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SURFACES OF CLASS VII_0 WITH CURVES

ICHIRO ENOKI

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0. Introduction. Let S denote a compact surface, i.e., a compact complex manifold of complex dimension 2. We write $b_i(S)$ for the i-th Betti number of S. For a divisor D on S, we write $(D)^2$ for its self-intersection number. A compact surface S is said to be *of Class* VII_0 if S is minimal and $b_1(S) = 1$.

Now let S be a surface of Class VII_0 with curves. Then S satisfies one of the following conditions:

(0.1) $\qquad S$ has a divisor $D \neq 0$ with $(D)^2 = 0$,

(0.2) $\qquad$ any divisor $D \neq 0$ on S has $(D)^2 < 0$.

Moreover, if $b_2(S) = 0$, Kodaira proved that S is either an elliptic surface or a Hopf surface. Note that $b_2(S) = 0$ implies (0.1). In this paper we shall complete the classification of surfaces of Class VII_0 which satisfy (0.1).

To state our result, we shall construct surfaces $S_{n,\alpha,t}$, $n > 0$, $0 < |\alpha| < 1$, $t \in \boldsymbol{C}^n$, with the following properties:

$$(0.3) \qquad \begin{cases} S_{n,\alpha,t} \text{ is a surface of Class } \mathrm{VII}_0 \text{ with } b_2 = n , \\ S_{n,\alpha,t} \text{ has a curve } D_{n,\alpha,t} \text{ with } (D_{n,\alpha,t})^2 = 0 , \end{cases}$$

(0.4) $\qquad S_{n,\alpha,t} - D_{n,\alpha,t}$ is an affine $\boldsymbol{C}$-bundle of degree $-n$ over an elliptic curve .

Clearly $S_{n,\alpha,t}$ satisfy (0.1) (cf. note (2) below). Our result is the following

MAIN THEOREM. *Let S be a surface of Class* VII_0 *with* $b_2(S) = n > 0$. *If S has a divisor $D \neq 0$ with $(D)^2 = 0$, then S is biholomorphic to $S_{n,\alpha,t}$ and $D = rD_{n,\alpha,t}$ for some $0 < |\alpha| < 1$, $t \in \boldsymbol{C}^n$ and $r \in \boldsymbol{Z}$.*

This Main theorem and some related results were announced in [2]. In subsequent papers, we shall study deformations of $S_{n,\alpha,t}$ (cf. [6]) and we shall give an application of the Main theorem to a study of compactifiable surfaces.

Here we recall some results on surfaces of Class VII_0.

(1) Class VII_0 was introduced by Kodaira. As for the significance of this class, we refer to his papers [8, I, IV]. He determined the structure of surface of Class VII_0 with $b_2 = 0$ which satisfy (0.1), as mentioned above.

(2) It was Inoue [4, 5] who first constructed examples of surfaces of Class VII_0 with $b_2 > 0$ which contain curves. In [4], he gave $S_{1,\alpha,0}$ as an example. $S_{1,\alpha,t}$ is contructed in [6]. We note that $S_{n,\alpha,0}$ is an n-fold unramified covering surface of $S_{1,\beta,0}$, $\alpha = \beta^n$, and $S_{n,\alpha,t}$ is a deformation of $S_{n,\alpha,0}$. In [5], he constructed examples satisfying (0.2).

(3) On the other hand, Kato discovered a series of surfaces of Class VII_0 with $b_2 > 0$ which contain global spherical shells and exactly b_2 rational curves (see [6; p. 74, Remark 4]). In this series, we find $S_{n,\alpha,t}$, Inoue's examples consrtucted in [5] and many other surfaces of Class VII_0 satisfying (0.2).

(4) We have divided surfaces of Class VII_0 with curves into two classes, those satisfying (0.1) and those satisfying (0.2). Our Main theorem, completing the classification of surfaces in the former, clarifies the difference between these two classes in the following way.

When a compact surface S satisfies (0.2), it is well known that for any curve C on S, any (complex analytic) compactification of $S - C$ is bimeromorphic to S (cf. Section 1). On the other hand, when a surface S of Class VII_0 satisfies (0.1), there exist a curve C on S and a compactification Σ of $S - C$ such that Σ is *not* bimeromorphic to S. Indeed we can take Σ to be a $\boldsymbol{P}^1$-bundle over an elliptic curve, where $\boldsymbol{P}^1$ is the complex projective line. When $b_2(S) = 0$, this fact is well known ([8, II; Sections 9–10]). When $b_2(S) > 0$, this fact is a direct consequence of (0.4) and our Main theorem.

The composition of this paper is as follows. In Sections 1–2, we shall collect together some known results. In Section 3, we shall construct the surfaces $S_{n,\alpha,t}$ and prove (0.3)–(0.4). Now let S and D be as in the Main theorem. Let C denote the support of D. In Section 4 we shall determine the structure of C and see there is a surjective holomorphic map ψ of $S - C$ onto an elliptic curve $\varDelta$. In Sections 5–6 we shall construct a compactification Σ of $S - C$ so that ψ extends to a holomorphic map Ψ of Σ onto $\varDelta$ and Ψ maps $\Gamma = \Sigma - (S - C)$ biholomorphically onto $\varDelta$. In Section 7 we shall prove that $\Psi: \Sigma \to \varDelta$ is a $\boldsymbol{P}^1$-bundle. In Sections 8–9, using Proposition 2.5 in Section 2, we shall complete the proof of our Main theorem.

The author would like to thank Dr. M. Inoue who kindly informed him of an alternative proof of Proposition 4.12 which is much simpler than the author's. Also the author would like to express appriciation to the referee for several suggestions that helped clarify the presentation.

1. Neighborhoods of curves. By a curve we shall mean a compact pure 1-dimensional analytic set. In this section, we collect together the results on neighborhoods of curves.

Let C be a curve on a surface and let $C = \sum_{i=1}^{n} \Theta_i$ denote the decomposition of the curve C into the irreducible components Θ_i of C ($\Theta_i \neq \Theta_j$ if $i \neq j$). We write $(\Theta_i \cdot \Theta_j)$ for the intersection number of Θ_i and Θ_j. The $n \times n$ matrix $[(\Theta_i \cdot \Theta_j)]$ of the intersection numbers is called the *intersection matrix* of the curve C. We quote a lemma from [13; p. 85, Lemma 2].

LEMMA 1.1. *Let $C = \sum_{i=1}^{n} \Theta_i$ be a curve on a surface. Assume that C is connected and the intersection matrix of $[(\Theta_i \cdot \Theta_j)]$ of C is negative semi-definite. Then we have*

(i) *if $(\sum_{i=1}^{n} m_i \Theta_i)^2 = 0$ for some integers m_i, then m_i are all positive, negative or zero simultaneously,*

(ii) $\operatorname{rank}[(\Theta_i \cdot \Theta_j)] \geqq n - 1$,

(iii) *if $\{j(1), \cdots, j(p)\} \subsetneqq \{1, \cdots, n\}$, then the intersection matrix of the curve $\bigcup_{k=1}^{p} \Theta_{j(k)}$ is negative definite.*

Let M be a surface. An open subset U of M is called *strongly pseudo-convex* if there exists a proper C^∞ map $\varphi: U \to [0, \infty)$ such that φ is strictly plurisubharmonic outside a compact subset of U. A curve C on M is called *exceptional* if there exists a normal analytic space M^* and a holomorphic map $\sigma: M \to M^*$ such that $\sigma(C)$ is a finite set of points on M^* and σ maps $M - C$ biholomorphically onto $M^* - \sigma(C)$. When C is an exceptional curve *of the first kind*, M^* is a manifold and M is a quadratic transform of M^* with respect to the point $\sigma(C)$. We recall the characterization of exceptional curves (cf. [3]).

PROPOSITION 1.2. *Let C be a curve on a surface M. Then the fol-*

lowing three conditions are mutually equivalent.

(a) *C is exceptional.*

(b) *The intersection matrix of C is negative definite.*

(c) *There exists a strongly pseudo-convex neighborhood of C in M.*

Let M be a non-compact surface. A compact surface S is called a *compactification* of M if M is an open submanifold of S and $S - M$ is a curve on S.

PROPOSITION 1.3. *Let S_1 and S_2 be minimal compactifications of the same surface M. If $C_i = S_i - M$ is a connected exceptional curve on S_i for each $i = 1, 2$, then S_1 is biholomorphic to S_2*

PROOF. Let $\sigma_i: S_i \to S_i^*$ be the holomorphic map of S_i onto the normal analytic space S_i^* so that $\sigma_i(C_i)$ is one point and σ_i maps $S_i - C_i$ biholomorphically onto $S_i^* - \sigma_i(C_i)$. Then the identity map $S_1 - C_1 \to S_2 - C_2$ extends to a biholomorphic map of S_1^* onto S_2^* (cf. [10; p. 118, Prop. 4]). Thus both S_1 and S_2 are the minimal desingularizations of the same space S_1^* and hence S_1 is biholomorphic to S_2. q.e.d.

Let C be a curve on a surface S. Assume that C is of normal crossing. Then, for each singular point p_i of C, we can choose a system of holomorphic coordinates (u_i, v_i) on a neighborhood U_i of p_i in S so that $C \cap U_i$ is defined by the equation: $u_i \cdot v_i = 0$ in U_i. Choose a Riemannian metric ds^2 on S such that $ds^2 = |du_i|^2 + |dv_i|^2$ on some neighborhood of p_i in U_i. Let $N_\varepsilon(C)$ denote the ε-neighborhood of C in S with respect to the distance determined by ds^2. From the arguments in [11; pp. 72–73], we infer

LEMMA 1.4. *Let C be a curve of normal crossing on a surface. For sufficiently small $\varepsilon > 0$, we have*

(i) *$N_\varepsilon(C)$ is homotopically equivalent to C,*

(ii) *$N_\varepsilon(C) - C$ is homotopically equivalent to the boundary $\partial N_\varepsilon(C)$ of $N_\varepsilon(C)$ in S,*

(iii) *$\partial N_\varepsilon(C)$ is a compact orientable topological manifold of real dimension 3.*

Moreover, if C is of simple normal crossing, then

(iv) *the topological structure of $\partial N_\varepsilon(C)$ is determined only by the intersection matrix and the topological structure of C.*

We call $N_\varepsilon(C)$ a *tubular neighborhood* of C ($\varepsilon > 0$ is sufficiently small). The proof of the following lemma is found in [13; pp. 83–84].

LEMMA 1.5. *Let S_i, $i = 1, 2$, be compactifications of the same surface*

M. Assume that $C_i = S_i - M$ *is connected and of normal crossing for each* $i = 1, 2$. *Let* N_i *be the tubular neighborhood of* C_i *in* S_i *given by Lemma* 1.4. *Then* ∂N_1 *is homotopically equivalent to* ∂N_2.

2. **Affine C-bundles over elliptic curves.** Let Δ be an elliptic curve. We write Δ as the quotient group $\Delta = \boldsymbol{C}^*/\langle\alpha\rangle$ of $\boldsymbol{C}^*$ by the multiplicative group $\langle\alpha\rangle$ generated by $\alpha \in \boldsymbol{C}^*$, $0 < |\alpha| < 1$.

Let $n \in \boldsymbol{N}$, $n \geqq 1$, and $t \in \boldsymbol{C}^n$. We identify $t = (t_0, \cdots, t_{n-1})$ with the polynomial $t(w) = \sum_{k=0}^{n-1} t_k w^k$. Define a holomorphic automorphism $g_{n,\alpha,t}$ of $\boldsymbol{C} \times \boldsymbol{C}^*$ by

$$g_{n,\alpha,t} : (z, w) \mapsto (w^n z + t(w), \alpha w) . \tag{2.1}$$

We write $A_{n,\alpha,t}$ for the quotient surface $\boldsymbol{C} \times \boldsymbol{C}^*/\langle g_{n,\alpha,t}\rangle$ of $\boldsymbol{C} \times \boldsymbol{C}^*$ by $g_{n,\alpha,t}$. Then $A_{n,\alpha,t}$ is an affine $\boldsymbol{C}$-bundle over Δ with the projection induced by $(z, w) \mapsto w$.

In general the *degree* of an affine $\boldsymbol{C}$-bundle over a curve is defined to be the degree of its linearization, e.g., $A_{n,\alpha,0}$ is the linearization of $A_{n,\alpha,t}$ and its degree is $-n$. We know

THEOREM 2.2. *Let A be an affine $\boldsymbol{C}$-bundle of degree $-n$ over $\Delta = \boldsymbol{C}^*/\langle\alpha\rangle$. Then A is equivalent to $A_{n,\alpha,t}$ as an affine $\boldsymbol{C}$-bundle for some $t \in \boldsymbol{C}^n$.*

For the proof of our Main theorem, we need a little more.

LEMMA 2.3. *Let $d(w)$ and $e(w)$ be holomorphic functions on $\boldsymbol{C}^*$ satisfying*

$$e(w) = \kappa w^n d(w) - d(\alpha w) \quad \text{for} \quad w \in \boldsymbol{C}^* \tag{2.4}$$

with $\kappa \in \boldsymbol{C}^*$, $n \geqq 1$, $0 < |\alpha| < 1$.

(i) *If $e(w)$ is holomorphic on $\boldsymbol{C}$, then $d(w)$ extends holomorphically to the whole $\boldsymbol{C}$.*

(ii) *If $e(w)$ is a polynomial of degree $< n$, then $d(w)$ and $e(w)$ vanish identically.*

The proof of the above lemma is elementary and hence we omit it. (Expand $d(w)$ and $e(w)$ into the power series in w and compare the coefficients of w^k in (2.4).)

PROPOSITION 2.5. *Let g be a holomorphic automorphism of $\boldsymbol{C} \times \boldsymbol{C}^*$ of the form*

$$g : (z, w) \mapsto (a(w)z + b(w), \alpha w) ,$$

where $a(w)$ and $b(w)$ are holomorphic functions on $\boldsymbol{C}^*$, and $\alpha \in \boldsymbol{C}^*$, $0 < |\alpha| < 1$. Assume that the quotient surface $A = \boldsymbol{C} \times \boldsymbol{C}^*/\langle g\rangle$ is an affine

C-bundle of degree $-n < 0$ over $\Delta = C^*/\langle\alpha\rangle$. Then there exists a holomorphic automorphism h of $C \times C^*$ of the form

$$h\colon (z, w) \mapsto (c(w)z + d(w), \beta w) \quad (\beta \in C^*) \tag{2.6}$$

satisfying

$$h \circ g \circ h^{-1}(z, w) = (w^n z + t(w), \alpha w) \tag{2.7}$$

for some polynomial $t(w)$ of degree $< n$. Moreover, if $a(w)$ and $b(w)$ are both holomorphic on C, then we can assume that h is a holomorphic automorphism of $C \times C$.

PROOF. By hypothesis, we can write $a(w)$ as $a(w) = w^n \exp u(w)$ where $u(w)$ is a holomorphic function on C^*. Expanding $u(w)$ into the Laurent power series $u(w) = \sum_{k \in Z} u_k w^k$, $u_k \in C$, in w, we define a holomorphic function $c(w)$ on C^* by $c(w) = \exp \sum_{k \neq 0} \{u_k w^k/(1 - \alpha^k)\}$. Then $c(w)$ is nowhere zero and satisfies

$$a(w)c(\alpha w)/c(w) = \kappa w^n\ , \quad \kappa = \exp u_0\ . \tag{2.8}$$

Let L denote the linearization of A. Then, for each $k \in Z$, the monomial w^k defines an element γ_k of $H^1(\Delta, \mathcal{O}(L))$. By Lemma 2.3 (ii) and the Riemann-Roch theorem, $\{\gamma_k\}_{k=0}^{n-1}$ forms a basis of $H^1(\Delta, \mathcal{O}(L))$. Thus we can write the element $\sigma \in H^1(\Delta, \mathcal{O}(L))$ determined by $c(\alpha w)b(w)$ as $\sigma = \sum_{k=0}^{n-1} s_k \gamma_k$ for some $s_k \in C$. This is equivalent to the existence of a holomorphic function $d(w)$ on C^* such that

$$\sum_{k=0}^{n-1} s_k w^k = -\kappa w^n d(w) + c(\alpha w)b(w) + d(\alpha w)\ . \tag{2.9}$$

Take $\beta \in C^*$ such that $\kappa = \beta^n$ and define h by (2.6). Then, by (2.8)-(2.9), we have (2.7) with $t(w) = \sum s_k \beta^{-k} w^k$. Now suppose that $a(w)$ and $b(w)$ are both holomorphic on C. Then $u(w)$ is holomorphic on C. Thus $c(w)$ extends to the whole C holomorphically so that $c(0) \neq 0$. By (2.9), we can apply Lemma 2.3 (i) to see that $d(w)$ is holomorphic on C. Thus h is a holomorphic automorphism of $C \times C$. q.e.d.

3. **Surfaces $S_{n,\alpha,t}$.** Let $n \geqq 1$, $0 < |\alpha| < 1$ and $t \in C^n$ ($n \in N$, $\alpha \in C$). We identify $t = (t_0, \cdots, t_{n-1})$ with the polynomial $t(w) = \sum_{k=0}^{n-1} t_k w^k$. We shall generalize the construction of $S_{1,\alpha,0}$ in [6; p. 57].

Let P^1 denote the complex projective line with the inhomogeneous coordinate z. Set $W_0 = P^1 \times C$, $\Gamma_\infty = \{\infty\} \times C$ and $C_0 = P^1 \times \{0\}$. Define a birational automorphism $g_{n,\alpha,t}$ of W_0 by

$$g_{n,\alpha,t}\colon (z, w) \mapsto (w^n z + t(w), \alpha w)\ . \tag{3.1}$$

By induction on k, we define blowings-up W_k, $k \geqq 0$, of W_0, curves $C_{\pm k}$ on W_k and points $p_k \in C_k$, $p_{-k-1} \in C_{-k}$ so that

(i)$_k$ $g_{n,\alpha,t}$ (resp. $g^{-1}_{n,\alpha,t}$) induces a birational automorphism of W_k, whose indeterminacy set consists of one point p_k (resp. p_{-k-1}),

(ii)$_k$ W_{k+1} is the blowing-up of W_k at p_k and p_{-k-1}; C_{k+1} and C_{-k-2} are total transforms of p_k and p_{-k-1} respectively.

