Tôhoku Math. Journ.
34 (1982), 425-435.

HELICOIDAL SURFACES WITH CONSTANT MEAN CURVATURE

MANFREDO P. DO CARMO AND MARCOS DAJCZER

(Received November 12, 1981)

Abstract. We describe the space Σ_H of all surfaces in R^3 that have constant mean curvature $H \neq 0$ and are invariant by helicoidal motions, with a fixed axis, of R^3. Similar to the case Σ_0 of minimal surfaces Σ_H behaves roughly like a circular cylinder where a certain generator corresponds to the rotation surfaces and each parallel corresponds to a periodic family of isometric helicoidal surfaces.

1. Introduction. 1.1. Rotation surfaces in the Euclidean space R^3 with constant mean curvature have been known for a long time (Delaunay [3]). A natural generalization of rotation surfaces are the helicoidal surfaces that can be defined as follows.

Let R^3 have coordinates (x, y, z). Consider the one-parameter subgroup $g_t: R^3 \to R^3$ of the group of rigid motions of R^3 given by

$$g_t(x, y, z) = (x \cos t + y \sin t, -x \sin t + y \cos t, z + ht), \quad t \in (-\infty, \infty).$$

The motion g_t is called a *helicoidal motion with axis Oz and pitch h*. A *helicoidal surface with axix Oz and pitch pitch h* is a surface that is invariant by g_t, for all t. When $h = 0$, they reduce to rotation surfaces.

The helicoidal minimal surfaces have also been known for quite a long time (see e.g. [6] for details). It is therefore mildly surprising that we do not find in the literature the helicoidal surfaces with constant nonzero mean curvature; in this paper we want to determine explicitly all of them.

Our interest in this question comes (aside its naturality) from the fact that there are very few explicit examples of surfaces with nonzero constant mean curvature. To understand certain aspects of such surfaces (behaviour of the Gauss map, stability, etc.) it might prove convenient to have at hand a reasonable supply of explicit examples. It should be mentioned that the techniques used here can also give a complete description of helicoidal surfaces with constant Gaussian curvature (Remark 3.16); this is, however, very simple and probably known.

Key words and sentences: mean curvature; helicoidal motions; rotation surfaces; associated surfaces, helicoidal surfaces.

The starting point of our paper is the following result of Lawson (the result is actually more general than the statement below).

THEOREM (Lawson [5, Theorem 8]). *Let M^2 be a simply-connected two-dimensional manifold and let $f: M^2 \to R^3$ be an immersion with constant mean curvature H. Then there exists a differentiable, 2π-periodic, family of immersions $f_\theta: M^2 \to R^3$, $\theta \in [0, 2\pi]$, $f_0 = f$, with the same induced metric and with constant mean curvature H. Furthermore, the family f_θ contains (the extensions of) all local isometric immersions with the given H.*

The family f_θ is called the *associated family* to f and generalizes the well known construction of associated minimal surfaces. In the case of minimal surfaces, the associated family to a rotation minimal surface is a family of helicoidal surfaces (see, e.g. [6]) and this generalizes to the above construction. Explicitly, it follows from Proposition 12.2 of Lawson [5] that if $f: M \to R^3$ is a rotation surface, the associated family f_θ is made up by helicoidal surfaces.

It will be convenient to rephrase Lawson's results as follows. Let M^2 be simply-connected, let the unit circle $S^1 \subset R^2$ be parametrized by θ, $\theta \in [0, 2\pi)$ and set

$$\Sigma_H = \{f \in \mathrm{Imm}(M^2 \to R^3);\ f \text{ is a helicoidal surface with a given axis and constant mean curvature } H \neq 0\}\,.$$

By using Kenmotsu's version [4] of Delaunay's results, we can parametrize the family of rotation surfaces with constant mean curvature $H \neq 0$ (and with a fixed axis) by a parameter B_0 that runs in $[0, \infty)$: the point 0 corresponds to the right circular cylinder, between 0 and 1 we find the onduloids, the point 1 represents the unit sphere, and in the interval $(1, \infty)$ are the nodoids. By Lawson's results, the closed half-cylinder $S^1 \times [0, \infty)$ represents then helicoidal surfaces, all with constant mean curvature H. Thus, there is a natural map $\phi: S^1 \times [0, \infty) \to \Sigma_H$ and we are reduced to proving the following statements:

(i) ϕ is a surjective map.

