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PINCHING DEFORMATIONS OF FUCHSIAN GROUPS

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Introduction. Limits of sequences of Kleinian groups have been investigated by many authors (cf. Abikoff [1], Bers [2], Chuckrow [3], Marden [4]). Let $\{w_n\}_{n=1}^{\infty}$ be a sequence of quasiconformal automorphisms of the extended complex plane $\hat{C}$ compatible with a Kleinian group Γ such that $w_n \Gamma w_n^{-1}$ converges to a group of Moebius transformations in the sense of generator convergence. Then there exists an isomorphism ϕ_G of Γ onto G (Chuckrow [3, Theorem 6]). The group G is called a cusp if there exists a loxodromic element γ of Γ such that $\phi_G(\gamma)$ is parabolic. Though cusps play important roles in the theory of Kleinian groups, even the existence of cusps is unknown in general. Let Γ be a finitely generated torsion free Fuchsian group of the first kind keeping U and L, the upper and the lower half planes, invariant. In this note we show the existence of cusps which are limits of sequences $\{w_n \Gamma w_n^{-1}\}_{n=1}^{\infty}$ of quasi-Fuchsian groups, where the automorphisms w_n of $\hat{C}$ are not conformal but quasi-conformal in $U \cup L$. This is an affirmative answer to the problem raised by Marden [4, p. 290]

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1. Preliminaries. Let G be a group of Moebius transformations acting on the Riemann sphere $\hat{C} = C \cup \{\infty\}$. The ordinary set $\Omega(G)$ of G is the maximal subset of $\hat{C}$ where G acts discontinuously. The group G is said to be Kleinian if $\Omega(G)$ is not void and if $\hat{C} - \Omega(G)$ contains more than two points. If a Kleinian group keeps the upper half plane U invariant, then the group is said to be Fuchsian. Throughout this note Γ denotes a finitely generated Fuchsian group of the first kind without elliptic elements. Let F be a quasi-conformal automorphism of $\hat{C}$ compatible with Γ, that is, $F\Gamma F^{-1}$ is again Kleinian. Then F induces a quasi-conformal homeomorphism f of the quotient space $\Omega(\Gamma)/\Gamma$ onto $\Omega(F\Gamma F^{-1})/F\Gamma F^{-1}$ with $\Pi \circ F = f \circ \pi$, where $\Pi: \Omega(F\Gamma F^{-1}) \to \Omega(F\Gamma F^{-1})/F\Gamma F^{-1}$ and $\pi: \Omega(\Gamma) \to \Omega(\Gamma)/\Gamma$ are natural projections.

A set $\{\alpha_i\}_{i=1}^{q}$ of simple analytic loops on a Riemann surface is said to be homotopically independent, if the following is satisfied:

(i) α_i and α_j are mutually disjoint, $1 \leqq i < j \leqq q$,

(ii) α_i is not freely homotopic to α_j, $1 \leqq i < j \leqq q$,

and

(iii) α_i bounds neither a disc nor a punctured disc, $1 \leqq i \leqq q$.

Let $\{\alpha_i\}_{i=1}^{p} \subset U/\Gamma$ and $\{\alpha_i\}_{i=p+1}^{q} \subset L/\Gamma$ be homotopically independent sets of loops, where L is the lower half plane. Then we can find a doubly connected region D_i containing α_i with $\mathrm{Cl}\, D_i \cap \mathrm{Cl}\, D_j = \emptyset$, $1 \leqq i < j \leqq q$. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of quasi-conformal automorphisms of $\hat{C}$ compatible with Γ and keeping three points in $\Omega(\Gamma) - \pi^{-1}(\bigcup_{i=1}^{q} \alpha_i)$ invariant such that on any set $E \Subset \Omega(\Gamma) - \pi^{-1}(\bigcup_{i=1}^{q} \alpha_i)$ F_n is uniformly $K(E)$-quasi-conformal and such that $f_n(D_i)$ is conformally equivalent to the annulus $1 < |z| < n$, $1 \leqq i \leqq q$. Then Γ is said to be pinched along $\{\alpha_i\}_{i=1}^{q}$ by $\{F_n\}_{n=1}^{\infty}$ if n tends to ∞.

A sequence of Moebius transformations $z \to (a_n z + b_n)/(c_n z + d_n)$ is said to converge to another $z \to (az + b)/(cz + d)$ if a_n, b_n, c_n and d_n converge to a, b, c and d, respectively. A group of Moebius transformations generated by $g_{1,n}, \cdots, g_{t,n}$ is said to converge to that generated by $g_1, \cdots, g_t$ in the sense of generator convergence, if $g_{s,n}$ converges to g_s, $1 \leqq s \leqq t$. Denote by $\tau(\alpha_i)$ the set of all elements of Γ keeping a component of $\pi^{-1}(\alpha_i)$ invariant.

2. Statement of Theorem. The purpose of this note is to prove the following, which gives an answer to a problem raised by Marden [4, p. 290].