In fact, we have (i)$_0$ with $p_0 = (\infty, 0)$, $p_{-1} = (t_0, 0)$. For $k \geqq 1$, (i)$_k$ follows from (i)$_j$ and (ii)$_j$, $j < k$.

In what follows, we denote each proper transform by the same symbol. Then we have $\{p_k\} = \Gamma_\infty \cap C_k$ and $p_{-k} \neq p_{-k-1}$ for $k \geqq 0$. Identifying $W_{k-1} - \Gamma_\infty - \{p_{-k}\}$ with the open submanifold of $W_k - \Gamma_\infty - \{p_{-k-1}\}$ canonically, we define a noncompact surface $\widetilde{S}_{n,\alpha,t}$ to be the inductive limit of $W_k - \Gamma_\infty - \{p_{-k-1}\}$: $\widetilde{S}_{n,\alpha,t} = \text{ind}\lim_k(W_k - \Gamma_\infty - \{p_{-k-1}\})$. Then we have infinitely many non-singular rational curves C_j, $j \in \boldsymbol{Z}$, with $(C_j)^2 = -2$ on $\widetilde{S}_{n,\alpha,t}$ so that

(3.2) C_j and C_{j+1} intersect transversally at p_j, C_j and C_k do not meet when $j \neq k \pm 1$.

$g_{n,\alpha,t}$ induces a holomorphic automorphism $\widetilde{g}_{n,\alpha,t}$ of $\widetilde{S}_{n,\alpha,t}$ such that

$$\widetilde{g}_{n,\alpha,t}(C_j) = C_{j-n} \quad \text{for} \quad j \in \boldsymbol{Z}\,. \tag{3.3}$$

By (3.1) and (3.3), $\widetilde{g}_{n,\alpha,t}$ generates a properly discontinuous group $\langle\widetilde{g}_{n,\alpha,t}\rangle$ of holomorphic automorphisms of $\widetilde{S}_{n,\alpha,t}$ free from fixed points. We define the surface $S_{n,\alpha,t}$ to be the quotient surface of $\widetilde{S}_{n,\alpha,t}$ by $\langle\widetilde{g}_{n,\alpha,t}\rangle$: $S_{n,\alpha,t} = \widetilde{S}_{n,\alpha,t}/\langle\widetilde{g}_{n,\alpha,t}\rangle$. Writing λ for the canonical projection of $\widetilde{S}_{n,\alpha,t}$ onto $S_{n,\alpha,t}$, set $D_{n,\alpha,t} = \bigcup_{i=0}^{n-1} \Theta_i$ with $\Theta_i = \lambda(C_i)$.

PROPOSITION 3.4. (i) *$D_{1,\alpha,t} = \Theta_0$ is a rational curve with one ordinary double point satisfying $(\Theta_0)^2 = 0$.*

(ii) *$D_{2,\alpha,t} = \Theta_0 \cup \Theta_1$; each Θ_i, $i = 0, 1$, is a non-singular rational curve with $(\Theta_i)^2 = -2$. Θ_0 and Θ_1 intersect transversally at two points.*

(iii) *$D_{n,\alpha,t} = \bigcup_{i=0}^{n-1} \Theta_i$ ($n \geqq 3$); each Θ_i is a non-singular rational curve with*

$$(\Theta_i \cdot \Theta_j) = \begin{cases} -2 & \textit{if} \quad i = j\,, \\ \ \ 1 & \textit{if} \quad i \equiv j \pm 1 \mod n\,, \\ \ \ 0 & \textit{otherwise}\,. \end{cases}$$

PROOF. It follows from (3.2) and (3.3). q.e.d.

PROPOSITION 3.5. (i) *$S_{n,\alpha,t}$ is a surface of Class VII_0 with $b_2(S_{n,\alpha,t}) = n$.*

(ii) *$(D_{n,\alpha,t})^2 = 0$.*

(iii) $S_{n,\alpha,t} - D_{n,\alpha,t}$ *is an affine* **C**-*bundle of degree* $-n$ *over an elliptic curve.*

PROOF (cf. [4]). (iii) Comparing (3.1) with (2.1), we see $S_{n,\alpha,t} - D_{n,\alpha,t} = A_{n,\alpha,t}$.

(ii) Proposition 3.4 implies this.

(i) First we shall show $b_1(S_{n,\alpha,t}) = 1$. By definition, W_k is simply connected. Using van Kampen's theorem, we see that $W_k - \Gamma_\infty - \{p_{-k-1}\}$ is simply connected. Hence their inductive limit $\tilde{S}_{n,\alpha,t}$ is also simply connected. Thus the fundamental group of the quotient space $S_{n,\alpha,t}$ of $\tilde{S}_{n,\alpha,t}$ is $\langle \tilde{g}_{n,\alpha,t} \rangle$ and hence infinite cyclic. In particular $b_1(S_{n,\alpha,t}) = 1$.

Next we shall show that $S_{n,\alpha,t}$ is compact. The coordinate w on W_0 induces a holomorphic function on $\tilde{S}_{n,\alpha,t}$, which will be denoted by the same symbol w, so that

$$\text{(3.6)} \qquad \begin{cases} \text{the divisor } (w) \text{ of } w \text{ is } \sum_{j \in \mathbf{Z}} C_j \,, \\ \tilde{g}^*_{n,\alpha,t} w = \alpha w \,. \end{cases}$$

Take a compact tubular neighborhood N_i of C_i for $0 \leqq i \leqq n-1$ and set

$$B = \bigcup_{\nu \leqq 0} \bigcup_{i=0}^{n-1} \{ y \in g^{\nu}_{n,\alpha,t}(N_i) \,|\, |w(y)| \leqq \varepsilon \} \,,$$
$$\Omega = \{ (z, w) \in W_0 - \Gamma_\infty - \{p_{-1}\} \,|\, |\alpha| \varepsilon \leqq |w| \leqq \varepsilon, |z| \leqq 1 \} \,.$$

Then we infer from (3.6) that $\lambda(\bigcup_{i=0}^{n-1} N_i)$ contains $\lambda(B)$ provided that $\varepsilon > 0$ is sufficiently small. Hence $\lambda(B)$ is a compact neighborhood of $D_{n,\alpha,t}$. Clearly $\lambda(\Omega)$ is a compact subset of $S_{n,\alpha,t} - D_{n,\alpha,t}$ and $\lambda(\Omega \cup B - \bigcup_j C_j) = S_{n,\alpha,t} - D_{n,\alpha,t}$. Therefore $S_{n,\alpha,t} = \lambda(B) \cup \lambda(\Omega)$. Thus $S_{n,\alpha,t}$ is compact.

By (3.6), we may assume that transition functions of the line bundle $[D_{n,\alpha,t}]$ determined by $D_{n,\alpha,t}$ are all constants. Hence the real first Chern class of $[D_{n,\alpha,t}]$ is zero. This implies that any irreducible curve on $S_{n,\alpha,t}$ is contained in either $D_{n,\alpha,t}$ or $S_{n,\alpha,t} - D_{n,\alpha,t}$, none of which contains exceptional curves of the first kind on $S_{n,\alpha,t}$. Thus $S_{n,\alpha,t}$ is of Class VII_0.

Finally we show $b_2(S_{n,\alpha,t}) = n$. Let $\chi(X)$ denote the Euler number of a topological space X. Note that $D_{n,\alpha,t}$ is of normal crossing and $\chi(D_{n,\alpha,t}) = n$ by Proposition 3.4. Let N be the tubular neighborhood of $D_{n,\alpha,t}$ given by Lemma 1.4. Then we have

$$\text{(3.7)} \qquad \chi(N) = n \,, \qquad \chi(N - D_{n,\alpha,t}) = 0 \,.$$

Since $S_{n,\alpha,t} - D_{n,\alpha,t}$ is an affine $\mathbf{C}$-bundle over an elliptic curve Δ, we have

$$\text{(3.8)} \qquad \chi(S_{n,\alpha,t} - D_{n,\alpha,t}) = \chi(\mathbf{C}) \cdot \chi(\Delta) = 0 \,.$$

Combining (3.7) and (3.8) with the Mayer–Vietoris exact sequence of the pair $(S_{n,\alpha,t}-D_{n,\alpha,t}, N)$, we obtain $\chi(S_{n,\alpha,t})=n$. Since $b_1(S_{n,\alpha,t})=1$, this implies $b_2(S_{n,\alpha,t})=n$.

REMARK. By Theorem 2.2, each affine $\boldsymbol{C}$-bundle of degree $-n<0$ over the elliptic curve $\varDelta=\boldsymbol{C}^*/\langle\alpha\rangle$ can be compactified into $S_{n,\alpha,t}$ for some $t\in\boldsymbol{C}^n$.

4. Surfaces of Class VII_0. Throughout this section, we let S denote a compact surface with $b_1(S)=1$ which has no meromorphic functions except constants. Let K denote the canonical bundle of S.

Since $b_1(S)=1$, it follows from Theorem 3 in [8, I; p. 755] that $q=\dim H^1(S,\mathcal{O})=1$. By Theorems 21 and 22 in [8, I; p. 789, p. 796], we have $p_g=\dim H^0(S,\mathcal{O})=0$ (see [8, I; p. 766, iii)]). Thus

$$\sum_{\nu=0}^{2}(-1)^\nu\dim H^\nu(S,\mathcal{O})=0\ . \tag{4.1}$$

Under the canonical identification: $H^4(S,\boldsymbol{R})=\boldsymbol{R}$, the cup product defines a non-degenerate symmetric bilinear form, $(\lambda\cdot\xi)$ for $\lambda,\xi\in H^2(S,\boldsymbol{R})$, on $H^2(S,\boldsymbol{R})$. Let b^+ denote the number of positive eigenvalues of this bilinear form $(\lambda\cdot\xi)$. Since $p_g=0$, it follows from Theorem 3 in [8, I; p. 755]

$$b^+=0\ . \tag{4.2}$$

For a line bundle $\varLambda$ over S, let $(\varLambda)\in H^2(S,\boldsymbol{R})$ denote the real first Chern class of $\varLambda$. For a divisor $\varXi$ on S, let $[\varXi]$ denote the line bundle over S determined by $\varXi$. We write $(\varXi)$ for $([\varXi])$. Then, the intersection number $(\varLambda\cdot\varXi)$ of $\varLambda$ and $\varXi$ is given by $((\varLambda)\cdot(\varXi))$. We write $(\varLambda)^2$ for $((\varLambda)\cdot(\varLambda))$. Then, by Noether's formula, (4.1) means

$$b_2(S)=-(K)^2\ . \tag{4.3}$$

LEMMA 4.4. *Let S' be a finite unramified covering surface of S. If S is minimal, then S' is a surface of Class VII_0 with no non-constant meromorphic function.*

PROOF. Suppose first that S' contains an exceptional curve E of the first kind. Let p denote the projection of S' onto S and let K' denote the canonical bundle of S'. Set $\Theta=p(E)$. We write $\{E\}$ and $\{\Theta\}$ for the homology class of E in S' and the homology class of Θ in S, respectively. Since (E) (resp. (Θ)) is the Poincaré dual of $\{E\}$ (resp. $\{\Theta\}$), we have

$$(K'\cdot E)=\langle(K'),\{E\}\rangle\ ,\qquad (K\cdot\Theta)=\langle(K),\{\Theta\}\rangle \tag{4.5}$$

where $\langle\ ,\ \rangle$ denotes the pairing of the cohomology and the homology.

Since E and Θ are irreducible, we have $p_*\{E\} = d\{\Theta\}$ for some integer d. Hence, using (4.5) and $(K') = p^*(K)$, we have $d(K\cdot\Theta) = (K'\cdot E) = -1$. Since the holomorphic map p preserves the orientation, d is non-negative. Therefore $(K\cdot\Theta) < 0$. In particular, $(\Theta) \neq 0$. Hence $(\Theta)^2 < 0$ by (4.2). Thus Θ is an exceptional curve of the first kind on S. This contradicts the minimality of S.

Now we know that S' contains no exceptional curves of the first kind. From our assumption on S, it follows that S' has no meromorphic functions except constants. Thus, by Theorem 11 in [8, I; p. 759], S' is either a $K3$ surface, a complex tours or a surface of Class VII_0. On the other hand, since $b_1(S) = 1$, the fundamental group $\pi_1(S)$ of S contains an infinite cyclic group. Therefore S' is not simply connected and hence S' is not a $K3$ surface. Since $b_1(S)$ is odd, S and hence S' are not Kählerian. In particular, S' is not a complex torus. Thus we conclude that S' is a surface of Class VII_0 with no non-constant meromorphic functions. q.e.d.

The following lemma is due to Inoue.

LEMMA 4.6. *If S contains a non-singular elliptic curve E, then there is a non-trivial line bundle F over S such that $(F) = 0$ and the restriction of F to E is trivial.*

PROOF. In the exact sequence

$$0 \to H^1(S, \boldsymbol{Z}) \to H^1(S, \boldsymbol{C}) \to H^1(S, \boldsymbol{C}^*)\ ,$$

all cohomology groups are (complex) Lie groups and the maps are homomorphisms of Lie groups. By $b_1(S) = 1$, $H^1(S, \boldsymbol{Z}) \cong \boldsymbol{Z}$ and $H^1(S, \boldsymbol{C}) \cong \boldsymbol{C}$. Thus $H^1(S, \boldsymbol{C}^*)$ contains $\boldsymbol{C}^* \cong \boldsymbol{C}/\boldsymbol{Z}$ as a Lie subgroup. On the other hand, $H^1(E, \boldsymbol{C}^*)$ is the Picard variety of E, which is isomorphic to E as a Lie group. Therefore, since the restriction map

$$r\colon H^1(S, \boldsymbol{C}^*) \to H^1(E, \boldsymbol{C}^*)$$

is also a homomorphism of Lie groups, there is a non-zero element f of $H^1(S, \boldsymbol{C}^*)$ such that $r(f) = 0$. Then the line bundle F over S corresponding to f is the desired one. q.e.d.

In the following, we assume that S has a divisor $D \neq 0$ with $(D)^2 = 0$. Let C denote the support of D. Applying Lemma 1.1 (i) to each connected component of C, we may assume that D is a positive divisor.

A multi-valued holomorphic function w on S is said to be a *multiplicative* holomorphic function on S if the analytic continuation along any closed (continuous) path γ transforms $w(x)$ into $\alpha(\gamma)w(x)$, where $\alpha(\gamma)$

is a constant depending on γ (cf. [8, II; p. 701]). We call $\alpha(\gamma)$ the *multiplier* of w (with respect to γ).

LEMMA 4.7. *There exists a multiplicative holomorpic function $w = w(x)$ on S whose divisor (w) is D.*

PROOF. We have the following commutative diagram:

$$\begin{array}{ccccccc} H^1(S, \boldsymbol{C}) & \longrightarrow & H^1(S, \boldsymbol{C}^*) & \longrightarrow & H^2(S, \boldsymbol{Z}) & \longrightarrow & H^2(S, \boldsymbol{C}) \\ \downarrow & & \downarrow & & \| & & \\ H^1(S, \mathcal{O}) & \longrightarrow & H^1(S, \mathcal{O}^*) & \longrightarrow & H^2(S, \boldsymbol{Z}) & & \end{array}$$

where the map $H^1(S, \boldsymbol{C}) \to H^1(S, \mathcal{O})$ is surjective by the formula (14) in [8, I; p. 756]. We have $(D) = 0$ by (4.2). Thus, using the above diagram, we see that the isomorphism class of the line bundle $[D]$ is in the image of the map $H^1(S, \boldsymbol{C}^*) \to H^1(S, \mathcal{O}^*)$. The rest of the proof is the same as that of Lemma 11 in [8, II; p. 701]. q.e.d.

Let $\pi_1(S)$ denote the fundamental group of S. For any closed path γ, the multiplier $\alpha(\gamma)$ of w depends only on the (free) homotopy class of γ. Thus the map $\gamma \mapsto \alpha(\gamma)$ induces a homomorphism $\mu(w)\colon \pi_1(S) \to \boldsymbol{C}^*$. Moreover, since $\boldsymbol{C}^*$ is abelian, $\mu(w)$ induces a homomorphism $H_1(S, \boldsymbol{Z}) \to \boldsymbol{C}^*$. We denote it by the same symbol $\mu(w)$.

The free part F of $H_1(S, \boldsymbol{Z})$ is infinite cyclic. Let σ be a generator of F. For any torsion cycle τ, $\mu(w)(\tau)$ is a root of unity. We have

$$|\mu(w)(\sigma)| \neq 1 \,. \tag{4.8}$$

In fact, if $|\mu(w)(\sigma)| = 1$, then $|w(x)|$ would be a single-valued continuous function on S and attains its maximum. This contradicts the fact that $w(x)$ is a non-constant holomorphic function. Due to Lemma 1.4 the proof of the following lemma is identical to that of Lemma 12 in [8, II; p. 702].

LEMMA 4.9. *Assume that C is of normal crossing. Then each connected component of C contains a closed path which represents a homotopy class of infinite order on S.*

Let $C = \sum_{i=0}^{m-1} \Theta_i$ denote the decomposition of C into the irreducible components Θ_i of C ($\Theta_i \neq \Theta_j$ if $i \neq j$). Note that the intersection matrix of C is negative semi-definite by (4.2).

PROPOSITION 4.10. *Assume that S is minimal and C is connected. Then C satisfies one of the following conditions I_b, $b \geqq 0$.*

I_0: *$C = \Theta_0$ is a non-singular elliptic curve.*

I_1: *$C = \Theta_0$ is a rational curve with one ordinary double point.*

I_2: *$C = \Theta_0 + \Theta_1$, where each Θ_i is a non-singular rational curve with*

$(\Theta_i)^2 = -2$. *Θ_0 and Θ_1 intersect transversally at two points.*

I_b $(b \geqq 3)$: *$C = \sum_{i=0}^{b-1} \Theta_i$, where each Θ_i is a non-singular rational curve with*

$$(\Theta_i \cdot \Theta_j) = \begin{cases} -2 & \text{if } i = j\,, \\ 1 & \text{if } i \equiv j \pm 1 \mod b\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

In particular, C is of nomal crossing and $(C)^2 = 0$.

PROOF. We blow up S, $\sigma: S^* \to S$, properly so that $C^* = \sigma^{-1}(C)$ is of normal crossing. Let D^* denote the total transform of D. Then we have $(D^*)^2 = 0$. Hence by Lemma 4.9 the support C^* of D^* is not simply connected.