(ii) The map ϕ can be explicitly determined.

We should make precise the meaning of (ii). For that, it is convenient to parametrize the domain M^2 of a helicoidal surace $f: M^2 \to R^3$ by parameters (s, t), where the images by f of the t-curves are the trajectories of the helicoidal motions, while the s-curves are their orthogonal trajectories parametrized by arclength in the induced metric; such a local parametrization will be called a *natural* parametrization of the helicoidal surface. Notice that the first fundamental form in such parameters can be written $d\sigma^2 = ds^2 + U^2dt^2$, were $U = U(s)$ is a function of s alone.

With the above notation, (ii) is to be understood as meaning that we can write $(\phi(\theta, B_0))(s, t)$ as a function of θ, B_0, s, t.

We summarize what is to be proved in the theorem below that is the main result of this paper.

THEOREM 1.2. *There exists a surjective map $\phi: S^1 \times [0, +\infty) \rightarrow \Sigma_H$ such that $\phi(0, [0, +\infty))$ are the rotation surfaces in Σ_H, and $\phi(\theta, B_0)$, $B_0 \in [0, +\infty)$, $0 \leqq \theta \leqq 2\pi$, is the associated family to $\phi(0, B_0)$. Furthermore, except for $(\theta, 1)$, the immersion $\phi(\theta, B_0) = f: M \rightarrow R^3$ is given explicitly, in a global natural parametrization of the immersion by putting together* (2.2), (3.11) *and* (4.12) *below.*

1.3. The idea of the proof is as follows. To prove that ϕ is surjective, we must show that given an arbitrary helicoidal surface $\bar{f} \in \Sigma_H$ there exists a rotation surface $f \in \Sigma_H$ and a number $\theta \in [0, 2\pi]$ such that $f_\theta = \bar{f}$.

By a result of Bour (cf. Lemma 2.3), given a helicoidal surface, there exists a two-parameter family of helicoidal surfaces isometric to it; such a family includes a rotation surface, the mean curvature of which, however, has no obvious relation with the mean curvature of the starting surface. The point here is twofold: First, we establish a condition for an element of Bour's family above to have constant mean curvature; such a condition depends on one parameter (3.11). We thus obtain a three-parameter family of helicoidal surfaces that have constant mean curvature but are not isometric (one of these parameters can actually be eliminated; however, the larger family is necessary for the next step). We then show, and this is the delicate point in the proof, that given one element of the above three-dimensional family, there exists a closed curve of isometric immersions connecting the given element to a rotation surface (Lemma 4.3); by Lawson's result, the immersions in the curve constitute the associated family to the rotation surface, and this proves the first part of Theorem 1.2. The explicit expressions mentioned in Theorem 1.2 are obtained in the process of proving steps one and two above (the exception mentioned in the statement comes from the fact that $\phi(\theta, 1)$ are spheres, the antipodal points of which cannot be covered by a fixed natural parametrization).

2. **Bour's lemma.** 2.1. Let $f: M \rightarrow R^3$ be an immersion and let $U \subset M$ be an open set. Assume, for the time being that the inter-

section of $f(U)$ with some plane $\Pi \subset R^3$ containing Oz is a curve which is a graph $z = \lambda(\rho)$ over the intersection of Π with the plane xy. If f is invariant by a helicoidal motion around Oz with pitch h, the restriction $f|U$ can be written as

$$f(\rho, \varphi) = (\rho\cos\varphi, \rho\sin\varphi, \lambda(\rho) + h\varphi) , \tag{2.2}$$

where ρ and φ are polar coordinates in the plane xy, and the plane xy has been rotated so that Ox is the origin of φ.

We need the following lemma that was first proved by Bour [1, p. 82, Theorem II].

LEMMA 2.3 (Bour). *Given a helicoidal surface of the form* (2.2), *there exists a two-parameter family of helicoidal surfaces isometric to* (2.2).

