is equivalent to $\sum_{n=1}^{\infty} 1/b(n) = \infty$. Put $\{s_n\} = \{n\}$ and $d = 1$ and choose $a_0 > 0$ satisfying $a_0 b(t) \geqq 1$ for all t. Then for any pair $(x(t), y(t))$ of bounded continuous functions on $[0, \infty)$, we have

$$a_0^{-1/(\alpha+1)} \sum_{n=1}^{\infty} \left[\int_{s_n}^{s_n+d} h(s, x(s), y(s))ds\right]^{-1/(\alpha+1)} \geqq M^{-1/(\alpha+1)} \sum_{n=1}^{\infty} \left[\int_n^{n+1} a_0 b(s)ds\right]^{-1/(\alpha+1)}$$
$$\geqq M^{-1/(\alpha+1)} \sum_{n=1}^{\infty} [a_0 b(n+1)]^{-1/(\alpha+1)} \geqq M^{-1/(\alpha+1)} a_0^{-1} \sum_{n=1}^{\infty} b(n+1)^{-1} = \infty ,$$

where M is a positive constant satisfying $M > \sup \phi(x(t), y(t))$. Therefore (H7) holds for such a choice of $\{s_n\}$ and d.

Thus the following is a generalization of Ballieu and Peiffer's result.

COROLLARY 9. *Let* (H1), (H2)$'$ *and* (H7) *hold. Moreover suppose that* $g = p = 0$ *and that* (i) $0 \leqq \psi(x) \leqq h(t, x, y)$ *and* $\psi(x)$ *is continuous, and* (ii) $\int_{-\eta}^{\eta} \psi(x)dx = m(\eta) > 0$ *for any* $\eta > 0$. *Then every solution of* (8) *tends to the origin as* $t \to \infty$.

PROOF. On account of Theorem 3, it suffices to prove that $y(t) \to 0$ as $t \to \infty$ for any solution $(x(t), y(t))$ of (8). Define $U(x, y) = y^2/2 + F(x)$. Then we have $\dot{U}_{(8)}(x, y) = -h(t, x, y)|y|^{\alpha+2} \leqq -\psi(x)|y|^{\alpha+2} \leqq 0$. Define the set $E = \{(x, y): y \cdot \psi(x) = 0\}$. Then we have $S(E) = E$ and hence $\Omega \subset E$, where Ω is the positive limit set of $(x(t), y(t))$. Furthermore, there exists a constant c such that $\Omega \subset \Gamma := \{(x, y): F(x) + y^2/2 = c\}$, and hence $\Omega \subset \Gamma \cap E$.

Suppose that $y(t) \not\to 0$ as $t \to \infty$. Then there exists an $(x_0, y_0) \in \Omega$ such that $y_0 \neq 0$, and consequently $F(x_0) < c$. We shall derive a contradiction. Consider the case $x_0 > 0$. By (ii), we can choose ξ satisfying $|\xi| < x_0$, $F(\xi) < c$ and $\psi(\xi) > 0$. Thus $\Omega \subset \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \{(x, y) \in \Gamma: x > \xi\}$ and $\Gamma_2 = \{(x, y) \in \Gamma: x < \xi\}$. Since Ω is connected and $(x_0, y_0) \in \Omega \cap \Gamma_1$, we have $\Omega \subset \Gamma_1$. Then for a sufficiently large number T, we have

$$(13) \qquad x(t) > \xi \quad \text{and} \quad U(x(t), y(t)) \geqq c \quad \text{for all} \quad t > T .$$

Furthermore, since $x(t)$ is bounded, we have $\Omega \cap R_x \neq \emptyset$. Choose a point $(x_1, 0) \in \Omega$. Then $F(x_0) < F(x_1) = c$, $\xi < x_1$ and consequently, by (H2)$'$, $x_0 < x_1$. Therefore it follows from (13) and the fact (x_0, y_0), $(x_1, 0) \in \Omega$ that there exists a number T_1, $T_1 > T$, such that $x_0 < x(T_1) < x_1$, $y_0 \cdot \dot{x}(T_1) \leqq 0$ and that $y_0 \cdot y(T_1) > 0$, which yields a contradiction, since $\dot{x}(t) = y(t)$. In the case $x_0 \leqq 0$, almost the same argument will lead us to a contradiction. q.e.d.

REMARK 4. In order to guarantee that every solution of (8) is bounded in the future, the assumption (H2)$'$ is too strong. In fact, this assumption may be replaced by the following weaker one [10]:

(H2)$''$ (i) $xf(x) > 0$ $(x \neq 0)$ and (ii) for any constant $B > 0$, there exists a nonnegative function $u(x)$ depending on B such that $\int_0^{\pm\infty}(u(x) + f(x))dx = \pm\infty$ and $\max(u(x) - r_{B,u}(x), 0) \in L_1(-\infty, \infty)$, where

$$r_{B,u}(x) = \begin{cases} u(x) & \text{if} \quad |f(x)| > B \cdot u(x) \quad \text{or} \quad x = 0\,, \\ \inf_y \{k(x, y) \colon |f(x)|/u(x) \leqq y \cdot \operatorname{sgn} x \leqq B\}\,. \end{cases}$$

Therefore, replacing (H2)$'$ by (H2)$''$, we obtain results similar to those in the above Theorem 3 and Corollaries 7, 8 and 9. Furthermore, we note that in the case $\alpha = 0$, (H7) is identical with the assumption (A5) in [10].