THEOREM. *Let Γ be a finitely generated Fuchsian group of the first kind without elliptic elements. Let $\{\alpha_i\}_{i=1}^{p} \subset U/\Gamma$ and $\{\alpha_i\}_{i=p+1}^{q} \subset L/\Gamma$ be homotopically independent sets of geodesic loops such that $(\bigcup_{i=1}^{p} \tau(\alpha_i)) \cap (\bigcup_{i=p+1}^{q} \tau(\alpha_i))$ consists only of the identity. If Γ is pinched along $\{\alpha_i\}_{i=1}^{q}$ by $\{F_n\}_{n=1}^{\infty}$, then there exists a subsequence $\{F_{n_k}\}_{k=1}^{\infty}$ of $\{F_n\}_{n=1}^{\infty}$ such that $F_{n_k}\Gamma F_{n_k}^{-1}$ converges to a Kleinian group in the sense of generator convergence. Moreover $F_{n_k}\gamma F_{n_k}^{-1}$ converges to a parabolic transformation for each $\gamma \in \bigcup_{i=1}^{q} \tau(\alpha_i)$.*

Bers [2] proved Theorem in the case $p(q-p) = 0$. See also Abikoff [1]. Therefore we give a proof of Theorem in the case $p(q-p) \neq 0$. Note that the second statement of Theorem is clear from Bers' $\log \lambda$ inequality (Bers [2]) and the first one.

3. Lemmas. For two points $x, y \in \hat{C}$, denote by $[x, y]$ the spherical distance between x and y. For a loxodromic Moebius transformation g, which may be hyperbolic, denote by $\xi(g)$ and $\xi'(g)$ the attracting and repelling fixed points, respectively. First we state a well-known result without proof.

LEMMA 1. *Let* $\{z_{i,n}\}_{n=1}^{\infty}$ *and* $\{z'_{i,n}\}_{n=1}^{\infty}$ *be sequences of points of* $\hat{C}$ *converging to* z_i *and* z'_i, *respectively,* $1 \leqq i \leqq 3$, *such that the sequences* $\{[z_{i,n}, z_{j,n}]\}_{n=1}^{\infty}$ *and* $\{[z'_{i,n}, z'_{j,n}]\}_{n=1}^{\infty}$ *are bounded away from zero,* $1 \leqq i < j \leqq 3$. *Let* $\{h_n\}_{n=1}^{\infty}$ *be a sequence of Moebius transformations with* $h_n(z_{i,n}) = z'_{i,n}$, $1 \leqq i \leqq 3$. *Then* h_n *converges to a Moebius transformation.*

LEMMA 2. *Let* G *be a Kleinian group keeping a region* Ω_0 *invariant. Let* $\{w_n\}_{n=1}^{\infty}$ *be a sequence of quasi-conformal automorphisms of* $\hat{C}$ *compatible with* G *such that for each region* $A \Subset \Omega_0$ *the restriction* $w_n|A$ *of* w_n *to* A *is uniformly* $K(A)$-*quasi-conformal. Assume the existence of a loxodromic element* g *of* G *and of a point* z_0 *of* Ω_0 *such that* $w_n g w_n^{-1}$ *converges to a parabolic transformation and such that* $\{[w_n(z_0), \xi(w_n g w_n^{-1})]\}_{n=1}^{\infty}$ *is bounded away from zero. Then, for each region* $B \Subset \Omega_0$ *there exists a subsequence* $\{w_{n_k}\}_{k=1}^{\infty}$ *such that* $w_{n_k}|B$ *converges to a* $K(B)$-*quasi-conformal homeomorphism uniformly on* B *and such that* $w_{n_k} G w_{n_k}^{-1}$ *converges to a group of Moebius transformations in the sense of generator convergence.*

PROOF. Since $w_n g w_n^{-1}$ converges to a parabolic transformation, $[\xi(w_n g w_n^{-1}), \xi'(w_n g w_n^{-1})]$ converges to zero. Therefore, both $\{[w_n(z_0), \xi(w_n g w_n^{-1})]\}_{n=1}^{\infty}$ and $\{[w_n(z_0), \xi'(w_n g w_n^{-1})]\}_{n=1}^{\infty}$ are bounded away from zero, and so is $\{[(w_n g w_n^{-1})^u(w_n(z_0)), (w_n g w_n^{-1})^v(w_n(z_0))]\}_{n=1}^{\infty} = \{[w_n(g^u(z_0)), w_n(g^v(z_0))]\}_{n=1}^{\infty}$, $0 \leqq u < v \leqq 2$. Let $\{g_1, \cdots, g_t\}$ be a system of generators for G. Let $\hat{B} \Subset \Omega_0$ be a region containing the set $B \cup \{g^u(z_0); 0 \leqq u \leqq 2\} \cup \{g_s^j(z_i); 1 \leqq i \leqq 3, 0 \leqq j \leqq 1, 1 \leqq s \leqq t\}$, where z_i is a point of Ω_0. Since $\{w_n | \hat{B}\}_{n=1}^{\infty}$ is a normal family (Lehto-Virtanen [5, p. 73]), there exists a subsequence $\{w_{n_k}\}_{k=1}^{\infty}$ of $\{w_n\}_{n=1}^{\infty}$ such that $w_{n_k}|B$ converges to a mapping W of B. Since $[W(g^u(z_0)), W(g^v(z_0))] > 0$, $0 \leqq u < v \leqq 2$, the mapping W is a $K(B)$-quasi-conformal homeomorphism of B (Lehto-Virtanen [5, p. 74]). Therefore $w_{n_k}|B$ converges to a $K(B)$-quasi-conformal homeomorphism of B. In Lemma 1, we put $z_{i,k} = w_{n_k}(z_i)$, $z'_{i_k} = w_{n_k}(g_s(z_i))$ and $h_j = w_{n_k} g_s w_{n_k}^{-1}$ to obtain that $w_{n_k} g_s w_{n_k}^{-1}$ converges to a Moebius transformation, $1 \leqq s \leqq t$, so that $w_{n_k} G w_{n_k}^{-1}$ converges to a group of Moebius transformations in the sense of generator convergence.