2.4. PROOF. We follow Darboux ([2, vol. I, pp. 129–130)]. The first fundamental form of (2.2) can be written

$$d\sigma^2 = (1 + \rho^2\lambda'^2(\rho^2 + h_o^2)^{-1})d\rho^2 + (\rho^2 + h_o^2)(d\varphi + h_o\lambda'(\rho^2 + h_o^2)^{-1}d\rho)^2 ,$$

where the prime denotes the derivative in ρ and we have set, for definiteness, $h = h_o$ in (2.2). We introduce new parameters (s, t) in (2.2) by functions $s = s(\rho, \varphi)$, $t = t(\rho, \varphi)$ that satisfy

$$ds = (1 + \rho^2\lambda'^2(\rho^2 + h_o^2)^{-1})^{1/2} , \quad dt = \varphi d + h_o\lambda'(\rho^2 + h_o^2)^{-1}d\rho .$$

Notice that the Jacobian $\partial(s, t)/\partial(p, \varphi)$ is nonzero and that (s, t) is a natural parametrization on $U \subset M$. By setting $U^2(s) = \rho^2(s) + h_o^2$, we can write, in the natural parametrization, $d\sigma^2 = ds^2 + U^2dt^2$.

We are now reduced to showing that, given a function $U = U(s)$, we can find functions ρ, λ and φ of s and t that satisfy:

$$ds^2 = d\rho^2 + \rho^2(\rho^2 + h^2)^{-1}d\lambda^2 , \tag{2.5}$$

$$Udt = \pm(\rho^2 + h)^{1/2}(d\varphi + h(\rho^2 + h^2)^{-1}d\lambda) , \tag{2.6}$$

for an arbitrary constant h.

We first observe, from (2.5), that ρ and λ do not depend on t. Then, from (2.6), we obtain

$$\frac{\partial\varphi}{\partial s} = -h(\rho^2 + h^2)^{-1}d\lambda/ds , \quad \frac{\partial\varphi}{\partial t} = \pm U/(\rho^2 + h^2)^{1/2} .$$

Thus $\partial^2\varphi/\partial tds = 0$, hence $U/(\rho^2 + h^2)^{1/2}$ does not depend on s. Therefore we can set

$$U/(\rho^2 + h^2)^{1/2} = 1/m \neq 0 , \quad m = \text{const.} , \tag{2.7}$$

and write (2.6) as

$$d\varphi = (1/m)dt - h(\rho^2 + h^2)^{-1}d\lambda .$$

Now, from (2.7), it follows that

$$\dot{\rho}^2 = m^4 U^2 \dot{U}^2 (m^2 U^2 - h^2)^{-1} , \tag{2.8}$$

where dot denotes the derivative in s. From (2.5) and (2.8) we obtain

$$(d\lambda)^2 = (m^2 U^2 (1 - m^2 \dot{U}^2) - h^2)(m^2 U^2 - h^2)^{-2} m^2 U^2 ds^2 . \tag{2.9}$$

It follows from (2.7), (2.8) and (2.9) that the helicoidal surface (2.2), where ρ, φ and λ are given by

$$\begin{cases} \rho = (m^2 U^2 - h^2)^{1/2} , \\ \varphi = m^{-1}\int dt - hm^{-1}\int (m^2 U^2 (1 - m^2 \dot{U}^2) - h^2)^{1/2} (U(m^2 U^2 - h^2))^{-1} ds , \\ \lambda = \int m U (m^2 U^2 - h^2)^{-1} (m^2 U^2 (1 - m^2 \dot{U}^2) - h^2)^{1/2} ds , \end{cases} \tag{2.10}$$

are all isometric with first fundamental form given by $d\sigma^2 = ds^2 + U^2 dt^2$. The constants of integration are easily eliminated by a rigid motion of the coordinate axis and by adjusting the origin of the parameter t. Thus there are essentially two parameters in the family described by (2.10). This proves Bour's lemma.

REMARK 2.11. Bour's family (2.10) contains the surface we started with for $m = 1$, $h = h_o$. In particular, Bour's lemma asserts the existence of a two-parameter family of helicoidal surfaces isometric to a given rotation surface ($m = 1$, $h = 0$).