Finally, as another application of Theorem 1, we consider the following system on $[0, \infty) \times R^{n+2}$,

$$\begin{aligned} \dot{x} &= -c_0 b_0(t)y - \sum_{i=1}^{n} c_i b_i(t) z_i \\ \dot{y} &= b_0(t)f(x) + e_0(t) \qquad\qquad (14) \\ \dot{z}_i &= -h_i(t, x, y, z)|z_i|^{\alpha_i} z_i + b_i(t)f(x) + e_i(t) \end{aligned}$$

$(i = 1, 2, \cdots, n)$, where $\alpha_i \geqq 0$, $c_i > 0$ are constants and $z = (z_1, \cdots, z_n)$. A system of this type was discussed by Levin and Nohel [5] and Miller [6] and others. Consider the following:

(A1) b_i, e_i, h_i and f are continuous, $e_i \in L_1[0, \infty)$ and each b_i is bounded on $[0, \infty)$,

(A2) $x \cdot f(x) > 0$ $(x \neq 0)$ and $F(x) = \int_0^x f(s)ds \to \infty$ as $|x| \to \infty$,

(A3) there exists a continuous function $k(x, y, z)$ such that $h_i(t, x, y, z) \geqq k(x, y, z) > 0$ $(z \neq 0)$ for all i,

(A4) there exists at least one index i_0 for which h_{i_0} satisfies (H7) with $\alpha = \alpha_{i_0}$,

(A5) for an index i_0 in (A4), $\liminf_{t\to\infty} |b_{i_0}(t)| \neq 0$,

and

(A6) $\liminf_{t\to\infty} |b_0(t)| \neq 0$.

THEOREM 4. *If the above hypotheses* (A1) *through* (A6) *are satisfied, then every solution of* (14) *is bounded in the future and tends to the origin as* $t \to \infty$.

PROOF. Define

$$V(t, x, y, z) = \exp(-E(t))[c_0 y^2/2 + \sum_{i=1}^{n} c_i z_i^2/2 + F(x) + 1]\,, \qquad (15)$$

where $E(t)=\sum_{i=0}^{n}\int_0^t |c_i^{1/2}e_i(s)|ds$. Then an easy computation shows that

$$(16)\qquad \dot{V}_{(14)}(t, x, y, z) \leqq -\exp(-E(\infty))\sum_{i=1}^{n} c_i h_i(t, x, y, z)|z_i|^{\alpha_i+2}$$
$$\leqq -\exp(-E(\infty))k(x, y, z)\sum_{i=1}^{n} c_i|z_i|^{\alpha_i+2} \leqq 0 .$$

From (A1), (A2), (15) and (16), it follows that every solution of (14) is bounded in the future. Clearly, the equation (14) satisfies Condition (C) with $I=\{1, 2\}$. Therefore, Theorem 1 implies that every solution $(x(t), y(t), z(t))$ of (14) tends to the set $S(E)=E=\{(x, y, 0): x, y \in R\}$ and hence

$$(17)\qquad z(t) \to 0 \quad \text{as} \quad t \to \infty .$$

Furthermore, from (16), it follows that

$$\sum_{i=1}^{n}\int^{\infty} h_i(s, x(s), y(s), z(s))|z_i(s)|^{\alpha_i+2}ds < \infty$$

and hence, by (A4) and Lemma 2, we conclude that there exists a sequence $\{t_m\}$, $t_m \to \infty$ as $m \to \infty$, such that

$$(18)\qquad \int_{t_m}^{t_m+d} h_{i_0}(s, x(s), y(s), z(s))|z_{i_0}(s)|^{\alpha_{i_0}+1}ds \to 0 \quad \text{as} \quad m \to \infty .$$

Taking a subsequence if necessary, without loss of generality, we may assume $(x(t_m), y(t_m)) \to (p, q)$ as $m \to \infty$ for a point (p, q). We shall show $(p, q)=(0, 0)$. Consider the functions $x_m(t)=x(t+t_m)$, $y_m(t)=y(t+t_m)$, $m=1, 2, \cdots$, defined for $t \in [0, d]$. Since $(x(t), y(t))$ is bounded, the functions $\{x_m(t)\}$ and $\{y_m(t)\}$ are uniformly bounded and equicontinuous on $[0, d]$. By Ascoli's theorem, taking a subsequence if necessary, we have $x_m(t) \to \phi(t)$ and $y_m(t) \to \psi(t)$ as $m \to \infty$ uniformly on $[0, d]$ for some continuous functions $\phi(t)$, $\psi(t)$. Integrating the equation of $\dot{z}_{i_0}$ in (14) over $[t_m+d_1, t_m+d_2]$ for any $d_1, d_2 \in [0, d]$, $d_1 < d_2$, and letting $m \to \infty$, by (17), (18) and (A1), we conclude