LEMMA 3. *Let* G *be a Kleinian group keeping a region* Ω_0 *invariant. Let* $\{w_n\}_{n=1}^{\infty}$ *be a sequence of quasi-conformal automorphisms of* $\hat{C}$ *compatible with* G *such that for each region* $A \Subset \Omega_0$ *the restriction* $w_n|A$ *of* w_n *to* A *is uniformly* $K(A)$-*quasi-conformal. Let* g_1 *and* g_2 *be loxodromic elements of* G *such that* g_1 *and* g_2 *are not commutative. Assume that* *both* $w_n g_1 w_n^{-1}$ *and* $w_n g_2 w_n^{-1}$ *converge to parabolic transformations. Then for each region* $B \Subset \Omega_0$ *there exists a subsequence* $\{w_{n_k}\}_{k=1}^{\infty}$ *of* $\{w_n\}_{n=1}^{\infty}$ *such that* $w_{n_k}|B$ *converges to a* $K(B)$*-quasi-conformal homeomorphism of* B *uniformly on* B *and such that* $w_{n_k} G w_{n_k}^{-1}$ *converges to group of Moebius transformations in the sense of generator convergence.*

PROOF. Since g_1 and g_2 are not commutative, neither are $\hat{g}_1 = \lim_{n\to\infty} w_n g_1 w_n^{-1}$ and $\hat{g}_2 = \lim_{n\to\infty} w_n g_2 w_n^{-1}$ (Chuckrow [3]). Therefore the fixed point of $\hat{g}_1$ and that of $\hat{g}_2$ are distinct from each other. So $\{[\xi(w_n g_1 w_n^{-1}), \xi(w_n g_2 w_n^{-1})]\}_{n=1}^{\infty}$ is bounded away from zero. Let z_0 be a point Ω_0. Then there exists a subsequence $\{w_{n_k}\}_{k=1}^{\infty}$ of $\{w_n\}_{n=1}^{\infty}$ such that at least one of $\{[w_{n_k}(z_0), \xi(w_{n_k} g_1 w_{n_k}^{-1})]\}_{k=1}^{\infty}$ and $\{[w_{n_k}(z_0), \xi(w_{n_k} g_2 w_{n_k}^{-1})]\}_{k=1}^{\infty}$ is bounded away from zero. Using Lemma 2, we obtain the desired conclusion.

Let $A \subset \hat{C}$ be a domain with more than two boundary points. Then, as is well known, the Poincaré metric $\rho_A(z)|dz|$ with the negative constant curvature -1 can be defined on A. We denote by $d(z', z''; A)$ the distance measured by $\rho_A(z)|dz|$.

LEMMA 4. *Let* $\{w_n\}_{n=1}^{\infty}$ *be a sequence of* K*-quasi-conformal homeomorphisms of a domain* A_0 *with more than two boundary points and let* z' *and* z'' *be points of* A_0. *Then* $d(w_n(z'), w_n(z''); A_n)$ *is bounded, where* $A_n = w_n(A_0)$.

PROOF. Let $\Delta_n = \{|\zeta| < 1\}$ be the universal covering surface of A_n with the natural projection $\tilde{\pi}_n$, $n = 0, 1, \cdots$. Let $\tilde{w}_n$ be the K-quasi-conformal homeomorphism of Δ_0 onto Δ_n keeping 0 and 1 invariant such that $w_n \tilde{\pi}_0 = \tilde{\pi}_n \tilde{w}_n$. Let ζ' and ζ'' be points of Δ_0 with $\tilde{\pi}_0(\zeta') = z'$ and $\tilde{\pi}_0(\zeta'') = z''$, respectively. Then $d(w_n(z'), w_n(z''); A_n) \leqq d(w_n(\zeta'), w_n(\zeta''); \Delta_n) \leqq \phi_K \cdot d(\zeta', \zeta''; \Delta_0)$, where ϕ_K is a positive constant depending only on K (Lehto-Virtanen [5, p. 65]). Thus we have proved Lemma 4.

4. **Proof of Theorem.** In this section we give the proof of Theorem, which are divided into Lemmas 5–14.

For the sake of simplicty, we merely say that a sequence $\{x_n\}_{n=1}^{\infty}$ converges when a subsequence of $\{x_n\}_{n=1}^{\infty}$ does. This convention will be valid from here to the end of this note.

Let Ω_1 be a component of $U - \pi^{-1}(\bigcup_{i=1}^{p} \alpha_i)$ and $\tilde{\alpha}$ a bounded component of $\pi^{-1}(\bigcup_{i=1}^{p} \alpha_i)$ lying on the boundary $\partial\Omega_1$ of Ω_1. Let δ be a hyperbolic element of the stabilizer subgroup $\mathrm{Stab}\,\tilde{\alpha} = \{\gamma \in \Gamma; \gamma(\tilde{\alpha}) = \tilde{\alpha}\}$ of $\tilde{\alpha}$ in Γ. Denote by Λ the anti-conformal automorphism of $\boldsymbol{C}$ mapping z into the complex conjugate of z. By the assumption that $(\bigcup_{i=1}^{p} \tau(\alpha_i)) \cap (\bigcup_{i=p+1}^{q} \tau(\alpha_i))$ consists only of the identity, we can find a component Ω_1^* of $L -$

$\pi^{-1}(\bigcup_{i=p+1}^{q} \alpha_i)$ with $\Lambda(\tilde{\alpha}) \cap \Omega_1^* \neq \emptyset$.

