

## SIEGEL MODULAR CUSP FORMS OF DEGREE TWO

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**Introduction.** The problem of the lifting of modular cusp forms has been discussed by several authors (cf. [3] and [12]).

Recently Oda [10] constructed, from modular cusp forms of one variable, modular cusp forms associated with indefinite quadratic forms with signature  $(2, n - 2)$ .

In this paper, we shall consider the case  $n = 5$  and we shall treat the relations between the Hecke operators of the space of modular forms of one variable and those of the space of Siegel modular forms of degree two. The preparatory section is § 1. In § 2, by using transformation formulas of theta series obtained by Shintani [13], we show the existence of a linear mapping  $\Psi$  of  $\mathfrak{S}_{2k-1}(\Gamma_0(4))$  into  $S_k(\Gamma_0^{(2)}(2))$ , where  $\mathfrak{S}_{2k-1}(\Gamma_0(4))$  (resp.  $S_k(\Gamma_0^{(2)}(2))$ ) denotes the space of cusp forms of weight  $(2k - 1)/2$  (resp. of weight  $k$  with respect to  $\Gamma_0^{(2)}(2)$ ). In § 3, by a method similar to that of Niwa [8], we determine explicitly Fourier coefficients of  $\Psi(f)$  at infinity, where  $f \in \mathfrak{S}_{2k-1}(\Gamma_0(4))$ . In the last section, applying the results in § 3, we show that  $\Psi(f)$  is a common eigen-function of Hecke operators on  $S_k(\Gamma_0^{(2)}(2))$ , if  $f$  is a common eigen-function of Hecke operators on  $\mathfrak{S}_{2k-1}(\Gamma_0(4))$ . Furthermore, we give a relation between Andrianov's zeta function associated with  $\Psi(f)$  and Shimura's one associated with  $f$ .

We note that our results are closely related with Maaß-Andrianov's results (cf. [2], [4], [5] and [6]).

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**1. Notations and preliminaries.** We denote, as usual, by  $\mathbf{Z}$ ,  $\mathbf{Q}$ ,  $\mathbf{R}$  and  $\mathbf{C}$  the ring of rational integers, the rational number field, the real number field and the complex number field. For a ring  $A$ , we denote by  $A_m^n$  the set of all  $n \times m$  matrices with entries in  $A$ , and denote  $A_1^n$  (resp.  $A_n^n$ ) by  $A^n$  (resp.  $M_n(A)$ ). For  $z \in \mathbf{C}$ , we put  $e[z] = \exp(2\pi iz)$  with  $i = \sqrt{-1}$  and define  $\sqrt{z} = z^{1/2}$  so that  $-\pi/2 < \arg(z^{1/2}) \leq \pi/2$ . Further we put  $z^{k/2} = (\sqrt{z})^k$  for every  $k \in \mathbf{Z}$ . This section is devoted to summarizing several fundamental facts which we need later.

Let  $Sp(n, \mathbf{R})$  be the real symplectic group of degree  $n$ , i.e.,

$$Sp(n, \mathbf{R}) = \{M \in M_{2n}(\mathbf{R}) \mid {}^t M J_n M = J_n\}, \quad \text{where } J_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$$

and  ${}^t M$  denotes the transpose of  $M$ . Let  $\mathfrak{H}_n$  be the complex Siegel upper half plane of degree  $n$ , i.e.,  $\mathfrak{H}_n = \{Z = X + iY \mid X, Y \in M_n(\mathbf{R}), {}^t Z = Z \text{ and } Y > 0\}$ . Define an action of  $Sp(n, \mathbf{R})$  on  $\mathfrak{H}_n$  by

$$Z \longrightarrow M\langle Z \rangle = (AZ + B)(CZ + D)^{-1} \quad \text{for all } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbf{R})$$

and for all  $Z \in \mathfrak{H}_n$ . Denote by  $K_n$  the group of stabilizers at  $iE_n \in \mathfrak{H}_n$ , i.e.,  $K_n = \{M \in Sp(n, \mathbf{R}) \mid M\langle iE_n \rangle = iE_n\}$ . It is well-known that  $K_n = Sp(n, \mathbf{R}) \cap O(2n)$ , where  $O(2n)$  denotes the orthogonal group. Clearly  $Sp(n, \mathbf{Z}) = Sp(n, \mathbf{R}) \cap M_{2n}(\mathbf{Z})$  is an arithmetic discrete subgroup of  $Sp(n, \mathbf{R})$ . For each positive integer  $N$ , put

$$\Gamma_0^{(n)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbf{Z}) \mid C \equiv 0 \pmod{N} \right\} \quad \text{and} \quad \Gamma_0(N) = \Gamma_0^{(1)}(N).$$

We call a holomorphic function  $F$  on  $\mathfrak{H}_n$  a Siegel modular cusp form of weight  $k$  with respect to  $\Gamma_0^{(n)}(N)$ , if the following conditions (i) and (ii) are satisfied:

(i) For all  $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n)}(N)$  and all  $Z \in \mathfrak{H}_n$ ,  $F(M\langle Z \rangle) = \det(CZ + D)^k F(Z)$ .