$$\int_{d_1}^{d_2} b_{i_0}(s+t_m)f(x_m(s))ds \to 0 \quad \text{as} \quad m \to \infty$$

and consequently

$$\int_{d_1}^{d_2} b_{i_0}(s+t_m)f(\phi(s))ds \to 0 \quad \text{as} \quad m \to \infty \quad \text{for any} \quad d_1, d_2 \in [0, d] ,$$

since $b_{i_0}(t)$ is bounded on $[0, \infty)$ and $x_m(t) \to \phi(t)$ as $m \to \infty$ uniformly on $[0, d]$. Therefore, by (A5), we have $f(\phi(t))=0$ on $[0, d]$ and hence, by (A2), $\phi(t)=0$ on $[0, d]$. Furthermore, integrating the equation of $\dot{x}$ in (14) over $[t_m+d_1, t_m+d_2]$ for any $d_1, d_2 \in [0, d]$, $d_1 < d_2$, and letting

$m \to \infty$, we have $\int_{d_1}^{d_2} b_0(s + t_m)y_m(s)ds \to 0$ as $m \to \infty$ by (A1), (17) and the fact that $x(t + t_m) \to \phi(t) = 0$ as $m \to \infty$ uniformly for $t \in [0, d]$. Then by (A6) and the same argument as in the case $\phi(t)$, we conclude $\psi(t) = 0$ on $[0, d]$. Hence $p = \phi(0) = 0$, $q = \psi(0) = 0$. Thus we have

$$(x(t_m),\ y(t_m),\ z(t_m)) \to (0, 0, 0) \quad \text{as} \quad m \to \infty\ . \tag{19}$$

Now, we shall prove that $(x(t), y(t), z(t)) \to (0, 0, 0)$ as $t \to \infty$. Define

$$U(t, x, y, z) = \left[c_0 y^2 + \sum_{i=1}^{n} c_i z_i^2 + 2F(x)\right]^{1/2} + \sum_{i=0}^{n} c_i^{1/2}\int_t^\infty |e_i(s)|\,ds\ .$$

Then an easy computation shows $\dot{U}_{(14)}(t, x, y, z) \leqq 0$. Then for $t \geqq t_m$, we have

$$0 \leqq U(t, x(t), y(t), z(t)) \leqq U(t_m, x(t_m), y(t_m), z(t_m))\ . \tag{20}$$

Since $U(t_m, x(t_m), y(t_m), z(t_m)) \to 0$ as $m \to \infty$ by (19) and (A1), it follows, from (20), that $U(t, x(t), y(t), z(t)) \to 0$ as $t \to \infty$ and hence by (A2), we conclude that $(x(t), y(t), z(t)) \to (0, 0, 0)$ as $t \to \infty$. q.e.d.

REMARK 5. One of the most essential assumptions given in [5], [6] is the following: All $h_i(t, x, y, z)$ are bounded when $x^2 + y^2 + |z|^2$ is bounded. Therefore Theorem 4 is a generalization of the results obtained in [5], [6].

REFERENCES

[1] Z. ARTSTEIN AND E. F. INFANTE, On the asymptotic stability of oscillations with unbounded damping, Quart. Appl. Math. 34 (1976), 195-199.

[2] Z. ARTSTEIN, The limiting equation of nonautonomous ordinary differential equations, J. Differential Equations 25 (1977), 184-202.

[3] R. J. BALLIEU AND K. PEIFFER, Attracting of the origin for the equation $\ddot{x} + f(t, x, \dot{x})|\dot{x}|^\alpha \dot{x} + g(x) = 0$, J. Math. Anal. Appl. 65 (1978), 321-332.

[4] J. P. LASALLE, Stability of nonautonomous systems, Nonlinear Analysis, Theory, Methods and Applications 1 (1976), 83-91.

[5] J. J. LEVIN AND J. A. NOHEL, Global asymptotic stability for nonlinear systems of differential equations and applications to reactor dynamics, Arch. Rational Mech. Anal. 5 (1960), 194-211.

[6] R. K. MILLER, Asymptotic behavior of solutions of nonlinear differential equations, Trans. Amer. Math. Soc. 115 (1965), 400-416.

[7] N. ONUCHIC, Invariance properties in the theory of ordinary differential equations with application to stability problems, SIAM J. Control 9 (1971), 97-104.

[8] N. ONUCHIC, Invariance and stability for ordinary differential equations, J. Math. Anal. Appl. 63 (1978), 9-18.

[9] N. ONUCHIC, Invariance properties for ordinary differential equations; stability and instability, Nonlinear Analysis, Theory, Methods and Applications 2 (1978), 69-76.

[10] L. H. THURSTON AND J. S. W. WONG, On global asymptotic stability of certain second

order differential equations with integrable forcing terms, SIAM J. Appl. Math. 24 (1973), 50–61.
[11] T. YOSHIZAWA, Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions, Springer-Verlag, Berlin-Heidelberg-New York, 1975.

DEPARTMENT OF GENERAL EDUCATION
HACHINOHE TECHNICAL COLLEGE
HACHINOHE, 031
JAPAN