REMARK 2.12. So far we have worked under the restriction that the helicoidal suraces $f: M \to R^3$ can be written in an open set $U \subset M$ in the form (2.2). However, in the natural parameters (s, t), f given by (2.10) is well defined as an immersion for all t, and all those s for which $\rho(s) \neq 0$. Notice that since Oz is a trajectory of the helicoidal motion, if $f(M)$ meets Oz at one point, the whole axis Oz is contained in $f(M)$. Thus, unless $f(M)$ is a cylinder, the natural parametrization includes at least that part of $f(M)$ swept by the "rays" of the orthogonal trajectories to the helices that pass through a fixed helix and start at Oz. We will see in the next section that for surfaces with constant mean curvature, $\rho(s)$ can only be zero in a very special case. Also, we will assume until Remark 3.14 below that $f(M)$ is not a cylinder.

3. Helicoidal surfaces of constant mean curvature. 3.1. A surface of Bour's family (2.10) is determined by giving a function $U(s)$ and constants m, h. For convenience, let us denote it by $[U, m, h]$.

LEMMA 3.2. *$[U, m, h]$ is a surface with constant mean curvature H if and only if $U(s)$ satisfies the equation*

$$m^2 U\ddot{U} + m^2\dot{U}^2 - 1 = 2H(m^2U^2)(1 - m^2\dot{U}^2) - h^2)^{1/2} . \tag{3.3}$$

PROOF. This follows easily by computing the second fundamental form

$$eds^2 + 2fdsdt + gdt^2$$

of $[U, m, h]$ in the parameters (s, t). We obtain from (2.2) that

$$(f_t, f_s, f_{tt}) = -\rho^2\dot{\lambda}/m^3 , \quad (f_t, f_s, f_{ts}) = (h\dot{\rho}^2 - \dot{\varphi}\lambda\rho^2)/m^2 ,$$

where (, ,) denotes the determinant of the enclosed vectors and dots denote derivatives in s. It follows that

$$g = -\rho^2\lambda(m^3U)^{-1} = -(m^2U^2(1 - m^2\dot{U}^2) - h^2)^{1/2}m^{-2} , \quad f = h(m^2U)^{-1} . \tag{3.4}$$

Furthermore, since the Gaussian curvature K is easily seen to be $K = -\ddot{U}/U$, we obtain that

$$e = (m^4U^3\ddot{U} - h^2)(m^2U^2(1 - m^2\dot{U}^2) - h^2)^{-1/2}(m^2U^2)^{-1} . \tag{3.5}$$

Finally, since $2H = e + g/U^2$, we obtain Equation (3.3), and this proves Lemma 3.2.

3.6. Equation (3.3) is easily integrated if we make the changes of variables: $x = mU$, $y = (x^2 - x^2\dot{x}^2 - h^2)^{1/2}$. Then (3.3) becomes $\dot{y} = -2Hx\dot{x}$, an integral of which is

$$y = -Hx^2 + a , \quad a = \text{const.} . \tag{3.7}$$

Returning to the variable x, we obtain

$$\dot{x}^2 = (x^2 - h^2 - (Hx^2 - a^2)/x^2 \tag{3.8}$$

which can be integrated by setting $z = x^2$ to transform it into

$$\{-H^2z^2 + (2Ha + 1)z - (a^2 + h^2)\}^{-1/2}dz = 2ds .$$

By assuming $H \neq 0$ and adjusting the origin of s, we obtain from the above

$$s = (2H)^{-1}\,\mathrm{sen}^{-1}((2H^2z - 2Ha - 1)((2Ha + 1)^2 - 4H^2(a^2 + h^2))^{-1/2}) .$$

Since $z = m^2U^2$ we finally arrive at

$$U(s)^2 = (1 + 2Ha + (1 - 4H^2h^2 + 4Ha)^{1/2}\sin 2Hs)/2m^2H^2 \tag{3.9}$$

that yields a one-parameter family of functions U_a such that $[U_a, m, h]$ is a helicoidal surface with constant mean curvature H.