We write $D = \sum_i m_i \Theta_i (m_i > 0)$. Since $(D) = 0$, we have $\sum_i m_i (K \cdot \Theta_i) = 0$. It follows that

$$(4.11) \qquad \sum_i m_i \{2\pi(\Theta_i) - 2 - (\Theta_i)^2\} = 0$$

where $\pi(\Theta_i)$ denotes the virtual genus of Θ_i. Now we can adapt the arguments in [7; pp. 567–568] as follows.

(A) *The case in which $C = \Theta_0$ is irreducible.* In this case, $(\Theta_0) = 0$ by (4.2). Hence $\pi(C) = 1$ by (4.11). When C is non-singular, it follows that C is an elliptic curve. When C has singular points, it follows that C is either a rational curve with one cusp or a rational curve with one ordinary double point. If C had a cusp, then C^* would be simply connected. Thus C satisfies I_0 or I_1.

(B) *The case in which C consists of at least two irreducible components.* Since $C = \bigcup \Theta_i$ is connected, we have $(\Theta_i)^2 < 0$ by Lemma 1.1 (iii), while by hypothesis Θ_i is not an exceptional curve of the first kind. Therefore, if $\pi(\Theta_i) = 0$, then $(\Theta_i)^2 \leqq -2$. Thus we conclude by (4.11) that each Θ_i is a non-singular rational curve with $(\Theta_i)^2 = -2$.

(B1) Suppose there is a pair Θ_0, Θ_1 with $(\Theta_0 \cdot \Theta_1) \geqq 2$. Then by (4.2) we have $0 \geqq (\Theta_0 + \Theta_1)^2 = 2(\Theta_0 \cdot \Theta_1) - 4$. Therefore, $(\Theta_0 \cdot \Theta_1) = 2$ and $(\Theta_0 + \Theta_1)^2 = 0$. Hence, by Lemma 1.1 (i), we have $C = \Theta_0 + \Theta_1$. Since $(\Theta_0 \cdot \Theta_1) = 2$, $\Theta_0 \cap \Theta_1$ consists of at most two points. If it consisted of one point, then C^* would be simply connected. Thus $\Theta_0 \cap \Theta_1$ consists of two points and hence C satisfies I_2.

(B2) Now we assume $(\Theta_i \cdot \Theta_j) \leqq 1$ for $i \neq j$.

($\mathrm{B2}_1$) Suppose there exist at least three irreducible components, say Θ_0, Θ_1 and Θ_2, which meet at one point. Then $(\Theta_0 + \Theta_1 + \Theta_2)^2 = 0$. Hence $C = \Theta_0 + \Theta_1 + \Theta_2$ by Lemma 1.1 (i). In this case, C^* would be simply connected.

($\mathrm{B2}_2$) Assume that $\Theta_i \cap \Theta_j \cap \Theta_k$ is empty for $i \neq j$, $j \neq k$, $i \neq k$. Then C itself is of normal crossing. Hence C is not simply connected, while each Θ_i is simply connected. Thus there exist irreducible components, say $\Theta_0, \Theta_1, \cdots, \Theta_{b-1}$ ($b \geqq 3$), such that $(\Theta_i \cdot \Theta_j) = 1$ if $i \equiv j \pm 1 \bmod b$. Then by (4.2) we have $(\sum_{i=0}^{b-1} \Theta_i)^2 = 0$, and $(\Theta_i \cdot \Theta_j) = 0$ unless $i = j$ or $i \equiv j \pm 1 \bmod b$. Thus $C = \sum_{i=0}^{b-1} \Theta_i$ by Lemma 1.1 (i) and hence C satisfies I_b. q.e.d.

The following proposition gives a characterization of Hopf surfaces among surfaces of Class VII_0 satisfying (0.1).

PROPOSITION 4.12. *Assume S is minimal. If C is disconnected or non-singular, then S is a Hopf surface.*

PROOF. We write the tensor product of line bundles in the additive form, e.g., $K + D = K \otimes [D]$. For any line bundle F over S, let F_C denote the restriction of F to C. We first prove

LEMMA 4.13. *Let F be a line bundle over S with $(F) = 0$. If $H^0(S, \mathcal{O}(K + F + rC)) \neq 0$ for some integer r, then S is a Hopf surface.*

PROOF OF LEMMA 4.13. Due to Theorem 34 in [8, II; p. 699], it suffices to show $b_2(S) = 0$. By hypothesis, there is a meromorphic section φ of $K + F$ over S whose polar cycle is contained in C. Thus $K + F$ is determined by the divisor of φ:

$$K + F = [\sum_i r_i \Xi_i - \sum_j s_j \Theta_j], \quad r_i > 0, \quad s_j \geqq 0 \tag{4.14}$$

where Ξ_i, Θ_j are irreducible curves and $C = \bigcup_j \Theta_j$.

We claim $(K \cdot \Xi_i) \geqq 0$, $(K \cdot \Theta_j) = 0$. In fact, Proposition 4.10 implies $(K \cdot \Theta_j) = 0$. Suppose $(K \cdot \Xi_i) < 0$. Then it follows that $(\Xi_i) \neq 0$ and hence $(\Xi_i)^2 < 0$ by (4.2). Thus Ξ_i is an exceptional curve of the first kind. This contradicts the minimality of S. Also $(K \cdot F) = 0$, by $(F) = 0$.

Then, using (4.14), we see $(K)^2 = \sum_i r_i (K \cdot \Xi_i) - \sum_j s_j (K \cdot \Theta_j) \geqq 0$. Combining this with (4.3), we obtain $b_2(S) = 0$ as desired.

Now we return to the proof of Proposition 4.12.

(A) *The case in which C is disconnected* (cf. [8, II; p. 702]). Take two connected components C_1 and C_2 of C. Then we have $(C_i)^2 = 0$. In view of Lemma 4.13, it suffices to show $H^0(S, \mathcal{O}(K + \sum_i C_i)) \neq 0$. By Lemma 4.7 each curve C_i determines a multiplicative holomorphic function w_i on S whose divisor (w_i) is C_i. We note that $w_i^{-1} dw_i$ is a meromorphic 1-form on S. Since C_1 and C_2 do not meet, w_1 is nowhere zero on C_2. By Lemma 4.9 and (4.8), C_2 contains a closed path γ such that $|\mu(w_1)(\gamma)| \neq 1$. It follows that w_1 is not constant on C_2, while w_1 is constant on any rational curve. Thus we conclude by Proposition 4.10 that C_2 is a non-singular elliptic curve and the restriction of $w_1^{-1}dw_1$ to C_2 is a non-trivial holomorphic 1-form. Similarly, the same is true for $w_2^{-1}dw_2$ and C_1. Therefore the meromorphic 2-form $w_1^{-1}dw \wedge w_2^{-1}dw_2$ defines a non-zero element of $H^0(S, \mathcal{O}(K + \sum_i C_i))$.

(B) *The case in which C is connected and non-singular.* In this case C is an elliptic curve by Proposition 4.10. Let F be a non-trivial line bundle on S given by Lemma 4.6 so that $(F) = 0$ and F_C is trivial. Then, since C is a non-singular elliptic curve, $[K + F + C]_C$ is trivial. Therefore we have the exact sequence

$$(4.15) \qquad 0 \to \mathcal{O}(K + F) \to \mathcal{O}(K + F + C) \to \mathcal{O}_C \to 0 .$$

By (4.2), $(C)^2 = 0$ implies $(C) = 0$. Also $(F) = 0$. Then by the Riemann-Roch theorem and (4.1):

$$(4.16) \qquad \begin{cases} \sum_{\nu=0}^{2} (-1)^\nu \dim H^\nu(S, \mathcal{O}(K + F + C)) = 0 \\ \sum_{\nu=0}^{2} (-1)^\nu \dim H^\nu(S, \mathcal{O}(K + F)) = 0 . \end{cases}$$

With the aid of (4.16) and the duality theorem, we infer from the exact cohomology sequence derived from (4.15) that either $H^0(S, \mathcal{O}(K + F + C)) \neq 0$ or $H^0(S, \mathcal{O}(-F)) \neq 0$. If $H^0(S, \mathcal{O}(K + F + C)) \neq 0$, then S is a Hopf surface by Lemma 4.13. Suppose therefore there is a non-identically-zero section φ of $-F$ over S. We write D' for the divisor (φ) of φ. Then $(D') = -(F) = 0$. Let C' denote the support of D'. Note that, since F is not trivial, C' is not empty.

(B1) Suppose $C \neq C'$. Then $C \cup C'$ is a disconnected curve with the self-intersection number zero. This is Case A.

(B2) Suppose $C = C'$. Then $D = rC$ $(r \geqq 1)$ and hence $[K + (r + 1)C]_C = [K + F + C]_C$ is trivial. Therefore we have the exact sequence

$$(4.17) \qquad 0 \to \mathcal{O}(K + rC) \to \mathcal{O}(K + (r + 1)C) \to \mathcal{O}_C \to 0 .$$

By the argument parallel to that for $K + F + C$, from (4.17) we can derive $H^0(S, \mathcal{O}(K + (r + 1)C)) \neq 0$. Thus S is a Hopf surface by Lemma 4.13. q.e.d.

NOTE. Inoue informed us that Proposition 4.12 Case B is easily obtained by means of Lemma 4.6. Another proof of Proposition 4.12, which does not use Lemma 4.6 and is more cumbersome, is in the authour's master's degree thesis (Univ. of Tokyo, 1981).

Finally we shall prove

Proposition 4.18. *Let S be a surface of Class* VII_0 *with* $b_2(S) > 0$, *which has a divisor* $D \neq 0$ *with* $(D)^2 = 0$. *Let C denote the support of D. Then there exist an unramified covering* $\lambda: \tilde{S} \to S$ *of S and a holomorphic function w on $\tilde{S}$ with the following properties*:

(i) $\lambda^{-1}(C)$ *consists of infinitely many non-singular rational curves* C_j, $j \in \boldsymbol{Z}$, *with* $(C_j)^2 = -2$.

(ii) C_j *and* C_{j+1} *intersect transversally at one point.* C_j *and* C_k *do not meet when* $j \neq k \pm 1$.

(iii) *The divisor (w) of w is* $\sum_{j \in \boldsymbol{Z}} C_j$.

(iv) *The covering transformation group of $\tilde{S}$ with respect to S is generated by a single element g such that*

$$g^*w = \alpha w \quad (0 < |\alpha| < 1)\,,$$
$$g(C_j) = C_{j-m} \quad \textit{for} \quad j \in \boldsymbol{Z} \quad (m \geqq 1)\,.$$

Proof. Since $b_2(S) > 0$, S is not a Hopf surface and S has no meromorphic functions except constants. Hence $(C) = 0$ by Proposition 4.10 and (4.2). By Lemma 4.7, we have a multiplicative holomorphic function w on S whose divisor is C. Let F and T denote respectively the free part and the torsion part of $H_1(S, \boldsymbol{Z})$. Take a generator σ of F and set $\alpha = \mu(w)(\sigma)$. In view of (4.8), taking $-\sigma$ instead of σ if necessary, we may assume $0 < |\alpha| < 1$. Notice that $\mu(w)(T)$ is a finite cyclic group generated by a root of unity ε. Thus the image of $\mu(w)$ is the multiplicative group $\langle \alpha, \varepsilon \rangle$ generated by α and ε. Let G denote the kernel of $\mu(w): \pi_1(S) \to \langle \alpha, \varepsilon \rangle$. Let W denote the universal covering surface of S. We identify $\pi_1(S)$ with the covering transformation group of W with respect to S. Define $\tilde{S}$ to be the quotient surface W/G of W by G. Let λ denote the canonical projection of $\tilde{S}$ onto S. Then $\lambda: \tilde{S} \to S$ is a covering and the covering transformation group of $\tilde{S}$ with respect to S is isomorphic to $\pi_1(S)/G \cong \langle \alpha, \varepsilon \rangle$. Let g and h be the covering transformations of $\tilde{S}$ corresponding to α and ε respectively. Then w induces a single-valued holomorphic function on $\tilde{S}$ so that $g^*w = \alpha w$ and $h^*w = \varepsilon w$. Moreover, since the divisor of w on S is C, we obtain (iii).

Since S is not a Hopf surface, it follows from Propositions 4.12 and 4.10 that $\pi_1(C) \cong \boldsymbol{Z}$. Let γ be a closed path representing a generator of $\pi_1(C)$. Then we can write

$$\mu(w)(\gamma) = \alpha^a \varepsilon^b \quad (a, b \in \boldsymbol{Z}) \tag{4.19}$$

where $a \neq 0$ by Lemma 4.9. Changing the orientation of γ if necessary, we may assume $a > 0$. We shall show $a = 1$, $\varepsilon = 1$ and h is the identity map. Consider the quotient surface $S' = \tilde{S}/\langle g^a \circ h^b \rangle$ of $\tilde{S}$ by the group

$\langle g^a \circ h^b \rangle$ generated by $g^a \circ h^b$. Let p denote the canonical projection of S' onto S. Then the covering transformation group of S' with respect to S is the quotient group $\langle g, h\rangle / \langle g^a \circ h^b \rangle$ of $\langle g, h\rangle$ by $\langle g^a \circ h^b \rangle$. Since h is of finite order and $a \neq 0$, the order of $\langle g, h\rangle / \langle g^a \circ h^b \rangle$ is finite, say d, i.e., S' is a d-fold unramified covering surface of S. Therefore, by Lemma 4.4, S' is a surface of Class VII_0 with no non-constant meromorphic functions. Moreover, since S is not a Hopf surface, S' is not a Hopf surface. Note that $(p^{-1}(C)) = p^*(C) = 0$ on S'. Hence $p^{-1}(C)$ is connected by Proposition 4.12. On the other hand we infer from (4.19) that $p^{-1}(C)$ consists of d connected components. Thus $d = 1$. This implies $\langle g, h\rangle = \langle g^a \circ h^b \rangle$. Therefore $a = 1$, h is the identity map and hence $\varepsilon = 1$. Now (4.19) means that the closed path γ corresponds to the covering transformation g and hence $\lambda^{-1}(C) \to C$ is the universal covering of C. Hence (i), (ii) and (iv) follow from Proposition 4.10. q.e.d.

5. Construction of Σ, I.

Let S be a compact surface free from exceptional curves of the first kind. Throughout Sections 5–8 we assume that S has a curve C and satisfies the following conditions (cf. Proposition 4.18):

(S-0) There are an unramified covering $\lambda: \tilde{S} \to S$ of S and a holomorphic function w on $\tilde{S}$.

(S-1) $\lambda^{-1}(C)$ consists of infinitely many non-singular rational curves C_j, $j \in \boldsymbol{Z}$, with $(C_j)^2 = -2$.

(5.1) C_j and C_{j+1} intersect transversally at one point p_j, C_j and C_k do not meet when $j \neq k \pm 1$,

(5.2) $$\text{the divisor } (w) \text{ of } w \text{ is } \sum_{j \in \boldsymbol{Z}} C_j .$$

(S-2) The covering transformation group of $\tilde{S}$ with respect to S is generated by a single element g such that

(5.3) $$g^*w = \alpha w \quad (0 < |\alpha| < 1) ,$$

(5.4) $$g(C_j) = C_{j-m} \quad (m \geqq 1) .$$

We set $\tilde{C} = \lambda^{-1}(C)$ and $C^+ = \bigcup_{j>0} C_j$.

In this section we shall construct on a neighborhood of C^+ a holomorphic 2-form which satisfies certain estimates. To state precisely and prove this result, we define coordinate charts $(U_{2j}, (\zeta_{2j}, w))$, $(U_{2j+1}, (\zeta^1_{2j+1}, \zeta^2_{2j+1}))$, $j \in \boldsymbol{Z}$, covering a neighborhood of $\tilde{C}$, with the following properties (where we set $j = \nu m + i$, $\nu \in \boldsymbol{Z}$, $0 \leqq i \leqq m-1$):

(i) U_{2j+1} is a neighborhood of p_j and identified with a polydisk by $(\zeta^1_{2j+1}, \zeta^2_{2j+1})$:

$$U_{2j+1} = \{(\zeta^1_{2j+1}, \zeta^2_{2j+1}) \,|\, |\zeta^e_{2j+1}| < \varepsilon_0,\ e = 1, 2\}\ ,$$

where $\varepsilon_0 > 0$ is independent of j. The equation: $\zeta^1_{2j+1} = 0$ defines C_{j+1} on U_{2j+1}. Moreover

$$\zeta^1_{2j+1}\zeta^2_{2j+1} = \alpha^\nu w\ . \tag{5.5}$$

(ii) U_{2j} is a neighborhood of $C_j - U_{2j-1} \cup U_{2j+1}$ and identified with the product of an annulus and a disk by (ζ_{2j}, w):

$$U_{2j} = \{(\zeta_{2j}, w) \,|\, r < |\zeta_{2j}| < r^{-1},\ |w| < |\alpha|^{-\nu}\varepsilon_1\}$$

where $0 < r < 1$ and $\varepsilon_1 > 0$ are independent of j. $\zeta_{2j}|C_j$ extends to the inhomogeneous coordinate of C_j such that $\zeta_{2j}(p_{j-1}) = \infty$, $\zeta_{2j}(p_j) = 0$.

(iii) We have

$$U_j \cap U_k = \varnothing \quad \text{if} \quad j \neq k \pm 1\ , \tag{5.6}$$

$$\begin{cases} U_{2j+1} = g^{-\nu}(U_{2i+1}) \\ U_{2j} = g^{-\nu}(U_{2i})\ , \end{cases} \tag{5.7}$$

$$\begin{cases} \zeta^e_{2j+1} = (g^\nu)^*\zeta^e_{2i+1} \quad \text{for} \quad e = 1, 2 \\ \zeta_{2j} = (g^\nu)^*\zeta_{2i}\ . \end{cases} \tag{5.8}$$

(iv) There is a holomorphic 2-form s_{2j+1} on U_{2j+1} so that it has no zero and satisfies

$$s_{2j+1} = \zeta^{-1}_{2j} d\zeta_{2j} \wedge dw \quad \text{on} \quad C_j \cap U_{2j} \cap U_{2j+1}\ , \tag{5.9}$$

$$s_{2j+1} = \alpha^{-\nu}(g^\nu)^* s_{2i+1}\ . \tag{5.10}$$

To define the above coordinate charts, let ξ_j be the inhomogeneous coordinate of C_j such that $\xi_j(p_j) = 0$ and $\xi_j(p_{j-1}) = \infty$. Let K denote the canonical bundle of $\tilde{S}$. Set $\sigma_j = \xi_j^{-1} d\xi_j \wedge dw$. Then σ_j defines a holomorphic section of K over $C_j - \{p_j\} - \{p_{j-1}\}$. Since ξ_j is determined uniquely up to constant multiples, σ_j is determined uniquely. Moreover, using (5.1)-(5.2), we see that σ_j extends to a holomorphic section of K over C_j so that it has no zero and satisfies $\sigma_j(p_j) = \sigma_{j+1}(p_j)$. Thus, σ_j's define a trivialization σ of K over $\tilde{C}$ by $\sigma|C_j = \sigma_j$.