(ii)  $|F(Z)|(\det(\text{Im}(Z)))^{k/2}$  is bounded on  $\mathfrak{H}_n$ .

We denote by  $S_k(\Gamma_0^{(n)}(N))$  the space of Siegel modular cusp forms of weight  $k$  with respect to  $\Gamma_0^{(n)}(N)$ . Every  $F \in S_k(\Gamma_0^{(n)}(N))$  has the Fourier expansion  $F(Z) = \sum_r a(T) e[\text{tr}(TZ)]$  at infinity, where  $T$  runs over the semi-integral positive definite matrices.

Let  $Q$  be a non-degenerate symmetric  $n \times n$  matrix. We denote by  $O(Q)$  (resp.  $O(Q)_0$ ) the real orthogonal group (resp. the connected component of the unity  $O(Q)$ ) for  $Q$ , i.e.,  $O(Q) = \{g \in GL_n(\mathbf{R}) \mid {}^t g Q g = Q\}$ . Now we treat

$$Q = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

with signature  $(2, 3)$ . It is well-known that  $O(Q)_0$  is isomorphic to

$Sp(2, \mathbf{R})/\{\pm E_4\}$  as a Lie group. We shall explicitly construct an isomorphism of  $Sp(2, \mathbf{R})/\{\pm E_4\}$  onto  $O(Q)_0$ .

First we summarize the fundamental facts on tensor algebras. Let  $(\mathbf{R}^4)^*$  be the dual space of  $\mathbf{R}^4$ . Let  $\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\}$  be the dual base of  $\{e_1, e_2, e_3, e_4\}$ , where  $e_i = {}^t(0, \dots, 0, 1, 0, \dots, 0)$  ( $1 \leq i \leq 4$ ). Put  $\otimes^2(\mathbf{R}^4)^* = \{f | f \text{ is a bilinear mapping of } \mathbf{R}^4 \times \mathbf{R}^4 \text{ into } \mathbf{R}\}$  and  $\wedge^2(\mathbf{R}^4)^* = \{f \in \otimes^2(\mathbf{R}^4)^* | f(X, Y) = -f(Y, X) \text{ for all } X, Y \in \mathbf{R}^4\}$ . We define a mapping  $\Phi$  of  $SL_4(\mathbf{R})$  into  $\text{End}_{\mathbf{R}}(\wedge^2(\mathbf{R}^4)^*)$  by  $(\Phi(g)f)(X, Y) = f(gX, gY)$  for all  $g \in SL_4(\mathbf{R})$ , all  $f \in \wedge^2(\mathbf{R}^4)^*$  and all  $X, Y \in \mathbf{R}^4$ . We note that  $\Phi(gg') = \Phi(g')\Phi(g)$  for all  $g, g' \in SL_4(\mathbf{R})$ . It can be easily seen that

$$(*) \quad Sp(2, \mathbf{R}) = \{g \in SL_4(\mathbf{R}) | \Phi(g)(\hat{e}_1 \wedge \hat{e}_3 + \hat{e}_2 \wedge \hat{e}_4) = \hat{e}_1 \wedge \hat{e}_3 + \hat{e}_2 \wedge \hat{e}_4\}.$$

Define an inner product  $\langle \alpha, \beta \rangle'$  ( $\alpha, \beta \in \wedge^2(\mathbf{R}^4)^*$ ) by  $\alpha \wedge \beta = \langle \alpha, \beta \rangle' \hat{e}_1 \wedge \hat{e}_2 \wedge \hat{e}_3 \wedge \hat{e}_4$ . Set  $V = \{\alpha \in \wedge^2(\mathbf{R}^4)^* | \langle \alpha, \hat{e}_1 \wedge \hat{e}_3 + \hat{e}_2 \wedge \hat{e}_4 \rangle = 0\}$ . We see that  $V = \mathbf{R}(f_1, \dots, f_5)$ , where  $f_1 = \hat{e}_1 \wedge \hat{e}_4, f_2 = \hat{e}_2 \wedge \hat{e}_3, f_3 = \hat{e}_1 \wedge \hat{e}_2, f_4 = \hat{e}_3 \wedge \hat{e}_4$  and  $f_5 = (\hat{e}_1 \wedge \hat{e}_3 - \hat{e}_2 \wedge \hat{e}_4)/\sqrt{2}$ . Through the mapping of  $V$  onto  $\mathbf{R}^5$  given by  $\alpha = \sum_{i=1}^5 x_i f_i \rightarrow x = {}^t(x_1, x_2, x_3, x_4, x_5)$ , we can identify  $V$  with  $\mathbf{R}^5$ . From (\*) and the above identification we get a mapping  $\Phi' = \Phi | Sp(2, \mathbf{R})$  of  $Sp(2, \mathbf{R})$  into  $GL_5(\mathbf{R})$ , where  $\Phi | Sp(2, \mathbf{R})$  denotes the restriction of  $\Phi$  to  $Sp(2, \mathbf{R})$ . Here it should be noted that  $\langle \alpha, \beta \rangle' = {}^t x(-Q)y$ . Now we define  $(\phi_{ij}(g))_{1 \leq i, j \leq 5}$  by  $\Phi'(g)f_i = \sum_{j=1}^5 \phi_{ji}(g)f_j$  ( $1 \leq i \leq 5$ ) for every  $g \in Sp(2, \mathbf{R})$ . The following lemma can be easily checked.