If $\Lambda(\tilde{\alpha}) \subset \Omega_1^*$, then we fix a point ζ_1 of $\Lambda(\tilde{\alpha})$. If $\Lambda(\tilde{\alpha}) \not\subset \Omega_1^*$, then we denote by η_1 the point on $\partial\Omega_1^* \cap \Lambda(\tilde{\alpha})$ such that the hyperbolic half line joining η_1 to the repelling fixed point of δ does not meet Ω_1^*. Let $\hat{\alpha}$ be the hyperbolic segment joining η_1 to $\delta(\eta_1)$. Let $\eta_1, \cdots, \eta_{t-1}, \eta_t = \delta(\eta_1)$ be the complete list of $\pi^{-1}(\bigcup_{i=p+1}^{q} \alpha_i) \cap \hat{\alpha}$ such that η_s separates η_{s-1} from η_{s+1}, $2 \leqq s \leqq t-1$. Denote by θ_s the component of $\pi^{-1}(\bigcup_{i=p+1}^{q} \alpha_i)$ containing η_s and by ζ_s the fixed point of some $\gamma_s \in \operatorname{Stab} \theta_s$ in the bounded domain surrounded by $\operatorname{Cl}(\tilde{\alpha} \cup \Lambda(\tilde{\alpha}))$, $1 \leqq s \leqq t$.

In either case, let w_n be the quasi-conformal automorphism of $\hat{C}$ keeping $\xi(\delta)$, ζ_1 and $\delta(\zeta_1)$ invariant with the same Beltrami coefficient on $\hat{C}$ as F_n.

LEMMA 5. *The loxodromic transformation $w_n \delta w_n^{-1}$ converges to a parabolic one.*

PROOF. Let h be the Moebius transformation mapping ζ_1, $\delta(\zeta_1)$ and $\xi(\delta)$ into $0, 1$ and ∞, respectively. Then $\delta_n = h w_n \delta w_n^{-1} h^{-1}$ is of the form $z \to a_n z + b_n$. Since $1 = \delta_n(0) = b_n$ and since $a_n \to 1$ (Bers [2]), δ_n converges to a parabolic transformation, so does $w_n \delta w_n^{-1} = h^{-1} \delta_n h$.

LEMMA 6. *If $\Lambda(\tilde{\alpha}) \subset \Omega_1^*$, then $w_n(\operatorname{Stab} \Omega_1^*) w_n^{-1}$ converges to a group of Moebius transformations in the sense of generator convergence.*

PROOF. Note that for each region $A \Subset \Omega_1^*$, the restriction of w_n to A is uniformly $K(A)$-quasi-conformal by the definition of pinching deformations. By Lemma 5, $w_n \delta w_n^{-1}$ converges to a parabolic transformation. Note that $w_n(\zeta_1) = \zeta_1 \in \Omega_1^*$ and $\xi(w_n \delta w_n^{-1}) = w_n(\xi(\delta)) = \xi(\delta)$. Then $\{[w_n(\zeta_1), \xi(w_n \delta w_n^{-1})]\}_{n=1}^{\infty}$ is bounded away from zero. So our assertion is evident from Lemma 2.

LEMMA 7. *If $\Lambda(\tilde{\alpha}) \not\subset \Omega_1^*$, then for each component Ω^* of $L - \pi^{-1}(\bigcup_{i=p+1}^{q} \alpha_i)$ with $\Omega^* \cap \Lambda(\tilde{\alpha}) \neq \emptyset$, $w_n(\operatorname{Stab} \Omega^*) w_n^{-1}$ converges to a group of Moebius transformations in the sense of generator convergence.*

PROOF. Let Ω_s^* be the component of $L - \pi^{-1}(\bigcup_{i=p+1}^{q} \alpha_i)$ whose closure contains both θ_s and θ_{s+1}. Then both γ_s and γ_{s+1} belong to $\operatorname{Stab} \Omega_s^*$. Since $\sum_{s=1}^{t-1} [w_n(\zeta_s), w_n(\zeta_{s+1})] \geqq [w_n(\zeta_1), w_n(\zeta_t)] = [\zeta_1, \zeta_t]$, there exists an integer $r \in \{1, \cdots, t-1\}$ such that $\{[w_n(\zeta_r), w_n(\zeta_{r+1})]\}_{n=1}^{\infty}$ is bounded away from zero. Let $\{h_n\}_{n=1}^{\infty}$ be a sequence of Moebius transformations converging to a Moebius transformation such that $h_n(w_n(\zeta_r)) = 0$ and $h_n(w_n(\zeta_{r+1})) = \infty$. Then $\gamma_{r,n} = h_n w_n \gamma_r w_n^{-1} h_n^{-1}$ is of the form $z \to a_n z/(c_n z + a_n^{-1})$ and $\gamma_{r+1,n} = h_n w_n \gamma_{r+1} w_n^{-1} h_n^{-1}$ is of the form $z \to (u_n z + v_n)/u_n^{-1}$. It was proved in Bers