We first take a coordinate chart $(U_{2i+1}, (\zeta^1_{2i+1}, \zeta^2_{2i+1}))$ around p_i for each $0 \leqq i \leqq m-1$. By (5.1)-(5.2), we may assume condition (i) for $0 \leqq i \leqq m-1$. We extend σ to a holomorphic 2-form s_{2i+1} on U_{2i+1}. Shrinking U_{2i+1} if necessary, we may assume s_{2i+1} has no zero. Take a real number $0 < r < 1$ so that the open set

$$U_{2i-1} \cup \{x \in C_i \,|\, r < |\xi_i(x)| < r^{-1}\} \cup U_{2i+1}$$

covers C_i for any $0 \leqq i \leqq m-1$, where $U_{-1} = g(U_{2m-1})$. According to

Siu [12], there is a Stein neighborhood T_i of $C_i - \{p_i\} - \{p_{i-1}\}$ in $\tilde{S} - \{p_i\} - \{p_{i-1}\}$. We extend ξ_i to a holomorphic function ζ_{2i} on T_i. Shrinking T_i if necessary, we may assume that (ζ_{2i}, w) forms a system of coordinates on T_i. Set

$$U_{2i} = \{x \in T_i \mid r < |\zeta_{2i}(x)| < r^{-1}, |w(x)| < \varepsilon_1\} .$$

Then coordinate charts $(U_{2i}, (\zeta_{2i}, w))$ $(0 \leqq i \leqq m-1)$ satisfy condition (ii) provide that ε_1 and r are chosen properly. Now we define coordinate charts $(U_{2j+1}, (\zeta^1_{2j+1}, \zeta^2_{2j+1}))$, $(U_{2j}, (\zeta_{2j}, w))$ by (5.7) and (5.8). Then they satisfy conditions (i)–(iii) as desired. Define holomorphic 2-forms s_{2j+1} by (5.10). Then they satisfy condition (iv).

Define an open neighborhood B^ε of C^+ by

$$B^\varepsilon = \bigcup_{j \geqq 0} \{x \in U_j \mid |w(x)| < \varepsilon\} , \quad \varepsilon > 0 .$$

PROPOSITION 5.11. *For sufficiently small $\varepsilon > 0$, there exists a holomorphic 2-form φ on B^ε such that φ has no zero on B^ε and its local expression*

$$\varphi = \varphi_{2j}\zeta_{2j}^{-1} d\zeta_{2j} \wedge dw$$

on $U_{2j} \cap B^\varepsilon$ satisfies

$$\varphi_{2j}(x) = 1 \quad \textit{for} \quad x \in \tilde{C} \cap U_{2j} , \quad j \geqq 0$$
$$|\varphi_{2j}(x) - 1| < 1/2 \quad \textit{for} \quad x \in B^\varepsilon \cap U_{2j} , \quad j \geqq 0 .$$

The following construction of the holomorphic 2-form φ is similar to that of the holomorphic map in [9]. However the noncompactness of C^+ forces us to make some alternations to the arguments in [9]. Namely, (i) while arbitrary coordinate charts could be used in [9], we have to use special coordinate charts such as (U_j, ζ_j), (ii) while the ordinary maximum-supremum norm of Čech cochains is used in [9], we shall use a *weighted* norm of Čech cochains defined on C^+. We divide our proof into five steps.

Step 1. We begin by introducing a norm of Čech cochains and proving a lemma which uses this norm. Let

$$V_j = U_j \cap \tilde{C}$$
$$z_{2j} = \zeta_{2j} | V_{2j}$$
$$z^1_{2j+1} = \zeta^1_{2j+1} | V_{2j+1} \cap C_j$$
$$z^2_{2j+1} = \zeta^2_{2j+1} | V_{2j+1} \cap C_{j+1} .$$

Then $(V_{2j-1} \cap C_j, z^2_{2j-1})$, (V_{2j}, z_{2j}) and $(V_{2j+1} \cap C_j, z^1_{2j+1})$ form coordinate charts covering C_j. Define a relatively compact subset V^δ_j of V_j by

$$V_{2j}^{\delta} = \{x \in V_{2j} \,|\, r + \delta < |z_{2j}(x)| < r^{-1} - \delta\}$$
$$V_{2j+1}^{\delta} = \{x \in V_{2j+1} \,|\, |z_{2j+1}^{e}(x)| < \varepsilon_0 - \delta,\ e = 1, 2\}$$

for sufficiently small $\delta > 0$. We may assume that $\bigcup_{i=0}^{2m+1} V_i^{\delta}$ covers a compact curve $\bigcup_{i=1}^{m} C_i$. Then, by (5.4) and (5.7), $\bigcup_{j\geq 0} V_j^{\delta}$ covers C^+.

Set $\mathscr{V} = \{V_j\}_{j\geq 0}$. Let $C^q(\mathscr{V}, \mathcal{O})$ denote the module of q-cochains on the covering $\mathscr{V}$ with the coefficients in $\mathcal{O}$. Let ρ be a positive constant. For any q-cochain $\eta = \{\eta_{i_0\cdots i_q}\}$, define the norm $\|\eta\|_\rho$ of η by

$$\|\eta\|_\rho = \sup\{\sup_x \rho^{-i_0}|\eta_{i_0\cdots i_q}(x)| \,|\, i_0, \cdots, i_q \geqq 0\} .$$

Let δ denote the coboundary map.

LEMMA 5.12. *Let $0 < \rho < 1$. Then, for any 1-cocycle γ, there exists a 0-cochain ψ satisfying*

$$\delta\psi = \gamma \qquad \text{and} \qquad \|\psi\|_\rho \leqq L_\rho \|\gamma\|_\rho$$

where L_ρ is a positive constant independent of γ.

PROOF. Let $\gamma = \{\gamma_{jk}\}$. Assume first $\|\gamma\|_\rho < \infty$. We expand $\gamma_{2j\,2j\pm1}(z_{2j})$ into the Laurent power series in z_{2j}:

$$\gamma_{2j\,2j\pm1}(z_{2j}) = a_{2j\,2j\pm1} + \sum_{\mu\neq 0} b_{j|\mu}^{\pm} z_{2j}^{\mu} ,$$

where $a_{2j\,2j\pm1}$, $b_{j|\mu}^{\pm} \in \boldsymbol{C}$. Set

$$f_j^{\pm}(z_{2j}) = \sum_{\mu>0} b_{j|\pm\mu}^{\pm} z_{2j}^{\pm\mu}$$
$$g_j^{\pm}(z_{2j}) = \sum_{\mu<0} b_{j|\mp\mu}^{\mp} z_{2j}^{\mp\mu} .$$

Then we have

$$\gamma_{2j\,2j\pm1} = a_{2j\,2j\pm1} + f_j^{\pm} + g_j^{\mp} . \tag{5.13}$$

Since z_{2j} extends to the inhomogeneous coordinate of C_j such that $z_{2j}(p_j) = 0$ and $z_{2j}(p_{j-1}) = \infty$, we can extend $f_j^{\pm}$ and $g_j^{\pm}$ to holomorphic functions on $V_{2j\pm1}$ and on $V_{2j\pm1} \cup V_{2j}$ respectively so that $f_j^{\pm}|C_{j\pm1} = 0$ and $g_j^{\pm}|C_{j\pm1} = 0$. By the definition of $\|\gamma\|_\rho$, we have

$$|\gamma_{jk}(x)| \leqq \rho^j \|\gamma\|_\rho \quad \text{for} \quad x \in V_j \cap V_k .$$

Hence, using Cauchy's inequality, we obtain the estimates

$$\begin{cases} |a_{jk}| \leqq R\rho^j \|\gamma\|_\rho \\ |f_j^{\pm}(x)| \leqq R\rho^{2j+1}\|\gamma\|_\rho \quad \text{for} \quad x \in V_{2j\pm1}^{\delta} \\ |g_j^{\pm}(x)| \leqq R\rho^{2j+1}\|\gamma\|_\rho \quad \text{for} \quad x \in V_{2j}^{\delta} \cup V_{2j\pm1} , \end{cases} \tag{5.14}$$

where R is a positive constant independent of j, k and γ. Combining (5.13) with (5.14), we obtain

$$(5.15)\qquad \begin{cases} |f_j^{\pm}(x)| \leqq (1+2R)\rho^{2j+1}\|\gamma\|_\rho & \text{for} \quad x \in V_{2j\pm 1} \\ |g_j^{\pm}(x)| \leqq (1+2R)\rho^{2j+1}\|\gamma\|_\rho & \text{for} \quad x \in V_{2j} \cup V_{2j\pm 1} . \end{cases}$$

Define a constant a_j by $a_j = -\sum_{i=j}^{\infty} a_{i\,i+1}$ for $j \geqq 0$. Then by (5.14),

$$(5.16)\qquad |a_j| \leqq (1-\rho)^{-1}R\rho^j\|\gamma\|_\rho \quad \text{for} \quad j \geqq 0 .$$

Now we define a 0-cochain $\psi = \{\psi_j\}$ by

$$\psi_{2j\pm 1} = a_{2j\pm 1} + f_j^{\pm} - g_j^{\pm} \quad \text{on} \quad V_{2j\pm 1} \cap C_j$$
$$\psi_{2j} = a_{2j} - g_j^{+} - g_j^{-} .$$

Then $\delta\psi = \gamma$. By (5.15) and (5.16),

$$|\psi_j(x)| \leqq L_\rho \rho^j \|\gamma\|_\rho \quad \text{for} \quad x \in V_j , \quad j \geqq 0 ,$$

where $L_\rho = 1 + 2R + R/(1-\rho)$. Therefore we obtain $\|\psi\|_\rho \leqq L_\rho\|\gamma\|_\rho$ as desired. When $\|\gamma\|_\rho = \infty$, we define a_j by

$$a_0 = 0 , \qquad a_j = \sum_{i=1}^{j} a_{i-1\,i} \quad (j \geqq 0) .$$

Then similarly we have $\delta\psi = \gamma$. q.e.d.

Step 2. We first introduce some notations. By (5.5), (ζ_{2j+1}^1, w) (resp. (ζ_{2j+1}^2, w)) is a system of coordinates on $U_{2j+1} \cap U_{2j}$ (resp. $U_{2j+1} \cap U_{2j+2}$). We write the coordinate changes as follows:

$$(\zeta_j^1, \zeta_j^2) = (g_{jk}^1(\zeta_k, w), g_{jk}^2(\zeta_k, w))$$
$$\zeta_k = g_{kj}(\zeta_j^{\tau}, w) , \qquad \zeta_j^{\sigma} = h_j^{\sigma\tau}(\zeta_j^{\tau}, w)$$

on $U_j \cap U_k$, where $k \equiv 0 \mod 2$, $(\sigma, \tau) = (2, 1)$ or $(1, 2)$ according as $j = k+1$ or $k-1$. For simplicity we write ζ_j and z_j for the vectors (ζ_j^1, ζ_j^2) and (z_j^1, z_j^2) respectively. Considering z_j, $j \in \boldsymbol{Z}$, as local coordinates for $\tilde{C}$, we write the coordinate changes as $z_j = b_{jk}(z_k)$. Let s_{2j+1} be the holomorphic 2-form on U_{2j+1} satisfying (5.9)–(5.10). Setting

$$s_{2j} = \zeta_{2j}^{-1} d\zeta_{2j} \wedge dw ,$$

define a holomorphic function f_{jk} on $U_j \cap U_k$ by $s_j = f_{kj}s_k$. Note that $f_{jk} = 1$ on $\tilde{C}$ by (5.9). We regard f_{jk} as a holomorphic function in two variables:

$$f_{jk} = \begin{cases} f_{jk}(\zeta_k, w) & \text{if} \quad k \equiv 0 \mod 2 \\ f_{jk}(\zeta_k^1, w) & \text{if} \quad k \equiv 1 \mod 2 , \quad j = k-1 \\ f_{jk}(\zeta_k^2, w) & \text{if} \quad k \equiv 1 \mod 2 , \quad j = k+1 . \end{cases}$$

In order to prove Proposition 5.11, it suffices to construct holomorphic functions φ_j, $j \geqq 0$, defined respectively on $U_j \cap B^{\varepsilon}$ such that

$$\text{(5.17)}\qquad \begin{cases}\varphi_j = f_{jk}\varphi_k & \text{on}\quad U_j \cap U_k \cap B^\varepsilon \\ \varphi_j = 1 & \text{on}\quad \tilde{C} \cap U_j\end{cases}$$

$$\text{(5.18)}\qquad |\varphi_j(x) - 1| < 1/2 \quad \text{for}\quad x \in U_j \cap B^\varepsilon .$$

We write φ_j in the form $\varphi_j = \sum_{\mu=0}^{\infty} \varphi_{j|\mu}(\zeta_j)w^\mu$ where $\varphi_{j|\mu}(\zeta_j)$ are holomorphic functions in ζ_j defined on U_j. Moreover, when j is odd, we assume that $\varphi_{j|\mu}$ is of the form

$$\varphi_{j|\mu}(\zeta_j) = a_{j|\mu} + f^1_{j|\mu}(\zeta^1_j) + f^2_{j|\mu}(\zeta^2_j)$$

where $a_{j|\mu}$ is a constant and each $f^e_{j|\mu}(\zeta^e_j)$, $e = 1$ or 2, is a holomorphic function in ζ^e_j, $|\zeta^e_j| < \varepsilon_0$, such that $f^e_{j|\mu}(0) = 0$. Let $\psi_{j|\mu}(z_j)$ denote the restriction of $\varphi_{j|\mu}$ to V_j. Then, corresponding to φ_j, we have a formal power series $\psi_j(z_j, w) = \sum_{\mu=0}^{\infty} \psi_{j|\mu}(z_j)w^\mu$ in w whose coefficients $\psi_{j|\mu}(z_j)$ are holomorphic functions on V_j. When $j = 2d + 1$ is odd, $\psi_{j|\mu}(z_j)$ is written as

$$\psi_{j|\mu}(z_j) = \begin{cases} a_{j|\mu} + f^1_{j|\mu}(z^1_j) & \text{for}\quad z_j \in V_j \cap C_d \\ a_{j|\mu} + f^2_{j|\mu}(z^2_j) & \text{for}\quad z_j \in V_j \cap C_{d+1} .\end{cases}$$

We regard the collection of $\psi_j(z_j, w)$, $j \geqq 0$, as a formal power series in w with coefficients $\psi_\mu = \{\psi_{j|\mu}\}$ in $C^0(\mathscr{V}, \mathscr{O})$. Let

$$\psi^\mu = \sum_{\nu=0}^{\mu} \psi_\nu w^\nu , \qquad \psi^\mu_j(z_j, w) = \sum_{\nu=0}^{\mu} \psi_{j|\nu}(z_j) w^\nu .$$

In what follows, we identify a holomorphic function with its power series expansion at a point on $\tilde{C}$. Define a formal power series $\Gamma(\psi^\mu)_{jk}(z_k, w)$ in w with coefficients in holomorphic functions on $V_j \cap V_k$ as follows:

$$\begin{aligned}\Gamma(\psi^\mu)_{jk}(z_k, w) &= \sum_{\nu=0}^{\mu} \{a_{j|\nu} + f^1_{j|\nu}(g^1_{jk}(z_k, w) + f^2_{j|\nu}(g^2_{jk}(z_k, w))\}w^\nu \\ &\qquad - f_{jk}(z_k, w)\psi^\mu_k(z_k, w) \quad \text{for}\quad j = k \pm 1 , \quad k \equiv 0 \mod 2 , \\ \Gamma(\psi^\mu)_{jk}(z_k, w) &= \psi^\mu_j(g_{jk}(z_k, w), w) \\ &\qquad - f_{jk}(z_k, w) \sum_{\nu=0}^{\mu} \{a_{k|\nu} + f^\sigma_{k|\nu}(h^{\sigma\tau}_k(z^\tau_k, w)) + f^\tau_{k|\nu}(z^\tau_k)\}w^\nu \\ &\qquad\qquad \text{for}\quad j = k \pm 1 , \quad k \equiv 1 \mod 2 ,\end{aligned}$$

where $(\sigma, \tau) = (2, 1)$ or $(1, 2)$ according as $k = j + 1$ or $j - 1$, and $\Gamma(\psi^\mu)_{jk}(z_k, w) = 0$ for $j = k$. For any power series $P(w)$, $Q(w)$ in w we indicate by $P(w) \equiv_\mu Q(w)$ that $P(w) - Q(w)$ contains no terms of degree $\leqq \mu$. With this notation, φ_j satisfy (5.17) if and only if $\psi_j(z_j, w)$ satisfy

$$(5.19)_\mu \qquad \Gamma(\psi^\mu)_{jk}(z_k, w) \equiv_\mu 0$$

for all $\mu \geqq 0$. In fact, identifying a holomorphic function $\varphi_j - f_{jk}\varphi_k$ with its power series expansion at $z_k \in V_j \cap V_k$ (with respect to the coordinates

((ζ_k, w) or (ζ_k^{τ}, w) on $U_j \cap U_k$ according as k is even or odd), we have

$$(\varphi_j - f_{jk}\varphi_k)(z_k, w) \equiv_{\mu} \Gamma(\psi^{\mu})_{jk}(z_k, w) .$$

We note that, in general, $\Gamma(\psi^{\mu})_{jk}$ is different from $\psi_j^{\mu} - f_{jk}\psi_k^{\mu}$ as a formal power series in w with coefficients in holomorphic functions on $V_j \cap V_k$.