It will be convenient to set

$$B = (1 - 4H^2h^2 + 4Ha)^{1/2} \neq 0 . \tag{3.10}$$

in (3.9). It is easily computed that, with this notation,

$$\dot{U} = B \cos 2Hs(2m^2HU)^{-1} ,$$

and

$$m^2U^2(1 - m^2\dot{U}^2) - h^2 = (1 + B \sin 2Hs)^2(4H^2)^{-1} ,$$
$$m^2U^2 - h^2 = (1 + B^2 + 2B \sin 2Hs)(4H^2)^{-1} ,$$

Thus the surface of Bour's family given by (2.10) has constant mean curvature $H \neq 0$ if and only if

$$\begin{cases} \rho = (1 + B^2 + 2B \sin 2Hs)^{1/2}(2H)^{-1} , \\ \varphi = \dfrac{t}{m} - 4H^2h \displaystyle\int \frac{(1+B \sin 2Hs)ds}{(1+4H^2h^2+B^2+2B \sin 2Hs)^{1/2}(1+B^2+2B \sin 2Hs)} , \\ \lambda = \displaystyle\int \frac{(1 + 4H^2h^2 + B^2 + 2B \sin 2Hs)^{1/2}(1 + B \sin 2Hs)ds}{1 + B^2 + 2B \sin 2Hs} , \end{cases} \tag{3.11}$$

where B is given by (3.10). In particular, if we start with an arbitrary helicoidal surface (2.2) with $m = 1, h = h_0$, then it has constant mean curvature $H \neq 0$ if and only if the functions $\rho = \rho(s)$, $\varphi = \varphi(s, t)$, $\lambda = \lambda(s)$ are given by setting $m = 1$, $h = h_0$ in (3.11).

We summarize the results obtained so far.

PROPOSITION 3.12. *The helicoidal surfaces of the form* (2.2) *that have constant mean curvature $H \neq 0$ constitute a two-parameter family with parameters B and h and are given in a natural parametrization if we replace ρ, φ and λ in* (2.2) *by their values obtained by setting $m = 1$ in* (3.11).

REMARK 3.13. By replacing in (3.11) t/m by $\bar{t}$, we see that the various surfaces with different m's in (3.11) have actually the same images with distinct parametrizations. This justifies why we do not need a further parameter in the description of Proposition 3.12 and can set $m = 1$. However, and this is a crucial point in the proof of Theorem 1.2, we *do* need the larger family to define new parameters that makes explicit the associated family to a given surface.

REMARK 3.14. It follows from (3.11) that $\rho(s) \neq 0$ for all s, except when $B = \pm 1$; even in this case singularities of the natural parametrization (cf. Remark 2.12) can only occur for the values of s that satisfy $\sin 2Hs = \pm 1$. We also see that cylinders can be included in (3.11) if we allow $B = 0$.

3.15. There are certain normalizations of signs that should be made now. Since $H \neq 0$ is constant, we can assume that $H > 0$. By (3.11) and Remark 3.14 we see, by changing s into $-s$, that we can assume $B \geqq 0$; by a similar argument, we can also assume $m > 0$. From now on we will assume these normalizations without further comment.

REMARK 3.16. By the same token, we can use Bour's family to determine all helicoidal surfaces with constant Gaussian curvature K. All we have to do is to replace in (2.10) the function $U = U(s)$ by the solutions of the equation $\ddot{U} + KU = 0$, $K =$ const.

4. **Proof of Theorem 1.2.** 4.1. We now consider the family given by (3.11) and introduce new parameters (a_o, θ) by setting:

$$\begin{aligned} a &= a_o \cos\theta/(1 + 2a_oH(1 - \cos\theta)) \,, \\ h &= a_o \sin\theta/(1 + 2a_oH(1 - \cos\theta)) \,, \\ m^2 &= (1 + 2a_oH(1 - \cos\theta))^{-1} \,. \end{aligned} \tag{4.2}$$

Let us denote by $U(a, h, m)$ the expression of U given by (3.9). Notice that for $a = a_o$, $h = 0$, $m = 1$, we obtain in (3.11) a rotation surface, to be denoted by $f(a_o)$, with constant mean curvature $H \neq 0$.

