Tôhoku Math. Journ.
33 (1981), 515-520.

ON A DECOMPOSITION OF BMO-MARTINGALES

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(Received November 4, 1980)

1. **Introduction.** Let f be a BMO-function defined on R^n, that is,

$$\sup_Q |Q|^{-1}\int_Q \left| f(x) - |Q|^{-1}\int_Q f(x)dx \right| dx < \infty ,$$

where the supremum is taken over all cubes Q in R^n. Recently Coifman and Rochberg [2] proved that f can be written in the form

$$(1) \qquad f = g_1 - g_2 + b$$

where g_i $(i = 1, 2)$ is a function of bounded lower oscillation (BLO), that is, there is a constant C_i such that for any cube Q in R^n,

$$|Q|^{-1}\int_Q g_i(x)dx - \inf_Q g_i(x) \leqq C_i \quad (i = 1, 2)$$

and b is a bounded function. Furthermore they showed that g is a BLO-function if and only if there is a nonnegative locally integrable function F with F^* finite a.e., a positive number α and a bounded function h such that

$$(2) \qquad g = \alpha \log F^* + h$$

where F^* is the Hardy-Littlewood maximal function of F, that is,

$$F^*(x) = \sup_{x \in Q} |Q|^{-1}\int_Q F(y)dy .$$

In this note we will consider a martingale version of these results. Let $(\Omega, F, P;\ (F_t)_{t \in R^+})$ be a probability system which satisfies the usual conditions. In the sequel assume that every martingale is continuous. Then, by definition, a uniformly integrable martingale $X = (X_t)$ is said to be a BMO-martingale if $\sup_t \operatorname{ess.sup} E[|X_\infty - X_t| \,|\, F_t] < \infty$.

2. **BLO-martingales.**

DEFINITION 1. A uniformly integrable martingale $X = (X_t)$ is said to be a BLO-martingale if there is a constant C such that for all t

$$(3) \qquad X_t - X_\infty \leqq C \quad \text{a.s.}$$

We denote by BLO the class of all BLO-martingales. BLO is a subclass of BMO. Indeed, if X satisfies (3), then

$$\begin{aligned} E[|X_t - X_\infty||F_t] &= E[(X_t - X_\infty)I_{\{X_t \geqq X_\infty\}}|F_t] + E[(X_\infty - X_t)I_{\{X_t < X_\infty\}}|F_t] \\ &\quad + E[(X_\infty - X_t)I_{\{X_t \geqq X_\infty\}}|F_t] - E[(X_\infty - X_t)I_{\{X_t \geqq X_\infty\}}|F_t] \\ &= 2E[(X_t - X_\infty)I_{\{X_t \geqq X_\infty\}}|F_t] + E[X_\infty - X_t|F_t] \leqq 2C . \end{aligned}$$

Thus X is in BMO. Note that BLO is not a linear space and that every bounded martingale is in BLO.

We will characterize the BMO-martingales by BLO-martingales.

THEOREM 1. *Any* BMO-*martingale X can be written in the form*

$$X = Y_1 - Y_2 + Z$$

where Y_i $(i = 1, 2)$ is in BLO *and Z is a bounded martingale.*

For the proof of (1), Coifman and Rochberg used the rather difficult theorem of Carleson [1]. In the martingale theory, Varopoulos [7] gave a decomposition of BMO-martingales for the proof of a martingale version of the Garnett-Jones theorem [4]. He used the concept of γ-graded sequences of stopping times.

REMARK. The refree suggested that the factorization theorem in Jones [5] might also have a martingale version. Namely, if W satisfies the A_p-condition $(p > 1)$, that is,

$$E[(W_t/W_\infty)^{1/(p-1)}|F_t]^{p-1} \leqq C \quad \text{a.s.} \quad \text{for every } t ,$$

then W_∞ can be written in the form $W_\infty = U_\infty V_\infty^{1-p}$, where U (resp. V) satisfies the A_1-condition (see § 3, Definition 3). This version was in the mean time proved by Varopoulos [7, Addendum III]. Though he defines the martingale satisfying the A_1-condition as a BLO-martingale satisfying the A_2-condition, it is easily seen that his definition is equivalent to that of ours. As a result of this factorization theorem, we see that any BMO-martingale is the difference of two BLO-martingales. Indeed, we have only to note that $E[\exp aM_\infty|F_t]$ satisfies the A_2-condition for some $a > 0$ for any BMO-martingale M.

DEFINITION 2. Let $(T_i)_{i=0,1,2,\cdots}$ be an increasing sequence of stopping times. We say that (T_i) is a γ-graded sequence if there is a constant γ between 0 and 1 such that

$$E[I_{\{T_{i+1} < \infty\}}|F_{T_i}] \leqq \gamma \quad \text{a.s.}$$

for $i = 0, 1, 2, \cdots$.