LEMMA 1.1. *Let  $g = (g_{ij})$  be an element of  $Sp(2, \mathbf{R})$ . Then it holds that*

- (1)  $\phi_{ij}(g)$  belongs to  $\mathbf{Z}[g_{ij}]_{1 \leq i, j \leq 5}$  for all  $i$  and  $j$  such that  $1 \leq i, j \leq 4$ ,
- (2)  $\phi_{3j}(g)$  belongs to  $\sqrt{2}\mathbf{Z}[g_{ij}]_{1 \leq i, j \leq 5}$  for all  $j$  such that  $1 \leq j \leq 4$ ,
- (3)  $\phi_{i5}(g)$  belongs to  $\sqrt{2}^{-1}\mathbf{Z}[g_{ij}]_{1 \leq i, j \leq 5}$  for all  $i$  such that  $1 \leq i \leq 4$ ,
- (4)  $\phi_{55}(g)$  belongs to  $\mathbf{Z}[g_{ij}]_{1 \leq i, j \leq 5}$ ,

where  $\mathbf{Z}[g_{ij}]_{1 \leq i, j \leq 5}$  denotes the polynomial ring over  $\mathbf{Z}$ . Moreover  $\phi_{ij}(g)$  satisfy the relations that  $\phi_{3j}(g) \equiv 0 \pmod{N}$  for all  $j = 1, 2$  and  $4$  and  $\sqrt{2}\phi_{35}(g) \equiv 0 \pmod{N}$  if  $g_{31}, g_{32}, g_{41}$  and  $g_{42}$  belong to  $N\mathbf{Z}$  and  $\phi_{33}(g) = g_{11}g_{22} - g_{12}g_{21}$ .

Set  $\rho(g) = \Phi'(g)^{-1}$ . Then we get the following lemma easily.

LEMMA 1.2. *Under the above notations,  $\rho$  gives an isomorphism of the Lie group  $Sp(2, \mathbf{R})/\{\pm E_4\}$  onto  $O(Q)_0$ ; moreover, it satisfies the property  $\rho(K_2) \subset O(E_5)$ .*

2. Weil representation and theta series. In the following, we as-

sume that  $k$  is even. In this section, we shall construct Siegel modular cusp forms of degree two from modular cusp forms of half integral weight. For this purpose, we need to derive transformation formulas of certain theta series (cf. [13]). Let  $Q$  be as in § 1. The Weil representation  $\gamma(*, Q)$  of  $SL_2(\mathbf{R})$  is defined by

$$(\gamma(\sigma, Q)f)(x) = \begin{cases} |c|^{-5/2} |\det(Q)|^{1/2} \int_{\mathbf{R}^5} e[(a\langle x, x \rangle - 2\langle x, y \rangle + d\langle y, y \rangle)/2c] f(y) dy & \text{if } c \neq 0, \\ |a|^{5/2} e[ab\langle x, x \rangle/2] f(ax) & \text{if } c = 0 \end{cases}$$

for every  $f \in L^2(\mathbf{R}^5)$  and for every

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{R}),$$

where  $\langle x, y \rangle = {}^t x Q y$ . The group  $GL_5(\mathbf{R})$  acts on  $L^2(\mathbf{R}^5)$  as follows:  $Tf(x) = |\det(T)|^{-1/2} f(T^{-1}x)$  for all  $T \in GL_5(\mathbf{R})$  and for all  $f \in L^2(\mathbf{R}^5)$ . Put  $f_k(x) = \langle x, {}^t(-i, i, 1, -1, 0) \rangle^k \exp(-\pi \sum_{i=1}^5 x_i^2)$  for all  $x \in \mathbf{R}^5$ . Then  $f_k$  satisfies the equalities

$$(2.1) \quad \rho(\kappa) f_k = (\det(A - Bi))^k f_k \quad \text{for all } \kappa = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K_2$$

and

$$\varepsilon(k(\theta)) \gamma(k(\theta), Q) f_k = \exp(-i\theta)^{-(2k-1)/2} f_k, \quad \text{where } k(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

and  $\varepsilon(k(\theta))$  is the symbol in [13]. Set  $L_1 = \{{}^t(x_1, x_2, \sqrt{2}x_3) \mid x_i \in \mathbf{Z}\}$  and  $L(N) = \{{}^t(x_1, x_2, Nx_3, x_4, \sqrt{2}x_5) \mid x_i \in \mathbf{Z}\}$ . We put

$$Q_1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Let  $\chi_0$  be the trivial character modulo 2. We can check that  $\rho$  satisfies the relations

$$(2.2) \quad \rho(M)L(1) = L(1), \quad \rho(M)L(2) = L(2) \quad \text{and} \quad \rho(M)x \equiv x \pmod{L(2)}$$

for all  $M \in \Gamma_0^{(2)}(2)$  and for all  $x \in L(1)$ .