LEMMA 4.3. *Fix a_o and θ, and let a, h, m be given by* (4.2). *Then for all θ, $0 \leqq \theta \leqq 2\pi$,*

$$U(a, h, m) = U(a_o, 0, 1) \,. \tag{4.4}$$

Thus for each rotation surface $f(a_o)$ given by (3.11), *there exists a one-parameter family $f(a_o, \theta)$ of isometric helicoidal surfaces with constant mean curvature $H \neq 0$. Furthermore,*

(i) *$f(a_o, \theta)$, $0 \leqq \theta \leqq 2\pi$, is the associated family to $f(a_o)$.*

(ii) *Given an arbitrary surface $\bar{f}$ of* (3.11), *there exists a rotation surface $f(a_o)$ in* (3.11) *and a number θ, $0 \leqq \theta \leqq 2\pi$, such that $\bar{f} = f(a_o, \theta)$.*

4.5. PROOF. The proof of (4.4) is a straightfoward verification, and (i) follows from (4.4) and the uniqueness part of Lawson [5, Theorem 8] (it can also be easily proved here). Thus we are left with the proof of (ii).

Let the given surface $\bar{f}$ have parameters $\bar{a}$, $\bar{h}$, $\bar{m}$ and metric given by $\bar{U} = U(\bar{a}, \bar{h}, \bar{m})$. The second fundamental form of $\bar{f}$ is easily computed to be (cf. (3.4), (3.5) and (3.7))

$$e = H + \bar{a}(\bar{m}^2\bar{U}^2)^{-1} \,, \quad f = \bar{h}(\bar{m}^2\bar{U})^{-1} \,, \quad g = H\bar{U}^2 - (\bar{a}/\bar{m}^2) \,. \tag{4.6}$$

We want to find a rotation surface $f(a_0)$ with parameters a_o, $h = 0$, $m = 1$ and and metric given by $U = \bar{U}$ such that $\bar{f}$ is associated to $f(a_o)$.

Assume for a moment that $f(a_0)$ exists, and let $e_0, f_0 = 0, g_0$ be the coefficients of its second fundamental form. By changing the parameter s of $f(a_0)$ into $\sigma = \int_0^s U^{-1}ds$, we see that the parameters (σ, t) are isothermal. We can then apply Lawson's construction [5] to obtain the associated family to $f(a_0)$. It turns out that the coefficients of the second fundamental form of $f(a_0, \theta)$ in the parametrization (s, t) are:

$$e = \cos\theta(e_0 - H) + H\,, \quad f = 2^{-1}U\sin\theta((g_0/U^2) - e_0)\,,$$
$$g = \cos\theta(g_0 - HU^2) + HU^2\,.$$

By comparing these values with (4.6) and using the fact that $U = \bar{U}$, one obtains

$$e_0 = H + \bar{a}(U^2\bar{m}^2\cos\theta)^{-1}\,, \quad g_0 = HU^2 - \bar{a}(\bar{m}^2\cos\theta)^{-1}\,, \tag{4.7}$$

$$\bar{h} = -\bar{a}\sin\theta/\cos\theta\,. \tag{4.8}$$

From (4.7) and the fact that

$$e_0 = H + a_0/U^2\,, \quad f_0 = 0\,, \quad g_0 = HU^2 - a_0\,,$$

it follows that

$$a_0 = \bar{a}(\bar{m}^2\cos\theta)^{-1}\,. \tag{4.9}$$

We claim that θ and a_0 given by (4.8) and (4.9) have the property that $f(a_0, \theta) = \bar{f}$, and this will complete the proof of (ii).

To prove the claim, we use the fact that (see (3.9))

$$U = \bar{U} \Leftrightarrow \begin{cases} (1 + 2\bar{a}H)/\bar{m}^2 = 1 + 2a_0H\,, \\ (1 - 4H^2\bar{h}^2 + 4H\bar{a})^{1/2}/\bar{m}^2 = (1 + 4Ha_0)^{1/2}\,. \end{cases}$$

By using (4.8) and (4.9), one can see that both of the above equations are satisfied by

$$1/\bar{m}^2 = 1 + 2a_0H(1 - \cos\theta)\,. \tag{4.10}$$

Now if we introduce (4.8), (4.9) and (4.10) into (4.2), we conclude that $f(a_0, \theta)$ is given by the parameters

$$m = \bar{m}\,, \quad a = \bar{a}\,, \quad h = \bar{h}\,,$$

thereby proving our claim and Lemma 4.3.