[2] that $\lim_{n\to\infty} a_n^2 = 1$ and $\lim_{n\to\infty} u_n^2 = 1$. By Lemma 5, $w_n\delta w_n^{-1}$ converges to a Moebius transformation, and so does $h_n w_n \delta w_n^{-1} h_n^{-1}$. Therefore, on applying a result of Chuckrow [3, Lemma 4] to the two generator groups $\langle \gamma_{r,n}, h_n w_n \delta w_n^{-1} h_n^{-1}\rangle$ and $\langle \gamma_{r+1,n}, h_n w_n \delta w_n^{-1} h_n^{-1}\rangle$, we see that $\lim_{n\to\infty} c_n \neq 0$ and $\lim_{n\to\infty} v_n \neq 0$. Assume that $\lim_{n\to\infty} c_n = \infty$. For a point $z^* \in \Omega_i^*$, at least one of $\{h_n w_n(z^*)\}_{n=1}^\infty$ and $\{\gamma_{r+1,n}(h_n w_n(z^*))\}_{n=1}^\infty = \{u_n^2 h_n w_n(z^*)) + u_n v_n\}_{n=1}^\infty$ is bounded away from zero. Denote the point by z_n. Then $\lim_{n\to\infty} \gamma_{r,n}(z_n) = 0$ since $\lim_{n\to\infty} c_n = \infty$ and since $\lim_{n\to\infty} |a_n| = 1$. Note that the point $z_0 = (h_n w_n)^{-1}(z_n) \in \Omega_1^*$ is constant. Let $\zeta_n \in \{z \in \boldsymbol{C};\ |z| = 1\}$ be a point in the limit set of the quasi-Fuchsian group $h_n w_n \Gamma w_n^{-1} h_n^{-1}$ which contains both 0 and ∞. Let $A \Subset \Omega^*$ be a region containing z_0 and $\gamma_r(z_0)$. By Lemma 2 we see that

$$\begin{aligned}\infty > M &\geqq d(h_n w_n(z_0),\, h_n w_n(\gamma_r(z_0));\, h_n w_n(A))\\ &\geqq d(z_n,\, \gamma_{r,n}(z_n);\, \boldsymbol{C} - \{0, \zeta_n\}) \to \infty\,, \quad \text{as} \quad n \to \infty\,.\end{aligned}$$

Because of this contradiction, we see that $\lim_{n\to\infty} c_n$ is a non-zero and finite complex number and that $\gamma_{r,n}$ converges to a parabolic transformation. In the same way as above, we can prove that $\gamma_{r+1,n}$ also converges to a parabolic transformation. By Lemma 3, $h_n w_n(\mathrm{Stab}\,\Omega_r^*) w_n^{-1} h_n^{-1}$ converges to a group of Moebius transformations in the sense of generator convergence, and so does $w_n(\mathrm{Stab}\,\Omega_r^*) w_n^{-1}$. Set $\gamma_{s+t-1} = \delta\gamma_s\delta^{-1}$ and $\Omega_{s+t-1}^* = \delta(\Omega_s^*)$, $s = 2, \cdots, r-1$. Note that $\sum_{s=r+1}^{t+r-2} [\xi(w_n\gamma_{s+1}w_n^{-1}), \xi(w_n\gamma_s w_n^{-1})] \geqq [\xi(w_n\gamma_{t+r-1}w_n^{-1}), \xi(w_n\gamma_{r+1}w_n^{-1})]$. Assume that the left hand side of the above inequality converges to zero. Then so does the right hand side. Since two parabolic transformations $\hat{\gamma}_{r+1} = \lim_{n\to\infty} w_n\gamma_{r+1}w_n^{-1}$ and $\hat{\gamma}_{t+r-1} = \lim_{n\to\infty} w_n\gamma_{t+r-1}w_n^{-1} = \lim_{n\to\infty} w_n\delta w_n^{-1}\cdot w_r\gamma_n w_n^{-1}\cdot w_n\delta w_n^{-1}$ have a common fixed point $\lim_{n\to\infty}\xi(w_n\gamma_{r+1}w_n^{-1})$, we see that $\hat{\gamma}_{r+1}$ and $\hat{\gamma}_{t+r-1}$ are commutative. On the other hand, since γ_{r+1} and γ_{t+r-1} are not commutative, neither are $\hat{\gamma}_{r+1}$ and $\hat{\gamma}_{t+r-1}$ (Chuckrow [3, Theorem 6]). This is a contradiction. So we can find some $\gamma_m \in \{\gamma_{r+1}, \cdots, \gamma_{t+r-2}\}$ such that $\{[\xi(w_n\gamma_m w_n^{-1}), \xi(w_n\gamma_{m+1}w_n^{-1})]\}_{n=1}^\infty$ is bounded away from zero. In the same way as above, we can prove that $w_n(\mathrm{Stab}\,\Omega_m^*)w_n^{-1}$ converges to a group of Moebius transformations in the sense of generator convergence. Therefore so does $w_n(\mathrm{Stab}\,\Omega_s^*)w_n^{-1}$, where $s = m$ if $1 \leqq m \leqq t-1$, and $s = m - t + 1$ if $t \leqq m \leqq t + r - 1$. Note that s is distinct from r. Repeat this procedure finitely many times. Then $w_n(\mathrm{Stab}\,\Omega_s^*)w_n^{-1}$ converges to a group of Moebius transformations in the sense of generator convergence, $1 \leqq s \leqq t-1$. Let Ω^* be a component of $L - \pi^{-1}(\bigcup_{i=p+1}^q \alpha_i)$ with $\Lambda(\tilde{\alpha}) \cap \Omega^* \neq \emptyset$. Then there exist some $s \in \{1, \cdots, t\}$ and some integer l with $\mathrm{Stab}\,\Omega^* = \delta^l(\mathrm{Stab}\,\Omega_s^*)\delta^{-l}$. Since $w_n(\mathrm{Stab}\,\Omega_s^*)w_n^{-1}$ converges to a group of Moebius transformations in the sense of generator convergence, so does

$w_n(\mathrm{Stab}\,\Omega^*)w_n^{-1} = w_n\delta^l w_n^{-1}\cdot w_n(\mathrm{Stab}\,\Omega_s^*)w_n^{-1}\cdot w_n\delta^{-l}w_n^{-1}$. Thus we complete the proof of Lemma 7.