Now we state the Varopoulos decomposition. For the proof see the article cited above.

LEMMA 1. *Let X be a* BMO-*martingale. Then there are two γ-graded sequences (T_i) and (S_i), a positive constant α, and a bounded random variable Z_∞ such that*

$$X_\infty = \alpha\Big(\sum_{i=1}^{\infty} I_{\{T_i<\infty\}} - \sum_{i=1}^{\infty} I_{\{S_i<\infty\}}\Big) + Z_\infty .$$

PROOF OF THEOREM 1. We show that the Varopoulos decomposition gives the required one. Let U be a martingale defined by $U_t = E[\sum_{i=1}^{\infty} I_{\{T_i<\infty\}}|F_t]$. We will see that U is in BLO. For that purpose, fix a t and let $A_n = \{\omega: T_n(\omega) \leqq t < T_{n+1}(\omega)\}$, $n = 0, 1, 2, \cdots$. Clearly $\Omega = \bigcup_{i=1}^{\infty} A_n$, $A_n \cap A_m = \varnothing$ $(n \neq m)$ and A_n is an element of both F_t and $F_{T_{n+1}}$ for every n. Then

$$\begin{aligned} U_t I_{A_n} &= E\Big[\sum_{i=1}^{\infty} I_{\{T_i<\infty\}} \,|\, F_t\Big] I_{A_n} = E\Big[I_{A_n}\sum_{i=1}^{\infty} I_{\{T_i<\infty\}} \,|\, F_t\Big] I_{A_n} \\ &= E\Big[I_{A_n}\sum_{i=1}^{n} I_{\{T_i\leqq t\}} + I_{A_n}\sum_{i=n+1}^{\infty} I_{\{T_i<\infty\}} \,|\, F_t\Big] I_{A_n} \\ &= I_{A_n}\sum_{i=1}^{n} I_{\{T_i\leqq t\}} + \sum_{i=n+1}^{\infty} E[I_{\{T_i<\infty\}}|F_t] I_{A_n} \\ &= I_{A_n}\sum_{i=1}^{n} I_{\{T_i<\infty\}} + \sum_{i=n+1}^{\infty} E[I_{\{T_i<\infty\}}|F_t] I_{A_n} . \end{aligned}$$

Thus

$$(U_t - U_\infty) I_{A_n} = \Big(\sum_{i=n+1}^{\infty} E[I_{\{T_i<\infty\}}|F_t] - \sum_{i=n+1}^{\infty} I_{\{T_i<\infty\}}\Big) I_{A_n} .$$

Now let $A \subset A_n$ be an element of F_t. Then we easily see that A is also an element of $F_{T_{n+1}}$. By this fact and the definition of γ-graded sequence, we have for every $k \geqq 1$

$$E[I_{\{T_{n+k}<\infty\}}|F_t] I_{A_n} = E[E[I_{\{T_{n+k}<\infty\}}|F_{T_{n+1}}]|F_t] I_{A_n} \leqq \gamma^{k-1} I_{A_n} .$$

Hence it follows that

$$(U_t - U_\infty) I_{A_n} \leqq \Big(\sum_{k=1}^{\infty} \gamma^{k-1} - \sum_{i=n+1}^{\infty} I_{\{T_i<\infty\}}\Big) I_{A_n} \leqq (1/(1 -))\gamma I_{A_n} .$$

Therefore U is in BLO. The same argument yields that the martingale V defined by $V_t = E[\sum_{i=1}^{\infty} I_{\{S_i<\infty\}}|F_t]$ is in BLO. Since αU and αV are also in BLO and the martingale Z defined by $Z_t = E[Z_\infty|F_t]$ is evidently bounded, our theorem is proved.

3. The A_1-condition.

DEFINITION 3. A positive uniformly integrable martingale $W = (W_t)$ is said to be in the class A_1 (or satisfy the A_1-condition) if there is a constant C such that for all t

$$W_t/W_\infty \leqq C \quad \text{a.s.}$$

LEMMA 2. *X is in* BLO *if and only if $E[\exp aX_\infty | F_t]$ is in the class A_1 for some positive number a.*

PROOF. Let X be in BLO. Then, by the John-Nirenberg type inequality for martingales (see Meyer [6] p. 479), we have

$$E[\exp a|X_\infty - X_t| \,|\, F_t] \leqq C$$

for some positive numbers a and C. Hence dropping the absolute value and the definition of BLO yield

$$E[\exp aX_\infty | F_t] \leqq C \exp aX_\infty \,. \tag{4}$$

Conversely assume (4). Taking logarithms in (4) and an application of Jensen's inequality show that X is in BLO.