4.11. It will be convenient to express the family given by (3.11) explicitly in terms of the parameters θ and a_0 defined by (4.2). To maintain a manageable notation, we will replace a_0 by a parameter B_0 defined as follows. By noticing that

$$B(a_0, \theta) = (1 - 4H^2h^2 + 4Ha)^{1/2} = (1 + 4a_0H)^{1/2}(1 + 2a_0H(1 - \cos\theta))^{-1}\,,$$

we see that

$$B(a_o, 0) = (1 + 4a_oH)^{1/2} .$$

We thus define $B_o = B(a_o, 0)$, hence $a_o = (B_o^2 - 1)/4H$. It follows by a straightforward computation that

$$(4.12) \qquad \begin{cases} B(B_o, \theta) = 2B_o/(2 + (B_o^2 - 1)(1 - \cos\theta)) , \\ h(B_o, \theta) = (B_o^2 - 1)\sin\theta/(2H(2 + (B_o^2 - 1)(1 - \cos\theta))) , \\ m^2(B_o, \theta) = 2/(2 + (B_o^2 - 1)(1 - \cos\theta)) . \end{cases}$$

4.13. Lemma 4.3 together with the uniqueness part of Lawson [5, Theorem 8] proves the first part of Theorem 1.2. To complete the proof, the only problem that remains is to consider the case where the natural parametrization may have a singularity, namely the case $B = 1$ (cf. Remarks 2.12 and 3.14). The rotation surface of the associated family, i.e., $\theta = 0$, is given, in this case, by $B_o = 1$. But then $h(1, \theta) = 0$, $B(1, \theta) = m^2(1, \theta) = 1$, for all θ. Thus $B_o = 1$ corresponds to the sphere, the associated family of which is the sphere itself. In this case the natural parametrization has actually a singularity at two points. This justifies the exception made in the statement of Theorem 1.2 and completes the proof.

REMARK 4.14. The expressions given in Theorem 1.2 generalize those obtained by Kenmotsu [4] for $h = 0$. It is easily checked that $B_o = 0$ gives $B(0, \theta) = 0$ for all θ, and $\rho = 1/2H$. This shows that associated family to the cylinder is the cylinder itself. The cylinder and the sphere are the only rotation surfaces of constant mean curvature with the property that the associated family is the surface itself.

REMARK 4.15. It follows from (3.11) that

$$(1 - b)/2H \leqq \rho(s) \leqq (1 + B)/2H .$$

Thus, except for the sphere ($B = 1$), the helicoidal surfaces are contained between two cylinders that depend on B_o and θ.

REFERENCES

[1] E. BOUR, Memoire sur le deformation de surfaces, Journal de l'École Polytechnique, XXXIX Cahier, 1862, 1-148.

[2] G. DARBOUX, Leçons sur la théorie des surfaces, Vol. I, Paris, 1914 (Reprinted by Chelsea Pub. Co., 1972).

[3] C. DELAUNAY, Sur la surface de revolution dont la courbure moyenne est constante, J. Math. Pures Appl. Series 1, 6 (1841), 309-320.

[4] K. KENMOTSU, Surfaces of revolution with prescribed mean curvature, Tôhoku Math. J. 32 (1980), 147-153.

[5] H. B. LAWSON JR., Complete minimal surfaces in S^3, Ann. of Math. 92 (1970), 335–374.
[6] W. WUNDERLICH, Beitrag zur Kentnis der Minimalscharaubflächen, Compositio Math. (1952), 297–311.

INSTITUTO DE MATEMÁTICA PURA E APLICADA (IMPA)
ESTRADA DONA CASTORINA, 110
22.460 RIO DE JANEIRO, RJ
BRASIL