Define a theta series  $\theta_k(z, g)$  on  $\mathfrak{S}_1 \times Sp(2, \mathbf{R})$  by

$$\theta_k(z, g) = v^{-(2k-1)/4} \sum_h \chi_0(h_3) (\rho(g) \gamma(\sigma_z, Q)) f_k(h), \quad \text{where } \sigma_z = \begin{pmatrix} \sqrt{v} & u\sqrt{v^{-1}} \\ 0 & \sqrt{v^{-1}} \end{pmatrix},$$

$z = u + iv$  and  $h = {}^i(\dots, h_3, \dots)$  runs over  $L(1)$ . By virtue of (2.1) and [13, Prop. 1.6], we have the following.

**LEMMA 2.1.** *The function  $\Theta_k(z, g)$  satisfies the following properties:*

(i)  $\Theta_k(\sigma(z), g) = j(\sigma, z)^{2k-1} \Theta_k(z, g)$  for all  $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$  and for  $j(\sigma, z) = \varepsilon_d^{-1} \left(\frac{c}{d}\right) (cz + d)^{1/2}$ ,

(ii)  $\Theta_k(z, \gamma g \kappa) = \det(A - Bi)^k \Theta_k(z, g)$  for all  $\gamma \in \Gamma_0^{(2)}(2)$  and for all  $\kappa = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K_2$ , where  $\varepsilon_d$  and  $\left(\frac{c}{d}\right)$  are the symbols in [12].

Let  $\mathfrak{S}_{2k-1}(\Gamma_0(4))$  be the space of modular cusp forms of weight  $(2k-1)/2$  with respect to  $\Gamma_0(4)$  (cf. [12]). The property (i) of Lemma 2.1 shows that, for a function  $f \in \mathfrak{S}_{2k-1}(\Gamma_0(4))$ , the integral

$$\int_{D_0(4)} v^{(2k-1)/2} f |[\tau_4]_{2k-1}(z) \overline{\Theta_k(z, g)} v^{-2} dudv$$

is well-defined with  $z = u + iv$ , where  $D_0(4)$  denotes the fundamental domain for  $\Gamma_0(4)$ ,  $f | [\tau_4]_{2k-1}(z) = (-2iz)^{-(2k-1)/2} f(-1/4z)$  and  $\overline{\Theta_k(z, g)}$  means the complex conjugate of  $\Theta_k(z, g)$ . Now we define a function  $\Psi(f)$  on  $\mathfrak{S}_2$  by

$$\Psi(f)(Z) = J(g, iE_2)^k \int_{D_0(4)} v^{(2k-1)/2} f |[\tau_4]_{2k-1}(z) \overline{\Theta_k(z, g)} v^{-2} dudv$$

with  $Z = g \langle iE_2 \rangle$ , where

$$J(g, iE_2) = \det(Ci + D) \quad \text{with} \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2, \mathbf{R}).$$

By virtue of Lemma 2.1 (ii), we have

$$\Psi(f)(M \langle Z \rangle) = (\det(CZ + D))^k \Psi(f)(Z) \quad \text{for every} \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(2)}(2).$$

Since  $\Psi(f)$  is holomorphic on  $\mathfrak{S}_2$ , we see that  $\Psi(f)$  is a Siegel modular form with respect to  $\Gamma_0^{(2)}(2)$  (cf. [10] and [11]).

**3. Explicit calculation of the Fourier coefficients of  $\Psi(f)$ .** For a positive matrix  $Y \in M_2(\mathbf{R})$ , set  $g = g(Y)$  and  $Y = yY_1$  with  $\det(Y_1) = 1$  and  $y > 0$ , where

$$g = \begin{pmatrix} \sqrt{Y} & 0 \\ 0 & \sqrt{Y}^{-1} \end{pmatrix} \quad \text{and} \quad Y_1 = \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix}.$$