Step 3. In this step, we prove the existence of a formal power series $\sum_{\mu=0}^{\infty} \psi_{\mu} w^{\mu}$ satisfying $(5.19)_{\mu}$ for all μ. We define ψ_{μ} by induction on μ. Set $\psi_{j|0} = 1$. Then $\psi_0 = \{\psi_{j|0}\}$ satisfies $(5.19)_0$. Suppose therefore we have defined $\psi^{\mu-1}$ satisfying $(5.19)_{\mu-1}$ for some $\mu \geqq 1$. We define $\gamma_{jk|\mu}(z_k)$ to be the coefficient of w^{μ} in $\Gamma(\psi^{\mu-1})_{jk}(z_k, w)$. The collection of $\gamma_{jk|\mu}$ forms an element $\gamma_{\mu} = \{\gamma_{jk|\mu}\}$ of $C^1(\mathscr{V}, \mathscr{O})$. In view of Lemma 5.12, the following lemma proves the existence of $\psi_{\mu} \in C^0(\mathscr{V}, \mathscr{O})$ such that $\psi^{\mu-1} + \psi_{\mu} w^{\mu}$ satisfies $(5.19)_{\mu}$.

LEMMA 5.20. *Assume* $\psi^{\mu-1}$ *satisfies* $(5.19)_{\mu-1}$. *Then*

(i) γ_{μ} *is a* 1-*cocycle of* $C^1(\mathscr{V}, \mathscr{O})$,

(ii) $\psi^{\mu} = \psi^{\mu-1} + \psi_{\mu} w^{\mu}$, $\psi_{\mu} \in C^0(\mathscr{V}, \mathscr{O})$, *satisfies* $(5.19)_{\mu}$ *if and only if* $\delta\psi_{\mu} = \gamma_{\mu}$ *in* $C^1(\mathscr{V}, \mathscr{O})$.

PROOF. By $(5.19)_{\mu-1}$, we have

$$\gamma_{jk|\mu}(z_k) w^{\mu} \equiv_{\mu} \Gamma(\psi^{\mu-1})_{jk}(z_k, w) . \tag{5.21}$$

Now, let $j = k \pm 1$, $k \equiv 0 \mod 2$. Let $z_j = b_{jk}(z_k)$. Furthermore, we let $(\sigma, \tau) = (2, 1)$ or $(1, 2)$ according as $j = k + 1$ or $k - 1$.

(i) By the definition we have $\gamma_{ii|\mu} = 0$, and by (5.6) we have $V_p \cap V_q \cap V_r = \varnothing$ for $p \neq q$, $q \neq r$, $r \neq p$. Hence it suffices to show the identities $\gamma_{jk|\mu} = -\gamma_{kj|\mu}$ on $V_j \cap V_k$. Since $z_k = g_{kj}(z_j^{\tau}, 0)$, we can rewrite (5.21) as

$$\gamma_{jk|\mu}(b_{kj}(z_j)) w^{\mu} \equiv_{\mu} \gamma_{jk|\mu}(g_{kj}(z_j^{\tau}, w)) w \equiv_{\mu} \Gamma(\psi^{\mu-1})_{jk}(g_{kj}(z_j^{\tau}, w), w) .$$

Multiply both hand sides of this formula by $f_{kj}(z_j^{\tau}, w)$. Then, since

$$\begin{aligned} g_{jk}^{\sigma}(g_{kj}(z_j^{\tau}, w), w) &= h_j^{\sigma\tau}(z_j^{\tau}, w) \\ g_{jk}^{\tau}(g_{kj}(z_j^{\tau}, w), w) &= z_j^{\tau} \\ f_{kj}(z_j^{\tau}, w) f_{jk}(g_{kj}(z_j^{\tau}, w), w) &= 1 , \end{aligned}$$

we obtain

$$\begin{aligned} f_{kj}(z_j, w)\gamma_{jk|\mu}(b_{kj}(z_j)) w^{\mu} \equiv_{\mu} f_{kj}(z_j, w) \sum_{\nu=0}^{\mu-1} \{a_{j|\nu} + f_{j|\nu}^{\sigma}(h_j^{\sigma\tau}(z_j^{\tau}, w)) + f_{j|\nu}^{\tau}(z_j^{\tau})\} w^{\nu} \\ - \psi_k^{\mu-1}(g_{kj}(z_j, w), w) . \end{aligned}$$

Comparing this with $\gamma_{kj|\mu}(z_j) w^{\mu}$ by (5.21), we see

$$f_{kj}(z_j, w)\gamma_{jk|\mu}(b_{kj}(z_j)) w^{\mu} \equiv_{\mu} -\gamma_{kj|\mu}(z_j) w^{\mu} .$$

Since $f_{kj}(z_j, 0) = 1$, it follows $\gamma_{jk|\mu} = -\gamma_{kj|\mu}$.

(ii) We regard the collection of $\Gamma(\psi^{\mu})_{jk}$ as a formal power series $\Gamma(\psi^{\mu}) = \{\Gamma(\psi^{\mu})_{jk}\}$ in w with coefficients in $C^1(\mathscr{Y}, \mathscr{O})$. For our purpose it suffices to show

$$\Gamma(\psi^{\mu}) \equiv_{\mu} \gamma_{\mu} w^{\mu} - \delta\psi_{\mu} w^{\mu} . \tag{5.22}$$

We write $\Gamma(\psi^{\mu})_{jk}(z_k, w)$ as

$$\begin{aligned}\Gamma(\psi^{\mu})_{jk}(z_k, w) = \Gamma(\psi^{\mu-1})_{jk}(z_k, w) + \{a_{j|\mu} + f^{\sigma}_{j|\mu}(g^{\sigma}_{jk}(z_k, w)) + f^{\tau}_{j|\mu}(g^{\tau}_{jk}(z_k, w))\}w^{\mu} \\ - f_{jk}(z_k, w)\psi_{k|\mu}(z_k)w^{\mu} ,\end{aligned}$$

while we have $g^{\tau}_{jk}(z_k, 0) = z^{\tau}_j$, $g^{\sigma}_{jk}(z_k, 0) = 0$, $f^{\sigma}_{j|\mu}(0) = 0$ and $f_{jk}(z_k, 0) = 1$. Taking these and (5.21) together, we see

$$\Gamma(\psi^{\mu})_{jk}(z_k, w) \equiv_{\mu} \gamma_{jk|\mu}(z_k)w^{\mu} + \{\psi_{j|\mu}(z_j) - \psi_{k|\mu}(z_k)\}w^{\mu} .$$

This means (5.22). q.e.d.

Step 4. Consider two power series

$$F(s) = \sum f_{\nu_1\cdots\nu_r} s_1^{\nu_1} \cdots s_r^{\nu_r} , \qquad G(s) = \sum g_{\nu_1\cdots\nu_r} s_1^{\nu_1} \cdots s_r^{\nu_r}$$

in $s = (s_1, \cdots, s_r)$ with coefficients in $\boldsymbol{C}$. We indicate by $F(s) \ll G(s)$ that $|f_{\nu_1\cdots\nu_r}| \leqq |g_{\nu_1\cdots\nu_r}|$. Let $A(w) = 16^{-1}bc^{-1}\sum_{\nu=1}^{\infty} \nu^{-2}c^{\nu}w^{\nu}$. In this step, we shall choose $\psi_{\mu} \in C^0(\mathscr{Y}, \mathscr{O})$ by induction on μ so that the power series $\sum_{\mu=0}^{\infty} \psi_{\mu} w^{\mu}$ satisfies $(5.19)_{\mu}$ and $\sum_{\mu=0}^{\infty} \|\psi_{\mu}\|_{\rho} w^{\mu}$ satisfies

$$\|\psi_1\|_{\rho} w + \cdots + \|\psi_{\mu}\|_{\rho} w^{\mu} \ll A(w) , \quad \mu \geqq 1 , \tag{5.23_{μ}}$$

for some constants ρ, b, $c > 0$ independent of μ.

We choose the constant ρ so that $|\alpha| \leqq \rho^{2m} < 1$. Set $j = 2\nu m + q$, $k = 2\nu m + r$ for $\nu = 0, 1, 2, 3, \cdots$, $0 \leqq q, r < 2m$. Then, by (5.3) and (5.8)-(5.10), we have

$$\begin{aligned} g^{e}_{jk}(\zeta, w) &= g^{e}_{qr}(\zeta, \alpha^{\nu} w) \quad (k \equiv 0 \bmod 2) \\ g_{jk}(\zeta, w) &= g_{qr}(\zeta, \alpha^{\nu} w) \quad (k \equiv 1 \bmod 2) \\ h^{\sigma\tau}_{k}(\zeta, w) &= h^{\sigma\tau}_{r}(\zeta, \alpha^{\nu} w) \quad (k \equiv 1 \bmod 2) \\ f_{jk}(\zeta, w) &= f_{qr}(\zeta, \alpha^{\nu} w) \end{aligned}$$

as holomorphic functions in two variables (ζ, w). Hence, estimating power series expansions in w of $g^{e}_{qr}(\zeta, w)$, $g_{qr}(\zeta, w)$, $h^{\sigma\tau}_{r}(\zeta, w)$ and $f_{qr}(\zeta, w)$ for $0 \leqq q, r < 2m$, we may assume

$$\begin{cases} g^{e}_{jk}(z_k, w) - g^{e}_{jk}(z_k, 0) \ll A_0(w) \quad (k \equiv 0 \bmod 2) \\ g_{jk}(z^{e}_k, e) - b_{jk}(z_k) \ll A_0(w) \quad (k \equiv 1 \bmod 2) \\ h^{\sigma\tau}_{k}(z^{\tau}_k, w) \ll A_0(w) \quad (k \equiv 1 \bmod 2,\ \sigma \neq \tau) \\ f_{jk}(z_k, w) - 1 \ll \rho^{j} A_0(w) \end{cases} \tag{5.24}$$

for all $j, k \geqq 0$, where $A_0(w)$ is the power series $A(w)$ in which the constants b, c are replaced by b_0, c_0. We fix a small positive number δ so that $\bigcup_{j\geqq 0} V_j^\delta$ covers C^+. Since (5.24) remains valid if we replace c_0 by a larger constant, we may assume

$$b_0/c_0\delta < 1/2 . \tag{5.25}$$

We define $\psi_0 = \{\psi_{j|0}\}$ by $\psi_{j|0} = 1$. Then $\psi^0 = \psi_0$ satisfies $(5.19)_0$ and we have $\Gamma(\psi^0)_{jk}(z_k, w) = 1 - f_{jk}(z_k, w)$. Since $\gamma_{jk|1}w$ is the linear part of $\Gamma(\psi^0)_{jk}$, it follows $\|\gamma_1\|_\rho w \ll A_0(w)$ by (5.24). By Lemma 5.12, we can choose $\psi_1 \in C^0(\mathscr{V}, \mathscr{O})$ so that $\|\psi_1\|_\rho \leqq L_\rho \|\gamma_1\|_\rho$ and $\delta\psi_1 = \gamma_1$. Then, by Lemma 5.20, $\psi^1 = \psi_0 + \psi_1 w$ satisfies $(5.19)_1$. We may assume $b \geqq L_\rho b_0$, $c \geqq c_0$. Then $(5.23)_1$ follows from this. Assume therefore we have chosen $\psi^{\mu-1}$ satisfying $(5.19)_{\mu-1}$ and $(5.23)_{\mu-1}$ for some $\mu \geqq 2$. To estimate ψ_μ, we need

LEMMA 5.26. *Assume* $(5.19)_{\mu-1}$ *and* $(5.23)_{\mu-1}$ *for some* $\mu \geqq 2$. *Then we have*

$$\|\gamma_\mu\|_\rho w^\mu \ll (K_0 b^{-1} + K_1 c^{-1} + K_2 c^{-2}) A(w)$$

where K_0, K_1 *and* K_2 *are positive constants independent of* γ_μ, b *and* c.

PROOF. Let $j = k \pm 1$, $k \equiv 0 \bmod 2$. For simplicity, we set

$$a_j(w) = 1 + a_{j|1}w + \cdots + a_{j|\mu-1}w^{\mu-1}$$
$$f_j^e(z_j^e, w) = f_{j|1}^e(z_j^e)w + \cdots + f_{j|\mu-1}^e(z_j^e)w^{\mu-1} \quad \text{for} \quad e = 1, 2 .$$

Then $\psi_j^{\mu-1}(z_j, w)$ is written as

$$\psi_j^{\mu-1}(z_j, w) = \begin{cases} a_j(w) + f_j^1(z_j^1, w) & \text{for} \quad z_j \in V_j \cap C_d \\ a_j(w) + f_j^2(z_j^2, w) & \text{for} \quad z_j \in V_j \cap C_{d+1} \end{cases}$$

where $d = k/2$. We note that $f_j^e(0, w) = 0$ for $e = 1, 2$.

By the induction assumption $(5.23)_{\mu-1}$, we have

$$\begin{cases} a_j(w) + f_j^e(z_j^e, w) - 1 \ll \rho^j A(w) & \text{for} \quad |z_j^e| < \varepsilon_0 , \quad e = 1, 2 , \\ \psi_k^{\mu-1}(z_k, w) - 1 \ll \rho^k A(w) & \text{for} \quad z_k \in V_k . \end{cases} \tag{5.27}$$

Let $R = \delta^{-1}$. Then, applying Cauchy's inequality to holomorphic functions $f_j^e(z_j^e + y, w) + a_j(w) - 1$ and $\psi_k^{\mu-1}(z_k + y, w) - 1$ in (y, w) with estimates (5.27), we obtain

$$f_j^e(z_j^e + y, w) - f_j^e(z_j^e, w) \ll \rho^j A(w) \sum_{\nu=1}^{\infty} (Ry)^\nu \quad \text{for} \quad |z_j^e| < \varepsilon_0 - \delta , \tag{5.28}$$

$$\psi_k^{\mu-1}(z_k + y, w) - \psi_k^{\mu-1}(z_k, w) \ll \rho^k A(w) \sum_{\nu=1}^{\infty} (Ry)^\nu \quad \text{for} \quad z_k \in V_k^\delta . \tag{5.29}$$

In particular, letting $z_j^e = 0$ in (5.28), we have

$$f_j^e(y, w) \ll \rho^j A(w) \sum_{\nu=1}^{\infty} (Ry)^\nu , \tag{5.30}$$

since $f_j^e(0, w) = 0$. Here we remark that

$$\begin{cases} A_0(w)^\nu \ll (b_0 c_0^{-1})^{\nu-1} A_0(w) & \text{for } \nu = 1, 2, 3, \cdots , \\ A_0(w) \ll b_0 b^{-1} A(w) & (\text{since } c \geqq c_0) \\ A_0(w)A(w) \ll b_0 c^{-1} A(w) . \end{cases} \tag{5.31}$$

First we estimate $\gamma_{jk|\mu}(z_k)w^\mu$ for $z_k \in V_j^\delta \cap V_k$. Let $(\sigma, \tau) = (2, 1)$ or $(1, 2)$ according as $j = k+1$ or $k-1$. For any power series $P(w)$ in w, let $[P(w)]_\mu$ denote its μ-th part. Then $\gamma_{jk|\mu} w^\mu$ is written as

$$\begin{aligned} \gamma_{jk|\mu}(z_k)w^\mu = [f_j^\sigma(g_{jk}^\sigma(z_k, w), w)]_\mu + [f_j^\tau(g_{jk}^\tau(z_k, w), w)]_\mu \\ - [f_{jk}(z_k, w)\psi_k^{\mu-1}(z_k, w)]_\mu . \end{aligned} \tag{5.32}$$

Let $z_j = b_{jk}(z_k)$, i.e., $g_{jk}^\tau(z_k, 0) = z_j^\tau$. In (5.28), we let $y = g_{jk}^\tau(z_k, w) - z_j^\tau$. Then, for $z_k \in V_j^\delta \cap V_k$, we have

$$\begin{aligned} f_j^\tau(g_{jk}^\tau(z_k, w), w) - f_j^\tau(z_j^\tau, w) &\ll \rho^j A(w) \sum_{\nu=1}^{\infty} R^\nu (g_{jk}^\tau(z_k, w) - z_j^\tau)^\nu \\ &\ll \rho^j A(w) \sum_{\nu=1}^{\infty} R^\nu A_0(w)^\nu \quad \text{by (5.24)} \\ &\ll \rho^j A(w) \sum_{\nu=1}^{\infty} R^\nu (b_0 c_0^{-1})^{\nu-1} A_0(w) \quad \text{by (5.31)} \\ &\ll 2R\rho^j A(w) A_0(w) \quad \text{by (5.25)} \\ &\ll 2R b_0 c^{-1} \rho^j A(w) \quad \text{by (5.31)} . \end{aligned}$$

Since $g_{jk}^\sigma(z_k, 0) = 0$, letting $y = g_{jk}^\sigma(z_k, w)$ in (5.30), we obtain similarly

$$f_j^\sigma(g_{jk}^\sigma(z_k, w), w) \ll 2R b_0 c^{-1} \rho^j A(w) \quad \text{for} \quad z_k \in V_j \cap V_k .$$

Thus for any $e = 1$ or 2 we have

$$[f_j^e(g_{jk}^e(z_k, w), w)]_\mu \ll 2R b_0 c^{-1} \rho^j A(w) \quad \text{for} \quad z_k \in V_j^\delta \cap V_k . \tag{5.33}$$

For $z_k \in V_j \cap V_k$, we have

$$\begin{aligned} &[f_{jk}(z_k, w)\psi_k^{\mu-1}(z_k, w)]_\mu \\ &\quad = [(f_{jk}(z_k, w) - 1)]_\mu + [(f_{jk}(z_k, w) - 1)(\psi_k^{\mu-1}(z_k, w) - 1)]_\mu \\ &\quad \ll \rho^j A_0(w) + \rho^j A_0(w) \rho^k A(w) \quad \text{by (5.24), (5.27)} \\ &\quad \ll (b_0 b^{-1} + b_0 c^{-1}) \rho^j A(w) \quad \text{by (5.31)} . \end{aligned} \tag{5.34}$$

Combining (5.33)-(5.34) with (5.32), we obtain

$$\gamma_{jk|\mu}(z_k)w^\mu \ll \{b_0 b^{-1} + (1 + 4R) b_0 c^{-1}\} \rho^j A(w) \quad \text{for} \quad z_k \in V_j^\delta \cap V_k . \tag{5.35}$$