LEMMA 8. *The Kleinian group $w_n(\mathrm{Stab}\,\Omega_1)w_n^{-1}$ converges to a group of Moebius transformations in the sense of generator convergence.*

PROOF. First we consider the case where $\Lambda(\pi^{-1}(\bigcup_{i=p+1}^q \alpha_i) \cap \partial\Omega_1^*) \cap \Omega_1 = \emptyset$. In this case it holds that $\Omega_1 \subset \Lambda(\Omega_1^*)$, so that $\mathrm{Stab}\,\Omega_1 \subset \mathrm{Stab}\,\Omega_1^*$. Then our assertion is clear by Lemmas 6 and 7.

Next we consider the other case. Then there exists a component α^* of $\pi^{-1}(\bigcup_{i=p+1}^q \alpha_i) \cap \partial\Omega_1^*$ with $\Lambda(\alpha^*) \cap \Omega_1 \neq \emptyset$. If $\Lambda(\alpha^*) \subset \Omega_1$, then a $\gamma^* \in \mathrm{Stab}\,\alpha^*$ belongs to $\mathrm{Stab}\,\Omega_1 \cap \mathrm{Stab}\,\Omega_1^*$. Since loxodromic transformations δ and γ^* are not commutative and since both $w_n\delta w_n^{-1}$ and $w_n\gamma^* w_n^{-1}$ converge to parabolic transformations by Lemmas 5 and 6, our assertion is evident from Lemma 3. If $\Lambda(\alpha^*) \not\subset \Omega_1$, then we use Lemma 7 here. Let h_n be the Moebius transformation mapping $\xi(w_n\delta w_n^{-1})$, $\xi(w_n\gamma^*\delta\gamma^{*-1}w_n^{-1})$ and $\xi(w_n\gamma^* w_n^{-1})$ into 0, 1 and ∞, respectively. Since all $w_n\delta w_n^{-1}$, $w_n\gamma^*\delta\gamma^{*-1}w_n^{-1}$ and $w_n\gamma^* w_n^{-1}$ converge to parabolic transformations by Lemmas 5, 6 and 7, all points $\lim_{n\to\infty}\xi(w_n\delta w_n^{-1})$, $\lim_{n\to\infty}\xi(w_n\gamma^*\delta\gamma^{*-1}w_n^{-1})$ and $\lim_{n\to\infty}\xi(w_n\gamma^* w_n^{-1})$ are distinct from one another (Chuckrow [3, Theorem 6]). Therefore h_n converges to a Moebius transformation by Lemma 1. In Lemma 7 we put $\tilde{\alpha} = \alpha^*$, $\Omega_1 = \Omega_1^*$ and $w_n = h_n w_n$ to obtain the conclusion that $h_n w_n(\mathrm{Stab}\,\Omega_1)w_n^{-1}h_n^{-1}$ converges to a group of Moebius transformations in the sense of generator convergence. Therefore $w_n(\mathrm{Stab}\,\Omega_1)w_n^{-1}$ also does, and we have proved Lemma 8.

Here we show an auxiliary lemma for our later use.

LEMMA 9. *Let $\{G_i\}_{i\in I}$ and $\{G'_j\}_{j\in J}$ be families of Kleinian groups. Let $\{w_n\}_{n=1}^\infty$ and $\{w'_n\}_{n=1}^\infty$ be sequences of quasi-conformal automorphisms of $\hat{C}$ compatible with each G_i and with each G'_j, respectively, such that w_n and w'_n have the same Beltrami coefficients and such that $w_n G_i w_n^{-1}$ and $w'_n G'_j w_n'^{-1}$ converge to groups of Moebius transformations for each $i \in I$ and $j \in J$. Assume that $G_1 \cap G'_1$ contains a non-elementary Kleinian group. Then $w_n G'_j w_n^{-1}$ converges to a group of Moebius transformations in the sense of generator convergence for each $j \in J$.*

PROOF. Set $h_n = w'_n w_n^{-1}$. Then h_n is a Moebius transformation. Let g_1, g_2 and g_3 be loxodromic elements of $G_1 \cap G'_1$ such that g_l and g_m are not commutative, $1 \leqq l < m \leqq 3$. Note that $\lim_{n\to\infty} w_n(\xi(g_l)) = \lim_{n\to\infty}\xi(w_n g_l w_n^{-1}) \neq \lim_{n\to\infty}\xi(w_n g_m w_n^{-1}) = \lim_{n\to\infty} w_n(\xi(g_m))$ and that $\lim_{n\to\infty} w'_n(\xi(g_l)) = \lim_{n\to\infty}\xi(w'_n g_l w_n'^{-1}) \neq \lim_{n\to\infty}\xi(w'_n g_m w_n'^{-1}) = \lim_{n\to\infty} w'_n(\xi(g_m))$ (Chuckrow [3, Theorem 6] and [6]). Then h_n mapping $w_n(\xi(g_m))$ into

$w'_n(\xi(g_m))$ converges to a Moebius transformation by Lemma 1. Since $w'_n G'_j w'^{-1}_n$ converges to a group of Moebius transformations in the sense of generator convergence, so does $w_n G'_j w_n^{-1} = h_n^{-1} \cdot w'_n G'_j w'^{-1}_n \cdot h_n$.

Now we return to the proof of Theorem.