For a non-negative integer  $\varepsilon$ , we define three theta series by

$$(3.1) \quad \Theta_{1,\varepsilon}(z; Y_1) = v^{(-\varepsilon+2)/2} \sum_l H_\varepsilon(\sqrt{2\pi v}(y_1, -y_3, -\sqrt{2}y_2)l) \\ \times e[(u^t l Q_1 l + iv^t l R(Y_1) l)/2],$$

$$(3.1)' \quad \Theta_{1,\varepsilon}^*(z; Y_1) = v^{(-\varepsilon+2)/2} \sum_{l'} H_\varepsilon(\sqrt{2\pi v} (y_1, -y_3, -\sqrt{2} y_2) l') \\ \times e[(u^t l' Q_1 l' + i v^t l' R(Y_1) l')/2]$$

and

$$(3.2) \quad \Theta_{2,\varepsilon}(z; y) = v^{(-\varepsilon+1)/2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \chi_0(m) \exp(-2\pi i m n u - \pi v (y^2 m^2 + y^{-2} n^2)) \\ \times H_\varepsilon(\sqrt{2\pi v} (m y - n y^{-1})),$$

where  $z = u + i v$ ,  $L_1^* = \{y \in \mathbf{R}^3 \mid {}^t y Q_1 x \in \mathbf{Z} \text{ for all } x \in L_1\}$ ,  $H_\varepsilon(x) = (-1)^\varepsilon \times \exp(x^2/2)(d^\varepsilon/dx^\varepsilon)(\exp(-x^2/2))$ ,

$$R(Y_1) = \begin{pmatrix} y_1^2 & -y_2^2 & -\sqrt{2} y_1 y_2 \\ -y_2^2 & y_3^2 & \sqrt{2} y_2 y_3 \\ -\sqrt{2} y_1 y_2 & \sqrt{2} y_2 y_3 & 1 + 2y_2^2 \end{pmatrix},$$

and  $l$  (resp.  $l'$ ) runs over  $L_1$  (resp.  $2L_1^*$ ). By the definition of  $\gamma(*, Q)$ , we have

$$(\gamma(\sigma_z, Q) f_k)(\rho(g)^{-1} l) = v^{(2k+5)/4} \langle l, \Phi'(g)^{-1} r_0 \rangle^k e[(u \langle l, l \rangle + i v^t l \Phi'(g^2) l)/2],$$

where  $r_0 = {}^t(-i, i, 1, -1, 0)$ . On the other hand, a direct calculation yields  $\Phi'(g)^{-1} r_0 = y^{-1} {}^t(-y y_3 i, y y_1 i, 1, -y^2, -\sqrt{2} y y_2 i)$  and

$$\Phi'(g^2) = \begin{pmatrix} y_1^2 & -y_2^2 & 0 & 0 & -\sqrt{2} y_1 y_2 \\ -y_2^2 & y_3^2 & 0 & 0 & \sqrt{2} y_2 y_3 \\ 0 & 0 & y^2 & 0 & 0 \\ 0 & 0 & 0 & y^{-2} & 0 \\ -\sqrt{2} y_1 y_2 & \sqrt{2} y_2 y_3 & 0 & 0 & 1 + 2y_2^2 \end{pmatrix}.$$

Noting that  $(x - i y)^k = \sum_{\varepsilon=0}^k {}_k C_\varepsilon (-i)^\varepsilon H_{k-\varepsilon}(x) H_\varepsilon(y)$ , we obtain

$$(3.3) \quad \Theta_k(z, g) = \sqrt{2\pi}^{-k} \sum_{\varepsilon=0}^k {}_k C_\varepsilon (-i)^\varepsilon \Theta_{1,\varepsilon}(z; Y_1) \Theta_{2,k-\varepsilon}(z; y),$$

The Poisson summation formula gives a different expression of  $\Theta_{2,\varepsilon}$  (cf. [8, p. 152]):

$$\Theta_{2,\varepsilon}(z; y) = \sqrt{2\pi}^\varepsilon i^\varepsilon y^{\varepsilon+1} v^{-\varepsilon} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \chi_0(m) (m \bar{z} + n)^\varepsilon \exp(-\pi y^2 v^{-1} |m z + n|^2),$$

which shows

$$(3.4) \quad \Theta_{2,\varepsilon}(-1/4z; y) = 2(\sqrt{2\pi} i y z v^{-1})^\varepsilon y \sum_{n=1}^{\infty} \chi_0(n) n^\varepsilon \sum_{\gamma} \overline{J(\gamma, z)}^\varepsilon k(\gamma(z), n, y),$$

where  $J\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) = (cz + d)$ ,  $k(z, n, y) = \exp(-\pi y^2 n^2/4v)$  and  $\gamma$  runs over

$\Gamma_\infty \backslash \Gamma_0(4)$ .

Next we derive several transformation formulas of  $\Theta_{1,\varepsilon}(z; Y_1)$ . For every  $x \in \mathbf{R}^3$ , put

$$g_\varepsilon(x) = H_\varepsilon(\sqrt{2\pi}(y_1, -y_3, -\sqrt{2}y_2)x)\exp(-\pi^t xR(Y_1)x).$$

Here we note that  $\sqrt{R(Y_1)}$  belongs to  $O(Q_1)$ . By a direct computation, we obtain

$$g_\varepsilon(\sqrt{R(Y_1)}^{-1}x) = H_\varepsilon(\sqrt{2\pi}(x_1 - x_2))\exp(-\pi(x_1^2 + x_2^2 + x_3^2))$$

with  $x = {}^t(x_1, x_2, x_3)$ . This equality shows that

$$\varepsilon(k(\theta))\gamma(k(\theta), Q_1)g_\varepsilon = \exp(-i\theta)^{-(2\varepsilon-1)/2}g_\varepsilon,$$

where  $\gamma(*, Q_1)$  is the Weil representation associated with  $Q_1$ . Therefore, by virtue of [13, Prop. 1.6], we obtain the following lemma.