Next we estimate $\gamma_{kj|\mu}(z_j)w^\mu$ for $z_j \in V_j \cap V_k^\delta$, which is defined by

$$(5.36)\qquad \gamma_{kj|\mu}(z_j)w^{\mu} = [\psi_k^{\mu-1}(g_{kj}(z_j, w), w)]_{\mu} - [f_{kj}(z_j, w)\{a_j(w) + f_j^{\tau}(z_j^{\tau}, w)\}]_{\mu} - [f_{kj}(z_j, w)f_j^{\tau}(h_j^{\sigma\tau}(z_j^{\tau}, w), w)]_{\mu} .$$

In (5.29), we let $y = g_{kj}(z_j^{\tau}, w) - z_k$. Then, in the same way as we derived (5.33), we have

$$(5.37)\qquad [\psi_k^{\mu-1}(g_{kj}(z_j, w), w)]_{\mu} \ll 2Rb_0c^{-1}\rho^k A(w) \quad \text{for} \quad z_j \in V_j \cap V_k^{\delta} .$$

Since $h_j^{\sigma\tau}(z_j^{\tau}, 0) = 0$, letting $y = h_j^{\sigma\tau}(z_j^{\tau}, w)$ in (5.30), we obtain similarly

$$f_j^{\sigma}(h_j^{\sigma\tau}(z_j^{\tau}, w), w) \ll 2Rb_0c^{-1}\rho^j A(w) \quad \text{for} \quad z_j \in V_j \cap V_k .$$

Hence, for $z_j \in V_j \cap V_k$,

$$\begin{aligned}(5.38)\qquad & [f_{kj}(z_j^{\tau}, w)f_j^{\tau}(h_j^{\sigma\tau}(z_j^{\tau}, w), w)]_{\mu} \\ &= [f_j^{\tau}(h_j^{\sigma\tau}(z_j^{\tau}, w), w)]_{\mu} + [(f_{kj}(z_j^{\tau}, w) - 1)f_j^{\sigma}(h_j^{\sigma\tau}(z_j^{\tau}, w), w)]_{\mu} \\ &\ll 2Rb_0c^{-1}\rho^j A(w) + \rho^k A_0(w)2Rb_0c^{-1}\rho^j A(w) \\ &\ll 2Rb_0c^{-1}\rho^j A(w) + 2Rb_0^{-2}c^{-2}\rho^k A(w) \quad \text{by (5.31)} .\end{aligned}$$

In the same way as we derived (5.34), we have

$$(5.39)\qquad [f_{kj}(z_j, w)\{a_j(w) + f_j(z_j, w)\}]_{\mu} \ll (b_0b^{-1} + b_0c^{-1})\rho^k A(w) \quad \text{for} \quad z_j \in V_j \cap V_k .$$

Combining (5.37)-(5.39) with (5.36), we obtain

$$(5.40)\qquad \gamma_{kj|\mu}(z_j)w^{\mu} \ll \{b_0b^{-1} + (1 + 2R)b_0c^{-1} + 2Rb_0^2c^{-2}\}\rho^k A(w) + 2Rb_0c^{-1}\rho^j A(w) \quad \text{for} \quad z_j \in V_j \cap V_k^{\delta} .$$

Now we recall that $\gamma_{\mu} = \{\gamma_{jk|\mu}\}$ is a 1-cocycle. In particular we have

$$(5.41)\qquad \gamma_{jk|\mu} = -\gamma_{kj|\mu} \quad \text{on} \quad V_j \cap V_k .$$

Since $V_i \cap V_j \cap V_k = \varnothing$ for $i \neq j$, $j \neq k$, $k \neq i$, we have

$$V_j \cap V_k = (V_j^{\delta} \cap V_k) \cup (V_j \cap V_k^{\delta}) .$$

Combining this with (5.35), (5.40) and (5.41), we obtain

$$\gamma_{jk|\mu}(z_j)w^{\mu} \ll (K_0b^{-1} + K_1c^{-1} + K_2c^{-2})\rho^j A(w)$$

for $z_k \in V_j \cap V_k$, $j, k \geqq 0$, where

$$K_0 = b_0/\rho , \qquad K_1 = b_0(1 + 4R)/\rho , \qquad K_2 = 2Rb_0^2/\rho$$

are positive constants independent of j, k, μ, b and c. We have thus the desired estimate for $\|\gamma_{\mu}\|_{\rho}$. q.e.d.

By Lemma 5.26, we have $\|\gamma_{\mu}\|_{\rho}w^{\mu} \ll K^*A(w)$ where $K^* = K_0b^{-1} + K_1c^{-1} + K_2c^{-2}$. Independently of μ, we choose the constants b, c sufficiently large so that $K^*L_{\rho} \leqq 1$. By Lemma 5.12, we can choose $\psi_{\mu} \in C^0(\mathscr{V}, \mathscr{O})$

so that $\delta\psi_\mu = \gamma_\mu$ and $\|\psi_\mu\|_\rho \leqq L_\rho\|\gamma_\mu\|_\rho$. Then, we have $\|\psi_\mu\|_\rho w^\mu \ll K^*L_\rho A(w) \ll A(w)$. Thus $\psi^\mu = \psi^{\mu-1} + \psi_\mu w^\mu$ satisfies the estimate $(5.23)_\mu$. Moreover, by Lemma 5.20, ψ^μ satisfies $(5.19)_\mu$. This completes our inductive choices of ψ_μ.

Final step. Let $\sum_{\mu=0}^\infty \psi_\mu w^\mu$ be the formal power series defined in Step 4. Let $\psi_\mu = \{\psi_{j|\mu}(z_j)\}$. We extend $\psi_{j|\mu}(z_j)$ to a holomorphic function $\varphi_{j|\mu}(\zeta_j)$ on U_j by setting

$$\varphi_{j|\mu}(\zeta_j) = \begin{cases} \psi_{j|\mu}(\zeta_j) & \text{if } j \text{ is even} \\ a_{j|\mu} + f^1_{j|\mu}(\zeta^1_j) + f^2_{j|\mu}(\zeta^2_j) & \text{if } j \text{ is odd.} \end{cases}$$

By the estimates $(5.23)_\mu$ we have

$$|\varphi_{j|\mu}(\zeta_j(x))|w^\mu \ll \rho^j A(w) \qquad (j \text{ is even})$$
$$|a_{j|\mu} + f^e_{j|\mu}(\zeta^e_j(x))|w^\mu \ll \rho^j A(w) \quad (j \text{ is odd})$$

for $\mu \geqq 1$, $x \in U_j$, $e = 1, 2$. Hence, for any $j \geqq 0$, we have

$$|\varphi_{j|\mu}(\zeta_j(x))|w^\mu \ll 3\rho^j A(w) \quad \text{for} \quad x \in U_j\,, \quad \mu \geqq 1\,.$$

Note that $A(w)$ converges absolutely for $|w| \leqq 1/c$ and $A(0) = 0$. Thus, for every $j \geqq 0$,

$$1 + \varphi_{j|1}(\zeta_j)w + \cdots + \varphi_{j|\mu}(\zeta_j)w^\mu + \cdots$$

converges to a holomorphic function φ_j absolutely and uniformly on $U_j \cap B^\varepsilon$ satisfying (5.18) provided that $\varepsilon > 0$ is sufficiently small. Then $(5.19)_\mu$, $\mu \geqq 0$, imply (5.17). This completes the proof of Proposition 5.11.

6. Construction of Σ, II. Let $\Delta = \boldsymbol{C}^*/\langle\alpha\rangle$ denote the quotient group of $\boldsymbol{C}^*$ by the multiplicative group generated by α. Then Δ is an elliptic curve since $0 < |\alpha| < 1$. By (5.2) and (5.3) the holomorphic function w on $\tilde{S}$ induces a surjective holomorphic map $\psi\colon S - C \to \Delta$. In this section, using the results of Section 5, we shall prove

PROPOSITION 6.1. *There exists a compactification Σ of $S - C$ such that*

(i) *ψ extends to a holomorphic map Ψ of Σ onto Δ,*

(ii) *Ψ maps $\Gamma = \Sigma - (S - C)$ biholomorphically onto Δ.*

First we derive several lemmas.

LEMMA 6.2. *Let X be a Riemann surface. Let Y be a relatively compact open subset of X with smooth boundary. Suppose that the closure $\bar{Y}$ of Y in X is homeomorphic to a closed annulus. Then there is a continuous function f on $\bar{Y}$ so that f is holomorphic on Y and f maps $\bar{Y}$ homeomorphically onto a closed annulus.*

When $X = \boldsymbol{C}$, this lemma is well known (e.g., [1; pp. 244–247]). The proof in [1] is valid verbatim in our situation. Briefly the argument goes as follows. The boundary ∂Y consists of two connected components γ_0, γ_1. Since ∂Y is smooth, we have a continuous function h on $\bar{Y}$ such that $h|\gamma_0 = 0$, $h|\gamma_1 = 1$, and h is harmonic on Y. Set

$$f(x) = \exp\int^x c\partial h \qquad (c \in \boldsymbol{R}) .$$

Then f is a single-valued function on $\bar{Y}$ and f maps $\bar{Y}$ homeomorphically onto a closed annulus provided that the constant c is chosen properly. Clearly f is holomorphic on Y.

In Section 5, we have defined the coordinate charts (U_j, ζ_j) on $\tilde{S}$ covering $\tilde{C}$, which will be used successively in this section. Set

$$\Pi_u = w^{-1}(u) \cap \bigcup_{j \leq 0} (U_j - C^+) \quad \text{for} \quad u \in \boldsymbol{C} .$$

Note that w is of maximal rank on $U_j - C^+$ for $j \geqq 0$. Thus Π_u is smooth for every u. Let D be the unit disk $\{t \in \boldsymbol{C} \,|\, |t| < 1\}$ and let $D^* = D - \{0\}$.

LEMMA 6.3. *There is a positive number ε so that, for each u, $0 < |u| < \varepsilon$, we have a biholomorphic map f of Π_u onto D^*, which satisfies*

$$\sup\{|f(x)| \,|\, x \in \Pi_u \cap \bigcup_{k \leq j} U_k\} \to 0 \quad as \quad j \to \infty . \tag{6.4}$$

PROOF. $\tilde{C} \cap (U_0 \cup \cdots \cup U_{2m})$ is a relatively compact subset whose boundary in $\tilde{C}$ consists of two circles (defined by the equations: $|\zeta_0| = 1/r$ and $|\zeta_{2m}| = r$). Hence there is a positive number ε so that the boundary of $w^{-1}(u) \cap (U_0 \cup \cdots \cup U_{2m})$ in $w^{-1}(u)$ consists of two circles for each u, $|u| < \varepsilon$. Fixing $u \in \boldsymbol{C}$ so that $0 < |u| < \varepsilon$, we shall show that Π_u is biholomorphic to the punctured disk D^*. Set

$$A_j = \Pi_u \cap \left(\bigcup_{k=0}^{2j} U_k\right) \quad \text{for} \quad j \geqq 0 .$$

Recalling that ζ_{2j} is defined on an open neighborhood of $\bar{U}_{2j}$, we define 1-cycles γ_j^σ, $\sigma = 1, 2$, on $\Pi_u \cap \bar{U}_{2j}$ by

$$\gamma_j^1\colon \theta \mapsto (\zeta_{2j}, w) = (e^{i\theta}/r, u)$$
$$\gamma_j^2\colon \theta \mapsto (\zeta_{2j}, w) = (re^{i\theta}, u) ,$$

where $\theta \in [0, 2\pi]$. We denote the image of any 1-cycle γ by the same symbol γ.

We divide the proof of Lemma 6.3 into four steps.

Step 1. We shall show that *A_j is biholomorphic to an annulus for* *each* $j \geqq 0$. By our choice of ε, we see from (5.3) and (5.7) that the boundary of A_j in $w^{-1}(u)$ consists of two circles, γ_0^1, γ_j^2, which are both smooth. Therefore, by Lemma 6.2, it suffices for our purpose to show that $\bar{A}_j$ is homeomorphic to a closed annulus by induction on j. Clearly $\bar{A}_0$ is identified with an annulus by the coordinate ζ_0. Suppose therefore that $\bar{A}_{j-1}$ is homeomorphic to an annulus for some $j \geqq 1$. Set $A_j^1 = A_j - A_{j-1} \cup U_{2j}$ and $A_j^2 = \bar{A}_j \cap \bar{U}_{2j}$. Then A_j^2 is an annulus whose boundary is $\gamma_j^1 \cup \gamma_j^2$. Note that $A_j^1 \subset U_{2j-1}$. Let p: $U_{2j-1} \to \boldsymbol{C}$ denote the projection to the first coordinate ζ_{2j-1}^1. Then, by (5.5), p maps $w^{-1}(u) \cap U_{2j-1}$ biholomorphically into $\boldsymbol{C}$. $p(A_j^1)$ is a compact set whose boundary consists of two disjoint circles, $p(\gamma_j^1)$ and $p(\gamma_j^2)$. Therefore A_j^1 is biholomorphic to an annulus. Since $\bar{A}_{j-1} \cap A_j^1 = \gamma_{j-1}^2$, the union $\bar{A}_{j-1} \cup A_j^1$ is (homeomorphic to) an annulus whose boundary is $\gamma_0^1 \cup \gamma_j^1$. Thus, for the same reason, $\bar{A}_j = (\bar{A}_{j-1} \cup A_j^1) \cup A_j^2$ is homeomorphic to a closed annulus.

By Lemma 6.2, we have a homeomorphism

$$f_j\colon \bar{A}_j \to \{t \in \boldsymbol{C} \mid r_j \leqq |t| \leqq 1\} \quad (0 < r_j < 1)$$

such that f_j is holomorphic on A_j. We may assume

$$\begin{cases} f_j(\gamma_0^1) = \{t \in \boldsymbol{C} \mid |t| = 1\} \\ f_j(\gamma_j^2) = \{t \in \boldsymbol{C} \mid |t| = r_j\}\ . \end{cases} \tag{6.5}$$

Step 2. Since f_j, $j \geqq 0$, are uniformly bounded on $\bar{\Pi}_u$, taking a subsequence if necessary, we may assume that the sequence $\{f_j\}$ converges to a continuous function f uniformly on each compact subset of $\bar{\Pi}_u$. In particular, f is holomorphic on Π_u. In this step we shall show that *f and ∂f are nowhere zero on Π_u.* For this purpose we define a number $\nu(t, \gamma, h)$ by

$$\nu(t, \gamma, h) = \frac{1}{2\pi i} \int_\gamma \frac{\partial h}{h - t}$$

for $t \in \boldsymbol{C}$, a 1-cycle γ and a holomorphic function h defined on a neighborhood of γ. Let γ be a 1-cycle on Π_u and let $\{t_k\}$ be a sequence of points $t_k \in \boldsymbol{C}$, $k = 0, 1, 2, \cdots$, with $t = \lim_{k\to\infty} t_k$. Then it follows from the compactness of γ that, if $t \notin f(\gamma)$, then

$$\nu(t, \gamma, f) = \lim_{k\to\infty} \nu(t_k, \gamma, f_k)\ . \tag{6.6}$$

By Step 1, γ_j^2, $j \geqq 0$, are all homologous to γ_0^1 in A_k for $j < k$, while f_k maps γ_0^1 homeomorphically onto a circle around the origin. Therefore it follows by the argument principle that

$$\nu(0, \gamma_j^2, f_k) = 1 \quad \text{for} \quad j < k\ . \tag{6.7}$$

We shall see first that f is not constant. By (6.5), $|f| = 1$ on γ_0^1, i.e., f does not vanish identically. Suppose therefore f is identically equal to a non-zero constant. Then $\nu(0, \gamma_j^2, f) = 0$. By (6.7), this contradicts (6.6).

Since f_k is a coordinate of A_k, for each $x \in \Pi_u$ there is a small 1-cycle γ_x around x on Π_u such that γ_x is homologous to zero in Π_u and $\nu(f_k(x), \gamma_x, f_k) = 1$ for sufficiently large k. Moreover we may assume $f(x) \notin f(\gamma_x)$. Now suppose $\partial f(x) = 0$ for some $x \in \Pi_u$. Then $\nu(f(x), \gamma_x, f) \geqq 2$. This contradicts (6.6). Suppose next $f(x)=0$ for some $x \in \Pi_u$. Then $\nu(0, \gamma_x, f) \geqq 1$. On the other hand, since f_k is nowhere zero, we have $\nu(0, \gamma_x, f_k) = 0$. This contradicts (6.6).

Step 3. In this step, we shall show

$$\sup\{|t| \,|\, t \in f(\Pi_u \cap A_j)\} \to 0 \quad \text{as} \quad j \to \infty . \tag{6.8}$$

Fixing $0 \leqq i < m$, set $j = \nu m + i$ for $\nu = 0, 1, 2, \cdots$. We recall that ζ_{2i} is defined on the neighborhood T_i of $C_i - \{p_i\} - \{p_{i-1}\}$ in $\tilde{S} - \{p_i\} - \{p_{i-1}\}$ and (ζ_{2i}, w) forms a system of coordinates on T_i. Set $T_j = g^{-\nu}(T_i)$. Then $\zeta_{2j} = (g^\nu)^*\zeta_{2i}$ extends to T_j and (ζ_{2j}, w) forms a system of coordinates on T_j for each j. In these coordinates, g^ν is written as

$$g^\nu: (\zeta_{2j}, w) \in T_j \mapsto (\zeta_{2j}, \alpha^\nu w) \in T_i .$$

Therefore, since $0 < |\alpha| < 1$, there exist real numbers R_j, $j \geqq 0$, such that $\lim_{j\to\infty} R_j = \infty$ and

$$\{x \in T_i \,|\, R_j^{-1} < |\zeta_{2i}(x)| < R_j,\ w(x) = \alpha^\nu u\} \subset g^\nu(\Pi_u \cap T_j) \quad \text{for} \quad j \geqq 0 .$$

Let $h_\nu = (g^{-\nu})^*f$. We identify $g^\nu(\Pi_u \cap T_j)$ with a domain in $\boldsymbol{C}$ by the coordinate ζ_{2i} and we regard h_ν as a holomorphic function on the annulus $\{\zeta_{2i} \in \boldsymbol{C} \,|\, R_j^{-1} < |\zeta_{2i}| < R_j\}$. Since h_ν, $\nu \geqq 0$, are uniformly bounded, there is a subsequence $\{h_{\nu'}\}$ of $\{h_\nu\}$ which converges to a bounded holomorphic function defined on $\boldsymbol{C}^*$. Therefore the sequence $\{h_{\nu'}\}$ converges to a constant uniformly on the compact set $\{\zeta_{2i} \,|\, r \leqq |\zeta_{2i}| \leqq r^{-1}\}$. Thus the sequence $\{d_{j'}\}$ of the diameters of $f(U_{2j'} \cap \Pi_u)$ in $\boldsymbol{C}$, $j' = \nu' m + i$, converges to zero. On the other hand, by (6.6)-(6.7) we have

$$\nu(0, \gamma_{j'}^2, f) = 1 . \tag{6.9}$$

This means that the convex hull of $f(U_{2j'} \cap \Pi_u)$ contains the origin. Therefore, by the maximum principle, $\lim_{j'\to\infty} d_{j'} = 0$ implies (6.8).