LEMMA 10. *Let $\tilde{\Omega}$ be a component of $L - \pi^{-1}(\bigcup_{i=p+1}^{q} \alpha_i)$ with $\Lambda(\tilde{\Omega}) \cap \Omega_1 \neq \emptyset$. Then $w_n(\mathrm{Stab}\,\tilde{\Omega})w_n^{-1}$ converges to a group of Moebius transformations in the sense of generator convergence.*

PROOF. First we consider the case where $\Lambda(\tilde{\Omega}) \subset \Omega_1$. Then $\mathrm{Stab}\,\tilde{\Omega} = \mathrm{Stab}\,\Lambda(\tilde{\Omega}) \subset \mathrm{Stab}\,\tilde{\Omega}_1$. Therefore our assertion is evident by Lemma 8.

Next we consider the other case. In this case there exists a component of $\pi^{-1}(\bigcup_{i=1}^{p} \alpha_i) \cap \partial\Omega_1$ whose image under Λ meets $\tilde{\Omega}$. Let $\tilde{\alpha}_1, \cdots, \tilde{\alpha}_m$ be a maximal list of non-conjugate components of $\pi^{-1}(\bigcup_{i=1}^{p} \alpha_i) \cap \partial\Omega_1$ under $\mathrm{Stab}\,\Omega_1$. As was shown in the proofs of Lemmas 6, 7 and 8, we can find a sequence $\{w_n^{(l)}\}_{n=1}^{\infty}$ of quasi-conformal automorphisms of $\hat{C}$ such that $w_n^{(l)}$ has the same Beltrami differential as w_n and such that for all components $\Omega^{(l)}$'s of $L - \pi^{-1}(\bigcup_{i=p+1}^{q} \alpha_i)$ with $\Omega^{(l)} \cap \Lambda(\tilde{\alpha}_l) \neq \emptyset$, $w_n^{(l)}(\mathrm{Stab}\,\Omega_1)w_n^{(l)-1}$ and $w_n^{(l)}(\mathrm{Stab}\,\Omega^{(l)})w_n^{(l)-1}$ converge to groups of Moebius transformations in the sense of generator convergence, $1 \leqq l \leqq m$. Using Lemma 9 finitely many times, we see that our assertion is true for each component $\hat{\Omega}$ of $L - \pi^{-1}(\bigcup_{i=p+1}^{q} \alpha_i)$ with $\hat{\Omega} \cap \Lambda(\bigcup_{l=1}^{m} \tilde{\alpha}_l) \neq \emptyset$. Let $\tilde{\Omega}$ be an arbitrary component of $L - (\bigcup_{i=p+1}^{q} \alpha_i)$ meeting $\Lambda(\pi^{-1}(\bigcup_{i=1}^{p} \alpha_i) \cap \partial\Omega_1)$. Then there exist an element $\gamma \in \mathrm{Stab}\,\Omega_1$ and a component $\hat{\Omega}$ of $L - \pi^{-1}(\bigcup_{i=p+1}^{q} \alpha_i)$ meeting $\Lambda(\bigcup_{i=1}^{m} \tilde{\alpha}_i)$ such that $\tilde{\Omega} = \gamma(\hat{\Omega})$. Since $w_n(\mathrm{Stab}\hat{\Omega})w_n^{-1}$ converges to a group of Moebius transformations in the sense of generator convergence, so does $w_n(\mathrm{Stab}\,\tilde{\Omega})w_n^{-1} = w_n \gamma w_n^{-1} \cdot w_n(\mathrm{Stab}\,\hat{\Omega})w_n^{-1} \cdot w_n \gamma^{-1} w_n^{-1}$. Thus we have proved Lemma 10.

Let Ω' be a component of $U - \pi^{-1}(\bigcup_{i=1}^{p} \alpha_i)$ such that $\Lambda(\Omega')$ meets some $\tilde{\Omega}$ which is a component of $L - \pi^{-1}(\bigcup_{i=p+1}^{q} \alpha_i)$ with $\Lambda(\tilde{\Omega}) \cap \Omega_i \neq \emptyset$. Then in the same way as in the proofs of Lemmas 6, 7, 8 and 10, there exists a sequence $\{w'_n)_{n=1}^{\infty}$ of quasi-conformal automorphisms of $\hat{C}$ such that w'_n has the same Beltrami coefficient as w_n and such that $w'_n(\mathrm{Stab}\,\tilde{\Omega})w'^{-1}_n$ and $w'_n(\mathrm{Stab}\,\Omega')w'^{-1}_n$ converge to groups of Moebius transformations in the sense of generator convergence. Using Lemma 9, we see that $w_n(\mathrm{Stab}\,\tilde{\Omega})w_n^{-1}$ and $w_n(\mathrm{Stab}\,\Omega')w_n^{-1}$ also do. Repeat this procedure finitely many times. Then we have the following.

LEMMA 11. *Let $\Omega^1, \cdots, \Omega^l$ be a finite number of components of $\Omega(\Gamma) - \pi^{-1}(\bigcup_{i=1}^{q} \alpha_i)$. Then $w_n(\mathrm{Stab}\,\Omega^k)w_n^{-1}$ converges to a group of Moebius transformations in the sense of generator convergence, $1 \leqq k \leqq l$.*

LEMMA 12. *The Kleinian group $w_n \Gamma w_n^{-1}$ converges to a group of Moebius transformations in the sense of generator convergence.*