LEMMA 3.1. *Let  $\sigma$  be an element of  $\Gamma_0(4)$ . Then*

$$\Theta_{1,\varepsilon}(\sigma(z); Y_1) = j(\sigma, z)^{2\varepsilon-1}\Theta_{1,\varepsilon}(z; Y_1)$$

and

$$(4z)^{-(2\varepsilon-1)/2}\Theta_{1,\varepsilon}(-1/4z; Y_1) = \sqrt{i/2}2^{2\varepsilon}\Theta_{1,\varepsilon}^*(z; Y_1).$$

Every  $f \in \mathfrak{S}_{2k-1}(\Gamma_0(4))$  has the Fourier expansion  $f(z) = \sum_{n=-1}^\infty a(n)e[nz]$  at  $\infty$ . For every semi-integral matrix  $T = \begin{pmatrix} t_1 & t_2/2 \\ t_2/2 & t_3 \end{pmatrix} > 0$  with integers  $t_1, t_2$  and  $t_3$ , we set  $T(t_1, t_2, t_3) = T$ ,  $e(T) = \text{g.c.m.}(t_1, t_2, t_3)$  and  $N(T) = 4 \det(T)$ . Define  $C_f(T)$  by  $C_f(T) = \sum \chi_0(m)m^{k-1}a(N(T)/m^2)$  ( $T > 0$ ), where  $m$  runs over all positive integers with  $m|e(T)$ .

THEOREM 1. *Let  $f$  be an element of  $\mathfrak{S}_{2k-1}(\Gamma_0(4))$ . Suppose  $k > 5$  is even. Then  $\Psi(f)$  has the Fourier expansion*

$$\Psi(f)(Z) = c \sum_T C_f(T)e[\text{tr}(TZ)]$$

at infinity, where  $c \neq 0$  is a constant not depending upon  $f$  and  $T$  runs over semi-integral matrices  $T = T(t_1, t_2, t_3) > 0$ .

PROOF. Since  $\Psi(f)$  is a Siegel modular form with respect to  $\Gamma_0^{(2)}(2)$ , we see that  $\Psi(f)$  has the Fourier expansion  $\Psi(f)(Z) = \sum_{T \geq 0} C(T)e[\text{tr}(TZ)]$  at infinity. Set  $Z = iY$  with  $Y > 0$ . Then

$$(*) \quad \Psi(f)(iY) = \sum_{T \geq 0} C(T)\exp(-2\pi \text{tr}(TY)).$$

On the other hand, we have

$$\Psi(f)(iY) = (\det(Y))^{-k/2} \int_{D_0(4)} f|[\tau_4]_{2k-1}(z)\bar{\Theta}_k \left( z, \begin{pmatrix} \sqrt{Y} & 0 \\ 0 & \sqrt{Y}^{-1} \end{pmatrix} \right) v^{(2k-1)/2} v^{-2} dudv.$$

The formulas (3.3), (3.4) and Lemma 3.1 imply that

$$\begin{aligned}
(**) \quad \Psi(f)(iY) &= y^{-k} \sqrt{2\pi}^{-k} \sum_{\varepsilon=0}^k {}_k C_{\varepsilon}(-i)^{\varepsilon} \int_{D_0(4)} f[[\tau_4]_{2k-1}(z) \bar{\Theta}_{1,\varepsilon}(z; Y_1) \bar{\Theta}_{2,k-\varepsilon}(z; y) \\
&\quad \times v^{(2k-1)/2} \nu^{-2} dudv \\
&= y^{-k} \sqrt{2\pi}^{-k} \sum_{\varepsilon=0}^k {}_k C_{\varepsilon}(-i)^{\varepsilon} \int_{D_0(4)} f[[\tau_4]_{2k-1}(-1/4z) \bar{\Theta}_{1,\varepsilon}(-1/4z; Y_1) \\
&\quad \times \bar{\Theta}_{2,k-\varepsilon}(-1/4z; y) (v/4|z|^2)^{(2k-1)/2} \nu^{-2} dudv \\
&= c' \sum_{\varepsilon=0}^k {}_k C_{\varepsilon} (2/\pi)^{\varepsilon/2} y^{-(\varepsilon-1)} \int_{D_0(4)} v^{\varepsilon-1/2} \sum_{m=1}^{\infty} \chi_0(m) m^{k-\varepsilon} \sum_{\gamma} J(\gamma, z)^{k-\varepsilon} \\
&\quad \times k(\gamma(z), m, y) \overline{\Theta_{1,\varepsilon}^*}(z; Y_1) f(z) v^{-2} dudv \\
&= c'' \sum_{m=1}^{\infty} \chi_0(m) m^{k-1} \sum_{T>0} a(N(T)) \exp(-2\pi m |\operatorname{tr}(TY)|) \\
&= c''' \sum_{T>0} C_f(T) \exp(-2\pi |\operatorname{tr}(TY)|) \\
&= c''' \sum_{T>0} C_f(T) \exp(-2\pi \operatorname{tr}(TY))
\end{aligned}$$