Step 4. We shall show that *$f: \Pi_u \to D^*$ is a proper map.* Suppose not. Then there is a sequence $\{x_\nu\}$ of points $x_\nu \in \Pi_u$, $\nu = 1, 2, 3, \cdots$, without accumulation points in Π_u such that the sequence $\{f(x_\nu)\}$ converges to a point y^* of D^*. Since $|y^*| > 0$, it follows from (6.8) that there exists $j \geqq 0$ such that $x_\nu \in A_j$ for all ν. Then the sequence $\{x_\nu\}$ converges to a point x^* on $\bar{A}_j$ and $y^* = f(x^*)$. From $x^* \notin \Pi_u$ it follows that $x^* \in \gamma_0^1$ and hence $|y^*| = |f(x^*)| = 1$. This contradicts $y^* \in D^*$.

By Steps 2 and 4, $f: \Pi_u \to D^*$ is a d-fold covering ($1 \leqq d < \infty$). Moreover, (6.9) shows that the degree of the map $f: \gamma_j^2 \to f(\gamma_j^2)$ is one. Since γ_j^2 is a generator of the fundamental group of Π_u, this implies $d = 1$. Thus f maps Π_u biholomorphically onto D^*. Now (6.4) follows from (6.8). q.e.d.

Take sufficiently small $\varepsilon > 0$ so that the conclusions of Proposition 5.11 and Lemma 6.3 hold. Then we have a holomorphic 2-form φ on B^ε such that φ is nowhere zero and its local expression $\varphi = \varphi_{2j}\zeta_{2j}^{-1}d\zeta_{2j} \wedge dw$ on $U_{2j} \cap B^\varepsilon$ satisfies

$$(6.10) \qquad \begin{cases} \varphi_{2j} | C_j = 1 \\ |\varphi_{2j}(x) - 1| < 1/2 \quad \text{for} \quad x \in U_{2j} \cap B^\varepsilon . \end{cases}$$

Define a holomorphic 1-form θ_u on Π_u by the formula $\varphi = \theta_u \wedge dw$ on Π_u. Then θ_u is nowhere zero on Π_u for any u, $|u| < \varepsilon$. Fix u so that $0 < |u| < \varepsilon$ and let $f: \Pi_u \to D^*$ be the biholomorphic map given by Lemma 6.3.

LEMMA 6.11. *$(f^{-1})^*\theta_u$ extends to a meromorphic 1-form on D so that the origin of D is a pole of order one.*

PROOF. In the standard coordinate t on D, we write θ_u as $\theta_u = f^*(hdt)$, where h is a holomorphic function on D^*. By the definition of θ_u, we have

$$(6.12) \qquad \varphi_{2j}\zeta_{2j}^{-1} = (f^*h)(\partial f/\partial\zeta_{2j}) \quad \text{on} \quad U_{2j} \cap \Pi_u .$$

For our purpose it suffices to show that h extends to a meromorphic function on D which has a pole of order one at the origin.

First we claim $\lim_{t \to 0} h(t)^{-1} = 0$. For simplicity let

$$L(j) = \sup \{|f(x)| \,|\, x \in U_{2j} \cap \Pi_u\} ,$$
$$U_{2j}^\delta = \{x \in U_{2j} \,|\, r + \delta < |\zeta_{2j}(x)| < r^{-1} - \delta\}$$

where $\varepsilon > 0$ is sufficiently small. Then by Cauchy's inequality we have

$$|(\partial f/\partial\zeta_{2j})(x)| \leqq L(j)/\delta \quad \text{for} \quad x \in U_{2j}^\delta \cap \Pi_u , \quad j \geqq 0 .$$

Combining this and (6.10) with (6.12), we have

$$|h(t)|^{-1} \leqq 2L(j)/r\delta \quad \text{for} \quad t \in f(U_{2j}^\delta \cap \Pi_u) , \quad j \geqq 0 .$$

By (6.4) and the maximum principle, it follows

$$(6.13) \qquad \sup \{|h(t)|^{-1} \,|\, t \in \bigcup_{k \geqq j} f(U_k \cap \Pi_u)\} \to 0 \quad \text{as} \quad j \to \infty .$$

Note that the collection of sets $\bigcup_{k \geqq j} f(U_k \cap \Pi_u) \cup \{0\}$, $j \geqq 0$, forms a neighborhood system of the origin in D. Therefore (6.13) implies $\lim_{t\to 0} h(t)^{-1} = 0$.

Now we know that h is meromorphic on D and not holomorphic at the origin. Next we claim *there is a sequence* $\{t_j\}$ *of points* $t_j \in D^*$, $j \geqq 0$, *such that* $|t_j h(t_j)|$ *are bounded with respect to* j *and* $\lim_{j\to\infty} t_j = 0$. This proves Lemma 6.11. By the argument principle we have

$$\int_{|\zeta_{2j}|=1} \left(\frac{\partial}{\partial \zeta_{2j}} \log f\right) d\zeta_{2j} = 2\pi i \quad \text{for} \quad j \geqq 0 .$$

Hence by the mean value theorem we can find a point x_j on $U_{2j} \cap \Pi_u$ such that

$$\left| f(x_j)^{-1}\left(\frac{\partial f}{\partial \zeta_{2j}}\right)(x_j)\right| \geqq 1 , \quad |\zeta_{2j}(x_j)| = 1 .$$

Let $t_j = f(x_j)$. Then, by (6.10) and (6.12), $|t_j h(t_j)| \leqq 3/2$ for all $j \geqq 0$. $\lim_{j\to\infty} t_j = 0$ follows from (6.4). q.e.d.

Let E denote the ε-disk $\{u \in \boldsymbol{C} \,|\, |u| < \varepsilon\}$.

LEMMA 6.14. *There are an open neighborhood* B *of* C^+ *in* $\tilde{S}$ *and a holomorphic function* τ *on* $B - C^+$ *such that* (τ, w) *maps* $B - C^+$ *biholomorphically onto* $D^* \times E$.

PROOF. We expand $\varphi_0(\zeta_0, w)$ into the Laurent power series in ζ_0: $\varphi_0(\zeta_0, w) = \sum_{\mu\in \boldsymbol{Z}} c_\mu(w)\zeta_0^\mu$ where $c_\mu(w)$ are holomorphic functions in w, $|w| < \varepsilon$. Since $c_0(0) = 1$ by (6.10), we have $c_0(u) \neq 0$ for any $u \in E$ provided that $\varepsilon > 0$ is sufficiently small. Thus

$$\int_{|\zeta_0|=1} c_0(u)^{-1}\theta_u = 2\pi i \quad \text{for} \quad u \in E . \tag{6.15}$$

Define a holomorphic map s of E into $U_0 \cap B^\varepsilon$ by

$$s\colon u \mapsto (\zeta_0, w) = (1, u) \quad \text{for} \quad u \in E .$$

For each $x \in B^\varepsilon - C^+$, set

$$\tau(x) = \exp\int_{s(u)}^{x} c_0(u)^{-1}\theta_u , \quad (u = w(x)) .$$

Then, since $c_0(u)^{-1}\theta_u$ depends on u holomorphically, $\tau = \tau(x)$ is a holomorphic function on $B^\varepsilon - C^+$. By (6.15) and Lemma 6.11, the restriction of τ to Π_u is a holomorphic coordinate of Π_u for each u, $0 < |u| < \varepsilon$. Note that (the extension of) ζ_0 maps $\Pi_0 = C_0 \cap (U_0 \cup U_1) - \{p_0\}$ biholomorphically onto a punctured disk. By the first line of (6.10), $\tau|\Pi_0 = \zeta_0|\Pi_0$. Thus

$$B = \{x \in B^\varepsilon - C^+ \,|\, |\tau(x)| < 1\} \cup C^+$$

is an open neighborhood of C^+ in $\widetilde{S}$ and (τ, w) maps $B - C^+$ biholomorphically onto $D^* \times E$ provided that $\varepsilon > 0$ is sufficiently small. q.e.d.

Let $E^* = E - \{0\}$. Form the union $W = w^{-1}(E^*) \cup (D \times E^*)$ by identifying each $x \in B - \widetilde{C}$ with $(\tau(x), w(x)) \in D \times E^*$. Then the map w extends to a holomorphic map ϖ of W onto E^*. First we shall show that *$\varpi: W \to E^*$ is a proper map.* Fix $u \in E^*$ arbitrarily and choose a real number ε_u such that $|\alpha|\varepsilon_u < |u| < \varepsilon_u < \varepsilon$. Let B' be the domain in B defined by the inequalities: $|\tau| < 1/2$, $|w| < \varepsilon_u$. Then $\lambda(B')$ is an open neighborhood of C in S by Lemma 6.14 and (5.4). Hence $S - \lambda(B')$ is compact. By the choice of ε_u and (5.3), we have

$$\lambda(w^{-1}(u) - B') = \lambda(w^{-1}(u)) - \lambda(B') ,$$

while λ embeds $w^{-1}(u)$ into $S - C$. Therefore $w^{-1}(u) - B'$ and hence

$$\varpi^{-1}(u) = (w^{-1}(u) - B') \cup \bar{D}_{1/2} \times \{u\}$$

are compact, where $\bar{D}_{1/2}$ is a closed disk of radius 1/2. Thus every fibre of ϖ is compact and hence ϖ is proper.

Next we shall show that *we can extend g to a biholomorphic map ρ of W into itself by setting*

$$(6.16) \qquad \begin{cases} \rho(x) = g(x) & \text{for} \quad x \in w^{-1}(E^*) \\ \rho(0, u) = (0, \alpha u) & \text{for} \quad (0, u) \in D \times E^* . \end{cases}$$

Let $\{x_\nu\}$ be a sequence of points $x_\nu \in W$, $\nu = 1, 2, \cdots$, which converges to $(0, u) \in D \times E^*$ in W. Then, since the sequence $\{\varpi(\rho(x_\nu))\}$ converges to $\alpha u \in E^*$ and ϖ is proper, the sequence $\{\rho(x_\nu)\}$ has some accumulation point only on $\varpi^{-1}(\alpha u)$. On the other hand, since ρ maps $w^{-1}(u)$ homeomorphically onto $w^{-1}(\alpha u)$, the sequence $\{\rho(x_\nu)\}$ has no accumulation points in $w^{-1}(\alpha u)$. Therefore, since $\varpi^{-1}(\alpha u) - w^{-1}(\alpha u)$ consists of one point $(0, \alpha u)$, the sequence $\{\rho(x_\nu)\}$ converges to $(0, \alpha u)$ and hence $\lim_\nu \rho(x_\nu) = \rho(\lim_\nu x_\nu)$. Thus ρ is continuous on W. Then, since $W - w^{-1}(E^*)$ is an analytic set of codimension 1 on W, it follows by Riemann's extension theorem that ρ is holomorphic on W. Moreover, since ρ is one-to-one, ρ maps W biholomorphically onto $\rho(W)$. We note that ρ preserves the fibres of ϖ.

Now we define Σ to be the complex manifold obtained from W by identifying each $y \in W$ with $\rho(y)$. Let Λ denote the canonical projection of W onto Σ. Then ϖ induces a holomorphic map Ψ of Σ onto Δ. Since ϖ is proper, Ψ is also proper and hence Σ is compact. In view of (5.2) and (5.3), we can identify $S - C$ with the open submanifold $\Lambda(w^{-1}(E^*))$ of Σ canonically. Let $\Gamma = \Sigma - (S - C)$. Then $\Gamma = \Lambda(\{0\} \times E^*)$ is a curve on Σ. Thus Σ is a compactification of $S - C$. We have $\Psi | S - C = \psi$ since they are both induced by w. Since $\Psi | \Lambda(D \times E^*)$ is induced by the projection $(t, u) \mapsto u$ of $D \times E^*$ onto E^*, it follows from (6.16) that Ψ maps $\Gamma = \Lambda(\{0\} \times E^*)$ biholomorphically onto $\Delta = \boldsymbol{C}^*/\langle\alpha\rangle$. This completes the proof of Proposition 6.1.

7. Structure of Σ and $S - C$. By (5.4) we know that C consists of m irreducible components ($m \geqq 1$). In this section, we shall prove

PROPOSITION 7.1. *$S - C$ has the structure of an affine $\boldsymbol{C}$-bundle of degree $-m$ over the elliptic curve Δ with the projection ψ.*

PROOF. We identify S with the quotient surface $\tilde{S}/\langle g\rangle$ of $\tilde{S}$ by the group generated by g. Hence for any positive integer k we have a k-fold unramified covering surface $S' = \tilde{S}/\langle g^k\rangle$ of S and a k-fold covering curve $\Delta' = \boldsymbol{C}^*/\langle\alpha^k\rangle$ of Δ. Let p denote the canonical projection of S' onto S and π that of Δ' onto Δ. Then the holomorphic function w on $\tilde{S}$ induces a holomorphic map ψ' of $S' - p^{-1}(C)$ onto Δ' such that $\psi \circ p = \pi \circ \psi'$. Suppose now that $\psi' \colon S' - p^{-1}(C) \to \Delta$ is an affine $\boldsymbol{C}$-bundle of degree d. Then, since $\pi \colon \Delta' \to \Delta$ is a k-fold covering, $\psi \colon S - C \to \Delta$ is an affine $\boldsymbol{C}$-bundle of degree d/k. Clearly $p^{-1}(C)$ is connected and consists of km irreducible components. Therefore, considering $S' - p^{-1}(C)$ instead of $S - C$, we may assume $m \geqq 3$.

Let Σ be the compactification of $S - C$ given by Proposition 6.1 so that $S - C = \Sigma - \Gamma$ and ψ extends to the holomorphic map Ψ of Σ onto Δ. The proof of Proposition 7.1 is divided into three steps.

Step 1. First we shall show $(\Gamma)^2 = m$. Note that C and $D_{m,\alpha,0}$ have the same intersection matrices and the same topological structure by (5.1)-(5.4) (cf. (3.2), (3.3) and (3.6)). Suppose now $(\Gamma)^2 < 0$. Then, since Γ is irreducible, it follows by Proposition 1.2 that Σ and hence C have strongly pseudo-convex neighborhoods in Σ and S respectively. Again by Proposition 1.2, this contradicts $(C)^2 = 0$. Thus it suffices to show $|(\Gamma)^2| = m$.

Let M be the tubular neighborhood of Γ. Then ∂M is a circle bundle of degree $\pm(\Gamma)^2$ over the elliptic curve Γ. Hence the Gysin exact homology sequence gives

$$H_1(\partial M, \boldsymbol{Z}) \cong \boldsymbol{Z} \oplus \boldsymbol{Z} \oplus (\boldsymbol{Z}/d\boldsymbol{Z}) , \quad d = |(\Gamma)^2| . \tag{7.2}$$

Let N and N_0, respectively, be the tubular neighborhoods of C in S and $D_{m,\alpha,0}$ in $S_{m,\alpha,0}$. We shall see

$$H_1(\partial M, \boldsymbol{Z}) \cong H_1(\partial N_0, \boldsymbol{Z}) . \tag{7.3}$$

In fact ∂M is homotopically equivalent to ∂N by Lemma 1.5. Since

$m \geqq 3$ by our hypothesis, C and $D_{m,\alpha,0}$ are of simple normal crossing. Therefore, according to Lemma 1.4, we may assume that ∂N and ∂N_0 are homotopically equivalent. Thus we obtain (7.3).

$S_{m,\alpha,0} - D_{m,\alpha,0}$ is a line bundle of degree $-m$ over an elliptic curve. Therefore, by Lemma 1.5, ∂N_0 is homotopically equivalent to a circle bundle of degree $\pm m$ over an elliptic curve. Hence $H_1(\partial N_0, \boldsymbol{Z}) \cong \boldsymbol{Z} \oplus \boldsymbol{Z} \oplus (\boldsymbol{Z}/m\boldsymbol{Z})$. Combining this with (7.2)-(7.3), we obtain $|(\Gamma)^2| = m$.

Step 2. Σ is obtained from a surface Σ^* free from exceptional curves of the first kind by successive quadratic transformations. Let σ denote the canonical projection of Σ onto Σ^*. Set $\Gamma^* = \sigma(\Gamma)$. Since Δ is an elliptic curve, Ψ is a constant map on each exceptional curve of the first kind on Σ. Hence Ψ induces a holomorphic map $\Psi^*\colon \Sigma^* \to \Delta$ satisfying $\Psi = \Psi^* \circ \sigma$. Note that, since $\Psi\colon \Gamma \to \Delta$ is biholomorphic, $\Psi^*\colon \Gamma^* \to \Delta$ is also biholomorphic. In this step we shall show that $\Psi^*\colon \Sigma^* \to \Delta$ is a $\boldsymbol{P}^1$-bundle.

Since $(\Gamma)^2 > 0$ and hence $(\Gamma^*)^2 > 0$, we see that Σ^* is algebraic ([8, I; p. 757, Th. 8]). Now let K denote the canonical divisor of Σ^*. Then, since Γ^* is a non-singular elliptic curve with $(\Gamma^*)^2 > 0$, we have $(\nu K - \mu\Gamma^* \cdot \Gamma^*) < 0$ for any $\nu > 0$, $\mu \geqq 0$. Hence all pluri-genera $P_\nu = \dim H^0(\Sigma^*, \mathcal{O}(\nu K))$ are zero ($\nu > 0$). On the other hand, since Ψ^* maps Σ^* onto the curve Δ holomorphically, Σ^* is not the projective plane $\boldsymbol{P}^2$. Therefore, by Enriques' theorem, Σ^* is a $\boldsymbol{P}^1$-bundle over a curve Δ' (cf. [8, IV; p. 1060, Th. 52]). Let $\Phi\colon \Sigma^* \to \Delta'$ denote the projection of the $\boldsymbol{P}^1$-bundle. Since Δ is an elliptic curve, Ψ^* is a constant map on each fibre $\Phi^{-1}(u) \cong \boldsymbol{P}^1$, $u \in \Delta'$. Hence there is a holomorphic map $\mu\colon \Delta' \to \Delta$ satisfying $\Psi^* = \mu \circ \Phi$. Moreover, since $\Psi^*\colon \Gamma^* \to \Delta$ is biholomorphic, μ is biholomorphic. Thus $\Psi^*\colon \Sigma^* \to \Delta$ is a $\boldsymbol{P}^1$-bundle.