Now we will consider a martingale version of (2).

LEMMA 3. *If W satisfies the A_1-condition, then there is a positive uniformly integrable martingale M, a number δ between* 0 *and* 1, *and a martingale H which is bounded and bounded away from zero such that $W_\infty = (M^*)^\delta H_\infty$, where $M^* = \sup_t |M_t|$.*

PROOF. Since W satisfies the A_1-condition, by the reverse Hölder inequality for martingales (see Doléans-Dade and Meyer [3] p. 320), there are two positive constants ε and C such that

$$E[W_\infty^{1+\varepsilon} | F_t] \leqq C W_t^{1+\varepsilon} \,.$$

An application of the A_1-condition to the right-hand side yields that

$$E[W_\infty^{1+\varepsilon} | F_t] \leqq C W_\infty^{1+\varepsilon} \,.$$

Now define a martingale M by $M_t = E[W_\infty^{1+\varepsilon} | F_t]$. We have

$$(M^*)^{1/(1+\varepsilon)} \leqq C W_\infty \tag{5}$$

and also have by Hölder's inequality

$$W_\infty \leqq (M^*)^{1/(1+\varepsilon)} \,. \tag{6}$$

Put $\delta = 1/(1+\varepsilon)$ and $H_t = E[(M^*)^{-\delta} W_\infty | F_t]$. Then $W_\infty = (M^*)^\delta H_\infty$ and by (5) and (6), H is a martingale such that $1/C \leqq H \leqq 1$. This completes the proof.

LEMMA 4. *Let M be a positive uniformly integrable martingale, δ a constant between* 0 *and* 1, *and H a martingale bounded and bounded away from zero. Then the martingale W defined by $W_t = E[(M^*)^\delta H_\infty | F_t]$*

satisfies the A_1-condition.

PROOF. By the assumption on H, we have only to deal with the case $W_t = E[(M^*)^\delta | F_t]$. Let N be a uniformly integrable martingale with $E[|N_\infty|] > 0$. Then Kolmogorov's weak type inequality says that $P(N^* > \lambda) \leqq (1/\lambda)E[|N_\infty|]$ for all $\lambda > 0$. By this inequality and the integration by parts, we have

$$(7) \qquad \begin{aligned} E[(N^*)^\delta] &= E[(N^*)^\delta : N^* > E[|N_\infty|]] + E[(N^*)^\delta : N^* \leqq E[|N_\infty|]] \\ &\leqq \delta \int_{E[|N_\infty|]}^\infty \lambda^{\delta-1} P(N^* > \lambda) d\lambda + E[|N_\infty|]^\delta \\ &\leqq (1/(1-\delta)) E[|N_\infty|]^\delta . \end{aligned}$$

Fix a t. If $E[|M_\infty - M_t|] = 0$, then $M_\infty = M_t$ and so M_∞ is F_t-measurable. Hence it is clear that

$$E[\sup_{s \geqq t} M_s^\delta | F_t] = E[\sup_{s \geqq t} E[M_\infty | F_s]^\delta | F_t] = E[M_\infty^\delta | F_t] = M_t^\delta .$$

Next consider the case $E[|M_\infty - M_t|] > 0$. An application of (7) to the martingale $M_s' = M_{t+s} - M_t$ with respect to $F_s' = F_{t+s}$ yields that

$$E[\sup_{s \geqq t} |M_s - M_t|^\delta | F_t] \leqq (1/(1-\delta)) E[|M_\infty - M_t| | F_t]^\delta .$$

Using the trivial inequalities $a^\delta - b^\delta \leqq (a-b)^\delta$ and $|a-b| \leqq |a| + |b|$ for $a, b > 0$ and the positivity of M, we have

$$E[\sup_{s \geqq t} M_s^\delta | F_t] \leqq C M_t^\delta$$

for some positive constant C which depends on δ. In both cases it follows that

$$\begin{aligned} E[(M^*)^\delta | F_t] &\leqq E[(\sup_{s \leqq t} M_s)^\delta + (\sup_{s \geqq t} M_s)^\delta | F_t] \\ &\leqq (2 + C)(\sup_{s \leqq t} M_s)^\delta \leqq (2 + C)(M^*)^\delta , \end{aligned}$$

which is to be proved.

Combining Lemmas 2, 3 and 4, the following theorem is established.

THEOREM 2. *X is in* BLO *if and only if there is a positive uniformly integrable martingale M, a positive constant α and a bounded martingale H such that $X_\infty = \alpha \log M^* + H_\infty$.*

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