where  $\gamma$  runs over  $\Gamma_{\infty} \backslash \Gamma_0(4)$ . Note that  $T > 0$  and  $Y > 0$ , hence  $|\operatorname{tr}(TY)| = \operatorname{tr}(TY)$ . Put  $T = T(n_1, n_2, n_3)$  and  $t_i = \exp(-2\pi y y_i)$  for  $i = 1, 2$  and  $3$ . We have

$$\sum C(T(n_1, n_2, n_3)) t_1^{n_1} t_2^{n_2} t_3^{n_3} = c''' \sum C_f(T(n_1, n_2, n_3)) t_1^{n_1} t_2^{n_2} t_3^{n_3}$$

for  $t_1, t_2$  and  $t_3$  with  $0 < t_1 < \exp(-\pi)$ ,  $\exp(-\pi) < t_2 < 1$  and  $0 < t_3 < \exp(-\pi)$ , where the summation is taken over all  $(n_1, n_2, n_3) \in \mathbf{Z}_3^1$  under the condition  $T(n_1, n_2, n_3) \geq 0$ . Here we note that both sides of the above equality are absolutely uniformly convergent in the cube considered above. Comparing the coefficients of Laurent expansions, we have

$$C(T) = \begin{cases} c''' C_f(T) & \text{if } T > 0, \\ 0 & \text{if not,} \end{cases}$$

which completes our proof of Theorem 1. By the same arguments as those in [10, § 6], we can show that  $\Psi(f)$  is a cusp form.

**4. An application to Andrianov's zeta functions.** Let  $n$  (resp.  $p$ ) be a positive integer (resp. a prime number). We denote by  $T_k^2(n; 2)$  (resp.  $T_{(2k-1)/2}(p^2; 4)$ ) the Hecke operator on  $S_k(\Gamma_0^{(2)}(2))$  (resp.  $\mathfrak{S}_{2k-1}(\Gamma_0(4))$ ) (see [7] and [12]).

Denote by  $U$  the set of all complex-valued functions  $\psi$  on the set  $\{T = T(n_1, n_2, n_3) | n_1, n_2 \text{ and } n_3 \in \mathbf{Z}, T > 0\}$  with the property  $\psi(\gamma T^t \gamma) = \psi(T)$  for all  $\gamma \in SL_2(\mathbf{Z})$ . For all  $\psi \in U$ , we define

$$T_a \left( SL_2(\mathbf{Z}) \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix} SL_2(\mathbf{Z}) \right) \psi(T) = \sum_{a=1}^i \psi(\sigma_a T^t \sigma_a),$$



where  $SL_2(\mathbf{Z})\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}SL_2(\mathbf{Z}) = \bigcup_{d=1}^l SL_2(\mathbf{Z})\sigma_d$  (a disjoint union). For every positive integer  $m$ , we also define operators  $\Delta^+(m)$ ,  $\Delta^-(m)$  and  $\Pi(m)$  by  $(\Delta^+(m)\psi)(T) = \psi(mT)$  and  $(\Delta^-(m)\psi)(T) = \psi(m^{-1}T)$  or 0 according as  $m|e(T)$  or  $m \nmid e(T)$  and

$$\Pi(m) = T_a \left( SL_2(\mathbf{Z}) \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix} SL_2(\mathbf{Z}) \right) \Delta^-(m).$$

The following theorem was proved by Andrianov [1] and Matsuda [7].

**THEOREM A.** *Let  $F(Z) = \sum_T a(T)e[\text{tr}(TZ)] \in S_k(\Gamma_0^{(2)}(2))$  and let  $p$  be a prime number. Then  $(T_k^2(p^n; 2))F(Z) = \sum_T a(p^n: T)e[\text{tr}(TZ)]$ . The coefficient  $a(p^n: T)$  has the following property:*

$$a(p^n: T) = \begin{cases} a(p^n T) & \text{if } p = 2, \\ \sum p^{(k-2)\beta + (2k-3)\gamma} (\Delta^-(p^\gamma)\Pi(p^\beta)\Delta^+(p^\alpha)a)(T) & \text{if } p \neq 2, \end{cases}$$

where the summation  $\sum$  is taken over all  $(\alpha, \beta, \gamma) \in \mathbf{Z}_3^1$  with  $\alpha, \beta, \gamma \geq 0$  and  $\alpha + \beta + \gamma = n$ .

Now we recall some results in the theory of lattices in quadratic fields. Let  $T$  be a semi-integral positive definite matrix. We denote by  $d$  the discriminant of the imaginary quadratic field  $\mathbf{Q}(\sqrt{-N(T)})$ . Clearly  $-N(T) = df^2$  with a positive integer  $f$ .