Step 3. Now we shall show $\Sigma = \Sigma^*$ and hence $\Psi = \Psi^*$. Suppose $\Sigma \neq \Sigma^*$. Then we can write $\Sigma = Q_k Q_{k-1} \cdots Q_1(\Sigma^*)$, $(k \geqq 1)$, where Q_ν denotes the quadratic transformation with respect to the point q_ν on $Q_{\nu-1} \cdots Q_1(\Sigma^*)$. We identify $Q_\nu \cdots Q_1(\Sigma^*) - Q_\nu(q_\nu)$ with $Q_{\nu-1} \cdots Q_1(\Sigma^*) - \{q_\nu\}$ canonically. Setting $\Gamma_0 = \Gamma^*$, inductively we define Γ_ν to be the proper transform of $\Gamma_{\nu-1}$ with respect to Q_ν, $\nu = 1, 2, \cdots, k$. Thus $\Gamma = \Gamma_k$. Since $S - C = \Sigma - \Gamma$ has no exceptional curve of the first kind, we have

$$q_{k-\mu} \in \Gamma_{k-\mu-1} \quad \text{for} \quad 0 \leqq \mu \leqq k-1\,. \tag{7.4}$$

Set $F_0 = \Psi^{*-1}(\Psi^*(q_1))$. Since Ψ^* is of maximal rank on Γ_0, it follows from (7.4) that F_0 intersects Γ_0 transversally at q_1. Therefore, since F_0 is a non-singular rational curve with $(F_0)^2 = 0$, the proper transform F_1 of F_0 with respect to Q_1 is an exceptional curve of the first kind on $Q_1(\Sigma^*) - \Gamma_1$.

Moreover, by (7.4), the proper transform of F_1 with respect to $Q_k \cdots Q_2$ is an exceptional curve of the first kind on $\Sigma - \Gamma_k = S - C$. This is a contradiction.

Thus $\Psi: \Sigma \to \Delta$ is a $\boldsymbol{P}^1$-bundle. Since $\Psi: \Gamma \to \Delta$ is biholomorphic, Γ is a holomorphic section of the $\boldsymbol{P}^1$-bundle Σ. We may regard Γ as an ∞-section. Hence $S - C = \Sigma - \Gamma$ is an affine $\boldsymbol{C}$-bundle with the projection $\psi = \Psi$. Note that the linearization of the affine $\boldsymbol{C}$-bundle $S - C$ is the dual of the normal bundle $[\Gamma]_\Gamma$. Therefore the degeree of $S - C$ is $-(\Gamma)^2 = -m$. q.e.d.

8. Structure of S. In this section we shall determine the structure of S. We begin with

LEMMA 8.1. *Let M be a noncompact surface and $w: M \to \boldsymbol{C}$ a holomorphic map. Assume*

(i) *w is of maximal rank at each point of M,*

(ii) *$M - w^{-1}(0)$ is an affine $\boldsymbol{C}$-bundle over $\boldsymbol{C}^*$ with the projection w,*

(iii) *$w^{-1}(0)$ is biholomorphic to $\boldsymbol{C}^*$.*

Then there exists a holomorphic function ξ on M so that (ξ, w) maps M biholomorphically onto $\boldsymbol{C}^2 - \{0\}$.

PROOF. Set $F = w^{-1}(0)$. Since every affine $\boldsymbol{C}$-bundle over $\boldsymbol{C}^*$ is trivial, there is a holomorphic function ξ_0 on $M - F$ so that (ξ_0, w) maps $M - F$ biholomorphically onto $\boldsymbol{C} \times \boldsymbol{C}^*$. Fix $x_0 \in F$. Let $(U_0, (z_0, w))$ be a coordinate chart around x_0 such that

$$U_0 = \{(z_0, w) \mid |z_0| < 1, |w| < \varepsilon_0\}, \qquad z_0(x_0) = 0,$$

where $\varepsilon_0 > 0$ is sufficiently small. Define a holomorphic map $s: w(U_0) \to U_0$ by

$$s: u \mapsto (z_0, w) = (0, u) \quad \text{for} \quad |u| < \varepsilon_0 .$$

Define holomorphic functions $a(u)$ and $b(u)$ on $w(U_0) - \{0\}$ respectively by

$$a(u) = \frac{\partial \xi_0}{\partial z_0}(s(u)), \quad b(u) = \xi_0(s(u)) \qquad \text{for} \quad u \in w(U_0) - \{0\} .$$

Note that $a(u)$ is nowhere zero. Set

$$\eta(x) = \{\xi_0(x) - b(w(x))\}/a(w(x)) \tag{8.2}$$

for $x \in w^{-1}(w(U_0)) - F$. Then η is holomorphic on $w^{-1}(w(U_0)) - F$.

First we shall show that *η extends to $w^{-1}(w(U_0))$ holomorphically so that η maps F biholomorphically onto $\boldsymbol{C}^*$.* Take $y \in F$ arbitrarily. Then we can find finitely many points $x_1, \cdots, x_\nu, \cdots, x_k$ on F and coordinate charts $(U_\nu, (z_\nu, w))$ around x_ν, $\nu = 1, \cdots, k$, such that $x_\nu \in U_{\nu-1} \cap U_\nu$ and $x_k = y$. Moreover we may assume that for each ν, $0 \leqq \nu \leqq k$,

$$U_\nu = \{(z_\nu, w) \mid |z_\nu| < 1, |w| < \varepsilon\} \quad (\varepsilon > 0) , \qquad z_\nu(x_\nu) = 0 .$$

By induction on ν, we prove

$(8.3)_\nu$ η extends to a holomorphic function on U_ν so that $\partial\eta/\partial z_\nu$ is nowhere zero.

By (8.2) we have

$$\begin{cases} \eta(0, w) = 0 \\ \dfrac{\partial\eta}{\partial z_0}(0, w) = 1 \end{cases} \qquad \text{for} \quad (0, w) \in U_0 - F . \tag{8.4}$$

Hence the distortion inequality holds:

$$|\eta(z_0, w)| \leqq |z_0|/(1 - |z_0|)^2 \quad \text{for} \quad 0 < |w| < \varepsilon_0 .$$

In particular η is locally bounded on U_0. Therefore by Riemann's extension theorem, η extends to U_0 holomorphically. Suppose $(\partial\eta/\partial z_0)(x) = 0$ for some $x \in U_0$. Then the equation: $\partial\eta/\partial z_0 = 0$ defines an analytic subset of pure dimension one, while $\partial\eta/\partial z_0$ is nowhere zero on $U_0 - F$. Hence $\partial\eta/\partial z_0$ is identically zero on $F \cap U_0$. On the other hand by (8.4) we have $(\partial\eta/\partial z_0)(x_0) = 1$. This is a contradiction. Thus $\partial\eta/\partial z_0$ is nowhere zero on U_0. This proves $(8.3)_0$. Assume therefore that $(8.3)_{\nu-1}$ holds for some $\nu \geqq 1$. Define a holomorphic map $s_\nu: w(U_\nu) \to U_\nu$ by

$$s_\nu: u \mapsto (z_\nu, w) = (0, u) \quad \text{for} \quad u \in w(U_\nu) .$$

We may assume that $U_{\nu-1} \cap U_\nu$ contains the whole $s_\nu(w(U_\nu))$. Define holomorphic functions $a_\mu(u)$ and $b_\nu(u)$ on $w(U_\nu)$ respectively by

$$a_\nu(u) = \frac{\partial\eta}{\partial z_\nu}(s_\nu(u)) , \quad b_\nu(u) = \eta(s_\nu(u)) \qquad \text{for} \quad u \in w(U_\nu) .$$

Then $a_\nu(u)$ is nowhere zero by $(8.3)_{\nu-1}$. As before, we define a holomorphic function η_ν on $U_\nu - F$ by

$$\eta_\nu(x) = \{\eta(x) - b_\nu(w(x))\}/a_\nu(w(x)) .$$

Repeating the same argument as that for $(8.3)_0$, we obtain that η_ν extends to U_ν holomorphically so that $\partial\eta_\nu/\partial z_\nu$ is nowhere zero. Therefore $(8.3)_\nu$ holds since we can write η as $\eta = a_\nu(w)\eta_\nu + b_\nu(w)$. Thus, since $y \in F$ is arbitrary, η extends to $w^{-1}(w(U_0))$ holomorphically so that for each $u \in w(U_0)$ the restriction of $d\eta$ to $w^{-1}(u)$ is nowhere zero. By (8.2), η is one-to-one on $w^{-1}(u)$ for $u \in w(U_0) - \{0\}$. Hence, using the argument principle, we obtain that η is one-to-one on F. Thus η maps F biholomorphically onto C^*.

Now we know that $(M - F, (\xi_0, w))$ and $(w^{-1}(w(U_0)), (\eta, w))$ are coordinate charts covering M. Since the coordinate change (8.2) is an affine transformation with respect to ξ_0, we can identify M with an open submanifold of some affine $\boldsymbol{C}$-bundle over $\boldsymbol{C}$ with the projection w. Since every affine $\boldsymbol{C}$-bundle over $\boldsymbol{C}$ is trivial, there is a holomorphic function ξ on M such that ξ defines the coordinate on $w^{-1}(u)$ for each $u \in \boldsymbol{C}$. Thus, taking $\xi - v$ instead of ξ for some constant v if necessary, (ξ, w) maps M biholomorphically onto $\boldsymbol{C}^2 - \{0\}$. q.e.d.

PROPOSITION 8.5. *Let S be a compact surface free from exceptional curves of the first kind. Assume that S satisfies the conditions* (S-0)-(S-2) *with a curve C. Then S is biholomorphic to $S_{m,\alpha,t}$ and $C = D_{m,\alpha,t}$ for some $m \geqq 1$, $0 < |\alpha| < 1$, $t \in \boldsymbol{C}^m$.*

PROOF. By our hypothesis we have the unramified covering $\lambda: \tilde{S} \to S$ of S, the holomorphic function w on $\tilde{S}$ and the covering transformation g satisfying (5.1)-(5.4). We write C_j, $j \in \boldsymbol{Z}$, for the irreducible components of $\lambda^{-1}(C)$ so that $g(C_j) = C_{j-m}$ ($m \geqq 1$). Set $\varDelta = \boldsymbol{C}^*/\langle\alpha\rangle$, where $g^*w = \alpha w$ ($0 < |\alpha| < 1$). Then w induces a holomorphic map ψ of $S - C$ onto the elliptic curve $\varDelta$. By Proposition 7.1, $\psi: S - C \to \varDelta$ is an affine $\boldsymbol{C}$-bundle of degree $-m$. Therefore $w: \tilde{S} - \lambda^{-1}(C) \to \boldsymbol{C}^*$ is also an affine $\boldsymbol{C}$-bundle. Set $M = \tilde{S} - \bigcup_{j \neq 0} C_j$ and $F = C_0 \cap M$. Then F is biholomorphic to $\boldsymbol{C}^*$ by (5.1). By (5.2), w is of maximal rank at each point of M. Hence, applying Lemma 8.1, we obtain a holomorphic function ξ on M such that (ξ, w) maps M biholomorphically onto $\boldsymbol{C}^2 - \{0\}$. Since $g^*w = \alpha w$, g is of the form

$$g: (\xi, w) \mapsto (a(w)\xi + b(w), \alpha w) \quad \text{for} \quad w \neq 0\,, \tag{8.6}$$

where $a(w)$, $b(w)$ are holomorphic functions on $\boldsymbol{C}^*$ and $a(w)$ is nowhere zero on $\boldsymbol{C}^*$.

First we prove that *$a(w)$ and $b(w)$ extend to $\boldsymbol{C}$ holomorphically.* By (5.1) and (5.4) we can choose a compact neighborhood N_j of C_j for each $j \in \boldsymbol{Z}$ so that

$$\begin{cases} g(N_j) = N_{j-m} \\ N_j \cap N_k = \varnothing \quad \text{if} \quad j \neq k \pm 1\,. \end{cases} \tag{8.7}$$

Fixing $0 < i < m$, set $j(\nu) = \nu m + i$ for $\nu \geqq 0$. Then, by $g^*w = \alpha w$, we have

$$\lambda(w^{-1}(u) \cap N_{j(\nu)}) = \lambda(w^{-1}(\alpha^\nu u) \cap N_i) \quad \text{for} \quad u \in \boldsymbol{C}\,.$$

Hence, from $|\alpha| < 1$ it follows that, for each $u \in \boldsymbol{C}$, the sequence of sets $\lambda(w^{-1}(u) \cap N_{j(\nu)})$, $\nu = 0, 1, 2, \cdots$, converges to $C \cap \lambda(N_i)$. Since $C \cap \lambda(w^{-1}(u)) = \varnothing$ for $u \neq 0$, this means that, for each $u \in \boldsymbol{C}^*$,

$$\inf\{|\xi(x)|\,|\,x \in w^{-1}(u) \cap N_j\} \to \infty \quad \text{as} \quad j \to +\infty\ . \tag{8.8}$$

For sufficiently small $\delta > 0$ and $\varepsilon > 0$, we set

$$B = \{x \in M\,|\,|\xi(x)| > 1/\delta,\ |w(x)| < \varepsilon\}\ .$$

Then, by (8.8) and the maximum principle we have

$$N_k \cap B = \varnothing \quad \text{for} \quad k < 0\ . \tag{8.9}$$

Take two points q_1, q_2 on F and define holomorphic maps $s_i\colon \boldsymbol{C} \to M$, $i = 1, 2$, by

$$s_i\colon u \mapsto (\xi, w) = (\xi(q_i), u) \quad \text{for} \quad u \in \boldsymbol{C}\ .$$

We may assume that $s_i(u) \in N_0$ for $|u| < \varepsilon$, $i = 1, 2$. Then, by (8.7) and (8.9), $g(s_i(u)) \notin B$ for $|u| < \varepsilon$. That is, by (8.6),

$$|a(w)\xi(q_i) + b(w)| < 1/\delta \quad \text{for} \quad 0 < |w| < \varepsilon\ .$$

Hence, by Riemann's extension theorem, $a(w)\xi(q_i) + b(w)$, $i = 1, 2$, extend to $\boldsymbol{C}$ holomorphically. Thus $a(w)$ and $b(w)$ extend to $\boldsymbol{C}$ holomorphically since $\xi(q_1) \neq \xi(q_2)$.

Now, applying Proposition 2.5 to the holomorphic automorphism g of $\tilde{S} - \lambda^{-1}(C)$, we obtain a holomorphic function z on M and a polynomial $t(w)$ of degree $< m$ such that (z, w) forms a system of coordinates on M and g is of the form

$$g\colon (z, w) \mapsto (w^m z + t(w), \alpha w)\ , \tag{8.10}$$

taking βw, $\beta \in \boldsymbol{C}^*$, instead of w if necessary.

By (5.1)-(5.4) (cf. (3.2), (3.3) and (3.6)), we know that C and $D_{m,\alpha,t}$ are homeomorphic and have the same intersection matrices. We have $C = \bigcup_{i=0}^{m-1} \lambda(C_i)$ by (5.4). Let D_i, $0 \leqq i < m$, denote the irreducible components of $D_{m,\alpha,t}$.

The case: $m = 1$. By (5.1) and (5.4), C (resp. $D_{1,\alpha,t}$) has the unique singular point p (resp. q). Comparing (8.10) with (3.1), we see from the construction of $S_{m,\alpha,t}$ in Section 3 that $S - \{p\}$ is biholomorphic to $S_{1,\alpha,t} - \{q\}$ and $C - \{p\} = D_{1,\alpha,t} - \{q\}$. Thus by Hartogs' extension theorem we conclude that S is biholomorphic to $S_{1,\alpha,t}$ and $C = D_{1,\alpha,t}$.

The case: $m > 1$. From (5.2) and (5.3) it follows that the real first Chern class of the line bundle $[C]$ and hence the real homology class of C are zero. This implies that S is not Kählerian. Therefore, since S has no exceptional curves of the first kind, S is minimal (cf. [8, IV; p. 1065, Th. 56]). Set $P = \bigcup_{i=1}^{m-1} \lambda(C_i)$ and $Q = \bigcup_{i=1}^{m-1} D_i$. Comparing (8.10) with (3.1), we see from the construction of $S_{m,\alpha,t}$ that $S - P$ is biholomorphic to $S_{m,\alpha,t} - Q$ and $C - P = D_{m,\alpha,t} - Q$, changing the indices of D_i if necessary.

Thus both S and $S_{m,\alpha,t}$ are minimal compactifications of the same surface $S-P$. Note that $P \subsetneqq C$, $Q \subsetneqq D_{m,\alpha,t}$ and the intersection matrices of C, $D_{m,\alpha,t}$ are negative semi-definite. Hence P and Q are both exceptional by Lemma 1.1 (iii) and Proposition 1.2. Also P, Q are connected. Thus we conclude by Proposition 1.3 that S is biholomorphic to $S_{m,\alpha,t}$. Since $S-P$ (resp. $C-P$) is identified with $S_{m,\alpha,t}-Q$ (resp. $D_{m,\alpha,t}-Q$), we have $C=D_{m,\alpha,t}$. q.e.d.

9. Proof of Main theorem. Let S and D be as in the Main theorem. Let C denote the support of D. Then S and C satisfy the conditions (S-0)-(S-2) by Proposition 4.18 (see the beginning of Section 5). Thus, by Proposition 8.5, $S=S_{m,\alpha,t}$ and $C=D_{m,\alpha,t}$. By Lemma 1.1 (ii) we have $D=rD_{m,\alpha,t}$ for some $r \in \boldsymbol{Z}$. Finally from $b_2(S)=n$ it follows $m=n$.

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DEPARTMENT OF MATHEMATICS
SOPHIA UNIVERSITY
KIOI-CHO 7, CHIYODA-KU
TOKYO, 102
JAPAN