PROOF. Let $\tilde{\beta}_1$, $\tilde{\beta}_2$ and $\tilde{\beta}_3$ be components of $\pi^{-1}(\bigcup_{i=1}^{p} \alpha_i) \cap \Omega_1$. Let $\xi(\delta_k)$ be the attracting fixed point of a loxodromic element δ_k of Stab $\tilde{\beta}_k$. Let $\{\tilde{\gamma}_1, \cdots, \tilde{\gamma}_t\}$ be a system of generators for Γ. Let ω_{3s+k} be a component of $\Omega(\Gamma) - \pi^{-1}(\bigcup_{i=1}^{q} \alpha_i)$ such that the point $\tilde{\gamma}_s(\xi(\delta_k)) = \xi(\tilde{\gamma}_s \delta_k \tilde{\gamma}_s^{-1})$ is kept invariant by an element of Stab ω_{3s+k}, $1 \leqq s \leqq t$, $1 \leqq k \leqq 3$. It follows from Lemma 11 that the group generated by $\{w_n(\text{Stab}\,\omega_u)w_n^{-1})\}_{u=4}^{3t+3}$ converges to a group of Moebius transformations in the sense of generator convergence. So $\{[w_n \tilde{\gamma}_s(\xi(\delta_k)), w_n \tilde{\gamma}_s(\xi(\delta_l))]\}_{n=1}^{\infty}$ is bounded away from zero, $1 \leqq s \leqq t$, $1 \leqq k < l \leqq 3$ (Chuckrow [3, Theorem 6]). Since $w_n(\text{Stab}\,\Omega_1)w_n^{-1}$ converges to a group of Moebius transformations in the sense of generator convergence, $\{[w_n(\xi(\delta_k)), w_n(\xi(\delta_l))]\}_{n=1}^{\infty} = \{[\xi(w_n \delta_k w_n^{-1}), \xi(w_n \delta_l w_n^{-1})]\}_{n=1}^{\infty}$ is bounded away from zero (Chuckrow [3, Theorem 6]). Therefore the Moebius tranformation $w_n \tilde{\gamma}_s w_n^{-1}$ mapping the point $w_n(\xi(\delta_k))$ to $w_n \tilde{\gamma}_s(\xi(\delta_k))$ converges to a Moebius transformation by Lemma 1, $1 \leqq s \leqq t$. Now we are done.

LEMMA 13. *The group $G = \lim_{n\to\infty} w_n \Gamma w_n^{-1}$ is Kleinian.*

PROOF. Let $V \Subset \Omega(\Gamma) - \pi^{-1}(\bigcup_{i=1}^{q} \alpha_i)$ be a region such that $\gamma(V) \cap V = \varnothing$ for each $\gamma \in \Gamma - \{\text{id}\}$. By Lemmas 3 and 12, the restriction of w_n to V converges uniformly to a $K(V)$-quasi-conformal homeomorphism W. Assume the existence of an element g of $G - \{\text{id}\}$ with $g(W(V)) \cap W(V) \neq \varnothing$. Let γ be the element of $\Gamma - \{\text{id}\}$ such that $\lim_{n\to\infty} w_n \gamma w_n^{-1} = g$. Then, for a sufficiently large integer n,

$$\varnothing \neq (w_n \gamma w_n^{-1})(w_n(V)) \cap w_n(V) = w_n(\gamma(V)) \cap w_n(V) = w_n(\gamma(V) \cap V) = \varnothing\ .$$

This contradiction implies that G is discontinuous, so that G is Kleinian.

LEMMA 14. *The Kleinian group $F_n \Gamma F_n^{-1}$ converges to a Kleinian group in the sense of generator convergence.*

PROOF. Set $h_n = w_n F_n^{-1}$, which is a Moebius transformation. Let ψ_1, ψ_2 and ψ_3 be the points of $\Omega(\Gamma) - \pi^{-1}(\bigcup_{i=1}^{q} \alpha_i)$ kept invariant under F_n. Then h_n maps ψ_k to $w_n(\psi_k)$, $1 \leqq k \leqq 3$. Let $V_k \Subset \Omega(\Gamma) - (\bigcup_{i=1}^{q} \alpha_i)$ be a region containing ψ_k, $1 \leqq k \leqq 3$. Then $w_n | \bigcup_{k=1}^{3} V_k$ converges to a $K(\bigcup_{k=1}^{3} V_k)$-quasi-conformal homeomorphism. So $\{[w_n(\psi_k), w_n(\psi_l)]\}_{n=1}^{\infty}$ is bounded away from zero, $1 \leqq k < l \leqq 3$. It follows from Lemma 1 that h_n mapping ψ_k into $w_n(\psi_k)$ converges to a Moebius transformation. Since $w_n \Gamma w_n^{-1}$ converges to a Kleinian group in the sense of generator convergence by Lemma 13, so does $F_n \Gamma F_n^{-1} = h_n^{-1} w_n \Gamma w_n^{-1} h_n$. Thus we complete the proof of Theorem.

REFERENCES

[1] W. ABIKOFF, On boundaries of Teichmüller spaces and on Kleinian groups III, Acta Math. 134 (1975), 211–237.

[2] L. BERS, On boundaries of Teichmüller spaces and on Kleinian groups I, Ann. of Math. 91 (1970), 570–600.

[3] V. CHUCKROW, On Schottky groups with applications to Kleinian groups, Ann. of Math. 88 (1968), 47–61.

[4] A. MARDEN, Geometrically finite Kleinian groups and their deformation spaces, in "Discrete groups and automorphic functions", Academic Press, London, 1977, 259–293.

[5] O. LEHTO AND K. VIRTANEN, Quasi-conformal mappings in the plane, Springer-Verlag, Berlin, 1973.

[6] H. YAMAMOTO, Squeezing deformations in Schottky spaces, J. Math. Soc. of Japan, 31 (1979), 227–243.

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