For a prime number  $p$ , we set

$$SL_2(\mathbf{Z})\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}SL_2(\mathbf{Z}) = \bigcup_{i=1}^{p+1} SL_2(\mathbf{Z})\sigma_i \quad (\text{a disjoint union}).$$

A slight modification of Shintani's arguments in [13] yields the following (cf. [13, Lemma 2.2, 2.3 and the proof of Lemma 2.8]).

**LEMMA 4.1.** *Suppose that  $T$  is primitive ( $e(T) = 1$ ). Then among  $(p + 1)$  matrices  $\{\sigma_i T^i \sigma_i\}_{i=1}^{p+1}$ , there are  $p - \left(\frac{d}{p}\right)$  matrices (resp.  $p$ ) with  $e(\sigma_i T^i \sigma_i) = 1$  and  $1 + \left(\frac{d}{p}\right)$  matrices (resp. 1) with  $e(\sigma_i T^i \sigma_i) = p$  (resp.  $e(\sigma_i T^i \sigma_i) = p^2$ ), if  $f$  is prime to (resp. divisible by)  $p$ .*

By using Theorem 1, we prove the following.

**THEOREM 2.** *Suppose that  $f \in \mathfrak{S}_{2k-1}(\Gamma_0(4))$  satisfies  $T_{(2k-1)/2}(p^2; 4)f = \omega_p f$  for all primes  $p$ . Then  $\Psi(f)$  is a common eigen-function of  $T_k^2(n; 2)$  for all integers  $n$ , i.e.,  $T_k^2(n; 2)\Psi(f) = \lambda(n)\Psi(f)$ . Furthermore,*

$$\begin{aligned} L(2s - 2k + 4, \chi_0) \sum_{n=1}^{\infty} \lambda(n)n^{-s} \\ = L(s - k + 1, \chi_0)L(s - k + 2, \chi_0) \prod_p (1 - \omega_p p^{-s} + \chi_0^2(p)p^{2k-3-2s})^{-1}, \end{aligned}$$

where  $L(s, \chi_0) = \sum_{n=1}^{\infty} \chi_0(n)n^{-s}$ .

PROOF. By Theorem 1, Theorem A, Lemma 4.1 and [12, Cor. 1.8], we can verify  $T_k^2(2; 2)\Psi(f) = \omega_2\Psi(f)$ ,  $T_k^2(p; 2)\Psi(f) = (\omega_p + p^{k-1} + p^{k-2})\Psi(f)$  and  $T_k^2(p^2; 2)\Psi(f) = (\omega_p^2 + (p^{k-1} + p^{k-2})\omega_p + p^{2k-2})\Psi(f)$  for all odd primes  $p$ . Since the proof of these is routine and long, we omit the details. For every positive integer  $n$ ,  $T_k^2(n; 2)$  belongs to the algebra generated by  $T_k^2(2; 2)$ ,  $T_k^2(p; 2)$  and  $T_k^2(p^2; 2)$  for all odd primes. From those we obtain the desired results (cf. [1] and [7]).

Let  $S$  be the Shimura mapping of  $\mathfrak{S}_{2k-1}(\Gamma_0(4))$  to  $S_{2k-2}(\Gamma_0(2))$  in [12] given by  $S(f)(z) = \sum_{n=1}^{\infty} A(n)e[nz]$ , where  $f(z) = \sum_{n=1}^{\infty} a(n)e[nz]$  and  $\sum_{n=1}^{\infty} A(n)n^{-s} = \sum_{n=1}^{\infty} a(n^2)n^{-s}L(s - k + 2, \left(\frac{-4}{*}\right))$ . By means of the trace formula, Niwa [9] showed that  $S$  is an isomorphic mapping between  $\mathfrak{S}_{2k-1}(\Gamma_0(4))$  and  $S_{2k-2}(\Gamma_0(2))$ . Let  $T_{2k-2}^1(n; 2)$  be the  $n$ -th Hecke operator on  $S_{2k-2}(\Gamma_0(2))$ . As a consequence of Theorem 2, we obtain the following corollary.

COROLLARY. *If  $f \in S_{2k-2}(\Gamma_0(2))$  satisfies  $T_{2k-2}^1(n; 2)f = \omega_n f$  for all  $n$ , then  $\Psi \circ S^{-1}(f)$  is a common eigen-function of  $T_k^2(n; 2)$  for all  $n$ , i.e.,  $T_k^2(n; 2)(\Psi \circ S^{-1}(f)) = \lambda(n)(\Psi \circ S^{-1}(f))$ . Furthermore,*

$$\begin{aligned} L(2s - 2k + 4, \chi_0) \sum_{n=1}^{\infty} \lambda(n)n^{-s} \\ = L(s - k + 1, \chi_0)L(s - k + 2, \chi_0) \prod_p (1 - \omega_p p^{-s} + \chi_0(p)^2 p^{2k-3-2s})^{-1}. \end{aligned}$$

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